J. Math. Sci. Univ. Tokyo
5 (1998), 477–506.

On the Uniqueness Theorem for Nonlinear Singular Partial Differential Equations

By Hidetoshi Tahara

Abstract. The paper proves a uniqueness theorem of the solution of nonlinear singular partial differential equations

$$(t\partial/\partial t)^m u = F\left(t, x, \{(t\partial/\partial t)^j (\partial/\partial x)^\alpha u\}_{j+|\alpha|\le m, j<m}\right).$$

If the characteristic exponents $\lambda_1(x), \dots, \lambda_m(x)$ of the equation satisfy the condition $\mathrm{Re}\,\lambda_i(0) < 0$ for $i = 1, \dots, m$, a uniqueness theorem was proved in Tahara [6]. The present paper discusses the case where $\mathrm{Re}\,\lambda_i(x) \le 0$ holds in a neighborhood of $x = 0$ for $i = 1, \dots, m$. The result is applied to the problem of removable singularities of the solution.

1. Introduction

Notations: $\mathbf{N} = \{0, 1, 2, \dots\}$, $\mathbf{N}^* = \{1, 2, \dots\}$, $m \in \mathbf{N}^*$, $n \in \mathbf{N}^*$, $t \in \mathbf{R}$, $x = (x_1, \dots, x_n) \in \mathbf{C}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $(\partial/\partial x)^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, $\alpha! = \alpha_1! \cdots \alpha_n!$ and

$$I_m = \{(j, \alpha) \in \mathbf{N} \times \mathbf{N}^n \,;\, j + |\alpha| \le m \text{ and } j < m\},$$
$$d(m) = \text{the cardinal of } I_m,$$
$$Z = \{Z_{j,\alpha}\}_{(j,\alpha)\in I_m} \in \mathbf{C}^{d(m)}.$$

Let $T > 0$, $r > 0$, $R > 0$, and let $F(t, x, Z)$ be a function on $\{(t, x, Z) \in \mathbf{R} \times \mathbf{C}^n \times \mathbf{C}^{d(m)} \,;\, 0 \le t \le T, |x| \le r \text{ and } |Z| \le R\}$ which is continuous in t and holomorphic in (x, Z). In this paper we will consider the following nonlinear singular partial differential equation:

$$\text{(E)} \qquad \left(t\frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{(j,\alpha)\in I_m}\right)$$

1991 *Mathematics Subject Classification.* Primary 35A07; Secondary 35A20, 35B40.

with an unknown function $u = u(t, x)$.

For (E) we define *the characteristic exponents* $\lambda_1(x), \dots, \lambda_m(x)$ by the roots of the equation in λ:

$$\lambda^m - \sum_{j<m} \frac{\partial F}{\partial Z_{j,0}}(0, x, 0)\lambda^j = 0.$$

A function $\mu(t)$ on $(0, T)$ is called *a weight function* if it satisfies the following conditions $\mu_1) \sim \mu_3)$:

$\mu_1)$ $\mu(t) \in C^0((0, T))$,

$\mu_2)$ $\mu(t) > 0$ on $(0, T)$ and $\mu(t)$ is increasing in t (in a weak sense),

$\mu_3)$ $\displaystyle\int_0^T \frac{\mu(s)}{s} ds < \infty$.

It follows from $\mu_2)$ and $\mu_3)$ that $\mu(t) \longrightarrow 0$ (as $t \longrightarrow +0$). The following functions are typical examples:

$$\mu(t) = t^a, \quad \frac{1}{(-\log t)^b}, \quad \frac{1}{(-\log t)(\log(-\log t))^c}$$

with $a > 0$, $b > 1$, $c > 1$.

Let us formulate the class of functions $\mathcal{S}_a(\varepsilon, \delta; \mu(t))$ or $\mathcal{S}_+(\varepsilon, \delta; \mu(t))$ in which we want to prove the uniqueness of the solution of (E).

DEFINITION 1. Let $\varepsilon > 0$, $\delta > 0$ and let $\mu(t)$ be a weight function.

(1) For $a > 0$, we denote by $\mathcal{S}_a(\varepsilon, \delta; \mu(t))$ the set of functions $u(t, x)$ satisfying the following conditions (i), (ii) and (iii):

(i) $u(t, x)$ is a function on $\{(t, x) \in \mathbf{R} \times \mathbf{C}^n ; 0 < t < \varepsilon \text{ and } |x| \le \delta\}$,

(ii) $u(t, x)$ is of C^m class in t and holomorphic in x,

(iii) for $j = 0, 1, \dots, m-1$ we have

$$\max_{|x| \le \delta} \left| \left(t\frac{\partial}{\partial t}\right)^j u(t, x) \right| = O(\mu(t)^a) \ \text{(as } t \longrightarrow +0).$$

(2) We define $\mathcal{S}_+(\varepsilon, \delta; \mu(t))$ by

$$\mathcal{S}_+(\varepsilon, \delta; \mu(t)) = \bigcup_{a>0} \mathcal{S}_a(\varepsilon, \delta; \mu(t)).$$

Definition 2. We say that the local uniqueness of the solution of (E) is valid in $\mathcal{S}_a(\varepsilon, \delta; \mu(t))$ if the following condition is satisfied: if $u_1(t,x)$ and $u_2(t,x)$ are solutions of (E) belonging to $\mathcal{S}_a(\varepsilon, \delta; \mu(t))$ we have $u_1(t,x) = u_2(t,x)$ on $\{(t,x) \in \mathbf{R} \times \mathbf{C}^n\,;\, 0 < t < \varepsilon_1 \text{ and } |x| \le \delta_1\}$ for some $\varepsilon_1 > 0$ and $\delta_1 > 0$.

Then, about the local uniqueness of the solution of (E) we already know the following results. Assume that $F(t,x,Z)$ is a function on $\{(t,x,Z) \in \mathbf{R} \times \mathbf{C}^n \times \mathbf{C}^{d(m)}\,;\, 0 \le t \le T, |x| \le r \text{ and } |Z| \le R\}$ and assume:

(C_1) $F(t,x,Z)$ is continuous in t and holomorphic in (x,Z);

(C_2) $\max_{|x|\le r} |F(t,x,0)| = O(\mu(t)^m)$ (as $t \longrightarrow +0$);

(C_3) $\max_{|x|\le r} \left|\dfrac{\partial F}{\partial Z_{j,\alpha}}(t,x,0)\right| = O(\mu(t)^{|\alpha|})$ (as $t \longrightarrow +0$) for any $(j,\alpha) \in I_m$.

Theorem 1. *Assume* (C_1), (C_2) *and* (C_3). *Then:*

(1) (Gérard-Tahara [2]) *In case* $\mu(t) = O(t^c)$ (*as* $t \longrightarrow +0$) *for some* $c > 0$, *if*

$$\operatorname{Re}\lambda_i(0) \le 0 \quad \textit{for } i = 1, \ldots, m \tag{1.1}$$

the local uniqueness of the solution of (E) *is valid in* $\mathcal{S}_+(\varepsilon, \delta; \mu(t))$.

(2) (Tahara [6]) *In case* $\mu(t)$ *is a general weight function, if*

$$\operatorname{Re}\lambda_i(0) < 0 \quad \textit{for } i = 1, \ldots, m \tag{1.2}$$

the local uniqueness of the solution of (E) *is valid in* $\mathcal{S}_+(\varepsilon, \delta; \mu(t))$.

Remark 1. In [6] we have assumed that the weight function $\mu(t)$ satisfies $\mu_1)$, $\mu_2)$, $\mu_3)$ and

$$\mu_4) \qquad \mu(t) \in C^1((0,T)) \quad \text{and} \quad t\frac{d\mu}{dt}(t) = O(\mu(t)) \ \ (\text{as } t \longrightarrow +0).$$

But, by the argument in this paper we can prove the result (2) in Theorem 1 without using the condition $\mu_4)$.

In this paper we want to study the following case: $\mu(t)$ is a general weight function and the characteristic exponents satisfy

$$\operatorname{Re}\lambda_i(x) \le 0 \quad \text{for } \ i = 1, \dots, m \tag{1.3}$$

in a neighborhood of $x = 0$.

The motivation comes from the following example:

Example 1. Let us consider

$$\Big(t\frac{\partial}{\partial t}\Big)^2 u = 6u\Big(\frac{\partial u}{\partial x}\Big) \tag{1.4}$$

where $(t,x) \in \mathbf{C}^2$. Then the characteristic exponents are $\lambda_1 = 0$ and $\lambda_2 = 0$. In this case we have:

1) $u(t,x) \equiv 0$ is the unique holomorphic solution of (1.4) satisfying $u(0,x) \equiv 0$.

2) (1.4) has a family of non-trivial solutions

$$u(t,x) = \frac{x+\alpha}{(C-\log t)^2} \qquad (\alpha, C \in \mathbf{C}).$$

This implies that the local uniqueness of the solution of (1.4) is not valid in $\mathcal{S}_+(\varepsilon,\delta;\mu(t))$ with $\mu(t) = 1/(-\log t)^c$ for any $c > 1$. Compare this with the result (2) of Theorem 1.

3) More precisely, if $0 < a < 2$ the local uniqueness is not valid in $\mathcal{S}_a(\varepsilon,\delta;\mu(t))$ for $\mu(t) = 1/(-\log t)^c$ with $1 < c \le 2/a$.

4) Nevertherless, the local uniqueness is valid in $\mathcal{S}_2(\varepsilon,\delta;\mu(t))$ for any weight function $\mu(t)$.

We want to generalize the result 4) in Example 1 to the general case.

