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Actions of Loop Groups on the Space of Harmonic Maps into Reductive Homogeneous Spaces

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Abstract. In this paper we study special affine harmonic maps into reductive homogeneous spaces and prove that there exist loop group actions on such harmonic maps.

Introduction

M. A. Guest and Y. Ohnita [5] clearly described the loop group actions on harmonic maps from Riemann surfaces into compact Lie groups. Later J. Dorfmeister, F. Pedit and H. Wu [4] generalized their result to harmonic maps into symmetric spaces of compact type. F. E. Burstall and F. Pedit [1],[2] still generalized this to the special harmonic maps into k-symmetric space, the so-called primitive maps from Riemann surfaces into a k-symmetric space G/K. They proved that any primitive map is harmonic with respect to the metric induced from a positive definite $\mathrm{Ad}(G)$-invariant quadratic form on the Lie algebra $\mathfrak{g}$ of G. They defined twisted loop group actions on the space of primitive maps.

Generalizing the harmonic map equation between Riemannian manifolds, we define the affine harmonic map ϕ from a Riemannian manifold (M, h) into an affine manifold (N, ∇) by

$$\mathrm{Tr}_h \nabla d\phi = 0.$$

As a special case, we define the ∇-harmonic map ϕ from a Riemann surface into an affine manifold (N, ∇) by

$$\nabla'' \partial\phi = 0.$$

In this paper we study particularly the case where the target manifold is a reductive homogeneous space G/H with the canonical connection ∇^{can}.

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We prove that any ∇^{can}-harmonic map is harmonic with respect to the invariant metric. Moreover we prove that there exist twisted loop group actions on the space of special ∇^{can}-harmonic maps from Riemann surfaces into reductive homogeneous spaces of compact type, generalizing the primitive maps due to F. E. Burstall and F. Pedit.

In §1 we review the canonical connection ∇^{can} on reductive homogeneous spaces. In §2 we define the equation of ∇-harmonic map which is a special class of maps from a Riemann surface Σ into an affine manifold (N, ∇). In §3 we express the equation of ∇^{can}-harmonic maps to a reductive homogeneous space $(G/H, \nabla^{\mathrm{can}})$ in terms of a framing which is a map from Σ into G. In §4 we define special ∇^{can}-harmonic maps and reformulate the equation by using the loop algebra. In §5 we prove the factorization theorem of loop groups. In §6 we define the action of loop groups on the space of special ∇^{can}-harmonic maps by using the factorization theorem in §5. In §7 we give some examples.

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§1. Canonical connection on reductive homogeneous spaces

Let G be a Lie group and H a closed subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{h}$ the Lie algebra of H, and $\mathfrak{m}$ an $\mathrm{Ad}(H)-$invariant summand. We define the G-invariant distribution $Q(\mathfrak{m})$ on G by the left translation L_g, namely by

$$Q(\mathfrak{m})_g = (dL_g)(\mathfrak{m}).$$

In the sequel we write Q for $Q(\mathfrak{m})$ for short.

The distribution $Q(\mathfrak{m})$ defines a G-invariant connection in the principal bundle

$$\pi : G \longrightarrow G/H.$$

We identify the tangent space at $x_0 = eH$ of $N = G/H$ with $\mathfrak{g}/\mathfrak{h}$ by

$$[\xi] \longmapsto \left.\frac{d}{dt}\right|_{t=0} \exp t\xi \cdot x_0.$$

This provides an isomorphism of the associated bundle $G \times_H \mathfrak{g}/\mathfrak{h}$ with TN by

$$[g, \xi] \longmapsto g_* \left(\frac{d}{dt}\bigg|_{t=0} \exp t\xi \cdot x_0 \right) = \frac{d}{dt}\bigg|_{t=0} \exp t \,\mathrm{Ad} g\, \xi \cdot x$$

where $x = \pi(g)$. Since the homogeneous space N is reductive we can identify $G \times_H \mathfrak{g}/\mathfrak{h}$ with $G \times_H \mathfrak{m}$. On the other hand we have a natural inclusion

$$G \times_H \mathfrak{m} \longrightarrow G \times_H \mathfrak{g}.$$

Moreover the associated bundle $G \times_H \mathfrak{g}$ is canonically identified with the trivial bundle $N \times \mathfrak{g}$ via

$$[g, \xi] \longmapsto (\pi(g), \mathrm{Ad} g\, \xi).$$

Thus we have an identification of TN with a subbundle of $N \times \mathfrak{g}$, which we denote by

$$\beta : TN \hookrightarrow N \times \mathfrak{g}.$$

In particular we may view β as a $\mathfrak{g}$ -valued 1-form on N.

If $X_x = g_* \left(\frac{d}{dt}\big|_{t=0} \exp t\xi \cdot x_0 \right) \quad (\xi \in \mathfrak{m}, \pi(g) = x)$ then we have

$$\beta_x(X) = \mathrm{Ad} g\, \xi \tag{1.1}$$

Using the covariant derivative in the associated bundle $G \times_H \mathfrak{m}$ induced by the connection Q in the principal bundle $\pi : G \longrightarrow N$, we define the covariant derivative ∇ in the tangent bundle TN. The following lemma was proved in Burstall and Rawnslay [3].

Lemma 1.1 [3].

$$\beta(\nabla_X Y) = X\beta(Y) - [\beta(X), \beta(Y)] \qquad X, Y \in \mathbf{\Gamma}(TN)$$

Invariant connections on reductive homogeneous spaces was studied by K. Nomizu [7].

