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Strict Convexity of Hypersurfaces in Spheres

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Abstract. This paper investigates compact, embedded, strictly convex hypersurfaces in the unit sphere and gives several conditions equivalent to the strict convexity of hypersurfaces. Particularly it is seen that the local strict convexity is equivalent to the strict convexity.

1. Introduction

Let M be a connected, compact, n-dimensional differentiable manifold($n \geqq 1$) and $\iota : M \to S^{n+1}$ an immersion of M into the unit $(n+1)$-sphere S^{n+1}. For any $p \in M$, we denote by S_p the totally geodesic hypersphere in S^{n+1} with $\iota(p) \in S_p$ and with $T_{\iota(p)}(S_p) = d\iota(T_p(M))$, and by H_p either of the two open hemispheres determined by S_p. An immersion ι is said to be *locally convex*(*resp. locally strictly convex*) *at a point* $p \in M$ if there exists some neighborhood U_p of p in M such that $\iota(U_p \setminus \{p\})$ is contained in $\overline{H_p}$(resp. H_p). Moreover, ι is said to be *convex*(*resp. strictly convex*) *at* p if $\iota(M \setminus \{p\})$ is contained in $\overline{H_p}$(resp. H_p). With respect to convexity of an immersion, the following is known([2]):

Theorem (do Carmo-Warner). *If M is orientable and if $n \geqq 2$, then the following are mutually equivalent:*

(1) *An immersion ι is locally convex at each point of M;*

(2) *An immersion ι is convex at each point of M;*

(3) *All sectional curvatures of M are greater than or equal to one.*

Moreover, one of the conditions above implies that ι is an embedding.

In this paper, we study the strict convexity of an embedding. We call a pair of two points $p, q \in M$ *an antipodal pair in relation to* ι if $\iota(q)$ is the

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antipodal point of $\iota(p)$ in S^{n+1}. Then our main result, proved throughout Section 2, is stated as follows:

THEOREM 1.1. *Let M be a connected, compact, n-dimensional differentiable manifold ($n \geqq 1$) and $\iota : M \to S^{n+1}$ an embedding of M into S^{n+1}. Then the following are mutually equivalent*:

(1) *An embedding ι is locally strictly convex at each point of M*;

(2) *An embedding ι is strictly convex at each point of M*;

(3) *A hypersurface M does not have any antipodal pair in relation to ι and we can choose either of two components of $S^{n+1} \setminus \iota(M)$ which, denoted by Ω, satisfies*

$$\gamma \setminus \{\iota(p), \iota(q)\} \subset \Omega$$

for $p, q \in M$ with $p \neq q$, where γ is the minimal geodesic segment in S^{n+1} joining $\iota(p)$ and $\iota(q)$;

(4) *A hypersurface M does not have any antipodal pair in relation to ι, and for $p, q \in M$ with $p \neq q$,*

$$\Gamma \cap \iota(M) = \{\iota(p), \iota(q)\},$$

where Γ is the great circle in S^{n+1} containing $\iota(p)$ and $\iota(q)$.

REMARK 1.1. We would like to express our thanks to Professor K. Enomoto for informing us of the existence of [2]. Considering the above theorem of do Carmo-Warner, we see that (1) and (2) in Theorem 1.1 are equivalent for $n \geqq 2$, even if $\iota : M \to S^{n+1}$ is an immersion. For, if an immersion ι is locally strictly convex at each point, then M is orientable(refer to Lemma 2.1 in Section 2).

REMARK 1.2. It is easily seen that an immersion ι with the definite second fundamental form at $p \in M$ is locally strictly convex at p. Therefore it follows from Remark 1.1 that if $n \geqq 2$ and if ι is an immersion with the definite second fundamental form at each point, then ι is strictly convex at each point. Notice that the local strict convexity of ι at p does not imply that the second fundamental form of ι is definite at p.

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2. Proof of Theorem 1.1

If there is not any danger of confusion, then we shall treat properties of an embedding ι as those of M and denote merely by M the image $\iota(M)$. For example, we call M a (locally) strictly convex hypersurface in S^{n+1} at $p \in M$ instead of saying that ι is (locally) strictly convex at p. Throughout this section, suppose that M is a connected, compact, embedded hypersurface in S^{n+1}. Then $S^{n+1} \setminus M$ has just two connected components.

If M satisfies (1), then we can choose one direction of normal vectors at each $p \in M$ by the following way: A normal vector $\xi(p)$ at p pointing to the direction is an inner normal vector of H_p. Moreover we immediately obtain the following, so we omit the proof.

