J. Math. Sci. Univ. Tokyo
5 (1998), 367–399.

The Spaces of Hilbert Cusp Forms for Totally Real Cubic Fields and Representations of $SL_2(\mathbb{F}_q)$

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Abstract. Let $S_{2m}(\Gamma(\mathfrak{p}))$ be the space of Hilbert modular cusp forms for the principal congruence subgroup with level $\mathfrak{p}$ of $SL_2(O_K)$ (here O_K is the ring of integers of K, and $\mathfrak{p}$ is a prime ideal of O_K). Then we have the action of $SL_2(\mathbb{F}_q)$ on $S_{2m}(\Gamma(\mathfrak{p}))$, where $q = N\mathfrak{p}$. When q is a power of an odd prime, for each $SL_2(\mathbb{F}_q)$ we have two irreducible characters which have conjugate values mutually. In the case where K is the field of rationals, M. Eichler gives a formula for the difference of multiplicites of these characters in the trace of the representation of $SL_2(\mathbb{F}_q)$ on $S_{2m}(\Gamma(\mathfrak{p}))$. In the case where K is a real quadratic field, H. Saito gives a formula analogous to that of Eichler for the difference. The purpose of this paper is to give a formula analogous to that of Eichler in the case where K is a totally real cubic field.

1. Introduction

In this paper, we consider the action of $SL_2(\mathbb{F}_q)$ ($\mathbb{F}_q$: a finite field consisting of q elements) on the space of Hilbert modular cusp forms. First, let us explain the motivation for the present paper.

Let K be a totally real number field of degree n, O_K the ring of integers of K, and $\mathfrak{p}$ a prime ideal of O_K. Let $\Gamma(\mathfrak{p})$ be the principal congruence subgroup of $SL_2(O_K)$, and $S_{2m}(\Gamma(\mathfrak{p}))$ the space of Hilbert cusp forms of weight $2m$ with respect to $\Gamma(\mathfrak{p})$. Since $SL_2(O_K)$ acts on $S_{2m}(\Gamma(\mathfrak{p}))$ and $\Gamma(\mathfrak{p})$ acts trivially on it, $SL_2(\mathbb{F}_q) \cong SL_2(O_K)/\Gamma(\mathfrak{p})$ acts on $S_{2m}(\Gamma(\mathfrak{p}))$ (we put $q := \#(O_K/\mathfrak{p})$). Let π be the representation of $SL_2(\mathbb{F}_q)$ on $S_{2m}(\Gamma(\mathfrak{p}))$. For a fixed power q of an odd prime number, there are two irreducible characters

1991 *Mathematics Subject Classification.* Primary 11F41; Secondary 10D21, 12A50.

χ_1, χ_2 of $SL_2(\mathbb{F}_q)$, whose values are conjugates mutually. Let y_i $(i = 1, 2)$ be the multiplicity of χ_i in the character tr π of π. We are interested in the difference $y_1 - y_2$.

There are two ways of considering the difference $y_1 - y_2$. The first way is to express it as the sum of the relative class numbers.

In the case where $n = 1$, $m = 1$, and $\mathfrak{p} = (p)$, Hecke [8] studied the action π of $SL_2(\mathbb{F}_p)$ on $S_2(\Gamma(p))$. He determined how tr π decomposes into irreducible characters. Above all, he showed that the difference $y_1 - y_2$ is expressed as $y_1 - y_2 = -\frac{1}{p}\sum_{i=1}^{p-1} i\left(\frac{i}{p}\right)$, where $\left(\frac{}{p}\right)$ is the quadratic residue symbol mod p. Using the formula of Dirichlet on the class number of an imaginary quadratic field, he showed that

$$(1) \qquad y_1 - y_2 = \begin{cases} 0 & (p \equiv 1 \pmod 4) \\ h_{\mathbb{Q}(\sqrt{-p})} & (p \equiv 3 \pmod 4) \end{cases},$$

where $h_{\mathbb{Q}(\sqrt{-p})}$ denotes the class number of $\mathbb{Q}(\sqrt{-p})$. S. Nakajima interpreted this result as that of Galois coverings of modular curves, and generalized it to the case of Galois coverings of algebraic curves.

In the case $n \geq 2$, H. Saito and H. Yoshida proved the following independently by using the Selberg trace formula: if $m \geq 2$, then we have

$$|y_1 - y_2| = 2^{n-1}\sum_{K_j}\frac{h_{K_j}}{h_K},$$

where K_j runs over totally imaginary quadratic extensions of K with the relative discriminant $\mathfrak{p}$, and h_{K_j} and h_K are the class numbers of K_j and K, respectively. This result is a generalization of Hecke's.

In the case $n = 2, m \geq 1$, W. Meyer and R. Sczech [10] got

$$y_1 - y_2 = -2\sum_{K_j}\frac{h_{K_j}}{h_K},$$

which is a refinement of the result of Saito-Yoshida in the case $n = 2$. They showed it by using the holomorphic Lefschetz formula. In his book [7], van der Geer generalized their result to the general Hilbert modular group.

Concerning this direction, T. Yamazaki, R. Tsushima, and K. Hashimoto studied the action of $Sp_2(\mathbb{F}_p)$ on the space of Siegel cusp forms of degree 2 with respect to $\Gamma(p)$. More precisely, R. Tsushima corrected the error in the result of T. Yamazaki, and presented a conjecture for the multiplicities of certain four irreducible representations of $Sp_2(\mathbb{F}_p)$. Finally, K. Hashimoto solved the conjecture by using the Selberg trace formula.

The second way is to write $y_1 - y_2$ by using the quadratic residue symbol and the intersection numbers of irreducible divisors obtained from the cusp resolution.