The paper is organized as follows. In the next section 2 we state our main results (Theorem 2 and Theorem 3). In sections 3∼5 we prove our results: in section 3 we present some preparatory discussions, in section 4 we prove Theorem 2 and in section 5 we prove Theorem 3. In the last section 6 we give an application of our result to the problem of removable singularities of the solution of (E).

2. Main results

The main results of this paper deal with the following case: for some p with $0 \leq p \leq m$ the characteristic exponents of (E) satisfy

$$\begin{cases} \operatorname{Re}\lambda_i(x) \leq 0 & \text{for } i = 1, \dots, p, \\ \operatorname{Re}\lambda_i(0) < 0 & \text{for } i = p+1, \dots, m \end{cases} \tag{2.1}$$

in a neighborhood of $x = 0 \in \mathbf{C}^n$.

Let p be as in (2.1) and let $\mu(t)$ be a weight function. Assume:

$(\mathrm{C}_3)_p$ The following 1) and 2) are valid:

1) for $j = 0, 1, \dots, m-1$ we have

$$\max_{|x| \leq r} \left| \frac{\partial F}{\partial Z_{j,0}}(t, x, 0) - \frac{\partial F}{\partial Z_{j,0}}(0, x, 0) \right| = O(\mu(t)^p) \ \ (\text{as } t \longrightarrow +0),$$

2) for $(j, \alpha) \in I_m$ with $|\alpha| > 0$ we have

$$\max_{|x| \leq r} \left| \frac{\partial F}{\partial Z_{j,\alpha}}(t, x, 0) \right| = O(\mu(t)^{\max\{p, |\alpha|\}}) \ \ (\text{as } t \longrightarrow +0).$$

Note that $(\mathrm{C}_3)_p$ with $p = 0$ is nothing but (C_3).

We have

Theorem 2. *Let p be an integer with $0 \leq p \leq m$ and let $\mu(t)$ be a weight function. Assume* (2.1), (C_1) *and* $(\mathrm{C}_3)_p$. *Then the local uniqueness of the solution of* (E) *is valid in* $\mathcal{S}_m(\varepsilon, \delta; \mu(t))$.

This proves the result 4) in Example 1.

In case $0 \le p \le m-1$ we can say more. Impose:

$(C_2)_p$ $\quad \max_{|x|\le r} |F(t,x,0)| = O(\mu(t)^{m+p})$ (as $t \longrightarrow +0$);

$(C_4)_p$ There is an $s>0$ such that for any $(j,\alpha) \in I_m$ with $|\alpha|>0$ we have

$$\max_{|x|\le r} \left| \frac{\partial F}{\partial Z_{j,\alpha}}(t,x,0) \right| = O(\mu(t)^{p+s}) \ \text{(as } t \longrightarrow +0).$$

THEOREM 3. *Let p be an integer with $0 \le p \le m-1$ and let $\mu(t)$ be a weight function. Assume* (2.1), (C_1), $(C_2)_p$, $(C_3)_p$ *and* $(C_4)_p$. *Then, if $a>p$ the local uniqueness of the solution of* (E) *is valid in* $\mathcal{S}_a(\varepsilon,\delta;\mu(t))$.

Since $(C_2)_0$ is nothing but (C_2), $(C_3)_0$ is nothing but (C_3) and $(C_4)_0$ follows from $(C_3)_0$, the case $p=0$ is already proved in Tahara [6]. Hence, in the proof of Theorems 2 and 3 we may assume $p \ge 1$.

Example 2. Let us consider

$$\Big(t\frac{\partial}{\partial t}\Big)^2 u + \Big(t\frac{\partial}{\partial t}\Big)u = (2u+x+1)\Big(\frac{\partial u}{\partial x}\Big)^2 \tag{2.2}$$

where $(t,x) \in \mathbf{C}^2$. Then the characteristic exponents are $\lambda_1 = 0$ and $\lambda_2 = -1$. In this case we have:

1) $u(t,x) \equiv 0$ is the unique holomorphic solution of (2.2) satisfying $u(0,x) \equiv 0$.

2) By Theorem 3 we see that if $a>1$ the local uniqueness of the solution of (2.2) is valid in $\mathcal{S}_a(\varepsilon,\delta;\mu(t))$ for any weight function $\mu(t)$.

3) Note that (2.2) has a family of non-trivial solutions

$$u(t,x) = \frac{x+1}{(C-\log t)} \qquad (C \in \mathbf{C}).$$

This implies that if $0<a<1$ the local uniqueness is not valid in $\mathcal{S}_a(\varepsilon,\delta;\mu(t))$ for $\mu(t) = 1/(-\log t)^c$ with $1<c\le 1/a$.

We note the following:

(1) Since $(\partial F/\partial Z_{j,\alpha})(0,x,0)$ is holomorphic on $\{x \in \mathbf{C}^n\,;\,|x| \le r\}$, it is easy to see that we can find an $x^0 \in \mathbf{C}^n$ sufficiently close to the origin such that all the characteristic exponents $\lambda_1(x),\ldots,\lambda_m(x)$ are holomorphic in a neighborhood of $x^0 \in \mathbf{C}^n$.

(2) Let $u_1(t,x)$ and $u_2(t,x)$ be solutions of (E) belonging to the class $\mathcal{S}_a(\varepsilon,\delta;\mu(t))$. If we prove $u_1(t,x) = u_2(t,x)$ on $\{(t,x) \in \mathbf{R}\times\mathbf{C}^n\,;\,0 < t < \varepsilon_1 \text{ and } |x - x^0| < \delta_1\}$ for some $\varepsilon_1 > 0$ and $\delta_1 > 0$, then by the analyticity in the x-variable we get the conclusion that $u_1(t,x) = u_2(t,x)$ on $\{(t,x) \in \mathbf{R}\times\mathbf{C}^n\,;\,0 < t < \varepsilon_1 \text{ and } |x| < \delta\}$.

Thus, in the proof of Theorems 2 and 3 we may assume the following condition:

(C_5) All the the characteristic exponents $\lambda_1(x),\ldots,\lambda_m(x)$ are holomorphic in a neighborhood of $x = 0 \in \mathbf{C}^n$.

Moreover we know that if a holomorphic function $\lambda(x)$ in a neighborhood D of $x = 0$ satisfies $\operatorname{Re}\lambda(0) = 0$ and $\operatorname{Re}\lambda(x) \le 0$ on D then we have $\lambda(x) = \sqrt{-1}\,\mu$ on D for some $\mu \in \mathbf{R}$.

Therefore, under (2.1) and (C_5) we may assume without loss of generality that in a neighborhood of $x = 0$ we have

$$\begin{cases} \lambda_i(x) = \sqrt{-1}\,\mu_i & \text{for } i = 1,\ldots,p, \\ \operatorname{Re}\lambda_i(0) < 0 & \text{for } i = p+1,\ldots,m \end{cases} \tag{2.3}$$

for some $\mu_i \in \mathbf{R}$ $(i = 1,\ldots,p)$.

Put

$$\begin{aligned} &\Theta_0 = 1, \\ &\Theta_1 = \Big(t\frac{\partial}{\partial t} - \lambda_1(0)\Big), \\ &\Theta_2 = \Big(t\frac{\partial}{\partial t} - \lambda_2(0)\Big)\Big(t\frac{\partial}{\partial t} - \lambda_1(0)\Big), \\ &\qquad\cdots\cdots\cdots \\ &\qquad\cdots\cdots\cdots \\ &\qquad\cdots\cdots\cdots \\ &\Theta_m = \Big(t\frac{\partial}{\partial t} - \lambda_m(0)\Big)\cdots\Big(t\frac{\partial}{\partial t} - \lambda_2(0)\Big)\Big(t\frac{\partial}{\partial t} - \lambda_1(0)\Big). \end{aligned} \tag{2.4}$$

The following fact will play an important role in the proof of Theorem 2: under (2.3) we have

$$\left(t\frac{\partial}{\partial t}\right)^m - \sum_{j<m} \frac{\partial F}{\partial Z_{j,0}}(0,x,0)\left(t\frac{\partial}{\partial t}\right)^j = \Theta_m - \sum_{p\le j\le m-1} a_j(x)\Theta_j \tag{2.5}$$

for some $a_j(x)$ $(p \le j \le m-1)$ holomorphic in a neighborhood of $x = 0$ and satisfying $a_j(0) = 0$ $(p \le j \le m-1)$.

3. Some discussions

Before the proof of Theorems 2 and 3 let us present some preparatory lemmas.

First, for a convergent power series

$$f(t,x) = \sum_{\alpha\in\mathbf{N}^n} f_\alpha(t)\, x^\alpha$$

with coefficients in $C^0((0,T))$ we define the norm $\|f(t)\|_\rho$ by

$$\|f(t)\|_\rho = \sum_{\alpha\in\mathbf{N}^n} |f_\alpha(t)|\,\frac{\alpha!}{|\alpha|!}\,\rho^{|\alpha|}$$

(which is a convergent power series in ρ with coefficients in $C^0((0,T))$). We write $\sum_k a_k\rho^k \ll \sum_k b_k\rho^k$ if $|a_k| \le b_k$ holds for all $k \in \mathbf{N}$.

LEMMA 1. *For $f(t,x)$ and $g(t,x)$ we have:*
(1) $\|(fg)(t)\|_\rho \ll \|f(t)\|_\rho\,\|g(t)\|_\rho$.
(2) $\left\|\dfrac{\partial f}{\partial x_i}(t)\right\|_\rho \ll \dfrac{\partial}{\partial \rho}\|f(t)\|_\rho$ *for* $i = 1,\dots,n$.

