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Uniqueness of Weak Solutions to the Phase-Field Model with Memory

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Abstract. The paper deals with a phase-field model based on the Gurtin-Pipkin heat flux law. A Volterra integrodifferential equation is coupled with a nonlinear parabolic equation in the resulting system, associated with a set of initial and Neumann boundary conditions. Uniqueness of the solution is proved when the convolution kernel is just supposed to be of positive type. Some regularity results are also derived.

1. Introduction

This note is concerned with a general version of the standard phase-field model for diffusive phase transitions (see e.g. [4, 10, 11]) in presence of memory effects for the heat flux. The related system of partial differential equations has been already discussed and investigated in some recent papers, among which we refer at once to [1] and [6] for a more detailed presentation of the physical model.

Letting ϑ denote the temperature and χ stand for a non-conserved order parameter, in this approach the classical Fourier law $\mathbf{q} = -k_0 \nabla\vartheta$ (k_0 constant) is replaced by the following nonlocal condition

$$\mathbf{q}(x,t) = -\int_{-\infty}^{t} k(t-s)\ \nabla\vartheta(x,s)\ ds, \tag{1.1}$$

for a kernel $k : (0,+\infty) \to \mathbb{R}$ of positive type. Then, assuming that the past evolution of ϑ is a known function ϑ_P up to $t = 0$,

$$\vartheta = \vartheta_P \quad \text{in} \quad \Omega \times (-\infty, 0), \tag{1.2}$$

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the initial and boundary value problem under consideration reads

$$\chi_t - \Delta\chi + \beta(\chi) \ni -\sigma'(\chi) + \lambda'(\chi)\,\vartheta \quad \text{in} \quad Q := \Omega \times (0,+\infty), \tag{1.3}$$

$$(\vartheta + \lambda(\chi))_t - \Delta(k * \vartheta) = g \quad \text{in} \quad Q, \tag{1.4}$$

$$\frac{\partial\chi}{\partial n} = \frac{\partial(k*\vartheta)}{\partial n} = 0 \quad \text{on} \quad \Sigma := \Gamma \times (0,+\infty), \tag{1.5}$$

$$\chi(0) = \chi_0, \quad \vartheta(0) = \vartheta_0 \quad \text{in} \quad \Omega, \tag{1.6}$$

where $*$ defines the usual convolution product with respect to time over $(0,t)$,

$$(k*\vartheta)(x,t) = \int_0^t k(t-s)\ \vartheta(x,s)\ ds, \qquad (x,t) \in Q,$$

and $\partial/\partial n$ represents the outward normal derivative.

From the previous analyses (cf. [1,5–9]) it remained unsolved the question of uniqueness for weak solutions to (1.3)-(1.6). In fact, the issue was only addressed and handled in special cases or under rather technical assumptions on k. Closing the related gap is the aim of this paper. Moreover, we provide a uniqueness proof which is very general since it also works for any maximal monotone graph $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ and for quadratic nonlinearities λ and σ. Thus, from the modelling viewpoint we cover not only the case of solid-liquid phase transitions but we deal with ferromagnetic transformations [11] as well.

Now, let us specify the setting for (1.3)-(1.6). Here, Ω is a bounded and connected open subset of $\mathbb{R}^3$ with smooth boundary Γ and the data β, σ, λ, k, χ_0, ϑ_0, g are supposed to fulfil the following requirements.

$$\begin{cases} \beta \text{ is a maximal monotone graph in } \mathbb{R}^2 \text{ with domain} \\ D(\beta) \text{ such that } \operatorname{int}(D(\beta)) \text{ is nonempty and } 0 \in \beta(0). \\ \text{We denote by } \widehat{\beta} \text{ a proper convex lower semicontinuous} \\ \text{function such that } \beta = \partial\widehat{\beta}. \end{cases} \tag{1.7}$$

$$\sigma\,, \lambda \in \mathcal{C}^2(\mathbb{R}) \quad \text{with } \sigma''\,, \lambda'' \in L^\infty(\mathbb{R}). \tag{1.8}$$

$$\text{The function } \widehat{\beta} + \sigma \text{ is nonnegative.} \tag{1.9}$$

$$
(1.10) \quad \begin{cases} k \in L^1(0,T) \ \text{ for each } \ T \in (0,+\infty) \ \text{ and} \\ k \ \text{ is of positive type, i.e.} \\ \displaystyle\int_0^T (v(t),(k * v)(t)) \ dt \geq 0, \quad \forall\, v \in L^2(0,T), \ \forall\, T \in (0,+\infty). \end{cases}
$$

$$
(1.11) \qquad \chi_0 \in H^1(\Omega), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega), \quad \vartheta_0 \in L^2(\Omega).
$$

$$
(1.12) \qquad g \in L^1(0,T;L^2(\Omega)) \quad \text{for each} \quad T \in (0,+\infty).
$$