Theorem 1.2 [7]. *There exists a one-to-one correspondence between the set of all invariant connections on a reductive homogeneous space G/H*

and the set of all bilinear functions γ *on* $\mathfrak{m} \times \mathfrak{m}$ *with values in* $\mathfrak{m}$ *which are invariant by* $Ad(H)$. *The correspondence is given by*

$$\gamma(X, Y) = (\nabla_{\widetilde{X}} \widetilde{Y})_{x_0}$$

where

$$\widetilde{X}_x = \left.\frac{d}{dt}\right|_{t=0} exp\, t\, Adg\, X \cdot x,$$
$$\widetilde{Y}_x = \left.\frac{d}{dt}\right|_{t=0} exp\, t\, Adg\, Y \cdot x \quad (X, Y \in \mathfrak{m},\ \pi(g) = x).$$

DEFINITION 1.3 [7]. The invariant connection defined by the zero function on $\mathfrak{m} \times \mathfrak{m}$ is called the canonical affine connection of the second kind on a reductive homogeneous space G/H with respect to a fixed reductive decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Now we state the relationship between our connection and the canonical connection of the second kind.

PROPOSITION 1.4. *The invariant connection defined by the distribution* $Q(m)$ *is the canonical affine connection of the second kind with respect to the decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

PROOF. We have

$$\beta(\nabla_{\widetilde{X}} \widetilde{Y}) = \widetilde{X} \beta(\widetilde{Y}) - [\beta(\widetilde{X}), \beta(\widetilde{Y})]$$

by Lemma 1.1.

First we calculate the left-hand side at $x_0 \in N$. By the definition of β and γ

$$\beta_{x_0}(\nabla_{\widetilde{X}} \widetilde{Y}) = (\nabla_{\widetilde{X}} \widetilde{Y})_{x_0} = \gamma(X, Y).$$

Next, by the definition of our connection and (1.1), we have

$$\begin{aligned}(\widetilde{X}\beta(\widetilde{Y}))_{x_0} &= (\pi_*\widetilde{X}^*)_{x_0}\beta(\widetilde{Y}) \\ &= \widetilde{X}^*_g(\beta(\widetilde{Y})\circ\pi) \\ &= \widetilde{X}^*_g(\mathrm{Ad}g\,Y) \\ &= \frac{d}{dt}\bigg|_{t=0}\big\{(\exp t\,\mathrm{Ad}g\,X)g\cdot Y\cdot g^{-1}(\exp(-t\,\mathrm{Ad}g\,X)\big\} \\ &= (\mathrm{Ad}g\,X)g\cdot Y\cdot g^{-1}+g\cdot Y\cdot g^{-1}(-\mathrm{Ad}g\,X) \\ &= \mathrm{Ad}g[X,Y]\end{aligned}$$

and

$$[\beta_{x_0}(\widetilde{X}),\beta_{x_0}(\widetilde{Y})] = \mathrm{Ad}_g[X,Y].$$

So we have

$$(\widetilde{X}\beta(\widetilde{Y}))_{x_0} - [\beta_{x_0}(\widetilde{X}),\beta_{x_0}(\widetilde{Y})] = 0. \ \square$$

Let ∇ be the invariant connection on G/H determined by a bilinear function α on $\mathfrak{m}\times\mathfrak{m}$. Since the torsion tensor of ∇ is invariant by G, it is determined by its value at $x_0 \in N$, where the torsion tensor is given by

$$T(\widetilde{X},\widetilde{Y}) = \nabla_{\widetilde{X}}\widetilde{Y} - \nabla_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}].$$

Evaluating at x_0, we obtain (1.2).

(1.2) $$T(X,Y) = \alpha(X,Y) - \alpha(Y,X) - [X,Y]_{\mathfrak{m}} \qquad (X,Y\in\mathfrak{m})$$

LEMMA 1.5. *The torsion tensor of the canonical connection is given by*

$$T(X,Y) = -[X,Y]_{\mathfrak{m}} \qquad (X,Y\in\mathfrak{m})$$

Now we give the formula for the Levi-Civita connection on G/H.

DEFINITION 1.6 [7]. The invariant connection defined by the function γ on $\mathfrak{m}\times\mathfrak{m}$ given by

$$\gamma(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}} \qquad X,Y\in\mathfrak{m},$$

is called the canonical affine connection of the first kind on a reductive homogeneous space G/H with respect to a fixed reductive decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

REMARK. From (1.2) the canonical affine connection of the first kind is torsion free.

THEOREM 1.7 [7]. *Let G be a compact connected Lie group and H a closed subgroup. Let B be a positive definite $Ad(G)$-invariant quadratic form on the Lie algebra $\mathfrak{g}$ and $\mathfrak{m}$ be the subspace orthogonal to the subalgebra $\mathfrak{h}$ of H with respect to this form. Let g be the Riemannian metric on G/H determined by the restriction of B to $\mathfrak{m}\times\mathfrak{m}$. Then the Levi-Civita connection ∇^g coincides with the canonical affine connection of the first kind.*

From Definition 1.6 and Theorem 1.7 we immediately obtain the following:

COROLLARY 1.8. *Let g be as in Theorem 1.7. Then the Levi-Civita connection ∇^g is given by*

$$\beta(\nabla^g_X Y) = X\beta(Y) - [\beta(X), \beta(Y)] + \frac{1}{2}P[\beta(X), \beta(Y)] \qquad X, Y \in \mathbf{\Gamma}(TN)$$

where P denotes the projection onto the tangent bundle

$$P : N \times \mathfrak{g} \longrightarrow G/H \times_H \mathfrak{m}.$$

§2. Harmonic maps into reductive homogeneous spaces

A map $\phi : M \longrightarrow N$ between Riemannian manifolds is harmonic if it extremizes the energy functional

$$E(\phi) = \int_D |d\phi|^2 \, dv_g$$

on all compact sub-domain $D \subset M$. The Euler-Lagrange equation of the energy functional is given by

$$\tau(\phi) = \mathrm{Trace}\nabla^L d\phi$$

where ∇^L is the connection in the vector bundle $T^*M \otimes \phi^{-1}TN$ which is induced by the Levi-Civita connection of M and N.

We generalize this equation for a map from a Riemannian manifold into an affine manifold.

Definition 2.1. Let $\phi : (M^n, h) \longrightarrow (N, \nabla^N)$ be a smooth map from a Riemannian manifold (M^n, h) into an affine manifold (N, ∇^N). We call ϕ an affine harmonic map if

$$\sum_{i=1}^{n} \nabla d\phi(e_i, e_i) = 0$$

where $\{e_i\}_{i=1,\cdots,n}$ is an orthonormal frame of TM.

Remark. This definition is independent of the choice of the framings.

Now we study the case where the domain manifold is a Riemannian surface Σ and the target manifold is an affine manifold (N, ∇^N). Let $z = x + iy$ be an isothermal coordinate in the Riemannian surface Σ. Then by the definition of the induced connection in $T^*M \otimes \phi^{-1}TN$

$$\begin{aligned}\nabla d\phi\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}\right) &= \nabla_{\frac{\partial}{\partial \bar{z}}} d\phi\left(\frac{\partial}{\partial z}\right) - d\phi\left(\nabla^{\Sigma}_{\frac{\partial}{\partial \bar{z}}} \frac{\partial}{\partial z}\right)\\ &= \nabla_{\frac{\partial}{\partial \bar{z}}} d\phi\left(\frac{\partial}{\partial z}\right)\end{aligned}$$

where ∇^Σ is the Levi-Civita connection of Σ and ∇ the induced connection by ∇^Σ and ∇^N.

The equation

$$\nabla_{\frac{\partial}{\partial \bar{z}}} d\phi\left(\frac{\partial}{\partial z}\right) = 0$$

is well-defined. We express this equation by

$$\nabla'' \partial \phi = 0.$$

Definition 2.2. Let $\phi : \Sigma \longrightarrow N$ be a smooth map from a Riemannian surface Σ into an affine manifold (N, ∇^N). We call ϕ a ∇^N-harmonic map if

$$\nabla'' \partial \phi = 0 \tag{2.1}$$

REMARK. Any ∇^N-harmonic map is an affine harmonic map.

If the connection in the target space ∇^N is torsion free then $\nabla d\phi$ is symmetric. So we have the following remark.

REMARK. Let N be a Riemannian manifold and ∇^L be the Levi-Civita connection. Then ϕ is a ∇^L-harmonic map if and only if ϕ is a harmonic map.

In connection with this remark, we study the case where the target manifold is a reductive homogeneous space.

PROPOSITION 2.3. *Let ∇^{can} be the canonical connection on a reductive homogeneous space G/H and g a Riemannian metric on G/H induced from an invariant metric on the Lie algebra $\mathfrak{g}$. Then any ∇^{can}-harmonic map $\phi : \Sigma \longrightarrow G/H$ is a harmonic map into $(G/H, g)$.*

PROOF. From Corollary 1.8 we have

$$\beta(Tr\nabla^g d\phi) = \beta(Tr\nabla^{\text{can}} d\phi). \ \square$$

§3. Framings of ∇^{can}-harmonic maps into reductive homogeneous spaces

Now we study a smooth map $\phi : \Sigma \longrightarrow G/H$ from a Riemannian surface into a reductive homogeneous space G/H, associated with the reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{3.1}$$

A framing of ϕ is a map $F : \Sigma \longrightarrow G$ satisfying $\pi \circ F = \phi$ where $\pi : G \longrightarrow G/H$ is the coset projection, which exists on a contractible domain. Now we set

$$\alpha = F^{-1} dF,$$

and decompose it into

$$\alpha = \alpha_{\mathfrak{h}} + \alpha_{\mathfrak{m}}$$

according to (3.1).

Let $\theta : TG \longrightarrow \mathfrak{g}$ be the left Maurer-Cartan form of G. Then with respect to (3.1) we can decompose θ as

$$\theta = \theta_{\mathfrak{h}} + \theta_{\mathfrak{m}}.$$

We have defined $\mathfrak{g}$-valued 1-form β on a reductive homogeneous space N in §1. The $\mathfrak{m}$-valued 1-form $\theta_{\mathfrak{m}}$ on G is related to β by

$$(\pi^*\beta)_g = Adg(\theta_{\mathfrak{m}})_g. \tag{3.2}$$

From this we have the relation between the 1-forms α and β

$$\phi^*\beta = AdF\alpha_{\mathfrak{m}}. \tag{3.3}$$

PROPOSITION 3.1. *Let $\phi : \Sigma \longrightarrow (G/H, \nabla^{\mathrm{can}})$ be a smooth map and $F : \Sigma \longrightarrow G$ a framing of ϕ. Set $\alpha = F^{-1}dF$. A map ϕ satisfies*

$$(\nabla^{\mathrm{can}})''\partial\phi = 0$$

if and only if

$$\bar{\partial}\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{m}}] = 0 \tag{3.4}$$

where $\alpha'_{\mathfrak{m}}$ is the (1,0)-part of $\alpha_{\mathfrak{m}}$.