Lemma 2.1. *If M satisfies* (1)*, then there exists some continuous normal vector field ξ defined globally on M such that $\xi(p)$ is the nonzero inner normal vector of the open hemisphere H_p at each $p \in M$. In particular, M is orientable.*

Now, we shall prove (3) from (2). Since M is locally strictly convex at each point of M, we can choose either of the two connected components of $S^{n+1} \setminus M$ as the domain of which $\xi(p)$ as in Lemma 2.1 is an inner normal vector. We denote by Ω the connected component. Then take arbitrary two points $p, q \in M$ with $p \neq q$ and let γ be the minimal geodesic segment joining p and q in S^{n+1}, and suppose that on $\gamma \setminus \{p, q\}$, there exists some point $x \in M$. If M and γ cross transversely in S^{n+1} at x, then we find that p is contained in one side of S_x and that q is contained in the other side of S_x. Therefore it is easily seen that M and S_x have at least two common points, which contradicts (2). If γ is tangent to M at x, then S_x contains γ. Therefore S_x contains three points p, q and x of M, which also contradicts (2). Thus (2) implies (3).

Next, to prove (4) from (3), we need the following lemma.

Lemma 2.2. *If M satisfies* (3) *and if there exists a great circle Γ such that $\Gamma \cap M$ consists of not less than three points, then $\Gamma \subset \overline{\Omega}$.*

Proof. There exist different three points $a, b, c \in \Gamma \cap M$. Then it follows from (3) that the minimal geodesic segment joining arbitrary two points of $\{a, b, c\}$ does not contain any point of $\Gamma \cap M$ except these two points. On the other hand, it immediately follows that these segments are contained in Γ. Therefore we immediately obtain that $\Gamma \cap M = \{a, b, c\}$ and that the sum of the three segments are just the great circle Γ. Therefore it follows that Γ is contained in $\overline{\Omega}$. $\square$

If M satisfies (3) and does not satisfy (4), then by Lemma 2.2 it is easily seen that Ω in (3) has some antipodal pair $\{p, -p\}$ of S^{n+1}. By the proof of Lemma 2.2, M and any great circle Γ_p through p and $-p$ have at most three common points, and Γ_p is contained in $\overline{\Omega}$ in spite of the number of the common points. Therefore it immediately follows that $\overline{\Omega} = S^{n+1}$, which causes the contradiction.

To prove (2) from (4), it suffices to show the following lemma.

Lemma 2.3. *If M satisfies* (4), *then* Γ *and* M *cross transversely in* S^{n+1} *at each point of* $\Gamma \cap M$.

Proof. Suppose that Γ is tangent to M at $p \in M$. We separate the proof of Lemma 2.3 into two case.

Case 1. $n = 1$.

Let C be a connected, simply closed curve C in S^2. In this case, we find that $\Gamma = S_p$. If C is locally strictly convex at p, then C is also locally strictly convex at q and the set $C \setminus \{p, q\}$ is contained in H_p. Since M does not have any antipodal pair, we can find the Euclidean coordinates (x_1, x_2, x_3) satisfying

$$S_p = S^2 \cap \{x_3 = 0\},$$

$$C \subset \{x_3 \geqq 0\},$$

$$p \in \{x_1 > 0\}$$

and

$$q = (0, 1, 0).$$

Hence, for a sufficiently small number $\theta > 0$, the great circle

$$\Gamma_\theta := S^2 \cap \{(\cos\theta) x_3 = (\sin\theta) x_1\}$$

contains q, and at least two points of C near p, that is,

$$\sharp\{\Gamma_\theta \cap C\} \geqq 3,$$

which contradicts (4).

If C is not locally strictly convex at p, then we set the Euclidean coordinates (x_1, x_2, x_3) satisfying

$$S_p = S^2 \cap \{x_3 = 0\}$$

and

$$p = (0, 1, 0).$$

Then Γ_θ as above contains p, and at least two points of C near p for some $\theta > 0$ or for some $\theta < 0$, which also contradicts (4).

Case 2. $n \geqq 2$.

As in Case 1, suppose that Γ is tangent to M at p. Taking the great circle $\Gamma^\perp$ perpendicular to M in S^{n+1} at p, we find that Γ and $\Gamma^\perp$ determine the two dimensional totally geodesic sphere S. Since S and M cross transversely in S^{n+1} at p, there exists some open smooth curve C such that $p \in C$ and $C \subset S \cap M$. Noticing that C is tangent to Γ and arguing as in Case 1, we can find a great circle Γ' in S such that Γ' contains p or q, and at least two points of C near p. □

By the above argument, we find that (2), (3) and (4) are mutually equivalent. Since (2) implies (1), we need show (2) from (1).