Concerning g, we point out that the right hand side g of (1.4) collects the two contributions of the heat supply and of the past history of ϑ (up to $t=0$) via the term (cf. (1.2))

$$
\int_{-\infty}^0 k(t-s)\ \Delta\vartheta_P(x,s)\ ds, \qquad (x,t) \in \Omega \times (0,+\infty).
$$

In fact, equation (1.4) reflects the balance of internal energy, where the heat flux is taken as in (1.1) following the linearized version of the Gurtin-Pipkin constitutive assumption (for a comparative discussion on heat flux laws we refer to the full review done in [12] and [13]). On the other hand, the inclusion in (1.3) describes the phase dynamics according to the Ginzburg-Landau theory of phase transitions. Actually, (1.3) is derived from the free energy expression, whose volumetric part reads

$$
-\vartheta \log \vartheta + \vartheta\widehat{\beta}(\chi) + \vartheta\sigma(\chi) + \lambda(\chi),
$$

by invoking the second law of thermodynamics and making a first order approximation around a critical temperature.

As far as we know, the first existence result for (1.3)-(1.6) has been reported in [1]. Making use of semigroup techniques, the authors of [1] treated the particular situation when $(\beta + \sigma')(\chi) = \chi^3 - \chi$ and λ' is a constant function, dealing with Dirichlet boundary conditions in place of (1.5). Uniqueness results are also obtained in [1] under the additional restriction that k is a nonnegative, decreasing, and convex function. On the other hand, in the case when the kernel k is smooth, strong and weak solutions to (1.3)-(1.6) are shown to exist and depend continuously on the data in [6] and [7], provided the function λ is either Lipschitz continuous or linear (i.e., λ' =const.). Finally, letting k just be of positive type, existence of weak solutions to (1.3)-(1.6) is proved in [9] for the general framework fixed by (1.7)-(1.12).

The purpose of this paper is to supplement the abovementioned existence theory with some further regularity and uniqueness properties of the solution (χ, ϑ) to (1.3)-(1.6) (see Theorem 2.2 below). The class of nonlinearities $\{\beta, \sigma, \lambda\}$ allowed is thus wider than the one considered in [1], and, since λ may have quadratic growth, our results also extend those of [6] and [7] along the direction of kernels of positive type. Moreover, we believe that a variation of our argument can be applied to the case of zero time relaxation (that is, when the term χ_t is missing in (1.3)) under suitable assumption on the function λ. In this regard, the possible open investigation would complement the result of [5] and [8].

The note is structured as follows. Next section is spent in giving the notation, notion of weak solution, and statement of our uniqueness result. In Section 3 we comment on the actual regularity of solutions in the framework of a bounded initial datum χ_0. Finally, Section 4 is devoted to the uniqueness proof.

2. Main result

We first set some notation and specify the definition of weak solutions to (1.3)-(1.6) used hereafter. We put

$$H = L^2(\Omega), \qquad V = H^1(\Omega), \qquad W = \left\{ v \in H^2(\Omega), \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma \right\},$$

identify H with its dual space H', and observe that $W \subset V \subset H \subset V' \subset W'$ with dense and compact injections. Also, we denote by $\ll \cdot, \cdot \gg$ the duality pairing between W' and W and by $< \cdot, \cdot >$ the duality pairing between V' and V. Moreover, $(\cdot, \cdot)$ stands for the usual scalar product in H.

Definition 2.1. A solution to (1.3)-(1.6) is a set of functions (χ, ξ, ϑ)

such that, for each $T \in (0, +\infty)$,

$$\chi \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,3/2}(\Omega)), \tag{2.1}$$

$$\vartheta \in \mathcal{C}([0,T];W') \cap L^\infty(0,T;H) \quad \left(\subset \mathcal{C}([0,T];V')\right), \tag{2.2}$$

$$\vartheta_t - g \in L^2(0,T;W'), \quad k * \vartheta \in \mathcal{C}([0,T];H), \tag{2.3}$$

$$\xi \in L^2(0,T;L^{3/2}(\Omega)), \tag{2.4}$$

and

$$\chi_t - \Delta\chi + \xi + \sigma'(\chi) = \lambda'(\chi)\,\vartheta \quad \text{a.e. in } \Omega \times (0,T), \tag{2.5}$$

$$\ll (\vartheta + \lambda(\chi))_t\,(t), v \gg - \int_\Omega (k * \vartheta)(t)\;\Delta v\;dx = \int_\Omega g(t)\;v\;dx, \tag{2.6}$$

for all $v \in W$ and almost every $t \in (0,T)$,

$$\xi \in \beta(\chi) \quad \text{a.e. in } \Omega \times (0,T), \tag{2.7}$$

$$\frac{\partial\chi}{\partial n} = 0 \quad \text{a.e. on } \Gamma \times (0,T), \tag{2.8}$$

$$\chi(0) = \chi_0 \quad \text{in } H, \quad \vartheta(0) = \vartheta_0 \quad \text{in } W'. \tag{2.9}$$

We may now state our result.