PROOF. By Lemma 1.1, we have

$$\beta((\nabla^{\mathrm{can}})''\partial\phi) = \bar{\partial}\beta(\partial\phi) - [(\phi^*\beta)'' \wedge \beta(\partial\phi)].$$

By (3.3), we have

$$\beta(\partial\phi) = (\phi^*\beta)' = \mathrm{Ad}F\alpha'_{\mathfrak{m}},$$

and hence

$$\begin{aligned}
\bar{\partial}\beta(\partial\phi) &= \bar{\partial}(\mathrm{Ad}F\alpha'_{\mathfrak{m}}) \\
&= \mathrm{Ad}F\{\bar{\partial}\alpha'_{\mathfrak{m}} + [\alpha'' \wedge \alpha'_{\mathfrak{m}}]\} \\
&= \mathrm{Ad}F\{\bar{\partial}\alpha'_{\mathfrak{m}} + [\alpha''_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{m}}] + [\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]\} \\
&= \mathrm{Ad}F\{\bar{\partial}\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{m}}] + [\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}]\}.
\end{aligned}$$

Moreover we have

$$\begin{aligned}[(\phi^*\beta)'' \wedge \beta(\partial\phi)] &= [(\mathrm{Ad}F\alpha''_{\mathfrak{m}}) \wedge (\mathrm{Ad}F\alpha'_{\mathfrak{m}})] \\ &= \mathrm{Ad}F[\alpha''_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}].\end{aligned}$$

Consequently

$$\beta((\nabla^{can})''\partial\phi) = \mathrm{Ad}F\{\bar{\partial}\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{m}}]\}. \ \square$$

On the other hand, we have the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$$

which is the integrable condition of α. Namely if a $\mathfrak{g}$-valued 1-form α on a Riemannian surface Σ satisfies the above equation and Σ is contractible, then there exists uniquely a map

$$F : \Sigma \longrightarrow G$$

up to the left G-transformations such that

$$\alpha = F^{-1}dF.$$

Moreover if α satisfies the equation (3.4) then

$$\phi = \pi \circ F : \Sigma \longrightarrow G/H$$

is a ∇^{can}-harmonic map.

§4. ω-maps and extended framings

Let G be a compact semisimple Lie group and $\mathfrak{g}$ the Lie algebra of G. Let τ be an automorphism of the Lie group G. The differential of τ, which we denote by the same notation τ, induces a direct sum decomposition of the complex Lie algebra $\mathfrak{g}^{\mathbf{C}}$

$$\mathfrak{g}^{\mathbf{C}} = \sum_{\omega:\text{eigenvalue of } \tau} \mathfrak{g}^{(\omega)}$$

where $\mathfrak{g}^{(\omega)}$ is the ω-eigenspace of τ. Since the Lie algebra automorphism τ preserves the Killing form of $\mathfrak{g}$, the eigenvalues of τ are elements of $S^1 = \{z \in \mathbf{C}; |z| = 1\}$. Then we have

$$\overline{\mathfrak{g}^{(\omega)}} = \mathfrak{g}^{(\overline{\omega})}. \tag{4.1}$$

We define the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by

$$\mathfrak{h}^{\mathbf{C}} = \mathfrak{g}^{(1)}$$

and $\mathfrak{m} \subset \mathfrak{g}$ by

$$\mathfrak{m}^{\mathbf{C}} = \sum_{\omega:\text{eigenvalue of } \tau,\, \omega \neq 1} \mathfrak{g}^{(\omega)}.$$

Then we have a reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Letting H be the Lie group corresponding to the Lie algebra $\mathfrak{h}$, we have a reductive homogeneous space

$$G/H.$$

Next we define an ω-map $\phi : \Sigma \longrightarrow G/H$ where Σ is a Riemannian surface. Let F be a framing of ϕ. Setting $\alpha = F^{-1}dF$, we decompose α into $\mathfrak{h}$ and $\mathfrak{m}$-part

$$\alpha = \alpha_{\mathfrak{h}} + \alpha_{\mathfrak{m}}$$

and $\alpha_{\mathfrak{m}}$ into (1,0) and (0,1)-form

$$\alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}.$$

Definition 4.1. Let ω be a non-real eigenvalue of τ. A map $\phi : \Sigma \longrightarrow G/H$ is defined to be an ω-map if $\alpha''_{\mathfrak{m}}$ is $\mathfrak{g}^{(\omega)}$-valued.

Remark. The above definition is independent of the choice of the framings.