Case 1. $n = 1$.

Let C be a connected, simply closed curve in S^2. If there exists some point $p \in C$ such that $S_p \cap C \supsetneqq \{p\}$, then there exists some point $q \in S_p \cap C$ such that one of two subarcs of C joining p and q, denoted by L, does not contain any point of S_p except p and q. Let s, t be arc-length parameters of C, S_p respectively such that

$$C(0) = S_p(0) = p, \qquad C'(0) = S_p'(0),$$

and that for some $s_0 > 0$, $L = C([0, s_0])$. We find that $C(s_0) = q$ and that for some $0 < t_0 < 2\pi$, $S_p(t_0) = q$. Then, with respect to t_0, exactly one of the following happens:

$$\text{(a)} \quad t_0 \in (0, \pi) \qquad \text{(b)} \quad t_0 = \pi \qquad \text{(c)} \quad t_0 \in (\pi, 2\pi).$$

Firstly, in Case (a), we can set the coordinates (x_1, x_2, x_3) of $\mathbf{R}^3$ such that

$$S_p = S^2 \cap \{x_3 = 0\},$$
$$p, q \in \{x_2 > 0\},$$

and

$$U_p \subset \{x_3 \geqq 0\},$$

where U_p is some open neighborhood of p in C. Let D_1, D_2 be two domains of the upper hemisphere $S^2 \cap \{x_3 > 0\}$ obtained from L such that $S_p([0, t_0]) \subset \overline{D_1}$. Then it immediately follows that for the normal vector field ξ of C in S^2 determined by Lemma 2.1, $\xi(p)$ is an inner normal vector of D_2 at p. For this Euclidean coordinates, set the polar coordinates (r, φ, ψ) as follows:

$$\begin{cases} x_1 = r \cos \psi, \\ x_2 = r \sin \varphi \sin \psi, \\ x_3 = r \cos \varphi \sin \psi. \end{cases}$$

In particular, it follows that $r \equiv 1$ on S^2. We can suppose that on L, $\varphi \in (-\pi/2, \pi/2]$ and $\psi \in (0, \pi)$. Then it follows that the points on L corresponding to $\varphi = \pi/2$ are only p and q. Therefore

$$\varphi_0 = \inf\{\varphi\, ;\, l \in L\}$$

can be attained, so take a point p_0 at which $\varphi = \varphi_0$. Then the great circle $\Gamma(\varphi_0)$ determined by $\varphi = \varphi_0$ is tangent to L at p_0. Hence, by the way of determining ξ, it is easily seen that $\xi(p_0)$ is an inner normal vector of D_1, which causes the contradiction.

In Case (b), if C is not tangent to S_p at q, then we can find a point of $S_p \cap C \setminus \{p\}$ which is not antipodal point of p in S_p. Therefore we can suppose that C is tangent to S_p at q and that

$$S_p \cap C = \{p, q\}.$$

However, in this case, we obtain the contradiction by the way as in Case (a). In Case (c), retaking parameters s, t brings our discussion to Case (a). Therefore we have proved (2) from (1) for $n = 1$.

Case 2. $n \geqq 2$.

Firstly, set

$$X^m(p) := \{\Pi^m \ ; \ \text{an } m\text{-dimensional totally geodesic sphere perpendicular to } S_p \text{ at } p \text{ in } S^{n+1}\}$$

for any $p \in M$ and for $2 \leqq m \leqq n$. To prove (2) from (1), we need some lemmas. The following lemma is immediately obtained.

LEMMA 2.4. *If M satisfies* (1), *then for any $\Pi^m \in X^m(p)$, the connected component $M_{\Pi^m}(p)$ of $M \cap \Pi^m$ containing p is a compact, embedded hypersurface in Π^m locally strictly convex at each point of $M_{\Pi^m}(p)$.*

Next, we shall prove

LEMMA 2.5. *For $p \in M$, there exists some point $q \in M \setminus \{p\}$ such that for any $\Pi_1, \Pi_2 \in X^2(p)$ with $\Pi_1 \neq \Pi_2$,*

$$M_{\Pi_1}(p) \cap M_{\Pi_2}(p) = \{p, q\}.$$

PROOF. We separate the proof of Lemma 2.5 into two cases.