In the case $n = 1$ and $m \geq 1$, by using his trace formula, Eichler [3] proved that

$$y_1 - y_2 = \frac{1}{\sqrt{(-1)^{(p-1)/2}p}} \sum_{i=1}^{p-1} \left(\frac{i}{p}\right) \nu(i), \tag{2}$$

where we put

$$\nu(i) := \frac{\mathbf{e}\left[\frac{i}{p}\right]}{1 - \mathbf{e}\left[\frac{i}{p}\right]}, \qquad \mathbf{e}[x] := \exp\left(2\pi\sqrt{-1}x\right).$$

He showed that the right hand side of this equation is equal to $-\frac{1}{p}\sum_{i=1}^{p-1} i \left(\frac{i}{p}\right)$, the Dirichlet expression for $h_{\mathbb{Q}(\sqrt{-p})}$. In this case, the cusps of $\Gamma(p)$ are not singulariites of the modular curve $X(p)$ with level p. As a result, the intersection numbers do not appear in $\nu(i)$.

In the case where $n = 2, h_K = 1, m = 1$, and $\mathfrak{p} = (\mu)$ (μ is a totally positive element of O_K), H. Saito [11] obtained the following, which is similar to the formula (2) of Eichler:

$$y_1 - y_2 = \frac{1}{\sqrt{(-1)^{(q-1)/2}q}} \cdot \frac{2}{[U : U(\mathfrak{p})]} \sum_{\alpha \bmod \mathfrak{p}} \left(\frac{\alpha}{\mathfrak{p}}\right) \nu(\alpha), \tag{3}$$

where $\left(\frac{\quad}{\mathfrak{p}}\right)$ is the quadratic residue symbol modulo $\mathfrak{p}$, and $\nu(\alpha)$ is expressed as $\mathbf{e}[\ ]$ and the self-intersection numbers of irreducible divisors obtained from the cusp resolution. He showed it by using the holomorphic Lefschetz formula.

The purpose of this paper is to gain a formula (see Theorem 4.4) similar to Eichler's formula for $y_1 - y_2$ in the case where $n = 3, h_K = 1$ and $\mathfrak{p} = (\mu)$

(μ is a totally positive element of O_K). We shall show it with the use of the holomorphic Lefschetz formula.

Let us explain the significance of our result. In the process of the proof of Saito-Yoshida's result, the difference $y_1 - y_2$ is expressed as a sum of values at 1 of some certain L-functions, and is proven to be equal to a sum of relative class numbers. Hence $y_1 - y_2$ can be written as an "infinite sum" by using the Selberg trace formula. On the other hand, Hecke and Eichler wrote $y_1 - y_2$ as a "finite sum" in the case $n = 1$. In the case $n = 2$, from this point of view Saito wrote $y_1 - y_2$ as a "finite sum" by some method, i.e., the holomorphic Lefschetz formula other than the Selberg trace formula. Our motivation to prove Theorem 4.4 arises from this point of view. Our result implies that in the case of $n = 3$ the difference $y_1 - y_2$ can be represented as a "finite sum".

The contents of this paper is as follows. In Section 2, we assemble some facts about Hilbert modular forms for the principal congruence subgroups. In Section 3, we review some facts about 3-dimensional Hilbert modular varieties. In Section 4, the statement of our main result is given. In Section 5, we shall prove it. First, Theorem 4.4 will be proven in the case where $m = 1$. And then the theorem will be proven for the general m. In Section 6, we give an example to our result.

Acknowledgement. The author expresses his sincere gratitude to Professor Hiroshi Saito, who suggested this problem and gave encouragement. He would also like to thank Professor Ryuji Tsushima sincerely for his helpful advice. Special thanks are also due to Professors Akira Fujiki, Hirotada Naito, and Hiroshi Saito for their helpful correspondence. He also thanks Professor Takayuki Oda for his interest and encouragement.

Notation. By $\#(S)$, we mean the cardinality of the set S. Put $\mathbf{e}[x] := \exp(2\pi\sqrt{-1}x)$. Let $\mathbb{C}, \mathbb{R}$, and $\mathbb{Q}$ be the field of complex, real, and rational numbers, respectively, and $\mathbb{F}_q$ the finite field consisting of q-elements.

2. Fundamental facts

1. Hilbert modular form

2.1. Let K be a totally real number field of degree n, O_K the ring of integers of K. Set $\mathfrak{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Let $\sigma_1, \cdots, \sigma_n$ be embeddings of K into $\mathbb{R}$. In particular, let σ_1 be a trivial embedding: $\sigma_1(x) = x$ for all $x \in K$. The group $SL_2(O_K)$ acts on $\mathfrak{H}^n$, the n-fold product of $\mathfrak{H}$ as follows: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(O_K)$ and $(z_1, \cdots, z_n) \in \mathfrak{H}^n$, we define

$$\gamma \cdot (z_1, \cdots, z_n) = \left(\frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_1(c)z_1 + \sigma_1(d)}, \cdots, \frac{\sigma_n(a)z_n + \sigma_n(b)}{\sigma_n(c)z_n + \sigma_n(d)} \right). \tag{4}$$

Let $\mathfrak{a}$ be an integral ideal of O_K. We set

$$\Gamma(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(O_K) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{a}} \right\}.$$

It is called the *principal congruence subgroup with level* $\mathfrak{a}$ of $SL_2(O_K)$. The group $\Gamma(\mathfrak{a})$ acts on $K \cup \{\infty\}$ by the linear fractional transformation. The orbits for the action are called the *cusps* for $\Gamma(\mathfrak{a})$.

An additive subgroup M of K which is a free group of rank n is called a *complete* $\mathbb{Z}$*-module* of K. We denote by U_M^+ the group of units u of K which are totally positive and satisfy $uM = M$. The group U_M^+ is a free group of rank $n-1$. For a subgroup V with rank $n-1$ of U_M^+, define

$$G(M, V) := \left\{ \begin{pmatrix} u & \alpha \\ 0 & 1 \end{pmatrix} \mid u \in V,\ \alpha \in M \right\}.$$

For each cusp x of $\Gamma(\mathfrak{a})$, let $\Gamma(\mathfrak{a})_x$ be the stabilizer of x in $\Gamma(\mathfrak{a})$. Then there exists an element ρ of $PGL_2^+(\mathbb{R})^n$ such that $\rho(x) = \infty$ and $\rho\Gamma(\mathfrak{a})_x\rho^{-1} = G(M, V)$. Then the cusp x is called *of type* (M, V). We say two complete $\mathbb{Z}$-modules M_1, M_2 *strictly equivalent* if there exists a totally positive element of u of K such that $uM_1 = M_2$. Then we have $U_{M_1}^+ = U_{M_2}^+$. The strictly equivalence class of M and the group V are completely determined by the cusp x and do not depend upon the choice of ρ.