Now let us fix any ω-map $\phi : \Sigma \longrightarrow G/H$, and take a framing F and let $\alpha = F^{-1}dF$ as above. Since $\alpha''_{\mathfrak{m}}$ is a $\mathfrak{g}^{(\omega)}$-valued 1-form, $\alpha'_{\mathfrak{m}}$ is a $\mathfrak{g}^{(\overline{\omega})}$-valued 1-form by (4.1). So $[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]$ is $\mathfrak{h}$-valued. Namely

$$[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]_{\mathfrak{m}} = 0.$$

We decompose the integrability condition

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$$

into $\mathfrak{g}^{(\omega)}$, $\mathfrak{g}^{(\overline{\omega})}$ and $\mathfrak{h}$-parts

$$\begin{aligned} &\bar{\partial}\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{m}}] = 0 \\ &\partial\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{h}} \wedge \alpha''_{\mathfrak{m}}] = 0 \\ &d\alpha_{\mathfrak{h}} + \frac{1}{2}[\alpha_{\mathfrak{h}} \wedge \alpha_{\mathfrak{h}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}] = 0. \end{aligned} \tag{4.2}$$

Thus we obtain the following lemma from (4.2) and (3.4).

Lemma 4.2. *Any ω-map is a ∇^{can}-harmonic map.*

Now we define a family of $\mathfrak{g}^{\mathbf{C}}$-valued 1-form on the Riemannian surface Σ by

$$\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{h}} + \lambda\alpha''_{\mathfrak{m}} \qquad (\lambda \in \mathbf{C}^*).$$

We observe that (4.2) is equivalent to

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \qquad \text{for any } \lambda \in \mathbf{C}^*.$$

We obtain the unique map

$$F_\lambda : \Sigma \longrightarrow G^{\mathbf{C}}$$

satisfying

$$F_\lambda^{-1} dF_\lambda = \alpha_\lambda, \qquad F_\lambda(p_0) = e$$

where p_0 is a point of Σ fixed once and for all. Since $\alpha'_{\mathfrak{m}}$ is $\mathfrak{g}^{(\varpi)}$-valued, α_λ satisfies

$$\tau\alpha_\lambda = \alpha_{\omega\lambda} \qquad \text{for any } \lambda \in \mathbf{C}^*$$

and

$$\overline{\alpha_\lambda} = \alpha_{1/\bar{\lambda}} \qquad \text{for any } \lambda \in \mathbf{C}^*.$$

Then by the uniqueness, we have

$$\tau F_\lambda = F_{\omega\lambda} \qquad \text{for any } \lambda \in \mathbf{C}^*$$

and

$$\overline{F_\lambda} = F_{1/\bar{\lambda}} \qquad \text{for any } \lambda \in \mathbf{C}^*$$

where conjugation is the Cartan involution of $G^{\mathbf{C}}$ fixing G. Moreover for each $p \in \Sigma$,

$$\lambda \longmapsto F_\lambda(p)$$

is holomorphic on $\mathbf{C}^*$. So we have defined a map

$$\widetilde{F} : \Sigma \longrightarrow \Lambda_{hol} G_{\tau,\omega}$$

where

$$\begin{aligned}&\Lambda_{hol} G_{\tau,\omega}\\ &= \left\{ g : \mathbf{C}^* \longrightarrow G^{\mathbf{C}}; g \text{ is holomorphic}, \tau g(\lambda) = g(\omega\lambda), \overline{g(\lambda)} = g(1/\bar{\lambda}) \right\}.\end{aligned}$$

Definition 4.3. A map $\widetilde{F} : \Sigma \longrightarrow \Lambda_{hol} G_{\tau,\omega}$ is called an extended framing if

$$\widetilde{F}^{-1} d\widetilde{F} = \lambda^{-1}\alpha'_{-1} + \alpha_o + \lambda\alpha''_1$$

where α'_{-1} is a 1-form of type (1,0) on Σ.

Remark. The above definition is equivalent to the following condition

$$\lambda \widetilde{F}^{-1} \partial \widetilde{F} \text{ is holomorphic with respect to } \lambda \text{ at } \lambda = 0.$$

Thus we observe that for any ω-map ϕ, we have an extended framing F_λ such that F_1 is a framing of ϕ.

§5. Factorization of loop groups

Let G be a compact semisimple Lie group, τ an automorphism of G and H the fixed point set of τ. We fix an Iwasawa decomposition of the complex Lie group $H^{\mathbf{C}}$

$$H^{\mathbf{C}} = H \cdot B$$

where B is a solvable subgroup of $H^{\mathbf{C}}$. In particular any element $h \in H^{\mathbf{C}}$ can be uniquely written as

$$h = h_H \cdot h_B$$

where $h_H \in H$ and $h_B \in B$. We fix a positive real number ε such that $0 < \varepsilon < 1$. Let C_ε and $C_{1/\varepsilon}$ denote the circles of radius ε and $1/\varepsilon$ about $0 \in \mathbf{C}$ in the Riemann sphere $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$. We define open sets in $\mathbf{P}^1$ by

$$I = \{\lambda \in \mathbf{P}^1; |\lambda| < \varepsilon \text{ or } |\lambda| > 1/\varepsilon\}$$
$$E = \{\lambda \in \mathbf{P}^1; \varepsilon < |\lambda| < 1/\varepsilon\}$$

and set $C^{(\varepsilon)} = C_\varepsilon \cup C_{1/\varepsilon}$. Let

$$G^{\mathbf{C}} = G \cdot \tilde{B}$$

be an Iwasawa decomposition of $G^{\mathbf{C}}$. We define a smooth loop group $\Lambda^\varepsilon G$ by

$$\Lambda^\varepsilon G = \{g : C^{(\varepsilon)} \longrightarrow G^{\mathbf{C}}; \overline{g(\lambda)} = g(1/\overline{\lambda}) \text{ for all } \lambda \in C^{(\varepsilon)}\}$$

and subgroups of $\Lambda^\varepsilon G$ by

$$\Lambda^\varepsilon_E G = \{g \in \Lambda^\varepsilon G; g \text{ extends holomorphically to } E\}$$
$$\Lambda^\varepsilon_{I,\tilde{B}} G = \{g \in \Lambda^\varepsilon G; g \text{ extends holomorphically to } I \text{ and } g(0) \in \tilde{B}\}.$$

For any $g \in \Lambda^\varepsilon G$, set $g_1 = g|_{C_\varepsilon}, g_2 = g|_{C_{1/\varepsilon}}$. Sometimes we write $g = (g_1, g_2)$ for g. The following theorem is given in [2].