Case 1. $n = 2$.

From (1) in Theorem 1.1 and the equation of Gauss(see [4, pp. 23]), we find that at each point of M, Gaussian curvature is not less than one, which implies that M is homeomorphic to S^2. And notice that for any $\Pi \in X^2(p)$, $\Gamma_0 := \Pi_1 \cap \Pi_2$ is a great circle in Π. Particularly, Γ_0 is a great circle in Π_1 and in Π_2. By Lemma 2.4, $M_{\Pi_i}(p)$ is simply closed and locally strictly convex at each point of $M_{\Pi_i}(p)$ in $\Pi_i (i = 1, 2)$. Hence noticing (4) in Theorem 1.1 for $n = 1$, we find that

$$\sharp\{M_{\Pi_i}(p) \cap \Gamma_0\} \leqq 2.$$

Since $M_{\Pi_i}(p)$ and Γ_0 cross transversely in Π_i at p, it follows that $\sharp\{M_{\Pi_i}(p) \cap \Gamma_0\} = 2$. On the other hand,

$$M_{\Pi_1}(p) \cap M_{\Pi_2}(p) \subset M_{\Pi_i}(p) \cap \Gamma_0 \subset \Gamma_0.$$

for $i = 1, 2$. Therefore, noticing that $M_{\Pi_1}(p)$ and $M_{\Pi_2}(p)$ cross transversely in M at p and that M is homeomorphic to S^2, we find that $M_{\Pi_1}(p) \cap$

$M_{\Pi_2}(p) = M_{\Pi_1}(p) \cap \Gamma_0$ and that there exists some point $q \in M \setminus \{p\}$ independent of Π_1 and Π_2 such that

$$M_{\Pi_1}(p) \cap M_{\Pi_2}(p) = M_{\Pi_1}(p) \cap \Gamma_0 = M_{\Pi_2}(p) \cap \Gamma_0 = \{p, q\}.$$

Case 2. $n \geqq 3$.

For $\Pi^3 \in X^3(p)$, $M_{\Pi^3}(p)$ is a compact, embedded surface locally strictly convex at each point of $M_{\Pi^3}(p)$ in Π^3 by Lemma 2.4. Therefore, by Case 1, there exists some point $q \in M \setminus \{p\}$ such that for any $\Pi_1, \Pi_2 \in X^2(p)$ in Π^3 with $\Pi_1 \neq \Pi_2$,

$$M_{\Pi_1}(p) \cap M_{\Pi_2}(p) = \{p, q\}.$$

From the argument in Case 1, it is easy to see that the way of choosing q does not depend on Π^3, but depends only on M. Hence it follows that

$$M_{\Pi_1}(p) \cap M_{\Pi_2}(p) = \{p, q\}$$

for any $\Pi_1, \Pi_2 \in X^2(p)$ in S^{n+1} with $\Pi_1 \neq \Pi_2$. □

Using Lemma 2.4, Theorem 1.1 for $n = 1$ and the following lemma, we can prove (2) from (1) for $n \geqq 2$.

Lemma 2.6. *For any* $p \in M$,

$$M = \sqcup_{\Pi \in X^2(p)} M_{\Pi}(p).$$

That is, for any p *and for any* $\Pi \in X^2(p)$,

$$M_{\Pi}(p) = M \cap \Pi.$$

Proof. Firstly, by Lemma 2.4 and by Theorem 1.1 for $n = 1$, if $p \in M$, then $-p \notin M_{\Pi}(p)$ for any $\Pi \in X^2(p)$. Therefore, the image of $M_{\Pi}(p)$ by the stereographic projection from $-p$

$$\pi_{-p} : S^{n+1} \setminus \{-p\} \to \mathbf{R}^{n+1}$$

is a simply closed curve in $\mathbf{R}^{n+1}$. Noticing $-p \in S_p$, we can set the coordinates $(x_1, \ldots, x_{n+1})$ of $\mathbf{R}^{n+1}$ such that

$$\pi_{-p}(S_p \setminus \{-p\}) = \{x_{n+1} = 0\}.$$

Notice that for any $\Pi \in X^2(p)$, there exists some nonzero vector $(r_1, \ldots, r_n) \in \mathbf{R}^n$, determined up to constant multiplication, such that a 2-plane $\pi_{-p}(\Pi \setminus \{p\})$ is generated by $(0, \ldots, 0, 1)$, $(r_1, \ldots, r_n, 0) \in \mathbf{R}^{n+1}$, and notice that any nonzero vector of $\mathbf{R}^n$ determines some Π in this way. Then it is easy to see that for p, there exists some open neighborhood U_p in M such that for any point $a \in U_p$, there exists some Π such that $a \in M_\Pi(p)$, i.e.,