Next, for $(j,k) \in \mathbf{N}^2$ with $j + k \le m-1$ we put

$$c(j,k) = \max\{j+k-p+1, 0\},$$
$$q(j,k) = \min\{m+j-p, m-k-1\},$$

where m and p are the ones in (2.1).

LEMMA 2. *Let m and p be integers with $0 \le p \le m$. We have:*

(1) $c(j,k)$ $(j+k \le m-1)$ *satisfy* $0 \le c(j,k) \le m-p$, $c(j+1,k-1) = c(j,k)$ *and*

$$c(j,0) = \begin{cases} 0 & \text{when } j \le p-1, \\ j-p+1 & \text{when } j \ge p-1. \end{cases}$$

(2) $q(j,k)$ $(j+k \le m-1)$ *satisfy* $0 \le q(j,k) \le m-1$, $q(j+1,k-1) = q(j,k)+1$ *and*

$$q(j,0) = \begin{cases} m+j-p & \text{when } j \le p-1, \\ m-1 & \text{when } j \ge p-1. \end{cases}$$

Let $\varepsilon > 0$ and $\mu(t)$ be a weight function on $(0,T)$. Define

$$\sigma_{j,k}(t) = \varepsilon^{c(j,k)} \mu(t)^{q(j,k)}. \tag{3.1}$$

The following result is a consequence of Lemma 2.

LEMMA 3. $\sigma_{j,k}(t)$ $(j+k \le m-1)$ *satisfy the following conditions:*

(1) $\sigma_{j,k}(t) > 0$ *on* $(0,T)$ *and* $(1/\sigma_{j,k}(t)) = O(\mu(t)^{-(m-1)})$ *(as* $t \longrightarrow +0$*).*

(2) $\sigma_{j,k}(t)$ *is increasing in* t *(in a weak sense).*

(3) *For* $j = 0, 1, \dots, m-2$ *we have*

$$\frac{\sigma_{j+1,0}(t)}{\sigma_{j,0}(t)} = \begin{cases} \mu(t) & \text{when } j+1 \le p-1, \\ \varepsilon & \text{when } j+1 \ge p. \end{cases}$$

(4) *For* $j \ge p-1$ *we have*

$$\frac{\sigma_{j,0}(t)}{\sigma_{m-1,0}(t)} = \frac{1}{\varepsilon^{m-j-1}}.$$

(5) *For* $j+k \le m-1$ *and* $k > 0$ *we have*

$$\frac{\sigma_{j+1,k-1}(t)}{\sigma_{j,k}(t)} = \mu(t).$$

(6) *For* $j+k \le m-1$ *we have*

$$\frac{\sigma_{j,k}(t)}{\sigma_{m-1,0}(t)} = \begin{cases} \dfrac{1}{\varepsilon^{m-p}\mu(t)^{p-j-1}} & \text{when } j+k \le p-1, \\ \dfrac{1}{\varepsilon^{m-j-k-1}\mu(t)^{k}} & \text{when } j+k \ge p-1. \end{cases}$$

Lastly, let us recall some results in the elementary calculus. For a real-valued function $\varphi(t) \in C^0((0,T))$ we define

$$\overline{D_t^+}\varphi(t) = \varlimsup_{h \to +0} \frac{\varphi(t+h)-\varphi(t)}{h},$$
$$\underline{D_t^+}\varphi(t) = \varliminf_{h \to +0} \frac{\varphi(t+h)-\varphi(t)}{h}.$$

It is clear that $\overline{D_t^+}\varphi(t) \ge \underline{D_t^+}\varphi(t)$ holds on $(0,T)$. Moreover we have:

Lemma 4. (1) *If* $\varphi(t) \in C^1((0,T))$ *we have*

$$\overline{D_t^+}\varphi(t) = \underline{D_t^+}\varphi(t) = \frac{d\varphi}{dt}(t).$$

(2) *If* $\varphi(t)$ *is decreasing in* t *(in a weak sense), we have* $\overline{D_t^+}\varphi(t) \le 0$ *on* $(0,T)$.

(3) *For* $f(t), g(t) \in C^0((0,T))$ *we have*

$$\overline{D_t^+}(fg)(t) \le (\overline{D_t^+}f(t))g(t) + f(t)(\overline{D_t^+}g(t)).$$

(4) *If* $\Phi(t,\rho) \in C^0((0,T) \times [0,\rho_0])$ *has a partial derivative in* ρ *with* $(\partial\Phi/\partial\rho)(t,\rho) \in C^0((0,T) \times [0,\rho_0])$ *and if* $\rho(t) \in C^1((0,T))$ *satisfies the condition* $\rho((0,T)) \subset [0,\rho_0]$, *for the composite function* $\varphi(t) = \Phi(t,\rho(t))$ *we have*

$$\overline{D_t^+}\varphi(t) = (\overline{D_t^+}\Phi)(t,\rho(t)) + \frac{\partial\Phi}{\partial\rho}(t,\rho(t))\frac{d\rho(t)}{dt}.$$

(5) *If* $\varphi(t) \in C^0((0,T))$ *satisfies* $\underline{D_t^+}\varphi(t) \le 0$ *on* $(0,T)$, *we have* $\varphi(a) \ge \varphi(b)$ *for any* $0 < a < b < T$.

For details, see Hukuhara [4]. For the convenience of readers, I will give a proof of (5).

Proof of (5) in Lemma 4. Assume that $\underline{D}_t^+\varphi(t) \le 0$ on $(0,T)$. If $\varphi(a) < \varphi(b)$ holds for some $0 < a < b < T$, we can derive a contradiction in the following way.

Choose ξ so that $\varphi(a) < \xi < \varphi(b)$ and set

$$\psi(t) = \varphi(t) - \varphi(a) - \frac{\xi - \varphi(a)}{b-a}(t-a),$$
$$\alpha = \inf\{c \in (0,b)\,;\, \psi(t) > 0 \text{ on } (c,b]\}.$$

Since $\psi(a) = 0$, $\psi(b) > 0$ hold, we have $a \le \alpha < b$, $\psi(\alpha) = 0$ and $\psi(t) > 0$ on $(\alpha, b]$. Hence, it is easy to see that

$$0 \le \overline{D}_t^+\psi(\alpha) = \overline{D}_t^+\varphi(\alpha) - \frac{\xi - \varphi(a)}{b-a},$$

that is

$$\overline{D}_t^+\varphi(\alpha) \ge \frac{\xi - \varphi(a)}{b-a} > 0$$

which contradicts the condition $\overline{D}_t^+\varphi(\alpha) \le 0$. □

4. Proof of Theorem 2

Let p be an integer with $1 \le p \le m$ and let $\mu(t)$ be a weight function on $(0,T)$. Assume (2.1), (C_1), $(\mathrm{C}_3)_p$ and (C_5). Without loss of generality we may assume that in a neighborhood of $x = 0$ we have

$$\text{(4.1)} \qquad \begin{cases} \lambda_i(x) = \sqrt{-1}\,\mu_i & \text{for } i = 1, \dots, p, \\ \operatorname{Re}\lambda_i(0) < -h & \text{for } i = p+1, \dots, m \end{cases}$$

for some $\mu_i \in \mathbf{R}$ $(i = 1, \dots, p)$ and some $h > 0$. If we write

$$\text{(4.2)} \qquad h_i = \begin{cases} 0 & \text{for } i = 1, \dots, p, \\ h & \text{for } i = p+1, \dots, m, \end{cases}$$

by (4.1) we have

$$\text{(4.3)} \qquad \operatorname{Re}\lambda_i(0) \le -h_i \quad \text{for } i = 1, \dots, m.$$

Let $u_1(t,x)$ and $u_2(t,x)$ be two solutions of (E) belonging to the class $\mathcal{S}_m(\varepsilon,\delta;\mu(t))$. Put

$$w(t,x)=u_2(t,x)-u_1(t,x). \tag{4.4}$$

We have $w(t,x)\in\mathcal{S}_m(\varepsilon,\delta;\mu(t))$ and by Cauchy's inequalities we see

$$\max_{|x|\le\delta_1}\Big|\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u_1(t,x)\Big|=O(\mu(t)^m)\ \ (\text{as } t\longrightarrow +0), \tag{4.5}$$

$$\max_{|x|\le\delta_1}\Big|\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha w(t,x)\Big|=O(\mu(t)^m)\ \ (\text{as } t\longrightarrow +0) \tag{4.6}$$

for any $(j,\alpha)\in I_m$ and $0<\delta_1<\delta$. Moreover, it is easy to see that $w(t,x)$ satisfies the following equation:

$$\begin{aligned}\Big(t\frac{\partial}{\partial t}\Big)^m w&=F\Big(t,x,\Big\{\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u_1+\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha w\Big\}_{(j,\alpha)\in I_m}\Big)\\ &\qquad -F\Big(t,x,\Big\{\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u_1\Big\}_{(j,\alpha)\in I_m}\Big)\\ &=\sum_{(j,\alpha)\in I_m}a_{j,\alpha}(t,x)\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha w\end{aligned}$$

where

$$\begin{aligned}&a_{j,\alpha}(t,x)\\ &=\int_0^1\frac{\partial F}{\partial Z_{j,\alpha}}\Big(t,x,\Big\{\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u_1+\theta\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha w\Big\}_{(j,\alpha)\in I_m}\Big)d\theta\\ &=\frac{\partial F}{\partial Z_{j,\alpha}}(t,x,0)+O(\mu(t)^m)\quad(\text{as } t\longrightarrow +0)\end{aligned}$$

(by (4.5) and (4.6)). Hence, by using the condition $(\mathrm{C}_3)_p$ we have

$$\begin{aligned}&\Big(t\frac{\partial}{\partial t}\Big)^m w-\sum_{j<m}\frac{\partial F}{\partial Z_{j,0}}(0,x,0)\Big(t\frac{\partial}{\partial t}\Big)^j w\\ &\quad=\sum_{j<m}\Big(\Big(\frac{\partial F}{\partial Z_{j,0}}(t,x,0)-\frac{\partial F}{\partial Z_{j,0}}(0,x,0)\Big)+O(\mu(t)^m)\Big)\Big(t\frac{\partial}{\partial t}\Big)^j w\end{aligned}$$

$$+\sum_{\substack{(j,\alpha)\in Im\\ |\alpha|>0}}\Big(\frac{\partial F}{\partial Z_{j,\alpha}}(t,x,0)+O(\mu(t)^m)\Big)\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha w$$

$$=\sum_{(j,\alpha)\in Im}\Big(O(\mu(t)^{\max\{p,|\alpha|\}})\Big)\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha w,$$

and by combining this with (2.5) we obtain

$$\Theta_m w=\sum_{p\le j\le m-1}a_j(x)\Theta_j w + \sum_{(j,\alpha)\in Im}O(\mu(t)^{\max\{p,|\alpha|\}})\,\Theta_j\Big(\frac{\partial}{\partial x}\Big)^\alpha w. \tag{4.7}$$

Recall that $a_j(0)=0$ holds and therefore $\|a_j\|_\rho=O(\rho)$ (as $\rho\longrightarrow+0$).