LEMMA 2.2. *Let* $\lambda = \alpha/\beta$ *be a cusp of* $\Gamma(\mathfrak{a})$ *such that* $O_K\alpha + O_K\beta = \mathfrak{b}$. *Then the stabilizer* $\Gamma(\mathfrak{a})_\lambda$ *of* λ *in* $\Gamma(\mathfrak{a})$ *is isomorphic to* $\left\{ \begin{pmatrix} e & m \\ 0 & e^{-1} \end{pmatrix} \mid e \in U(\mathfrak{a}),\ m \in \mathfrak{a}\mathfrak{b}^{-2} \right\}$, *where* $U(\mathfrak{a})$ *is the group of units of* K *congruent to 1 modulo* $\mathfrak{a}$.

PROOF. Since the proof is essentially the same as that of Lemma 2 in Saito [11], we omit it. □

2.3. Let m be a positive integer. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(O_K)$ and $z = (z_1, \cdots, z_n) \in \mathfrak{H}^n$, put

$$J_{2m}(\gamma, z) := \prod_{i=1}^{n} (\sigma_i(c)z_i + \sigma_i(d))^{-2m} .$$

A *Hilbert cusp form* f of weight $2m$ with respect to $\Gamma(\mathfrak{a})$ is a holomorphic function on $\mathfrak{H}^n$ satisfying

a) $f(\gamma z)J_{2m}(\gamma, z) = f(z)$ for any $\gamma \in \Gamma(\mathfrak{a})$,

b) $f(z)$ is holomorphic at every cusp of $\Gamma(\mathfrak{a})$ (This condition automatically holds if $n \geq 2$).

c) $f(z)$ vanishes at every cusp of $\Gamma(\mathfrak{a})$.

We denote by $S_{2m}(\Gamma(\mathfrak{a}))$ the space of Hilbert cusp forms of weight $2m$ for $\Gamma(\mathfrak{a})$. For $\gamma \in SL_2(O_K)$ and $f \in S_{2m}(\Gamma(\mathfrak{a}))$, we have $f|[\gamma]_{2m} := f(\gamma z)J_{2m}(\gamma, z) \in S_{2m}(\Gamma(\mathfrak{a}))$. Hence by the map $\gamma \mapsto [\gamma]_{2m}$, we obtain a representation π of $SL_2(O_K)/\Gamma(\mathfrak{a})$ on $S_{2m}(\Gamma(\mathfrak{a}))$. In particular, if $\mathfrak{a}$ is a prime ideal $\mathfrak{p}$ and $\#(O_K/\mathfrak{a}) = q$, then we have $SL_2(O_K)/\Gamma(\mathfrak{a}) \cong SL_2(\mathbb{F}_q)$. We thus have the representation π of $SL_2(\mathbb{F}_q)$ on $S_{2m}(\Gamma(\mathfrak{p}))$.

2.4. An element γ of $SL_2(O_K)$ is called *elliptic* if it satisfies $\mathrm{tr}(\sigma_i(\gamma))^2 - 4 \cdot \det(\sigma_i(\gamma)) < 0 \quad (i = 1, \cdots, n)$. A point $z \in \mathfrak{H}^n$ which is a fixed point of an elliptic element of $\Gamma(\mathfrak{a})$ is called *elliptic fixed point* of $\Gamma(\mathfrak{a})$.

LEMMA 2.5. *Let $\mathfrak{a}$ be an integral ideal of K such that $\mathfrak{a}$ is prime to $6 \cdot d_K$ (here d_K is the discriminant of K). Then $\Gamma(\mathfrak{a})$ has no elliptic fixed points.*

PROOF. Since the proof is essentially the same as that of Remark 1 in Saito [11], we omit it. See also Yoshida [17], page 11. □

2. Representations of $SL_2(\mathbb{F}_q)$

2.6. Let q be a power of an odd prime. There are two pairs of irreducible characters whose values are conjugate mutually. We give a list of values at $\epsilon = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\epsilon' = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$ (η is a nonsquare element of $\mathbb{F}_q^*$) of such pairs (W', W'') and (X', X'') as follows:

	ϵ	ϵ'
W'	$\frac{1+\sqrt{q}}{2}$	$\frac{1-\sqrt{q}}{2}$
W''	$\frac{1-\sqrt{q}}{2}$	$\frac{1+\sqrt{q}}{2}$
X'	$\frac{-1+\sqrt{-q}}{2}$	$\frac{-1-\sqrt{-q}}{2}$
X''	$\frac{-1-\sqrt{-q}}{2}$	$\frac{-1+\sqrt{-q}}{2}$

If $q \equiv 1 \pmod 4$, then X' and X'' do not appear. If $q \equiv 3 \pmod 4$, then W' and W'' do not appear. Let us consider a representation π of $SL_2(\mathbb{F}_q)$ on $S_{2m}(\Gamma(\mathfrak{p}))$, which is treated in 2.3. Let y_1 be a multiplicity of W' (resp. X') in tr π when $q \equiv 1 \pmod 4$ (resp. $q \equiv 3 \pmod 4$), and y_2 a multiplicity of W'' (resp. X'') in tr π when $q \equiv 1 \pmod 4$ (resp. $q \equiv 3 \pmod 4$). Since the values at ϵ and ϵ' of irreducible characters of $SL_2(\mathbb{F}_q)$ other than these characters are equal, we have

$$\text{tr } \pi(\epsilon) - \text{tr } \pi(\epsilon') = \sqrt{(-1)^{(q-1)/2} q}(y_1 - y_2).$$

Hence we obtain

$$y_1 - y_2 = \frac{1}{\sqrt{(-1)^{(q-1)/2} q}} \left(\text{tr } \pi(\epsilon) - \text{tr } \pi(\epsilon') \right).$$