THEOREM 2.2. *Assume that (1.7)-(1.12) are fulfilled and that*

$$\chi_0 \in L^\infty(\Omega). \tag{2.10}$$

Then there is a unique solution (χ, ξ, ϑ) to (1.3)-(1.6) (in the sense of Definition 2.1). In addition, for each $T \in (0, +\infty)$ there holds

$$\chi \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T;W), \qquad \xi \in L^2(0,T;H), \tag{2.11}$$

$$\vartheta \in \mathcal{C}([0,T];H). \tag{2.12}$$

Under the only assumptions (1.7)-(1.12), existence of triplets (χ, ξ, ϑ) solving (2.1)-(2.9) is provided by [9, Theorem 2.3 (i)]. In Section 3, we prove that all these solutions to (1.3)-(1.6) enjoy the regularity properties (2.11)-(2.12) as soon as χ_0 satisfies (2.10). In the final section we verify that the solutions are at most one.

Remark 2.3. Let us point out that condition (2.10) is ensured by (1.11) and (1.7) whenever $D(\beta)$ is bounded. Actually, the boundedness of $D(\beta)$ occurs in some typical examples for phase-field models. For instance, one can think of β coinciding with the subdifferential of the indicator function of the interval $[0,1]$,

$$\beta(\chi) = \begin{cases} (-\infty, 0] & \text{if } \chi = 0 \\ 0 & \text{if } 0 < \chi < 1 \\ [0, +\infty) & \text{if } \chi = 1 \\ \emptyset & \text{otherwise} \end{cases},$$

if χ represents a phase fraction in a two-phase system, or one may consider the case of the logarithmic potential $\widehat{\beta}(\chi) = \chi \log \chi + (1-\chi)\log(1-\chi)$ as in the Ising model for (solid-solid) martensitic phase transitions.

From now on, we assume that $(\beta, \sigma, \lambda, k)$ are fixed and fulfil (1.7)-(1.10). We also put

$$K(t) = \int_0^t k(s)\, ds, \qquad t \in [0, +\infty), \tag{2.13}$$

and note that $K \in W^{1,1}(0,T)$ for each $T > 0$, with $K' = k$ of positive type. Let A denote the operator

$$< Av_1, v_2 > := \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx, \qquad v_1,\, v_2 \in V, \tag{2.14}$$

which extends $-\Delta$ with homogeneous Neumann boundary conditions to elements of V. One easily sees that A is linear, densely defined and unbounded from V' to V' (and from H to H as well) with $D(A) = V$ (resp.

$D(A) = W$), being selfadjoint and positive semidefinite too. Then, owing to [15, Corollary 1.2], we infer that the scalar Volterra equation

$$y(t) + A(K * y)(t) = z(t), \qquad t \in (0, +\infty), \tag{2.15}$$

with datum $z \in \mathcal{C}([0, +\infty); V')$ and solution $y \in \mathcal{C}([0, +\infty); V')$, admits a resolvent $S(t)$ in V' according to [15, Definition 1.3]. That is, there exists a family $\{S(t)\}_{t\geq 0} \subset \mathcal{L}(V')$ of bounded linear operators on V' such that $S(t)$ is continuous on $[0, +\infty)$, $S(0) = I$, $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and any $t \geq 0$, and finally $\|S(t)\|_{\mathcal{L}(V')} \leq 1$.

In the sequel, we let C denote a positive constant that may depend on Ω and vary from line to line. In carrying out the proof, we use the following Gagliardo-Nirenberg inequality (recall that Ω is a three-dimensional domain)

$$|v|_{L^p(\Omega)} \leq C\, |v|_H^{\eta}\, |v|_V^{1-\eta} \qquad \forall\, v \in V, \qquad \text{for } p \in [2, 6] \text{ and } \frac{1}{p} = \frac{\eta}{2} + \frac{1-\eta}{6}, \tag{2.16}$$

and the Young theorem

$$|a * b|_{L^r(0,T;X)} \leq |a|_{L^p(0,T)}\, |b|_{L^q(0,T;X)} \qquad \forall\, a \in L^p(0,T),\ \forall\, b \in L^q(0,T;X), \tag{2.17}$$

where $1 \leq p, q, r \leq \infty$, $r^{-1} = p^{-1} + q^{-1} - 1$, and X is a real Banach space. Besides, we notice that the same symbol $|\cdot|_X$ is employed for the norm in the space of scalar functions and for the norm in the space of corresponding vectors in X^3.

3. Improved regularity

Let (χ, ξ, ϑ) be a solution to (1.3)-(1.6) in the sense of Definition 2.1. Assuming that (1.11) and (2.10) hold throughout the Section, we start by proving that χ is uniformly bounded in $\Omega \times (0, T)$ for any $T > 0$.