From now, let us show Theorem 2 by proving that $w(t,x)=0$ holds on $\{(t,x)\,;\,0<t<\varepsilon_1$ and $|x|\le\delta_1\}$ for some $\varepsilon_1>0$ and $\delta_1>0$.

First, for $(j,k)\in\mathbf{N}^2$ with $j+k\le m-1$ we define $\phi_{j,k}(t,\rho)$ by

$$\phi_{j,k}(t,\rho)=\sum_{|\alpha|=k}\int_0^t\Big(\frac{\tau}{t}\Big)^{h_{j+1}}\Big\|\Theta_{j+1}\Big(\frac{\partial}{\partial x}\Big)^\alpha w(\tau)\Big\|_\rho\frac{d\tau}{\tau} \tag{4.8}$$

where h_{j+1} is the constant in (4.2) and Θ_{j+1} is the operator in (2.4). By (4.6) and (4.7) we easily see

$$\Big\|\Theta_{j+1}\Big(\frac{\partial}{\partial x}\Big)^\alpha w(t)\Big\|_\rho=O(\mu(t)^m)\ \ (\text{as } t\longrightarrow+0) \tag{4.9}$$

for any $j=0,1,\dots,m-1$.

Lemma 5. *$\phi_{j,k}(t,\rho)$ $(j+k\le m-1)$ are well-defined and satisfy the following conditions* (1)$\sim$(5) *on $\{(t,\rho)\,;\,0\le t\le T_0$ and $0\le\rho\le\rho_0\}$ for some $T_0>0$ and $\rho_0>0$.*

(1) *$\phi_{j,k}(t,\rho)$ is of C^1 class in $t\in(0,T_0]$ and analytic in $\rho\in[0,\rho_0]$; moreover we have*

$$\phi_{j,k}(t,\rho)=\begin{cases}O(\mu(t)^{m-1})\times\displaystyle\int_0^t\frac{\mu(\tau)}{\tau}\,d\tau, & \textit{when } j\le p-1,\\ O(\mu(t)^m), & \textit{when } j\ge p\end{cases} \tag{4.10}$$

(as $t \longrightarrow +0$) uniformly for $0 \le \rho \le \rho_0$.

(2) *For any (j,k) we have*

$$\sum_{|\alpha|=k} \Big\| \Theta_j \Big(\frac{\partial}{\partial x}\Big)^{\alpha} w(t) \Big\|_{\rho} \ll \phi_{j,k}(t,\rho).$$

(3) *When $k>0$, we have*

$$\Big(t\frac{\partial}{\partial t} + h_{j+1}\Big)\phi_{j,k}(t,\rho) \le n \frac{\partial}{\partial \rho}\phi_{j+1,k-1}(t,\rho).$$

(4) *When $k=0$ and $j=0,1,\cdots,m-2$, we have*

$$\Big(t\frac{\partial}{\partial t} + h_{j+1}\Big)\phi_{j,0}(t,\rho) \le \phi_{j+1,0}(t,\rho).$$

(5) *When $k=0$ and $j=m-1$ there are constants $C_1>0$ and $C_2>0$ such that*

$$\begin{aligned}&\Big(t\frac{\partial}{\partial t} + h_m\Big)\phi_{m-1,0}(t,\rho)\\ &\le C_1\,\rho \sum_{j\ge p} \phi_{j,0}(t,\rho) + C_2 \sum_{j+k\le m-1} \mu(t)^{\max\{p,k+1\}} \Big(1+\frac{\partial}{\partial \rho}\Big)\phi_{j,k}(t,\rho).\end{aligned}$$

Proof. The former half of (1) is clear from the definition of $\phi_{j,k}(t,\rho)$. When $h_{j+1}=0$ we have

$$\int_0^t \Big(\frac{\tau}{t}\Big)^{h_{j+1}} \mu(\tau)^m \frac{d\tau}{\tau} = \int_0^t \mu(\tau)^m \frac{d\tau}{\tau} \le \mu(t)^{m-1} \int_0^t \frac{\mu(\tau)}{\tau} d\tau;$$

when $h_{j+1}=h>0$ we have

$$\begin{aligned}\int_0^t \Big(\frac{\tau}{t}\Big)^{h_{j+1}} \mu(\tau)^m \frac{d\tau}{\tau} &= \int_0^t \Big(\frac{\tau}{t}\Big)^{h} \mu(\tau)^m \frac{d\tau}{\tau}\\ &\le \mu(t)^m \int_0^t \Big(\frac{\tau}{t}\Big)^{h} \frac{d\tau}{\tau} = \frac{\mu(t)^m}{h}.\end{aligned}$$

Combining this with (4.2) and (4.9) we can get the latter half of (1).

Note the following fact: if $\operatorname{Re}\lambda \le 0$ and $v(t) \in C^1((0,T_0])$ satisfy $v(t) = o(1)$ (as $t \longrightarrow +0$) and $(t\partial/\partial t - \lambda)v = O(\mu(t))$ (as $t \longrightarrow +0$), by solving the equation $(t\partial/\partial t - \lambda)v = g$ with $g = (t\partial/\partial t - \lambda)v$ we have

$$\begin{aligned} v(t) &= \int_0^t \Big(\frac{\tau}{t}\Big)^{-\lambda} g(\tau)\frac{d\tau}{\tau} \\ &= \int_0^t \Big(\frac{\tau}{t}\Big)^{-\lambda} \Big(\Big(\tau\frac{\partial}{\partial\tau} - \lambda\Big)v\Big)\frac{d\tau}{\tau}. \end{aligned}$$

By applying this to $\Theta_j(\partial/\partial x)^\alpha w$ we see that

$$\Theta_j\Big(\frac{\partial}{\partial x}\Big)^\alpha w(t,x) = \int_0^t \Big(\frac{\tau}{t}\Big)^{-\lambda_{j+1}(0)} \Theta_{j+1}\Big(\frac{\partial}{\partial x}\Big)^\alpha w(\tau,x)\,\frac{d\tau}{\tau}$$

and hence by using (4.3) we have

$$\Big\|\Theta_j\Big(\frac{\partial}{\partial x}\Big)^\alpha w(t)\Big\|_\rho \ll \int_0^t \Big(\frac{\tau}{t}\Big)^{h_{j+1}} \Big\|\Theta_{j+1}\Big(\frac{\partial}{\partial x}\Big)^\alpha w(\tau)\Big\|_\rho\,\frac{d\tau}{\tau}$$

which implies the result (2).

When $k > 0$ we have

$$\begin{aligned} \Big(t\frac{\partial}{\partial t} + h_{j+1}\Big)\phi_{j,k}(t,\rho) &= \sum_{|\alpha|=k} \Big\|\Theta_{j+1}\Big(\frac{\partial}{\partial x}\Big)^\alpha w(t)\Big\|_\rho \\ &\ll \sum_{|\beta|=k-1}\sum_{i=1}^n \Big\|\Theta_{j+1}\Big(\frac{\partial}{\partial x_i}\Big)\Big(\frac{\partial}{\partial x}\Big)^\beta w(t)\Big\|_\rho \\ &\ll n\frac{\partial}{\partial\rho}\sum_{|\beta|=k-1} \Big\|\Theta_{j+1}\Big(\frac{\partial}{\partial x}\Big)^\beta w(t)\Big\|_\rho. \end{aligned}$$

Hence, by using (2) we obtain the result (3).

When $k = 0$ and $j = 0, 1, \cdots, m-2$, we have

$$\Big(t\frac{\partial}{\partial t} + h_{j+1}\Big)\phi_{j,0} = \Big\|\Theta_{j+1}w(t)\Big\|_\rho \le \phi_{j+1,0}$$

which implies the result (4).