3. Holomorphic Lefschetz formula

2.7. Let X be a compact complex manifold, $\mathcal{V}$ a holomorphic vector bundle over X, and G a finite group of automorphisms of the pair $(X, \mathcal{V})$. For an element g of G, we denote by X^g the fixed subvariety of g in X. Let $X^g = \sum_\alpha X^g_\alpha$ be the irreducible decomposition of X^g, and $\mathcal{N}^g_\alpha = \sum_\theta \mathcal{N}^g_\alpha(\theta)$ the decomposed normal bundle of X^g_α corresponding to the eigenvalues $\exp(\sqrt{-1}\theta)$ of g. If the Chern class of $\mathcal{N}^g_\alpha(\theta)$ is $c\left(\mathcal{N}^g_\alpha(\theta)\right) = \prod_\beta (1 + x_\beta)$, then put

$$\mathcal{U}^\theta\left(\mathcal{N}^g_\alpha(\theta)\right) = \prod_\beta \left(\frac{1 - \exp(-x_\beta - \sqrt{-1}\theta)}{1 - \exp(-\sqrt{-1}\theta)} \right)^{-1}.$$

Let $\mathcal{T}(X_\alpha^g)$ be the Todd class of X_α^g, and $\mathrm{ch}(\mathcal{V}|X_\alpha^g)(g)$ the Chern character of $\mathcal{V}|X_\alpha^g$ with g-action. Put

$$\tau\,(g, X_\alpha^g) = \left\{ \frac{\mathrm{ch}(\mathcal{V}|X_\alpha^g)(g) \cdot \prod_\theta \mathcal{U}^\theta(\mathcal{N}_\alpha^g(\theta)) \cdot \mathcal{T}(X_\alpha^g)}{\det(1 - g|(\mathcal{N}_\alpha^g)^*)} \right\} [X_\alpha^g]\,,$$

where $[X_\alpha^g]$ denotes the fundamental class of X_α^g. Moreover, put $\tau(g) = \sum_\alpha \tau\,(g, X_\alpha^g)$.

THEOREM 2.8 (Holomorphic Lefschetz formula [1]). *Notation being as above, we have*

$$\tau(g) = \sum_{i \geq 0} (-1)^i tr(g|H^i(X, \mathcal{O}(\mathcal{V})).$$

3. Hilbert modular 3-folds

In this section, we remember some facts on Hilbert modular 3-folds. We refer to Ehlers [4], van der Geer [7], and Hirzebruch [9] for details. From now on, all totally real number fields we consider are totally real cubic fields.

3.1. Let K be a totally real cubic number field, O_K the ring of integers of K, and $\mathfrak{a}$ an integral ideal of O_K. Since $\Gamma(\mathfrak{a})$ acts on $\mathfrak{H}^3$, we have the quotient space $\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3$ of $\mathfrak{H}^3$ by $\Gamma(\mathfrak{a})$. The space $\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3$ can be compactified by adding all cusps of $\Gamma(\mathfrak{a})$. We denote by $\overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$ the resulting space. The space $\overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$ is a normal compact space with a finite number of isolated singularities, i.e., quotient singularities arising from elliptic fixed points of $\Gamma(\mathfrak{a})$ and cusp singularities arising from cusps of $\Gamma(\mathfrak{a})$. By Hironaka's general theory, there exists a proper morphism $X(\mathfrak{a}) \to \overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$ resolving the singularities. The space $X(\mathfrak{a})$ is a 3-dimensional nonsingular projective variety. We call it *Hilbert modular 3-fold* obtained from $\Gamma(\mathfrak{a})$.

3.2. Let γ be an element of $SL_2(O_K)$. Since $\Gamma(\mathfrak{a})$ is a normal subgroup of $SL_2(O_K)$, γ induces an automorphism of $\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3$ given by

$$(z_1, z_2, z_3) \mapsto (\sigma_1(\gamma)z_1, \sigma_2(\gamma)z_2, \sigma_3(\gamma)z_3),$$

and moreover this automorphism can be extended to that of $\overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$. Take any element a of O_K, and let f_γ be the automorphism of $\overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$ defined by $\gamma = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. By the same argument as the proof of Lemma 2.5, we can easily see that f_γ has no fixed points in $\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3$ under the same assumption as that of Lemma 2.5.

LEMMA 3.3. *Let f_ϵ, $f_{\epsilon'}$ be automorphisms of $\overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$ defined by ϵ, ϵ' given in 2.6, respectively. Suppose that $h_K = 1$ and $\mathfrak{a}$ is a prime ideal $\mathfrak{p}$ generated by μ. Then the fixed points of f_ϵ are the cusps which are $\Gamma(\mathfrak{p})$-equivalent to the cusps of the form α/μ, $(\alpha \in O_K, O_K\alpha + O_K\mu = O_K)$. The same thing holds for $f_{\epsilon'}$.*

PROOF. We refer to Remark 3 in Saito [11]. □

LEMMA 3.4. *Let the notation and the assumption be as in Lemma 3.3. Let $\widetilde{U}$ be the image of U in $(O_K/\mathfrak{p})^\times$. If $\{\alpha_i\}$ is a complete system of the representatives of $(O_K/\mathfrak{p})^\times/\widetilde{U}$, then $\{\alpha_i/\mu\}$ is the set of all fixed points of f_ϵ (resp. $f_{\epsilon'}$).*

PROOF. Since the proof is essentially the same as that of Lemma 1 in Saito [11], we omit it. □

3.5. We assume that $\mathfrak{a}$ is prime to $6 \cdot d_K$. Then $\overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$ has no quotient singularities by Lemma 2.5. Hence it suffices to consider the cusp resolution in this case. We shall describe the cusp resolution of $\overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$ in the rest of this section.

3.6. Let W be a n-dimensional vector space over $\mathbb{R}$, and M a rank n free $\mathbb{Z}$-module in W. Let $v_1, \cdots, v_r$ be linearly independent elements of M, and set

$$\sigma = \langle v_1, \cdots v_r \rangle := \left\{ \sum_{i=1}^{r} c_i v_i \mid c_i \geq 0 \right\}.$$

The set σ is called *r-simplex* if $M/\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_r$ is torsion-free. For any subset $\{v_{i_1}, \cdots, v_{i_k}\}$ of $\{v_1, \cdots, v_r\}$, we call $\langle v_{i_1}, \cdots, v_{i_k} \rangle$ the *k-face* of σ. By abuse of notation, we may write w for a 1-simplex $\langle w \rangle$. We consider $\{0\}$ as a 0-simplex.