THEOREM 5.1 [2].

$$\Lambda^\varepsilon G = \Lambda^\varepsilon_E G \cdot \Lambda^\varepsilon_{I,\tilde{B}} G, \quad \Lambda^\varepsilon_E G \cap \Lambda^\varepsilon_{I,\tilde{B}} G = \{e\}.$$

Let ω be an eigen value of the automorphism $\tau : \mathfrak{g}^{\mathbf{C}} \longrightarrow \mathfrak{g}^{\mathbf{C}}$. We define subgroups of the loop group $\Lambda^\varepsilon G$ as follows

$$\begin{aligned}
&\Lambda^\varepsilon G_{\tau,\omega} = \{g \in \Lambda^\varepsilon G; \tau g(\lambda) = g(\omega\lambda)\}\\
&\Lambda^\varepsilon_E G_{\tau,\omega} = \{g \in \Lambda^\varepsilon G_{\tau,\omega}; g \text{ extends holomorphically to } E\}\\
&\Lambda^\varepsilon_{I,B} G_{\tau,\omega}\\
&\quad = \{g \in \Lambda^\varepsilon G_{\tau,\omega}; g \text{ extends holomorphically to } I \text{ and } g(0) \in B\}.
\end{aligned}$$

Let

$$\Lambda^\varepsilon \mathfrak{g} = \{\xi : C^{(\varepsilon)} \longrightarrow \mathfrak{g}^{\mathbf{C}}; \overline{\xi(\lambda)} = \xi(1/\overline{\lambda}) \text{ for all } \lambda \in C^{(\varepsilon)}\}$$

be the loop algebra of the loop group $\Lambda^\varepsilon G$. Let

$$\begin{aligned}
&\Lambda^\varepsilon \mathfrak{g}_{\tau,\omega} = \{\xi \in \Lambda^\varepsilon \mathfrak{g}; \tau\xi(\lambda) = \xi(\omega\lambda)\}\\
&\Lambda^\varepsilon_E \mathfrak{g}_{\tau,\omega} = \{\xi \in \Lambda^\varepsilon \mathfrak{g}_{\tau,\omega}; \xi \text{ extends holomorphically to } E\}\\
&\Lambda^\varepsilon_{I,\tilde{\mathfrak{b}}} \mathfrak{g}_{\tau,\omega} = \{\xi \in \Lambda^\varepsilon \mathfrak{g}_{\tau,\omega}; \xi \text{ extends holomorphically to } I \text{ and } \xi(0) \in \tilde{\mathfrak{b}}\}
\end{aligned}$$

be the corresponding loop algebras of the loop groups $\Lambda^\varepsilon G_{\tau,\omega}$, $\Lambda^\varepsilon_E G_{\tau,\omega}$ and $\Lambda^\varepsilon_{I,\tilde{B}} G_{\tau,\omega}$.

COROLLARY 5.4.

$$\Lambda^\varepsilon G_{\tau,\omega} = \Lambda^\varepsilon_E G_{\tau,\omega} \cdot \Lambda^\varepsilon_{I,B} G_{\tau,\omega}, \quad \Lambda^\varepsilon_E G_{\tau,\omega} \cap \Lambda^\varepsilon_{I,B} G_{\tau,\omega} = \{e\}.$$

PROOF. We only have to show that the above factorization is possible. Take any element g of $\Lambda^\varepsilon G_{\tau,\omega}$ and factorize g as

$$g = g_E \cdot g_I$$

by Theorem 5.1. Then we have

$$\begin{aligned}
g(\omega\lambda) &= g_E(\omega\lambda) \cdot g_I(\omega\lambda)\\
\tau g(\lambda) &= \tau g_E(\lambda) \cdot \tau g_I(\lambda).
\end{aligned}$$

We prove

$$g_E \in \Lambda^\varepsilon_E G_{\tau,\omega} \quad \text{and} \quad g_I \in \Lambda^\varepsilon_{I,\tilde{B}} G_{\tau,\omega}.$$

By the uniqueness of the factorization in Theorem 5.1, it is sufficient to prove

$$\tau g_E \in \Lambda^\varepsilon_E G \quad \text{and} \quad \tau g_I \in \Lambda^\varepsilon_{I,\tilde{B}} G.$$

These are shown easily since

$$\tau : \Lambda^\varepsilon \mathfrak{g}_{\tau,\omega} \longrightarrow \Lambda^\varepsilon \mathfrak{g}_{\tau,\omega}$$

preserves $\Lambda^\varepsilon_E \mathfrak{g}_{\tau,\omega}$ and $\Lambda^\varepsilon_{I,\tilde{\mathfrak{b}}} \mathfrak{g}_{\tau,\omega}$, hence τ preserves $\Lambda^\varepsilon_E G_{\tau,\omega}$ and $\Lambda^\varepsilon_{I,\tilde{B}} G_{\tau,\omega}$.