LEMMA 3.1. *For each $T > 0$, there holds*

$$\chi \in L^\infty(\Omega \times (0, T)). \tag{3.1}$$

Proof. Let $T > 0$ and put $F = \chi - \sigma'(\chi) + \lambda'(\chi)\,\vartheta$. In view of (2.1), (2.2), (1.8), and of the continuous embedding $V \subset L^6(\Omega)$, it is not difficult to check that

$$F \in L^\infty(0,T;L^{3/2}(\Omega)). \tag{3.2}$$

Now, the first step consists in adding χ to both sides of (2.5), multiplying by χ^7, and integrating by parts over Ω. Obviously, in the framework of (2.1) this is only a formal computation, but it can be made rigorous by taking a sequence of bounded real functions approximating $r \mapsto r^7$, for instance. Let us proceed formally. Note that $\xi\chi^7 \geq 0$ due to (2.7) and $0 \in \beta(0)$. Hence, the Hölder inequality and (2.16) enable us to obtain

$$\frac{1}{8}\frac{d}{dt}\left|\chi^4(t)\right|_H^2 + \frac{7}{16}\left|\chi^4(t)\right|_V^2 \leq C\,|F(t)|_{L^{3/2}(\Omega)}\,\left|\chi^4(t)\right|_H^{1/8}\,\left|\chi^4(t)\right|_V^{13/8}$$

and consequently, by the Young inequality,

$$\frac{d}{dt}\left|\chi^4(t)\right|_H^2 \leq C\,|F(t)|_{L^{3/2}(\Omega)}^{16/3}\,\left|\chi^4(t)\right|_H^{2/3} \leq \left|\chi^4(t)\right|_H^2 + C\,|F(t)|_{L^{3/2}(\Omega)}^8$$

for a.a. $t \in (0,T)$. Then, we integrate from 0 to some $\bar{t} \in (0,T)$, observing that $\chi_0 \in L^8(\Omega)$ because of (2.10). On account of (3.2), an application of the Gronwall lemma yields $\chi \in L^\infty(0,T;L^8(\Omega))$, whence we are able to improve the regularity of F up to

$$F \in L^\infty(0,T;L^{8/5}(\Omega)). \tag{3.3}$$

Summing up, from (2.5) and (2.7)-(2.9) it follows that χ solves the problem

$$\begin{aligned} \chi_t + \chi - \Delta\chi + \beta(\chi) \ni F \quad & \text{a.e. in } \Omega \times (0,T), \\ \frac{\partial\chi}{\partial n} = 0 \quad & \text{a.e. on } \Gamma \times (0,T), \\ \chi(0) = \chi_0 \quad & \text{a.e. in } \Omega, \end{aligned}$$

with $\chi_0 \in L^\infty(\Omega)$, $\chi \in L^\infty(0,T;L^1(\Omega))$ and $F \in L^\infty(0,T;L^{8/5}(\Omega))$ (cf. (2.10), (2.1) and (3.3)). At this point, to achieve (3.1) it suffices to exploit the lemma stated just below. $\square$

Lemma 3.2. *Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary Γ and let γ be a maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \gamma(0)$. Consider $z_0 \in L^\infty(\Omega)$, $f \in L^\infty(0,T;L^q(\Omega))$, $q \in [1,\infty]$ and denote by z the solution to*

$$\begin{aligned} z_t - \Delta z + z + \gamma(z) \ni f \quad & a.e.\ in\ \Omega \times (0,T), \\ \frac{\partial z}{\partial n} = 0 \quad & a.e.\ on\ \Gamma \times (0,T), \\ z(0) = z_0 \quad & a.e.\ in\ \Omega. \end{aligned}$$

If $z \in L^\infty(0,T;L^1(\Omega))$ and $q > 3/2$, then $z \in L^\infty(\Omega \times (0,T))$.

Proof. It relies on a Moser iteration technique and can be adapted from that of [14, Part I, Lemma 6.6] with minor changes. □

As a consequence of (3.1), we obtain some more regularity properties for χ and $\lambda(\chi)$ (the latter will play an important role in the proof of (2.12)).

Lemma 3.3. *For each $T > 0$, there holds*

$$\chi \in L^2(0,T;W), \qquad \xi \in L^2(0,T;H), \tag{3.4}$$

$$\lambda(\chi) \in \mathcal{C}([0,T];H) \cap L^\infty(0,T;V) \cap L^1(0,T;W). \tag{3.5}$$