A set Σ of simplices is called *complex* when it satisfies the following:

i) If $\sigma, \sigma' \in \Sigma, \sigma \neq \sigma'$, then we have $\overset{\circ}{\sigma} \cap \overset{\circ}{\sigma'} = \phi$ and $\sigma \cap \sigma' \in \Sigma$, where $\overset{\circ}{\sigma}$ is the interior of σ. Take any element $\sigma \in \Sigma$. If τ is a face of σ, then $\tau \in \Sigma$.

ii) For any element $\tau \in \Sigma$, the set $\{\sigma \in \Sigma \mid \tau \text{ is a face of } \sigma\}$ is finite.

iii) If $\tau \in \Sigma$ satisfies dim $\tau < n$, then τ is a face of certain n-simplex in Σ.

For each complex Σ, we obtain a n-dimensional complex manifold X_Σ. We call it a torus embedding associated to Σ (cf. [18]). There exists a 1-1 correspondence between the coordinate charts $(\mathbb{C}^n)_\sigma$ of X_Σ and the n-simplices σ of Σ. Let $\Sigma^{(k)}$ be the set of k-simplices in Σ. Each element σ of $\Sigma^{(k)}$ corresponds to a codimension k submanifold F_σ of X_Σ. Set $F_\Sigma = \bigcup_{\sigma \in \Sigma^{(1)}} F_\sigma$.

3.7. Let U^+ be the group of totally positive units of K. Let M be a rank 3 complete $\mathbb{Z}$-module in K, and V a subgroup of rank 2 of U^+ such that $V \cdot M = M$. Set

$$G(M, V) := \left\{ \begin{pmatrix} u & m \\ 0 & 1 \end{pmatrix} \mid u \in V,\ m \in M \right\}.$$

The group $G(M, V)$ acts on $\mathfrak{H}^3$ by the same way as (4) in Section 2. The space $\mathcal{H}(M, V) := G(M, V) \setminus \mathfrak{H}^3 \cup \{\infty\}$ is a normal space with an isolated singularity at ∞, which is of type (M, V). The space $\mathcal{H}(M, V)$ has the following properties:

(i) $\mathcal{H}(M, V)$ is locally compact.

(ii) $G(M, V) \setminus \mathfrak{H}^3$ is open dense in $\mathcal{H}(M, V)$.

(iii) For any positive real number c, set

$$U_c := \{z \in \mathfrak{H}^3 \mid \mathrm{Im}(z_1) \cdot \mathrm{Im}(z_2) \cdot \mathrm{Im}(z_3) > c\}.$$

Then $G(M, V)$ acts on U_c, and $\{G(M, V) \setminus U_c \cup \{\infty\} \mid c > 0\}$ forms a fundamental system of neighbourhoods of ∞.

Each cusp singularity x of Hilbert modular 3-fold $\overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$ is analytically equivalent to ∞ on some $\mathcal{H}(M, V)$.

Let $\widehat{M}$ be the dual $\mathbb{Z}$-module of M, i.e.,

$$\widehat{M} := \{x \in K \mid \mathrm{tr}(xy) \in \mathbb{Z}, \text{for all } y \in M\}.$$

Here we used the notation $\mathrm{tr}(xy) := \sigma_1(xy) + \sigma_2(xy) + \sigma_3(xy)$. Let $\widehat{M}_+$ be the set of totally positive elements of $\widehat{M}$. For a cusp ∞ of type (M, V), let $\mathcal{O}(M, V)$ be the ring of holomorphic functions f at a neighbourhood of ∞ satisfying the following conditions:

(a) Each $f \in \mathcal{O}(M, V)$ has the Fourier expansion

$$f(z) = a_0 + \sum_{x \in \widehat{M}_+} a_x \mathbf{e}[\mathrm{tr}(xz)]$$

such that $a_x = a_{ux}$ for all $u \in V$ (here we put $\mathrm{tr}(xz) := \sigma_1(x)z_1 + \sigma_2(x)z_2 + \sigma_3(x)z_3$).

(b) Each $f \in \mathcal{O}(M, V)$ converges on U_c for some $c > 0$ depending on f.

3.8. In this subsection, we recall resolutions of cusp singularities of Hilbert modular 3-folds. We here construct a cusp resolution of $\mathcal{H}(M, V)$.

Let M be a rank 3 complete $\mathbb{Z}$-module in K. The module M acts on $\mathbb{C}^3$ by $(z_1, z_2, z_3) \mapsto (z_1 + \sigma_1(m), z_2 + \sigma_2(m), z_3 + \sigma_3(m)) \quad (m \in M)$. The quotient $M \setminus \mathbb{C}^3$ is an algebraic torus. Let $\{u, v, w\}$ be a basis of M. Then there exists an isomorphism

$$\varphi(u, v, w) : M \setminus \mathbb{C}^3 \to (\mathbb{C}^*)^3, \quad z \bmod M \mapsto (t_1, t_2, t_3),$$

where t_1, t_2, t_3 are determined by

$$\begin{cases} 2\pi\sqrt{-1}z_1 \equiv \sigma_1(u)\log\, t_1 + \sigma_1(v)\log\, t_2 + \sigma_1(w)\log\, t_3 & (\bmod\ 2\pi\sqrt{-1}M) \\ 2\pi\sqrt{-1}z_2 \equiv \sigma_2(u)\log\, t_1 + \sigma_2(v)\log\, t_2 + \sigma_2(w)\log\, t_3 & (\bmod\ 2\pi\sqrt{-1}M) \\ 2\pi\sqrt{-1}z_3 \equiv \sigma_3(u)\log\, t_1 + \sigma_3(v)\log\, t_2 + \sigma_3(w)\log\, t_3 & (\bmod\ 2\pi\sqrt{-1}M). \end{cases}$$