Put

$$c = g_I(0) \in H^{\mathbf{C}}$$

and decompose c as

$$c = c_H \cdot c_B$$

where $c_H \in H$ and $c_B \in B$. If we set

$$\tilde{g_E} = g_E \cdot c_H$$
$$\tilde{g_I} = c_H^{-1} \cdot g_I$$

then

$$\tilde{g_E} \in \Lambda^\varepsilon_E G_{\tau,\omega}, \qquad \tilde{g_I} \in \Lambda^\varepsilon_{I,B} G_{\tau,\omega}$$

and

$$g = \tilde{g_E} \cdot \tilde{g_I}.$$

So the above factorization is possible. □

§6. Actions of loop groups

By Corollary 5.4 the loop group $\Lambda^\varepsilon_{I,B} G_{\tau,\omega}$ acts on $\Lambda^\varepsilon_E G_{\tau,\omega}$ as follows. For $g \in \Lambda^\varepsilon_{I,B} G_{\tau,\omega}$ and $h \in \Lambda^\varepsilon_E G_{\tau,\omega}$ we define $g \sharp h \in \Lambda^\varepsilon_E G_{\tau,\omega}$ by

$$g \sharp h = (g \cdot h)_E,$$

where $(g \cdot h)_E$ is the $\Lambda^\varepsilon_E G_{\tau,\omega}$-component of $g \cdot h$ with respect to the decomposition of Corollary 5.4.

For $g \in \Lambda^{\varepsilon}_{I,B} G_{\tau,\omega}$ and an extended framing $\widetilde{F} : \Sigma \longrightarrow \Lambda_{hol} G_{\tau,\omega}$ we define $g \sharp \widetilde{F} : \Sigma \longrightarrow \Lambda_{hol} G_{\tau,\omega}$ by

$$(g \sharp \widetilde{F})(p) = g \sharp \widetilde{F}(p) \qquad \text{for } p \in \Sigma.$$

PROPOSITION 6.1. *$g \sharp \widetilde{F}$ is an extended framing.*

PROOF. We decompose $g \cdot \widetilde{F}$ by Corollary 5.4 as

$$g \cdot \widetilde{F} = a \cdot b$$

where $a = g \sharp \widetilde{F} : \Sigma \longrightarrow \Lambda^{\varepsilon}_{E} G_{\tau,\omega}$ and $b : \Sigma \longrightarrow \Lambda^{\varepsilon}_{I,B} G_{\tau,\omega}$. Then we have

$$a^{-1} da = \mathrm{Ad} b(\widetilde{F}^{-1} d\widetilde{F} - b^{-1} db).$$

So we have

$$\lambda a^{-1} \partial a = \mathrm{Ad} b(\lambda \widetilde{F}^{-1} \partial \widetilde{F} - \lambda b^{-1} \partial b).$$

By the definition of an extended framing, $\lambda \widetilde{F}^{-1} \partial \widetilde{F}$ is holomorphic with respect to λ at $\lambda = 0$. Since b is $\Lambda^{\varepsilon}_{I,B} G_{\tau,\omega}$-valued, $\lambda b^{-1} \partial b$ is also holomorphic with respect to λ at $\lambda = 0$. Consequently $\lambda a^{-1} \partial a$ is holomorphic at $\lambda = 0$. This implies that $g \sharp \widetilde{F}$ is an extended framing. □

PROPOSITION 6.2. *The action of $\Lambda^{\varepsilon}_{I,B} G_{\tau,\omega}$ on $\Lambda^{\varepsilon}_{E} G_{\tau,\omega}$ induces the action on $\Lambda^{\varepsilon}_{E} G_{\tau,\omega} / H$.*

PROOF. First we prove that the action preserves H. Let g be any element of $\Lambda^{\varepsilon}_{I,B} G_{\tau,\omega}$ and k any element of H. Putting $g_0 = g(0) \in B$, we apply the Iwasawa decomposition to $g_0 \cdot k$ as

$$g_0 \cdot k = (g_0 \cdot k)_H \cdot (g_0 \cdot k)_B$$

where $(g_0 \cdot k)_H \in H$ and $(g_0 \cdot k)_B \in B$. Then we have

$$g \cdot k = (g_0 \cdot k)_H \cdot \{(g_0 \cdot k)_H^{-1} g k\}$$

where $(g_0 \cdot k)_H^{-1} g k \in \Lambda^{\varepsilon}_{I,B} G_{\tau,\omega}$. Consequently we obtain

$$g \sharp k = (g_0 \cdot k)_H \in H.$$

Let h be any element of $\Lambda^{\varepsilon}_{E}G_{\tau,\omega}$. Then we have

$$\begin{aligned} g\sharp(hk) &= (ghk)_E \\ &= \{(gh)_E \cdot (gh)_I k\}_E \quad (\text{where } (gh)_I \in \Lambda^{\varepsilon}_{I,B}G_{\tau,\omega}) \\ &= (gh)_E \cdot \{(gh)_I k\}_E \\ &= (g\sharp h) \cdot \{(gh)_I \sharp k\} \end{aligned}$$

where $(gh)_I\sharp k = \{(gh)_I(0) \cdot k\}_H \in H$. $\square$

Now we fix a point p_0 of Σ. Let ω be an eigenvalue of the induced automorphism $\tau : \mathfrak{g}^{\mathbf{c}} \longrightarrow \mathfrak{g}^{\mathbf{c}}$. Let