Proof. Let $T > 0$. For almost any $t \in (0,T)$, $\chi(t)$ is a solution to

$$\chi(t) - \Delta\chi(t) + \xi(t) = \mathcal{F}(t) \quad \text{a.e. in } \Omega, \tag{3.6}$$

$$\frac{\partial \chi}{\partial n}(t) = 0 \quad \text{a.e. on } \Gamma, \tag{3.7}$$

where $\mathcal{F}(t) = \chi(t) - \sigma'(\chi(t)) + \lambda'(\chi(t))\,\vartheta(t) - \chi_t(t)$, due to (2.5) and (2.8). On the one hand, thanks to (2.7) and (1.7), the use of a standard monotonicity argument (see e.g. [3]) in (3.6)-(3.7) leads to the bound

$$|\xi(t)|_H \le |\mathcal{F}(t)|_H \qquad \text{for a.a. } t \in (0,T), \tag{3.8}$$

which in turn entails, together with (3.6)-(3.7) and classical elliptic estimates,

$$|\chi(t)|_W \leq C\ |\mathcal{F}(t)|_H \quad \text{for a.a. } t \in (0,T). \tag{3.9}$$

On the other hand, from (2.1)-(2.2), (1.8), and (3.1) we infer that

$$\mathcal{F} \in L^2(0,T;H). \tag{3.10}$$

We then deduce (3.4) from (3.8)-(3.10). Finally, (3.5) is a consequence of (2.1), (1.8), (3.1), (3.4), and of the continuous embedding $W \subset W^{1,4}(\Omega)$. □

We are now ready to prove (2.12).

LEMMA 3.4. *There holds*

$$\vartheta \in \mathcal{C}([0,+\infty);H). \tag{3.11}$$

PROOF. Let $T > 0$. We put

$$G(x,t) = \int_0^t g(x,s)\ ds, \qquad (x,t) \in \Omega \times (0,T),$$

and $u = \vartheta + \lambda(\chi)$. Owing to (3.5), (2.2), (1.10) and the continuity of K (cf. (1.10) and (2.13)), u enjoys the properties

$$u \in \mathcal{C}([0,T];V') \cap L^\infty(0,T;H), \qquad K * u \in \mathcal{C}([0,T];H). \tag{3.12}$$

For $t \in [0,T]$ and $v \in W$, an integration of (2.6) over $(0,t)$ yields

$$\begin{aligned} \int_\Omega u(t)v\,dx = &\int_\Omega (K*u)(t)\,\Delta v\,dx \\ &+ \int_\Omega \left((u_0 + G(t))\,v - (K*\lambda(\chi))\,(t)\,\Delta v\right)\,dx, \end{aligned} \tag{3.13}$$

where $u_0 := \vartheta_0 + \lambda(\chi_0)$. Observe that (3.13) holds for any $t \in [0,T]$, since u is weakly continuous from $[0,T]$ to H. As $K \in \mathcal{C}([0,T])$, (3.5) ensures that

$$K * \lambda(\chi) \in \mathcal{C}([0,T];W). \tag{3.14}$$

Setting $\mathcal{G} = (K * u) + u_0 + G - u - \Delta(K * \lambda(\chi))$, with the help of (1.8), (1.11)-(1.12), (3.12), and (3.14) we deduce that $\mathcal{G}$ belongs to $L^\infty(0,T;H)$. In addition, for each $t \in [0,T]$ we have that $(K * u)(t) \in H$ and, by virtue of (3.13),

$$\int_\Omega (K * u)(t)\,(v - \Delta v)\,dx = \int_\Omega \mathcal{G}(t)\,v\,dx \qquad \forall\ v \in W.$$

Consequently, one recovers $K * u \in L^\infty(0,T;W)$, and this and (3.12) yield

$$K * u \in \mathcal{C}([0,T];V) \cap L^\infty(0,T;W). \tag{3.15}$$

Thus, recalling (3.12) and (3.14)-(3.15), we may rewrite (3.13) as

$$u(t) + A(K * u)(t) = u_0 + G(t) + A\,(K * \lambda(\chi))\,(t) \tag{3.16}$$

in V', for each $t \in [0,T]$. Being $u \in \mathcal{C}([0,T];V')$ and $K * u \in \mathcal{C}([0,T];V)$, it turns out that u is a mild solution in V' of the Volterra equation (3.16), in the sense of [15, Definition 1.1]. However, concerning the right hand side of (3.16), on account of (1.8), (1.10)-(1.12), (2.17), (3.5), and (3.14) we realize that

$$\begin{aligned}
&u_0 + G + A\,(K * \lambda(\chi)) \in \mathcal{C}([0,T];H),\\
&\left(u_0 + G + A\,(K * \lambda(\chi))\right)_t = g + A\,(k * \lambda(\chi)) \in L^1(0,T;H),
\end{aligned}$$

whence $u_0 + G + A\,(K * \lambda(\chi)) \in W^{1,1}(0,T;H)$. We then infer from (1.10) and [15, Proposition 1.2 & Corollary 1.2] that (3.16) has a mild solution $\tilde{u}$ in H, which is also a mild solution to (3.16) in V'. Uniqueness of mild solutions (see [15, Proposition 1.2]) now implies that $\tilde{u} = u$. Therefore, we conclude that $u \in \mathcal{C}([0,T];H)$ and, in view of (3.5) too, we obtain (3.11). □