Take another basis $\{u', v', w'\}$ of M. Then we have a commutative diagram:

$$\begin{array}{ccc} M\backslash\mathbb{C}^3 & \overset{\varphi(u,v,w)}{\to} & (\mathbb{C}^*)^3 \\ \| & & \downarrow \psi \\ M\backslash\mathbb{C}^3 & \overset{\varphi(u',v',w')}{\to} & (\mathbb{C}^*)^3, \end{array}$$

where we put $\psi = \varphi(u', v', w') \circ \varphi(u, v, w)^{-1}$. If a matrix $g = (g_{ij}) \in GL_3(\mathbb{Z})$ transforms (u, v, w) into (u', v', w'), then ψ is expressed as

$$\psi(t_1, t_2, t_3) = ({t_1}^{g_{11}}{t_2}^{g_{12}}{t_3}^{g_{13}}, {t_1}^{g_{21}}{t_2}^{g_{22}}{t_3}^{g_{23}}, {t_1}^{g_{31}}{t_2}^{g_{32}}{t_3}^{g_{33}}).$$

The quotient $M \setminus \mathbb{C}^3$ contains $M \setminus \mathfrak{H}^3$ as an open subset. If $\mathrm{Im}(z_1)\mathrm{Im}(z_2)\mathrm{Im}(z_3)$ tends to ∞, then t_1, t_2, or t_3 appeared in the above isomorphism tends to 0. We consider the inclusion $(\mathbb{C}^*)^3 \subset \mathbb{C}^3$ for any basis of M. Take any element $\sigma = \langle u, v, w\rangle$ of $\Sigma^{(3)}$. By the construction of Σ, $\{u, v, w\}$ is a basis of M. Let $\left(\mathbb{C}^3\right)_\sigma$ be a copy of $\mathbb{C}^3$. We can glue these copies $\left(\mathbb{C}^3\right)_\sigma$ $(\sigma \in \Sigma^{(3)})$ by using biholomorphic maps ψ appeared in the above diagram. Then we obtains a three dimensional complex manifold X_Σ. Let $\Phi : M \setminus \mathbb{C}^3 \hookrightarrow X_\Sigma$ be an embedding defined by $M \setminus \mathbb{C}^3 \to (\mathbb{C}^*)^3 \hookrightarrow X_\Sigma$. The map Φ is independent of a choice of a basis of M by the construction of X_Σ. Put $X := \Phi\left(M \setminus \mathfrak{H}^3\right) \cup F_\Sigma$, where $F_\Sigma := X_\Sigma - \Phi\left(M \setminus \mathbb{C}^3\right)$. Since there is an exact sequence $0 \to M \to G(M, V) \to V \to 0$, V acts on $M \setminus \mathfrak{H}^3$. Take an element $\sigma = \langle u, v, w\rangle$ of $\Sigma^{(3)}$. From the construction of Σ, $\sigma' := \langle eu, ev, ew\rangle \in \Sigma^{(3)}$ for any element $e \in V$. By sending a point with coordinates u, v, w in $\left(\mathbb{C}^3\right)_\sigma$ to the point with coordinates eu, ev, ew in $\left(\mathbb{C}^3\right)_{\sigma'}$, V acts on X_Σ. The map $\Phi : M \setminus \mathfrak{H}^3 \to X$ is compatible with the action of V. According to Ehlers, V acts on X freely and properly discontinuously (Ehlers [4], section 2, Lemma 1, 2). The quotient $Y(M, V) := V \setminus X$ is a three dimensional complex manifold, and Φ^{-1} induces a surjective morphism $p : Y(M, V) \to \mathcal{H}(M, V)$ satisfying $p^{-1}(\infty) = V \setminus F_\Sigma$. The complex 3-fold $Y(M, V)$ is a resolution of the cusp ∞.

3.9. We keep the notation of 3.7. We consider a cusp of type (M, V). Let e be a unit element of K such that $eM = M$, and m an element of K such that $(e-1)m \in M$ for all $e \in V$. Then e and m define maps

$$\begin{aligned} &(z_1, z_2, z_3) \mapsto (\sigma_1(e)^2 z_1, \sigma_2(e)^2 z_2, \sigma_3(e)^2 z_3), \\ &(z_1, z_2, z_3) \mapsto (z_1 + \sigma_1(m), z_2 + \sigma_2(m), z_3 + \sigma_3(m)), \end{aligned}$$

respectively. The neighbourhoods U_c of ∞ are stable under these maps. These maps define automorphisms g_e, g_m of $\mathcal{H}(M, V)$, respectively. Moreover, we have two automorphisms g_e^*, g_m^* of $\mathcal{O}(M, V)$ induced by g_e, g_m, respectively:

$$\begin{aligned} g_e^* &: \mathcal{O}(M, V) \to \mathcal{O}(M, V), \\ &\quad (\mathbf{e}[z_1], \mathbf{e}[z_2], \mathbf{e}[z_3]) \mapsto (\mathbf{e}[e^2 z_1], \mathbf{e}[e^2 z_2], \mathbf{e}[e^2 z_3]), \\ g_m^* &: \mathcal{O}(M, V) \to \mathcal{O}(M, V), \\ &\quad (\mathbf{e}[z_1], \mathbf{e}[z_2], \mathbf{e}[z_3]) \mapsto (\mathbf{e}[z_1 + \sigma_1(m)], \mathbf{e}[z_2 + \sigma_2(m)], \mathbf{e}[z_3 + \sigma_3(m)]). \end{aligned}$$

PROPOSITION 3.10. *The maps g_e and g_m can be extended to a cusp resolution $Y(M,V)$ of $\mathcal{H}(M,V)$.*

PROOF. We use the notation in 3.8. First, let us show the claim for g_e. Let $\widetilde{g_e} : X_\Sigma \to X_\Sigma$ be a map with the property that a point with coordinates u, v, w in $(\mathbb{C}^3)_\sigma$ is mapped to the point with coordinates u', v', w' in $(\mathbb{C}^3)_{\sigma'}$. Here we put $u' = eu, v' = ev, w' = ew$, and $\sigma' = \langle u', v', w' \rangle$. Put $W_c := \Phi(M \setminus U_c) \cup F_\Sigma$ for any $c > 0$. The set W_c is open in X, and is stable under the map $\widetilde{g_e}$ for any element $e \in V$, and $\widetilde{g_e}$ induces a map $V \setminus W_c \to V \setminus W_c$. Also, F_Σ is stable under $\widetilde{g_e}$, and $\widetilde{g_e}$ induces in $\mathcal{O}(M,V)$ the map g_e^* from the relation

$$(5)\qquad \begin{cases} 2\pi\sqrt{-1}z_1 = \sigma_1(u)\log t_1 + \sigma_1(v)\log t_2 + \sigma_1(w)\log t_3 \\ 2\pi\sqrt{-1}z_2 = \sigma_2(u)\log t_1 + \sigma_2(v)\log t_2 + \sigma_2(w)\log t_3 \\ 2\pi\sqrt{-1}z_3 = \sigma_3(u)\log t_1 + \sigma_3(v)\log t_2 + \sigma_3(w)\log t_3 \end{cases}$$

between the coordinates of $(\mathbb{C}^3)_\sigma \cap X$ and those of $\mathcal{H}(M,V)$. This proves the claim for g_e.