When $k=0$ and $j=m-1$, by (4.7) we have

$$\begin{aligned}
\Big(t\frac{\partial}{\partial t}+h_m\Big)\phi_{m-1,0} &= \|\Theta_m w(t)\|_\rho \\
&\ll \sum_{p\le j\le m-1} \|a_j\|_\rho \,\|\Theta_j w(t)\|_\rho \\
&\qquad\qquad + \sum_{(j,\alpha)\in Im} O(\mu(t)^{\max\{p,|\alpha|\}}) \left\|\Theta_j \Big(\frac{\partial}{\partial x}\Big)^\alpha w(t)\right\|_\rho \\
&= \sum_{p\le j\le m-1} O(\rho)\,\|\Theta_j w(t)\|_\rho + \sum_{j<m} O(\mu(t)^p)\,\|\Theta_j w(t)\|_\rho \\
&\quad + \sum_{j+|\beta|\le m-1}\sum_{i=1}^{n} O(\mu(t)^{\max\{p,|\beta|+1\}}) \left\|\Theta_j\Big(\frac{\partial}{\partial x_i}\Big)\Big(\frac{\partial}{\partial x}\Big)^\beta w(t)\right\|_\rho \\
&\ll \sum_{p\le j\le m-1} O(\rho)\,\phi_{j,0} + \sum_{j<m} O(\mu(t)^p)\,\phi_{j,0} \\
&\qquad + \sum_{j+k\le m-1} O(\mu(t)^{\max\{p,k+1\}})\, n\,\frac{\partial}{\partial\rho}\phi_{j,k}
\end{aligned}$$

which implies the result (5). □

Next, let $h>0$ be the one in (4.2) and $C_1>0$ be the constant in (5) of Lemma 5. Choose $\varepsilon>0$ so that

$$\varepsilon < \frac{h}{2} \tag{4.11}$$

and then choose $\rho_1>0$ so that

$$\frac{C_1\rho_1}{\varepsilon^{m-j-1}} < \frac{h}{2} \quad \text{for} \quad p\le j\le m-1. \tag{4.12}$$

By using this ε we define $\sigma_{j,k}(t)$ $(j+k\le m-1)$ by (3.1) and put

$$\Phi(t,\rho) = \sum_{j+k\le m-1} \frac{1}{\sigma_{j,k}(t)}\,\phi_{j,k}(t,\rho). \tag{4.13}$$

Then we have

Lemma 6. (1) $\Phi(t,\rho) = o(1)$ (*as* $t \longrightarrow +0$) *uniformly for* $0 \le \rho \le \rho_1$.
(2) *There is a constant* $C > 0$ *such that*

$$t\,\overline{D}_t^+ \Phi(t,\rho) \le C\mu(t)\Big(1 + \frac{\partial}{\partial\rho}\Big)\Phi(t,\rho) \tag{4.14}$$

on $\{(t,\rho)\,;\, 0 < t \le T_0$ *and* $0 \le \rho \le \rho_1\}$.

Proof. By (4.10) we know

$$\phi_{j,k}(t,\rho) = o(\mu(t)^{m-1}) \quad (\text{as } t \longrightarrow +0)$$

and therefore by using the condition (1) of Lemma 3 we obtain the result (1).

Since $(1/\sigma_{j,k}(t))$ is decreasing in t we have $\overline{D}_t^+(1/\sigma_{j,k}(t)) \le 0$ on $(0,T)$ and therefore from (4.13) we have

$$\begin{aligned} t\,\overline{D}_t^+ \Phi(t,\rho) &\le \sum_{j+k\le m-1} \Big(t\,\overline{D}_t^+\Big(\frac{1}{\sigma_{j,k}}\Big)\phi_{j,k} + \frac{1}{\sigma_{j,k}}\,t\frac{\partial}{\partial t}\phi_{j,k}\Big) \\ &\le \sum_{j+k\le m-1} \frac{1}{\sigma_{j,k}}\,t\frac{\partial}{\partial t}\phi_{j,k}. \end{aligned} \tag{4.15}$$

By using (4.2) and Lemma 5 we get from (4.15) that

$$\begin{aligned} t\,\overline{D}_t^+ \Phi &\le \sum_{j+k\le m-1} \frac{1}{\sigma_{j,k}}\Big(t\frac{\partial}{\partial t} + h_{j+1}\Big)\phi_{j,k} - h\sum_{j\ge p}\frac{1}{\sigma_{j,0}}\,\phi_{j,0} \\ &\le \sum_{\substack{j+k\le m-1\\ k>0}} \frac{1}{\sigma_{j,k}}\,n\,\frac{\partial}{\partial\rho}\,\phi_{j+1,k-1} + \sum_{j\le m-2}\frac{1}{\sigma_{j,0}}\,\phi_{j+1,0} \\ &\quad + \frac{1}{\sigma_{m-1,0}}\Big(C_1\,\rho\sum_{j\ge p}\phi_{j,0} \\ &\qquad + C_2\sum_{j+k\le m-1}\mu(t)^{\max\{p,k+1\}}\Big(1+\frac{\partial}{\partial\rho}\Big)\phi_{j,k}\Big) \\ &\quad - h\sum_{j\ge p}\frac{1}{\sigma_{j,0}}\,\phi_{j,0}. \end{aligned} \tag{4.16}$$

Recall that the conditions (3) $\sim$ (6) of Lemma 3 imply the following:

1) if $k>0$, $(1/\sigma_{j,k}) = (\mu(t)/\sigma_{j+1,k-1})$;
2) if $j+1 \leq p-1$, $(1/\sigma_{j,0}) = (\mu(t)/\sigma_{j+1,0})$;
3) if $j+1 \geq p$, $(1/\sigma_{j,0}) = (\varepsilon/\sigma_{j+1,0})$;
4) if $j \geq p-1$, $(1/\sigma_{m-1,0}) = (1/\varepsilon^{m-j-1}\sigma_{j,0})$;
5) $(\mu(t)^{\max\{p,k+1\}}/\sigma_{m-1,0}) = O(\mu(t)/\sigma_{j,k})$ (as $t \longrightarrow +0$).

Hence, applying these to (4.16) we obtain

$$
\begin{aligned}
t\,\overline{D}_t^+\Phi \leq & \sum_{\substack{j+k\leq m-1\\ k>0}} \mu(t)\, n\, \frac{\partial}{\partial\rho}\Big(\frac{1}{\sigma_{j+1,k-1}}\,\phi_{j+1,k-1}\Big) \\
& + C_2 \sum_{j+k\leq m-1} O(\mu(t))\Big(1+\frac{\partial}{\partial\rho}\Big)\Big(\frac{1}{\sigma_{j,k}}\,\phi_{j,k}\Big) \\
& + \sum_{j+1\leq p-1} \mu(t)\Big(\frac{1}{\sigma_{j+1,0}}\,\phi_{j+1,0}\Big) \\
& + \sum_{j\geq p}\Big(\varepsilon+\frac{C_1\rho}{\varepsilon^{m-j-1}}-h\Big)\Big(\frac{1}{\sigma_{j,0}}\,\phi_{j,0}\Big).
\end{aligned} \tag{4.17}
$$

Since (4.11) and (4.12) are assumed, we see that

$$
\varepsilon+\frac{C_1\rho}{\varepsilon^{m-j-1}}-h \leq 0 \tag{4.18}
$$

for any $0 \leq \rho \leq \rho_1$. Thus, by (4.17) and (4.18) we obtain

$$
t\,\overline{D}_t^+\Phi \leq C\,\mu(t)\,\Big(1+\frac{\partial}{\partial\rho}\Big)\sum_{j+k\leq m-1}\frac{1}{\sigma_{j,k}}\,\phi_{j,k}
$$

for some $C>0$. This completes the proof of (2) of Lemma 6. □

Completion of the proof of Theorem 2. Since $\|w(t)\|_\rho \leq \phi_{0,0}(t,\rho)$ holds (by (2) of Lemma 5), to complete the proof of Theorem 2 it is sufficient to prove that $\Phi(t,\rho)=0$ on $\{(t,\rho)\,;\, 0<t\leq\varepsilon_0 \text{ and } 0\leq\rho\leq\delta_0\}$ for some $\varepsilon_0>0$ and $\delta_0>0$. Let us show this now.

Let $\rho_1>0$ and $C>0$ be the same as in Lemma 6 and choose $T_1>0$ so that $T_1<T_0$ and

$$
C\int_0^{T_1}\frac{\mu(s)}{s}ds < \rho_1.
$$

Set

$$\rho(t) = C \int_t^{T_1} \frac{\mu(s)}{s} ds, \quad 0 < t \le T_1,$$
$$\varphi(t) = e^{\rho(t)} \Phi(t, \rho(t)), \quad 0 < t \le T_1.$$

Then we have $\rho(\varepsilon) = O(1)$ (as $\varepsilon \longrightarrow +0$) and by using the condition (1) of Lemma 6 we see that $\varphi(\varepsilon) = o(1)$ (as $\varepsilon \longrightarrow +0$). Since $t(d\rho/dt) = -C\mu(t)$, by (4) of Lemma 4 and (4.14) we have for $t > 0$

$$\begin{aligned} t\,\underline{D}_t^+ \varphi \le t\,\overline{D}_t^+ \varphi &\le e^{\rho(t)} \left(t\frac{d\rho}{dt}\Phi + t\,\overline{D}_t^+ \Phi + \frac{\partial \Phi}{\partial \rho} t \frac{d\rho}{dt} \right)\Big|_{\rho=\rho(t)} \\ &= e^{\rho(t)} \left(-C\mu(t)\Phi + t\,\overline{D}_t^+ \Phi - C\mu(t)\frac{\partial \Phi}{\partial \rho} \right)\Big|_{\rho=\rho(t)} \\ &\le 0, \end{aligned}$$

that is

$$\underline{D}_t^+ \varphi(t) \le 0 \quad \text{on } (0, T_1).$$

Thus, by applying (5) of Lemma 4 we have $\varphi(t) \le \varphi(\varepsilon)$ for any $0 < \varepsilon < t \le T_1$ and therefore by letting $\varepsilon \longrightarrow +0$ we obtain

$$\varphi(t) \le 0 \quad \text{for } 0 < t \le T_1.$$

Since $\varphi(t) \ge 0$ is trivial, we have $\varphi(t) = 0$ for $0 < t \le T_1$ and therefore $\Phi(t, \rho(t)) = 0$ for $0 < t \le T_1$. Since $\Phi(t, \rho)$ (≥ 0) is increasing in ρ we obtain $\Phi(t, \rho) = 0$ on $\{(t, \rho)\,;\, 0 < t \le T_1 \text{ and } 0 \le \rho \le \rho(t)\}$. This completes the proof of Theorem 2. □

5. Proof of Theorem 3

Theorem 3 follows from Theorem 2 and the following proposition.