4. Uniqueness

Actually, we show a continuous dependence result. Let $(\chi_i, \xi_i, \vartheta_i)$, $i = 1,2$, be two solutions to (1.3)-(1.6) corresponding to the data $(\chi_{0i}, \vartheta_{0i}, g_i)$, $i = 1,2$. We put $u_i = \vartheta_i + \lambda(\chi_i)$, $i = 1,2$, and

$$\mathcal{X} = \chi_1 - \chi_2, \quad \Theta = \vartheta_1 - \vartheta_2, \quad U = u_1 - u_2, \quad h = g_1 - g_2.$$

For a fixed $T > 0$, henceforth we denote by R_i, $i \in \mathbb{N}$, positive constants (possibly varying from line to line and) depending only on Ω, T, σ, λ, $|k|_{L^1(0,T)}$, $|\chi_{01}|_{L^\infty(\Omega)}$, $|\chi_{02}|_{L^\infty(\Omega)}$, $|\chi_1|_{L^\infty(\Omega\times(0,T))}$, $|\chi_2|_{L^\infty(\Omega\times(0,T))}$, $|\chi_1|_{L^2(0,T;W)}$, $|\chi_2|_{L^2(0,T;W)}$, $|\vartheta_1|_{L^\infty(0,T;H)}$ and $|\vartheta_2|_{L^\infty(0,T;H)}$. Observe that all these quantities are finite thanks to (3.1), (3.4) and (2.2). We finally set

$$B(t) = |\chi_1(t)|_W^2 \quad \text{for a.a. } t \in (0,T),$$

and note that $B \in L^1(0,T)$ by (3.4).

From the previous section (cf. (3.16)) we already know that both u_1 and u_2 are mild solutions in V' to the Volterra equation (2.15) with

$$z(t) = u_i(0) + \int_0^t g_i(s)\,ds + A\left(K * \lambda(\chi_i)\right)(t), \qquad t \in (0,T),$$

for $i = 1$ and $i = 2$, respectively. We take the difference and exploit the formula of variation of constants for Volterra equations (see e.g. [15, Proposition 1.2]) in order to derive the equality

$$U(t) = S(t)U(0) + \int_0^t S(t-s)\left(h + A\left(k * (\lambda(\chi_1) - \lambda(\chi_2))\right)\right)(s)\,ds$$

for all $t \in [0,T]$, where $S(t)$ is the resolvent associated with A and discussed below (2.15). Since $\|S(s)\|_{\mathcal{L}(V')} \le 1$ for any $s \in [0,+\infty)$, for $t \in (0,T)$ we deduce

$$\begin{aligned}(4.1)\quad |U(t)|_{V'} &\le |U(0)|_{V'} \\ &\quad + \int_0^t \left(|h(s)|_{V'} + \left|A\left(k * (\lambda(\chi_1) - \lambda(\chi_2))\right)(s)\right|_{V'}\right) ds.\end{aligned}$$

We may estimate the last term of the right hand side of (4.1) as follows. Using (2.16)-(2.17) and the continuous embedding $W \subset W^{1,6}(\Omega)$ it is straightforward to verify that

$$
\begin{aligned}
&\int_0^t |A\,(k*(\lambda(\chi_1)\;-\;\lambda(\chi_2)))\,(s)|_{V'}\,ds \\
&\quad \le |k|_{L^1(0,t)} \int_0^t |\nabla(\lambda(\chi_1)-\lambda(\chi_2))(s)|_H\,ds \\
&\quad \le R_1 \int_0^t \big(|\nabla\chi_1(s)|_{L^6(\Omega)}\,|\mathcal{X}(s)|_{L^3(\Omega)} + |\lambda'(\chi_2)(s)|_{L^\infty(\Omega)}\,|\nabla\mathcal{X}(s)|_H\big)\,ds \\
&\quad \le R_1 \int_0^t \Big(B(s)^{1/2}\,|\mathcal{X}(s)|_H^{1/2}\,|\mathcal{X}(s)|_V^{1/2} + |\mathcal{X}(s)|_V\Big)\,ds \\
&\quad \le R_1 \int_0^t (|\mathcal{X}(s)|_V + B(s)\;|\mathcal{X}(s)|_H)\,ds.
\end{aligned}
$$