We next show the claim for g_m. For any element $\sigma = \langle u, v, w \rangle \in \Sigma$, we define a map $(\mathbb{C}^3)_\sigma \to (\mathbb{C}^3)_\sigma$ by

$$(t_1, t_2, t_3) \mapsto \left(\mathbf{e}\left[\frac{d(m,v,w)}{d(u,v,w)}\right] t_1, \mathbf{e}\left[\frac{d(u,m,w)}{d(u,v,w)}\right] t_2, \mathbf{e}\left[\frac{d(u,v,m)}{d(u,v,w)}\right] t_3 \right),$$

where we put

$$(6)\qquad d(a,b,c) := \begin{vmatrix} \sigma_1(a) & \sigma_1(b) & \sigma_1(c) \\ \sigma_2(a) & \sigma_2(b) & \sigma_2(c) \\ \sigma_3(a) & \sigma_3(b) & \sigma_3(c) \end{vmatrix}$$

for $a, b, c \in K$. For a 3-simplex $\langle u, v, w \rangle$, we may assume $d(u,v,w) > 0$ by reordering u, v, w. Then we have $d(u,v,w) = \sqrt{d_K}$. This map is compatible with the glueing of $(\mathbb{C}^3)_\sigma$ $(\sigma \in \Sigma)$ by ψ's, and therefore induces an automorphism $\widetilde{g_m} : X_\Sigma \to X_\Sigma$. By the construction of W_c, W_c is stable under $\widetilde{g_m}$. Also, $\widetilde{g_m}$ makes stable F_Σ, and $\widetilde{g_m}$ induces in $\mathcal{O}(M,V)$ the map g_m^* by (5). This proves the claim for g_m. $\square$

3.11. We take an element γ of $SL_2(O_K)$. Since $\Gamma(\mathfrak{a})$ is a normal subgroup of $SL_2(O_K)$, γ induces an automorphism of $\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3$ defined by

$(z_1, z_2, z_3) \mapsto (\sigma_1(\gamma)z_1, \sigma_2(\gamma)z_2, \sigma_3(\gamma)z_3)$. This automorphism can be extended to that of $\overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$. We denote the resulting map by f_γ. Then we have the following:

PROPOSITION 3.12. *The map f_γ can be extended to an automorphism of $X(\mathfrak{a})$.*

PROOF. Let $\psi : X(\mathfrak{a}) \to \overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$ be a morphism resolving the singularities of $\overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$. The morphism ψ induces an isomorphsm $X(\mathfrak{a}) - \psi^{-1}(S) \to \Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3$. Here S denotes the set of cusp singularities of $\overline{\Gamma(\mathfrak{a}) \setminus \mathfrak{H}^3}$. Thus f_γ can be extended to $X(\mathfrak{a}) - \psi^{-1}(S)$ as an automorphism of $X(\mathfrak{a}) - \psi^{-1}(S)$. Let λ be a cusp for $\Gamma(\mathfrak{a})$, and put $\gamma(\lambda) = \lambda'$. By our assumption, λ and λ' are of type $(\mathfrak{a}, U(\mathfrak{a}))$. There exist $\gamma_\lambda, \gamma_{\lambda'} \in SL_2(O_K)$ such that $\gamma_\lambda(\infty) = \lambda, \gamma_{\lambda'}(\infty) = \lambda'$. The matrix $\gamma_{\lambda'}^{-1}\gamma\gamma_\lambda$ has the form $\begin{pmatrix} e & m \\ 0 & e^{-1} \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} 1 & e^{-1}m \\ 0 & 1 \end{pmatrix}$ for some element $e \in U$ and for some element $m \in O_K$. We see that e and $e^{-1}m$ satisfy the condition in 3.9. By Proposition 3.10, maps g_e and $g_{e^{-1}m}$ can be extended to $X(\mathfrak{a})$ as automorphisms of $X(\mathfrak{a})$. Since $\gamma_{\lambda'}^{-1}\gamma\gamma_\lambda$ is expressed as $g_e \cdot g_{e^{-1}m}$, f_γ can be extended to $X(\mathfrak{a})$ as an automorphism of $X(\mathfrak{a})$. $\square$

4. The main result

In this section, we present a formula for $y_1 - y_2$, which is an analogue of a formula of Eichler.

4.1. In this subsection, we prepare for some definitions and notations needed in the next theorem. Let K be a totally real cubic field with $h_K = 1$, and $\mathfrak{p}$ a prime ideal of K with the conditions that $\mathfrak{p}$ is generated by a totally positive element μ and that $\mathfrak{p}$ is prime to $6 \cdot d_K$ (here d_K is the discriminant of K). Let Σ be a complex which describe the cusp resolution of a cusp with type $(O_K, U(\mathfrak{p})^2)$. Take a 2-simplex $\langle v, w\rangle \in \Sigma^{(2)}$. Let $a(v, w)$ be the selfintersection number of $F_{\langle v,w\rangle}$ on $F_{\langle v\rangle}$, and $a(w, v)$ be that of $F_{\langle v,w\rangle}$ on $F_{\langle w\rangle}$. Take a 1-simplex $\langle w\rangle \in \Sigma^{(1)}$. Let $\{\sigma_1, \cdots, \sigma_s\}$ be the set of all 2-simplices in Σ with the property $\langle \sigma_i, w\rangle \in \Sigma^{(3)}$. There exist elements $u_1, \cdots, u_s \in \Sigma^{(1)}$ such that

$$\sigma_i = \langle u_i, u_{i+1}\rangle \qquad (1 \leq i \leq s), \qquad u_{s+1} = u_1.$$