PROPOSITION 1. *Let $\varepsilon > 0$, $\delta > 0$ be sufficiently small, let p be an integer with $1 \le p \le m - 1$ and let $\mu(t)$ be a weight function. Assume* (2.1), (C_1), $(C_2)_p$, $(C_3)_p$-1), *and* $(C_4)_p$. *Then, if $u(t, x)$ is a solution of* (E) *belonging to $\mathcal{S}_a(\varepsilon, \delta; \mu(t))$ and if $a > p$ we have $u(t, x) \in \mathcal{S}_m(\varepsilon, \delta_1; \mu(t))$ for any $0 < \delta_1 < \delta$.*

Let us show this from now. By (2.1) we may assume that

$$\begin{cases} \operatorname{Re}\lambda_i(x) \le 0 \ \text{ on } D_\delta & \text{for } i = 1, \dots, p, \\ \operatorname{Re}\lambda_i(x) \le -h \ \text{ on } D_\delta & \text{for } i = p+1, \dots, m \end{cases} \tag{5.1}$$

for some $h > 0$, where $D_\delta = \{x \in \mathbf{C}^n \,;\, |x| \le \delta\}$.

Definition 3. Let $k \in \mathbf{N}$, $\varepsilon > 0$, $\delta > 0$ and let $\mu(t)$ be a weight function.

(1) For $a \ge 0$, we denote by $\mathcal{X}_a^k(\varepsilon, \delta; \mu(t))$ the set of functions $u(t,x)$ satisfying the following conditions (i), (ii) and (iii):

(i) $u(t,x)$ is a function on $(0,\varepsilon) \times D_\delta$,
(ii) $u(t,x)$ is of C^k class in t and continuous in x,
(iii) for $j = 0, 1, \dots, k$ we have

$$\max_{|x| \le \delta} \left| \Big(t \frac{\partial}{\partial t} \Big)^j u(t,x) \right| = O(\mu(t)^a) \ \ (\text{as } t \longrightarrow +0).$$

(2) We define $\mathcal{X}_+^k(\varepsilon, \delta; \mu(t))$ by

$$\mathcal{X}_+^k(\varepsilon, \delta; \mu(t)) = \bigcup_{a>0} \mathcal{X}_a^k(\varepsilon, \delta; \mu(t)).$$

First we note:

Lemma 7. *Let* $\lambda(x) \in C^0(D_\delta)$ *and let us consider*

$$\Big(t \frac{\partial}{\partial t} - \lambda(x) \Big) u = f \quad \text{on } (0,\varepsilon) \times D_\delta. \tag{5.2}$$

(1) *Assume that* $\operatorname{Re}\lambda(x) \le 0$ *on* D_δ*. Then, for any* $f \in \mathcal{X}_a^0(\varepsilon, \delta; \mu(t))$ *with* $a > 1$ *the equation* (5.2) *has a unique solution* $u \in \mathcal{X}_{a-1}^1(\varepsilon, \delta; \mu(t))$*. If* f *satisfies* $|f(t,x)| \le C\mu(t)^a$ *on* $(0,\varepsilon) \times D_\delta$ *we have the estimate*

$$|u(t,x)| \le C\mu(t)^{a-1} \times \int_0^t \frac{\mu(\tau)}{\tau} d\tau \quad \text{on } (0,\varepsilon) \times D_\delta.$$

Moreover, the uniqueness of the solution of (5.2) *is valid in* $\mathcal{X}_+^1(\varepsilon, \delta; \mu(t))$*.*

(2) *Assume that* $\mathrm{Re}\,\lambda(x) \le -h$ *on* D_δ *for some* $h > 0$. *Then, for any* $f \in \mathcal{X}_a^0(\varepsilon, \delta; \mu(t))$ *with* $a \ge 0$ *the equation* (5.2) *has a unique solution* $u \in \mathcal{X}_a^1(\varepsilon, \delta; \mu(t))$. *If* f *satisfies* $|f(t,x)| \le C\mu(t)^a$ *on* $(0,\varepsilon) \times D_\delta$ *we have the estimate*

$$|u(t,x)| \le \frac{C}{h}\mu(t)^a \quad \text{on } (0,\varepsilon) \times D_\delta.$$

Moreover, the uniqueness of the solution of (5.2) *is valid in* $\mathcal{X}_0^1(\varepsilon, \delta; \mu(t))$.

Proof. It is easy to see that in both cases (1) and (2) the unique solution of the equation (5.2) is given by

$$u(t,x) = \int_0^t \left(\frac{\tau}{t}\right)^{-\lambda(x)} f(\tau, x)\,\frac{d\tau}{\tau}.$$

The estimate of the solution is verified as follows: if $\mathrm{Re}\,\lambda(x) \le 0$ on D_δ and $|f(t,x)| \le C\mu(t)^a$ on $(0,\varepsilon) \times D_\delta$ for some $a > 1$ we have

$$\begin{aligned} |u(t,x)| &\le \int_0^t \left(\frac{\tau}{t}\right)^{-\mathrm{Re}\lambda(x)} C\mu(\tau)^a\,\frac{d\tau}{\tau} \\ &\le C\int_0^t \mu(\tau)^a\,\frac{d\tau}{\tau} \le C\mu(t)^{a-1}\int_0^t \frac{\mu(\tau)}{\tau}d\tau; \end{aligned}$$

if $\mathrm{Re}\,\lambda(x) \le -h < 0$ on D_δ and $|f(t,x)| \le C\mu(t)^a$ on $(0,\varepsilon) \times D_\delta$ for some $a \ge 0$ we have

$$\begin{aligned} |u(t,x)| &\le \int_0^t \left(\frac{\tau}{t}\right)^{-\mathrm{Re}\lambda(x)} C\mu(\tau)^a\,\frac{d\tau}{\tau} \\ &\le C\mu(t)^a\int_0^t \left(\frac{\tau}{t}\right)^h \frac{d\tau}{\tau} = C\mu(t)^a\,\frac{1}{h}. \ \square \end{aligned}$$

Put

$$\begin{aligned} C(\lambda, x) &= \lambda^m - \sum_{j<m} \frac{\partial F}{\partial Z_{j,0}}(0,x,0)\,\lambda^j \\ &= (\lambda - \lambda_m(x))\cdots(\lambda - \lambda_2(x))(\lambda - \lambda_1(x)) \end{aligned}$$

and let us consider

$$C\Big(t\frac{\partial}{\partial t},x\Big)u=f. \tag{5.3}$$

Applying Lemma 7 m-times to (5.3) we obtain

Lemma 8. *Assume* $1\le p\le m$, (5.1) *and* (C_1). *Then, for any* $f\in\mathcal{X}_a^0(\varepsilon,\delta;\mu(t))$ *with* $a>p$ *the equation* (5.3) *has a unique solution* $u\in\mathcal{X}_{a-p}^m(\varepsilon,\delta;\mu(t))$. *If* f *satisfies* $|f(t,x)|\le C\mu(t)^a$ *on* $(0,\varepsilon)\times D_\delta$ *we have the estimates*

$$|\Theta_j u(t,x)|\le C\mu(t)^{a-p}\times\varphi(t)\quad \textit{on } (0,\varepsilon)\times D_\delta$$
$$\textit{for } j=0,1,\dots,m-1$$

where Θ_j $(j=0,1,\dots,m-1)$ *are the operators in* (2.4) *and*

$$\varphi(t)=\max_{\substack{1\le i\le m-p\\ 1\le l\le p}}\left\{\frac{\mu(t)^p}{h^i},\ \frac{\mu(t)^{p-l}}{h^{m-p}}\Big(\int_0^t\frac{\mu(\tau)}{\tau}d\tau\Big)^l\right\}.$$

Moreover, the uniqueness of the solution of (5.3) *is valid in* $\mathcal{X}_+^m(\varepsilon,\delta;\mu(t))$.