Hence, we have

$$
|U(t)|_{V'} \le |U(0)|_{V'} + R_1 \int_0^t (|h(s)|_{V'} + |\mathcal{X}(s)|_V + B(s)\,|\mathcal{X}(s)|_H)\,ds
$$

and, thanks to (3.4) and the Hölder inequality,

$$
\begin{aligned}
|U(t)|_{V'}^2 &\le 2\,|U(0)|_{V'}^2 + R_1\,|h|_{L^1(0,t;V')}^2 \\
&\quad + R_1\left(\int_0^t |\mathcal{X}(s)|_V^2\,ds + |B|_{L^1(0,T)} \int_0^t B(s)\,|\mathcal{X}(s)|_H^2\,ds\right),
\end{aligned}
$$

that is,

$$
\begin{aligned}
(4.2)\quad |U(t)|_{V'}^2 &\le 2\,|U(0)|_{V'}^2 \\
&\quad + R_2\left(|h|_{L^1(0,t;V')}^2 + \int_0^t \big(|\mathcal{X}(s)|_V^2 + B(s)\,|\mathcal{X}(s)|_H^2\big)\,ds\right).
\end{aligned}
$$

Next, subtract the equations (2.5) satisfied by χ_1 and χ_2 and take the scalar product in $L^2(\Omega\times(0,t))$ of the resulting identity with $\mathcal{X}$. Due to (1.7), (1.8), and (2.7), we easily obtain

$$
\begin{aligned}
(4.3)\qquad |\mathcal{X}(t)|_H^2 + \int_0^t |\nabla\mathcal{X}(s)|_H^2\,ds &\le |\mathcal{X}(0)|_H^2 + R_3 \int_0^t |\mathcal{X}(s)|_H^2\,ds \\
&\quad + I_1(t) + I_2(t),
\end{aligned}
$$

where

$$I_1(t) := \int_0^t \int_\Omega \left|\lambda'(\chi_1)\,(\vartheta_1 - \vartheta_2)\,\mathcal{X}\right|\,dxds,$$
$$I_2(t) := \int_0^t \int_\Omega \left|\vartheta_2\left(\lambda'(\chi_1) - \lambda'(\chi_2)\right)\,\mathcal{X}\right|\,dxds,$$

for all $t \in [0,T]$. To handle I_1, note that (1.8), (2.16), (3.1), the continuous embedding $W \subset W^{1,6}(\Omega)$, and the Young inequality allow us to achieve

$$\begin{aligned}
I_1(t) &\le \int_0^t |\Theta(s)|_{V'}\,\left|(\lambda'(\chi_1)\,\mathcal{X})(s)\right|_V\,ds \\
&\le R_4 \int_0^t |\Theta(s)|_{V'}\,\big(|\lambda'(\chi_1)(s)|_{L^\infty(\Omega)}\,|\mathcal{X}(s)|_V \\
&\qquad\qquad + |(\lambda''(\chi_1)\,\mathcal{X}\,\nabla\chi_1)(s)|_H\big)\,ds \\
&\le R_4 \int_0^t |\Theta(s)|_{V'}\,\big(|\mathcal{X}(s)|_V + |\lambda''|_{L^\infty(\mathbb{R})}\,|\nabla\chi_1(s)|_{L^6(\Omega)}\,|\mathcal{X}(s)|_{L^3(\Omega)}\big)\,ds \\
&\le R_4 \int_0^t |\Theta(s)|_{V'}\,\Big(|\mathcal{X}(s)|_V + B(s)^{1/2}\,|\mathcal{X}(s)|_H^{1/2}\,|\mathcal{X}(s)|_V^{1/2}\Big)\,ds \\
&\le R_4 \Big(\int_0^t |\Theta(s)|_{V'}\,|\mathcal{X}(s)|_V\,ds \\
&\qquad + \int_0^t B(s)\,|\Theta(s)|_{V'}^2\,ds + \int_0^t |\mathcal{X}(s)|_H\,|\mathcal{X}(s)|_V\,ds\Big) \\
&\le \frac{1}{4}\int_0^t |\nabla\mathcal{X}(s)|_H^2\,ds \\
&\quad + R_5 \int_0^t \big(|\Theta(s)|_{V'}^2 + B(s)\,|\Theta(s)|_{V'}^2 + |\mathcal{X}(s)|_H^2\big)\,ds.
\end{aligned}$$

Thus, in virtue of the simple remark

$$|\Theta(s)|_{V'} \le |U(s)|_{V'} + R_6\,|\mathcal{X}(s)|_H \quad \forall\ s \in [0,T], \tag{4.4}$$

it turns out that

$$\begin{aligned}
I_1(t) &\le \frac{1}{4}\int_0^t |\nabla\mathcal{X}(s)|_H^2\,ds \\
&\quad + R_7 \int_0^t (1 + B(s))\,\big(|U(s)|_{V'}^2 + |\mathcal{X}(s)|_H^2\big)\,ds.
\end{aligned} \tag{4.5}$$

Concerning I_2, from (1.8), (2.2), (2.16), and the Young inequality we infer that

$$
\begin{aligned}
(4.6)\qquad I_2(t) &\le R_8 \int_0^t |\vartheta_2(s)|_H \, |\mathcal{X}(s)|^2_{L^4(\Omega)} \, ds \\
&\le R_8 \int_0^t |\mathcal{X}(s)|_H^{1/2} \, |\mathcal{X}(s)|_V^{3/2} \, ds \\
&\le \frac{1}{4} \int_0^t |\nabla \mathcal{X}(s)|^2_H \, ds + R_9 \int_0^t |\mathcal{X}(s)|^2_H \, ds.
\end{aligned}
$$