$$U_p \subset \sqcup M_\Pi(p).$$

Similarly, it follows that for q in Lemma 2.5, there exists some open neighborhood U_q in M such that

$$U_q \subset \sqcup M_\Pi(p).$$

On the other hand, take any $\Pi_0 \in X^2(p)$ and any point $a_0 \in M_{\Pi_0}(p) \setminus \{p, q\}$. Corresponding to a_0, there exists some vector $(a_1, \ldots, a_{n+1}) \in \mathbf{R}^{n+1}$ such that $\pi_{-p}(a_0) = (a_1, \ldots, a_{n+1})$. Since

$$\pi_{-p}(\Gamma_0 \setminus \{-p\}) = \{(0, \ldots, 0, r) \in \mathbf{R}^{n+1} \ ; \ r \in \mathbf{R}\},$$

where Γ_0 is in the proof of Lemma 2.5, and since $M_{\Pi_0}(p) \cap \Gamma_0 = \{p, q\}$ by the proof of Lemma 2.5, it follows that $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$. So set

$$r = (r_1, \ldots, r_n) = \frac{1}{\sqrt{a_1^2 + \ldots + a_n^2}}(a_1, \ldots, a_n) \in S_0^{n-1}$$

where S_0^{n-1} is the unit sphere centered at the origin in a hyperplane $\{x_{n+1} = 0\}$, and let $(U_r \ ; \ (\theta_1, \ldots, \theta_{n-1}))$ be a local coordinate neighborhood of r in S_0^{n-1}. Functions $\theta_1, \ldots, \theta_{n-1}$ can be extended to the domain

$$\{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \ ; \ (x_1, \ldots, x_n) \neq (0, \ldots, 0), \quad \frac{1}{\sqrt{x_1^2 + \ldots + x_n^2}}(x_1, \ldots, x_n) \in U_r\}$$

in $\mathbf{R}^{n+1}$. Then, since M and Π_0 cross transversely in S^{n+1} at a_0, it is easy to see that there exist some neighborhood U'_{a_0} of a_0 in M and some smooth funciton θ_n on U'_{a_0} such that $(U'_{a_0} \ ; \ (\theta_1, \ldots, \theta_n))$ is a local coordinate eighborhood of a_0 in M. From U'_{a_0}, we obtain a 'rectangular neighborhood' U_{a_0}, i.e.,

$$U_{a_0} = \{x \in U'_{a_0} \ ; \ b_i < \theta_i < c_i\}$$

where b_i and $c_i (i = 1, \ldots, n)$ are real numbers satisfying $a_0 \in U_{a_0}$. Notice that for any

$$(\theta_1, \ldots, \theta_{n-1}) \in (b_1, c_1) \times \ldots \times (b_{n-1}, c_{n-1}),$$

the set

$$\{(\theta_1, \ldots, \theta_{n-1}, r_n) \ ; \ r_n \in (b_n, c_n)\} \subset U_{a_0}$$

is contained in some $\Pi \in X^2(p)$. Since $M_{\Pi_0}(p)$ is a compact set of M, there exists a finite set $\{U_i\}_{i=1}^k$ where each U_i is such an open set as U_{a_0}, such that the set

$$\{U_p, U_q, U_1, \ldots, U_k\}$$

is an open covering of $M_{\Pi_0}(p)$. Then, we easily find that for $a_0 \in M_{\Pi_0}(p) \setminus \{p, q\}$, there exists some open neighborhood V_{a_0} such that for any $a \in V_{a_0}$, there exists some Π such that $a \in M_\Pi(p)$, i.e.,

$$V_{a_0} \subset \sqcup M_\Pi(p).$$

Moreover, as V_{a_0}, we can take an open neighborhood whose closure is still contained in $\sqcup M_\Pi(p)$. Hence it follows that the subset $\sqcup M_\Pi(p)$ of M is open, closed and, needless to say, not empty in M. Since M is connected, it follows that

$$M = \sqcup M_\Pi(p). \ \square$$

Remark 2.1. The local strict convexity of ι implies the local convexity of ι. Therefore Lemma 2.6 can also be obtained from the theorem of do Carmo-Warner in Section 1. However, notice that our proof of Lemma 2.6 is independent of the theorem of do Carmo-Warner.

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