Next, put

$$L\Big(t,x,t\frac{\partial}{\partial t}\Big)=\Big(t\frac{\partial}{\partial t}\Big)^m-\sum_{j<m}\frac{\partial F}{\partial Z_{j,0}}(t,x,0)\Big(t\frac{\partial}{\partial t}\Big)^j$$

and let us consider

$$L\Big(t,x,t\frac{\partial}{\partial t}\Big)u=f. \tag{5.4}$$

Lemma 9. *Assume* $1\le p\le m$, (5.1), (C_1) *and* $(C_3)_p$-1). *Let* $\mu(t)$ *be a weight function and let* $\varepsilon_0>0$ *be a sufficiently small number* (*depending on* $\mu(t)$ *and the equation*). *Then, for any* $f\in\mathcal{X}_a^0(\varepsilon_0,\delta;\mu(t))$ *with* $a>p$

the equation (5.4) *has a unique solution* $u \in \mathcal{X}^m_{a-p}(\varepsilon_0, \delta; \mu(t))$*. If* f *satisfies* $|f(t,x)| \le C\mu(t)^a$ *on* $(0, \varepsilon_0) \times D_\delta$ *we have the estimates*

$$|\Theta_j u(t,x)| \le C\mu(t)^{a-p} \times 2\varphi(t) \quad \text{on } (0, \varepsilon_0) \times D_\delta$$
$$\text{for } j = 0, 1, \dots, m-1.$$

Moreover, the uniqueness of the solution of (5.4) *is valid in* $\mathcal{X}^m_+(\varepsilon_0, \delta; \mu(t))$*.*

Proof. Put

$$\begin{aligned} K\Big(t, x, t\frac{\partial}{\partial t}\Big) &= \sum_{j<m} \Big(\frac{\partial F}{\partial Z_{j,0}}(t,x,0) - \frac{\partial F}{\partial Z_{j,0}}(0,x,0)\Big)\Big(t\frac{\partial}{\partial t}\Big)^j \\ &= \sum_{j<m} O(\mu(t)^p)\Theta_j. \end{aligned}$$

Then the equation (5.4) is expressed in the form

$$C\Big(t\frac{\partial}{\partial t}, x\Big)u = K\Big(t, x, t\frac{\partial}{\partial t}\Big)u + f$$

and therefore to solve (5.4) we can use the method of successive approximations:

$$\begin{aligned} &C\Big(t\frac{\partial}{\partial t}, x\Big)u_0 = f, \\ &C\Big(t\frac{\partial}{\partial t}, x\Big)u_1 = K\Big(t, x, t\frac{\partial}{\partial t}\Big)u_0, \\ &C\Big(t\frac{\partial}{\partial t}, x\Big)u_2 = K\Big(t, x, t\frac{\partial}{\partial t}\Big)u_1, \\ &\qquad\cdots\cdots\cdots \\ &\qquad\cdots\cdots\cdots \\ &\qquad\cdots\cdots\cdots \end{aligned}$$

It is easy to see that $u_0, u_1, u_2, \dots$ are well-defined and they satisfy the estimates

$$|\Theta_j u_k| \le C\mu(t)^{a-p} \times (M\varphi(t))^k \varphi(t) \tag{5.5}$$
$$\text{for } j = 0, 1, \dots, m-1 \text{ and } k = 0, 1, 2, \dots$$

for some $M > 0$.

Since $p \geq 1$ is assumed, by the definition of $\varphi(t)$ we have $\varphi(t) = o(1)$ (as $t \longrightarrow +0$) and therefore we can choose $\varepsilon_0 > 0$ so that $M\varphi(\varepsilon_0) < 1/2$. Then, by (5.5) we have

$$\sum_{k=0}^{\infty} |\Theta_j u_k| \leq C\mu(t)^{a-p} \times 2\varphi(t) \quad \text{on } (0, \varepsilon_0) \times D_\delta$$
$$\text{for } j = 0, 1, \dots, m-1.$$

Thus, we can easily conclude that the sum $u = \sum_{k=0}^{\infty} u_k$ is a solution of (5.4) in $\mathcal{X}_{a-p}^m(\varepsilon_0, \delta; \mu(t))$.

The uniqueness may be proved in the same way. □

Corollary to Lemma 9. *Assume* $1 \leq p \leq m$, (5.1), (C_1) *and* $(C_3)_{p\text{-}1}$. *If* $u(t,x)$ *belongs to* $\mathcal{S}_+(\varepsilon, \delta; \mu(t))$ *and satisfies* $L(t, x, t\partial/\partial t)u(t,x) = O(\mu(t)^a)$ *(as* $t \longrightarrow +0$*) uniformly on* D_δ *for some* $a > p$*, then we have* $u(t,x) \in \mathcal{S}_{a-p}(\varepsilon, \delta; \mu(t))$.

Now, we write

$$R\,[u] = F\Big(t, x, \Big\{\Big(t\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u\Big\}_{(j,\alpha)\in I_m}\Big) - \sum_{j<m} \frac{\partial F}{\partial Z_{j,0}}(t, x, 0)\Big(t\frac{\partial}{\partial t}\Big)^j u.$$

The equation (E) is then written as

$$L\Big(t, x, t\frac{\partial}{\partial t}\Big)u = R[u]. \tag{5.6}$$

The following lemma is a consequence of our assumptions (C_1), $(C_2)_p$ and $(C_4)_p$:

Lemma 10. *Assume* (C_1), $(C_2)_p$, $(C_4)_p$ *and let* $s > 0$ *be the one in* $(C_4)_p$. *Then, if* $u(t,x)$ *belongs to* $\mathcal{S}_a(\varepsilon, \delta; \mu(t))$ *for some* $a > 0$ *we have* $R[u](t,x) = O(\mu(t)^b)$ *(as* $t \longrightarrow +0$*) uniformly on* D_{δ_1} *for any* b *satisfying*

$$0 < b \leq \min\{2a, a+p+s, m+p\}$$

and any δ_1 *with* $0 < \delta_1 < \delta$.

Proof. Let $u(t,x) \in \mathcal{S}_a(\varepsilon,\delta;\mu(t))$ for some $a > 0$. Then we have

$$\max_{|x|\le\delta_1} \left| \left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u(t,x) \right| = O(\mu(t)^a) \ \ (\text{as } t \longrightarrow +0) \tag{5.7}$$

for any $(j,\alpha) \in I_m$ and any $0 < \delta_1 < \delta$.

By the Taylor expansion in Z we see

$$\begin{aligned} R\,[u] = F(t,x,0) &+ \sum_{\substack{(j,\alpha)\in I_m \\ |\alpha|>0}} \frac{\partial F}{\partial Z_{j,\alpha}}(t,x,0)\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u \\ &+ \sum_{(j,\alpha),(k,\beta)\in I_m} O\bigg(\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u \times \Big(t\frac{\partial}{\partial t}\Big)^k\Big(\frac{\partial}{\partial x}\Big)^\beta u\bigg). \end{aligned}$$

Thus, by applying $(\mathrm{C}_2)_p$, $(\mathrm{C}_4)_p$ and (5.7) we obtain

$$R\,[u] = O(\mu(t)^{m+p}) + O(\mu(t)^{p+s})O(\mu(t)^a) + O(\mu(t)^a \times \mu(t)^a)$$

which leads us to the conclusion of Lemma 10. □

Finally, let us give a proof of Proposition 1 by using Corollary to Lemma 9 and Lemma 10.

Proof of Proposition 1. Let $u(t,x)$ be a solution of (E) belonging to $\mathcal{S}_a(\varepsilon,\delta;\mu(t))$ for some $a > p$. Let $s > 0$ be the one in $(\mathrm{C}_4)_p$. Take a sequence $a_0, a_1, \dots, a_N$ which satisfies the following:

i) $p < a = a_0 < a_1 < \cdots < a_N = m$;

ii) $a_{i+1} + p \le \min\{2a_i, a_i + p + s, m + p\}$ $\quad (i = 0, 1, \dots, N-1)$.

Since $a_0 = a$, we have $u \in \mathcal{S}_{a_0}(\varepsilon,\delta;\mu(t))$. Therefore, by Lemma 10 and the above condition ii) we have $R[u] = O(\mu(t)^{a_1+p})$ (as $t \longrightarrow +0$) uniformly on D_{δ_1} for any $0 < \delta_1 < \delta$. Since u is a solution of (5.6) we see $L(t,x,t\partial/\partial t)u = R[u] = O(\mu(t)^{a_1+p})$ (as $t \longrightarrow +0$) and therefore by Corollary to Lemma 9 we obtain $u \in \mathcal{S}_{a_1}(\varepsilon,\delta_1;\mu(t))$ for any $0 < \delta_1 < \delta$.

By using Lemma 10 again to $u \in \mathcal{S}_{a_1}(\varepsilon,\delta_1;\mu(t))$ we have $L(t,x,t\partial/\partial t)u = R[u] = O(\mu(t)^{a_2+p})$ (as $t \longrightarrow +0$) and hence by Corollary to Lemma 9 we obtain $u \in \mathcal{S}_{a_2}(\varepsilon,\delta_1;\mu(t))$ for any $0 < \delta_1 < \delta$.

Thus, by repeating the same argument N-times we can conclude that $u(t,x) \in \mathcal{S}_{a_N}(\varepsilon,\delta_1;\mu(t))$ holds for any $0 < \delta_1 < \delta$. Since $a_N = m$, this completes the proof of Proposition 1. □

6. Application

Lastly, let us apply Theorem 2 to the problem of removable singularities of solutions of (E) (see also Tahara [5]).

Let $t \in \mathbf{C}$, $x = (x_1, \dots, x_n) \in \mathbf{C}^n$, $Z = \{Z_{j,\alpha}\}_{(j,\alpha)\in I_m} \in \mathbf{C}^{d(m)}$, let $F(t,x,Z)$ be a function in (t,x,Z) and let us consider the following equation:

$$(6.1) \qquad \Big(t\frac{\partial}{\partial t}\Big)^m u = F\bigg(t, x, \Big\{\Big(t\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u\Big\}_{(j,\alpha)\in I_m}\bigg).$$

In this section we impose the following conditions on $F(t,x,Z)$:

(A_1) $F(t,x,Z)$ is a holomorphic function in a neighborhood of $(t,x,Z) = (0,0,0)$;

(A_2) $F(0,x,0) \equiv 0$ near $x = 0$;

(A_3) $\dfrac{\partial F}{\partial Z_{j,\alpha}}(0,x,0) \equiv 0$ near $x = 0$, if $|\alpha| > 0$.