Combining (4.3) and (4.5)-(4.6) yields

$$
\begin{aligned}
(4.7)\quad & |\mathcal{X}(t)|^2_H + \frac{1}{2} \int_0^t |\nabla \mathcal{X}(s)|^2_H \, ds \\
&\le |\mathcal{X}(0)|^2_H + R_{10} \int_0^t (1 + B(s)) \left(|U(s)|^2_{V'} + |\mathcal{X}(s)|^2_H \right) ds.
\end{aligned}
$$

We now multiply (4.2) by $1/(4R_2)$ and add the result to (4.7). Setting

$$
\Psi(t) = |\mathcal{X}(t)|^2_H + \frac{1}{4R_2} |U(t)|^2_{V'} + \frac{1}{4} \int_0^t |\nabla \mathcal{X}(s)|^2_H \, ds,
$$

we obtain

$$
\Psi(t) \le R_{11} \left(\Psi(0) + |h|^2_{L^1(0,t;V')} + \int_0^t (1 + B(s)) \, \Psi(s) \, ds \right) \qquad \forall \, t \in [0, T].
$$

Since $B \in L^1(0, T)$, the Gronwall lemma and (3.4) entail

$$
\Psi(t) \le R_{12} \left(\Psi(0) + |h|^2_{L^1(0,t;V')} \right) \qquad \forall \, t \in [0, T].
$$

Therefore, recalling (4.4) and noting that

$$
\Psi(0) \le R_{13} \left(|\Theta(0)|^2_{V'} + |\mathcal{X}(0)|^2_H \right),
$$

we have deduced the following result.

Proposition 4.1. *Assume that (1.7)-(1.10) are satisfied and let* $\{\chi_{0i}, \vartheta_{0i}, g_i\}$, $i = 1,2$, *be two sets of data fulfilling (1.11)-(1.12) and (2.10). Denote by* $(\chi_i, \xi_i, \vartheta_i)$, $i = 1,2$, *two corresponding solutions to (1.3)-(1.6) (in the sense of Definition 2.1). Then, for each* $T > 0$, *there is a constant* $R_0(T)$ *depending only on* Ω, T, σ, λ, $|k|_{L^1(0,T)}$, $|\chi_{01}|_{L^\infty(\Omega)}$, $|\chi_{02}|_{L^\infty(\Omega)}$, $|\chi_1|_{L^\infty(\Omega\times(0,T))}$, $|\chi_2|_{L^\infty(\Omega\times(0,T))}$, $|\chi_1|_{L^2(0,T;W)}$, $|\chi_2|_{L^2(0,T;W)}$, $|\vartheta_1|_{L^\infty(0,T;H)}$ *and* $|\vartheta_2|_{L^\infty(0,T;H)}$ *such that*

$$
\begin{aligned}
(4.8)\quad & |\vartheta_1 - \vartheta_2|_{\mathcal{C}([0,T];V')} + |\chi_1 - \chi_2|_{\mathcal{C}([0,T];H)} + |\chi_1 - \chi_2|_{L^2(0,T;V)} \\
& \quad \le R_0(T)\, \left(|\vartheta_{01} - \vartheta_{02}|_{V'} + |\chi_{01} - \chi_{02}|_H + |g_1 - g_2|_{L^1(0,T;V')}\right).
\end{aligned}
$$

The uniqueness statement in Theorem 2.2 is then a straightforward consequence of Proposition 4.1.

Remark 4.2. One can easily check that Theorem 2.2 and Proposition 4.1 still hold for one-dimensional and bi-dimensional domains Ω, with minor efforts. In this case, the Gagliardo-Nirenberg inequality (2.16) must be adapted to the dimension of the space. However, computations are easier and, for instance, if $\Omega \subset \mathbb{R}$ then (2.10) is ensured by (1.11).

Remark 4.3. In the case when the function λ is Lipschitz continuous (which includes various physically interesting examples), the arguments of this paper can be reproduced almost exactly without using the assumption (2.10). In particular, the conclusions of Theorem 2.2 are still valid provided (2.11) is replaced by (3.4), and the constant $R_0(T)$ in (4.8) depends only on Ω, T, σ, λ, $|k|_{L^1(0,T)}$, $|\chi_1|_{L^2(0,T;W)}$, $|\chi_2|_{L^2(0,T;W)}$, $|\vartheta_1|_{L^\infty(0,T;H)}$ and $|\vartheta_2|_{L^\infty(0,T;H)}$. In this respect, the above proofs complemented with those of [9] (in which the existence part of Theorem 2.2 is proved) turn out to provide an actual improvement of the results of [6] and [7].

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