$$\mathcal{H}_\omega = \{\phi : \Sigma \longrightarrow G/H;\ \omega\text{-map}\ ,\ \phi(p_0) = e \cdot H\}$$

be the space of based ω-maps,

$$\mathcal{E}_\omega = \{F : \Sigma \longrightarrow \Lambda_{hol}G_{\tau,\omega};\ \text{extended framing}\ ,\ F(p_0) \in H\}$$

the space of based extended framings, and

$$\mathcal{K} = C^\infty(\Sigma, H)$$

the space of smooth functions from Σ to H. Then the group $\mathcal{K}$ acts on $\mathcal{E}_\omega$ by point-wise multiplication on the right. Moreover we have a bijective correspondence

$$\mathcal{H}_\omega \cong \mathcal{E}_\omega/\mathcal{K}.$$

From Proposition6.1 and Proposition6.2, we see that the group $\Lambda^{\varepsilon}_{I,B}G_{\tau,\omega}$ acts on $\mathcal{E}_\omega/\mathcal{K}$. Consequently we have the following theorem.

Theorem 6.3. *The group $\Lambda^{\varepsilon}_{I,B}G_{\tau,\omega}$ acts on $\mathcal{H}_\omega$.*

§7. Examples

Let $G = SO(5)$ and $\mathfrak{g}$ be the Lie algebra of G. Let H_1 and H_2 be the elements of $\mathfrak{g}$ given by

$$H_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For any $a, b \in \mathbf{R}$, we define the automorphism $\tau_{a,b}$ of G as

$$\tau_{a,b} = \mathrm{e}^{\mathrm{ad}(aH_1 + bH_2)}.$$

Then fixed point set of generic $\tau_{a,b}$ is $H = SO(2) \times SO(2)$. The group H is a subgroup of G

$$H = SO(2) \times SO(2) \ni (A, B) \longmapsto \begin{bmatrix} A & & \\ & B & \\ & & 1 \end{bmatrix} \in SO(5).$$

We define a subspace $\mathfrak{m}$ of $\mathfrak{g}$ by

$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & 0 & 0 & a_{35} \\ -a_{14} & -a_{24} & 0 & 0 & a_{45} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 \end{bmatrix} ; a_{13}, a_{14}, \cdots a_{45} \in \mathbf{R} \right\}$$

Then we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

where $\mathfrak{h}$ is the Lie algebra of H

$$\mathfrak{h} = \left\{ \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ -a_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{34} & 0 \\ 0 & 0 & -a_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ; a_{12}, a_{34} \in \mathbf{R} \right\}.$$

and G/H is a reductive homogeneous space but not a symmetric space. The automorphism $\tau_{a,b} : G \longrightarrow G$ induces the Lie algebra automorphism

$$\tau_{a,b} : \mathfrak{g}^{\mathbf{C}} \longrightarrow \mathfrak{g}^{\mathbf{C}}.$$

The eigenvalues of $\tau_{a,b}$ are

$$1\ ,\ \mathrm{e}^{\pm ai}\ ,\ \mathrm{e}^{\pm bi}\ ,\ \mathrm{e}^{\pm(a+b)i}\ ,\ \mathrm{e}^{\pm(a-b)i}.$$

Now we remark that $\Lambda^{\varepsilon}\mathfrak{g} \ni \xi = \Sigma_k \lambda^k \xi_k$ is an element of $\Lambda^{\varepsilon}\mathfrak{g}_{\tau,\omega}$ if and only if any ξ_k is an element of ω^k-eigenspace of $\tau_{a,b}$.

We take $a = \frac{2}{3}\pi$ and $b = 1$ then the eigenvalues of $\tau_{\frac{2}{3}\pi,1}$ are

$$1\ ,\ \mathrm{e}^{\pm\frac{2}{3}\pi i}\ ,\ \mathrm{e}^{\pm i}\ ,\ \mathrm{e}^{\pm(\frac{2}{3}\pi+1)i}\ ,\ \mathrm{e}^{\pm(\frac{2}{3}\pi-1)i}.$$

Set $\omega = \mathrm{e}^{\frac{2}{3}\pi i}$, the eigenvalues are

$$\omega\ ,\ \omega^2\ ,\ \omega^3 = 1\ ,\ \mathrm{e}^{\pm i}\ ,\ \omega\cdot\mathrm{e}^{\pm i}\ ,\ \omega^2\cdot\mathrm{e}^{\pm i}.$$

Then the infinite dimensional loop group $\Lambda^{\varepsilon}_{I,B}G_{\tau,\omega}$ acts on the space of ω-maps.

We take $a = \sqrt{2}$ and $b = 1$ then the eigenvalues of $\tau_{\sqrt{2},1}$ are

$$1\ ,\ \mathrm{e}^{\pm\sqrt{2}i}\ ,\ \mathrm{e}^{\pm i}\ ,\ \mathrm{e}^{\pm(\sqrt{2}+1)i}\ ,\ \mathrm{e}^{\pm(\sqrt{2}-1)i}.$$

Set $\omega = \mathrm{e}^{\sqrt{2}i}$, then the finite dimensional loop group $\Lambda^{\varepsilon}_{I,B}G_{\tau,\omega}$ acts on the space of ω-maps. There is a similar example of a reductive homogeneous space $SP(2)/U(1) \times U(1)$.

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