Denote by $\lambda_1(x), \dots, \lambda_m(x)$ the roots of the equation in λ:

$$\lambda^m - \sum_{j<m} \frac{\partial F}{\partial Z_{j,0}}(0,x,0)\lambda^j = 0.$$

For $\delta > 0$ we set

$$N(\lambda_i;\delta) = \{x \in \mathbf{C}^n\,;\, |x| \le \delta \ \text{ and } \ \mathrm{Re}\,\lambda_i(x) \le 0\},$$
$$N(\lambda_1,\dots,\lambda_m;\delta) = \bigcap_{i=1}^m N(\lambda_i;\delta)$$

and denote by $N^0(\lambda_1,\dots,\lambda_m;\delta)$ the interior of $N(\lambda_1,\dots,\lambda_m;\delta)$. Set:

(B) $N^0(\lambda_1,\dots,\lambda_m;\delta) \ne \emptyset$ holds for any sufficiently small $\delta > 0$.

Denote by $\mathcal{R}(\mathbf{C}\setminus\{0\})$ the universal covering space of $\mathbf{C}\setminus\{0\}$ and set: $S_\theta = \{t \in \mathcal{R}(\mathbf{C}\setminus\{0\})\,;\, |\arg t| < \theta\}$, $S_\theta(\varepsilon) = \{t \in S_\theta\,;\, |t| < \varepsilon\}$, and $D_r = \{x \in \mathbf{C}^n\,;\, |x| \le r\}$. For a weight function $\mu(t)$ satisfying $\mu_1)$, $\mu_2)$, $\mu_3)$ we impose another condition

$\mu^*)$ $\mu(t+ct) = O(\mu(t))$ (as $t \longrightarrow +0$) for some $c > 0$.

THEOREM 4. *Assume* (A_1)*,* (A_2)*,* (A_3) *and* (B)*. Then, if* $u(t,x)$ *is a solution of* (6.1) *holomorphic on* $S_\theta(\varepsilon) \times D_r$ *for some* $\theta > 0$*,* $\varepsilon > 0$*,* $r > 0$ *and satisfying*

$$\max_{|x|\le r} \left| u(t,x) \right| = O\left(\mu(|t|)^m\right) \quad (\textit{as } t \longrightarrow 0 \ \textit{ in } S_\theta) \tag{6.2}$$

for some weight function $\mu(t)$ *with* $\mu^*)$*,* $u(t,x)$ *is holomorphic in a full neighborhood of* $(0,0) \in \mathbf{C} \times \mathbf{C}^n$*.*

REMARK 2. (1) Note that the condition (B) implies

$$\operatorname{Re} \lambda_i(0) \le 0 \quad \text{ for } \ i = 1, \dots, m. \tag{6.3}$$

But, the converse is not true for $m \ge 2$ in general.

(2) When $m = 1$, (6.3) implies (B); in this case Theorem 4 is already proved in Gérard-Tahara [1].

(3) The author believes that Theorem 4 is valid even if we replace the condition (B) by (6.3). But, at present, he has no idea to prove this conjecture in the case $m \ge 2$.

PROOF OF THEOREM 4. Assume (A_1), (A_2), (A_3) and (B). Then we have (6.3) and therefore we know by [2] that the equation (6.1) has a unique holomorphic solution $u_0(t,x)$ in a neighborhood of $(0,0) \in \mathbf{C}\times\mathbf{C}^n$ satisfying $u_0(0,x) \equiv 0$. Note that $t^{1/m}$ is a weight function and that

$$u_0(t,x) \in \mathcal{S}_m(\varepsilon_1, \delta_1; t^{1/m}) \tag{6.4}$$

for some $\varepsilon_1 > 0$ and $\delta_1 > 0$.

Let $u(t,x)$ be a holomorphic solution of (6.1) on $S_\theta(\varepsilon) \times D_\delta$ for some $\theta > 0$, $\varepsilon > 0$, $\delta > 0$, and assume

$$\left| u(t,x) \right| \le A\mu(|t|)^m \ \text{ on } S_\theta(\varepsilon) \times D_\delta \tag{6.5}$$

for some constant $A > 0$ and some weight function $\mu(t)$ with $\mu^*)$. Our aim is to prove that $u(t,x)$ is holomorphic in a full neighborhood of $(0,0) \in \mathbf{C}\times\mathbf{C}^n$.

We may assume $0 < \theta < \pi/2$, let $c > 0$ be the constant in μ^*), take $r > 0$ sufficiently small so that $r < \min\{\sin\theta, c\}$, and put $T_0 = \varepsilon/(1+r)$. Then, for any real number $t \in (0, T_0)$ we have $\{\tau \in \mathbf{C}\,;\, |\tau - t| = rt\} \subset S_\theta(\varepsilon)$ and by Cauchy's integral formula we see that

$$\frac{\partial^k u}{\partial t^k}(t,x) = \frac{k!}{2\pi\sqrt{-1}} \int_{|\tau-t|=rt} \frac{u(\tau,x)}{(\tau-t)^{k+1}}\, d\tau \tag{6.6}$$

for any $k \in \mathbf{N}$. Hence, by (6.5) and (6.6) we obtain

$$\left| t^k \frac{\partial^k u}{\partial t^k}(t,x) \right| \le \frac{k!}{r^k} A\mu(t+rt)^m = O(\mu(t)^m) \ \ (\text{as } t \longrightarrow +0);$$

this implies

$$u(t,x) \in \mathcal{S}_m(\varepsilon, \delta; \mu(t)). \tag{6.7}$$

Now, put

$$\mu_0(t) = t^{1/m} + \mu(t);$$

then $\mu_0(t)$ is a weight function. It is easy to see that the equation (6.1) satisfies (C_1) and $(\mathrm{C}_3)_p$ with $p = m$ and $\mu(t) = \mu_0(t)$. Moreover, by (6.4) and (6.7) we know that $u_0(t,x)$ and $u(t,x)$ belong to the class $\mathcal{S}_m(\varepsilon_0, \delta_0; \mu_0(t))$ for some $\varepsilon_0 > 0$ and $\delta_0 > 0$. Thus, if the condition (2.1) is valid we can apply Theorem 2 to this case.

By the condition (B) we can take $x^* \in N^0(\lambda_1, \cdots, \lambda_m; \delta_0)$. Then, in a neighborhood of x^* we have

$$\mathrm{Re}\,\lambda_i(x) \le 0 \quad \text{for } i = 1, \ldots, m$$

and hence by applying Theorem 2 we obtain the result that $u_0(t,x) = u(t,x)$ on $\{(t,x) \in \mathbf{R} \times \mathbf{C}^n\,;\, 0 < t < T_1 \text{ and } |x - x^*| \le r_1\}$ for some $T_1 > 0$ and $r_1 > 0$. Since $u_0(t,x)$ is a holomorphic function in a full neighborhood of $(0,0) \in \mathbf{C} \times \mathbf{C}^n$, by the unique continuation property of holomorphic functions we can conclude that $u(t,x)$ is holomorphic in a full neighborhood of $(0,0) \in \mathbf{C} \times \mathbf{C}^n$. This completes the proof of Theorem 4. □

The following lemma gives a sufficient condition for the condition μ^*).

Lemma 11. *If a weight function $\mu(t)$ satisfies $\mu(t) \in C^1((0,T))$ and $(td\mu/dt)(t) = O(\mu(t))$ (as $t \longrightarrow +0$), then $\mu(t)$ satisfies the condition μ^*).*

Proof. By the condition $t\mu_t'(t) = O(\mu(t))$ (as $t \longrightarrow +0$) we have $t\mu_t'(t) \leq A\mu(t)$ for some $A > 0$. Then, if we take a $c > 0$ such that $cA < 1$ holds, the condition μ^*) is verified in the following way.

For $t > 0$ and $0 \leq \theta \leq c$ we have

$$\mu_t'(t + t\theta)\, t \leq (t\mu_t')(t + t\theta) \leq A\mu(t + t\theta) \leq A\mu(t + ct)$$

and therefore

$$\begin{aligned} \mu(t + ct) &= \mu(t) + \int_0^c \mu_t'(t + t\theta)\, t\, d\theta \\ &\leq \mu(t) + \int_0^c A\mu(t + ct)\, d\theta \\ &= \mu(t) + cA\mu(t + ct). \end{aligned}$$

Since $cA < 1$ is assumed we obtain

$$\mu(t + ct) \;\leq\; \frac{1}{1 - cA}\, \mu(t).$$

This implies the condition μ^*). □

Acknowledgement. The author got the idea of this paper while he was staying in Wuhan University (China) in March, 1997. He expresses his thanks to Wuhan University and NNSFC for supporting his stay in Wuhan, and to Professor Chen Hua for his kind hospitality during his stay in Wuhan.

References

[1] Gérard, R. and H. Tahara, Holomorphic and singular solutions of nonlinear singular first order partial differential equations, Publ. RIMS, Kyoto Univ. **26** (1990), 979–1000.

[2] Gérard, R. and H. Tahara, Solutions holomorphes et singulières d'équations aux dérivées partielles singulières non linéaires, Publ. RIMS, Kyoto Univ. **29** (1993), 121–151.

[3] Gérard, R. and H. Tahara, Singular nonlinear partial differential equations, Vieweg, 1996.
[4] Hukuhara, M., Ordinary differential equations, 2nd edition, Iwanami-Shoten, 1980 (Japanese).
[5] Tahara, H., Removable singularities of solutions of nonlinear singular partial differential equations, Banach Center Publications **33** (1996), 395–399.
[6] Tahara, H., Uniqueness of the solution of non-linear singular partial differential equations, J. Math. Soc. Japan **48** (1996), 729–744.

(Received October 29, 1997)

Department of Mathematics
Sophia University
Kioicho, Chiyoda-ku
Tokyo 102, JAPAN
E-mail: tahara@mm.sophia.ac.jp