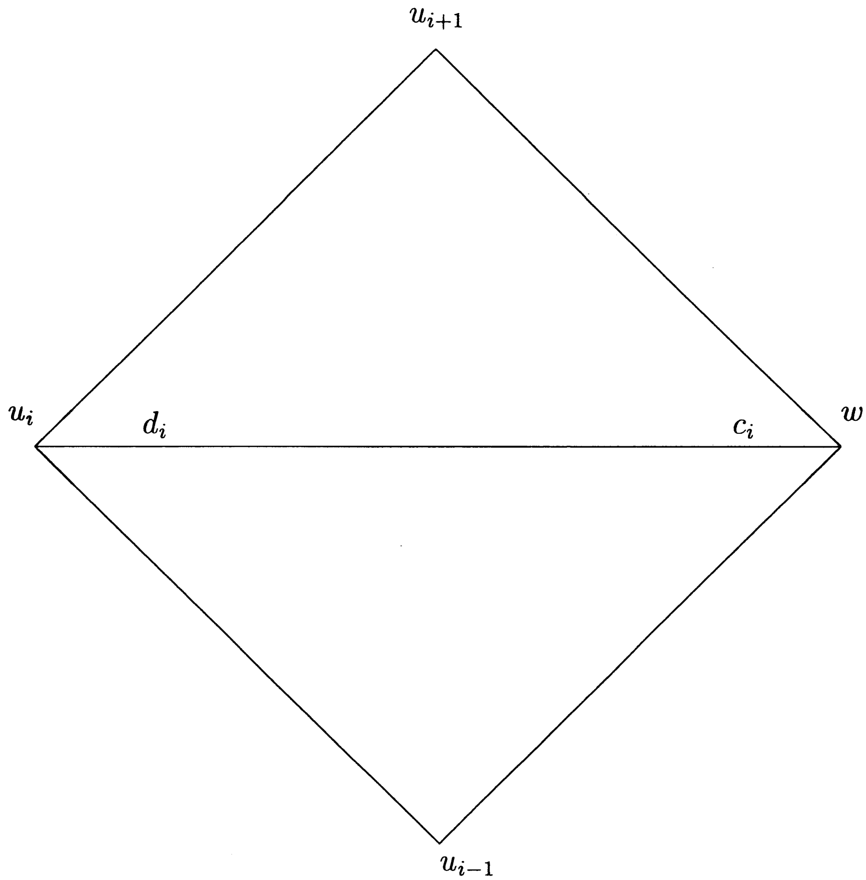

Fig. 1

Then we have

$$u_{i+1} + u_{i-1} = c_i w + d_i u_i \qquad (1 \leq i \leq s)$$

for certain integers $c_i,\ d_i \in \mathbb{Z}$ (cf. Thomas-Vasquez [14], page 177). We write the integers c_i and d_i on the sides of w and u_i as Fig. 1.

The numbers $c_i,\ d_i\ \ (1 \leq i \leq s)$ are three dimensional analogues of periodic continued fractions. It is known that $-c_i = a(u_i, w),\ -d_i = a(w, u_i)$ (cf. Tsuchihashi [15], page 628). Then we put $c(w) := -\sum_{i=1}^{s} c_i$. Let

$d(w) := {F_{\langle w\rangle}}^3$ be the triple intersection number of $F_{\langle w\rangle}$. Using c_i and d_i, we define

$$A_0 = 0, \quad A_1 = c_s,$$
$$A_{i+1} = c_i + d_i A_i - A_{i-1}, \quad (1 \le i \le s-3)$$

recursively. Then we have $d(w) = \sum_{i=1}^{s-2} c_i A_i$. For the definition of $d(\cdot,\cdot,\cdot)$, see (6).

DEFINITION 4.2. Let the notation be as above. For each element α of O_K, let f_{γ_α} be the automorphism of $X(\mathfrak{p})$ for $\gamma_\alpha := \begin{pmatrix} 1 & \alpha/\mu \\ 0 & 1 \end{pmatrix}$ (cf. 3.9, 3.12). Then for each $\alpha \in O_K$, we define

$$\begin{aligned}
\nu(\alpha) := &-\sum_{(1)} \frac{\mathbf{e}\left[\frac{d(\alpha/\mu,v,w)}{d(u,v,w)}\right]\cdot\mathbf{e}\left[\frac{d(u,\alpha/\mu,w)}{d(u,v,w)}\right]\cdot\mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]}{\left(1-\mathbf{e}\left[\frac{d(\alpha/\mu,v,w)}{d(u,v,w)}\right]\right)\left(1-\mathbf{e}\left[\frac{d(u,\alpha/\mu,w)}{d(u,v,w)}\right]\right)\left(1-\mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]\right)} \\
&+\sum_{(2)} \frac{\mathbf{e}\left[\frac{d(u,\alpha/\mu,w)}{d(u,v,w)}\right]\cdot\mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]}{\left(1-\mathbf{e}\left[-\frac{d(u,\alpha/\mu,w)}{d(u,v,w)}\right]\right)\left(1-\mathbf{e}\left[-\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]\right)} \\
&\quad\times\left\{-1-\frac{a(v,w)}{1-\mathbf{e}\left[-\frac{d(u,\alpha/\mu,w)}{d(u,v,w)}\right]}-\frac{a(w,v)}{1-\mathbf{e}\left[-\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]}\right\} \\
&+\sum_{(3)} \frac{\mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]}{1-\mathbf{e}\left[-\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]}\cdot\left\{1-\frac{c(w)+d(w)}{1-\mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]}\right\},
\end{aligned}$$

where the sum $\sum_{(1)}$ runs over the elements $\langle u,v,w\rangle$ of $\sum^{(3)}$ corresponding to the components of 0-dimensional fixed subvariety of f_{γ_α}, the sum $\sum_{(2)}$ runs over the elements $\langle v,w\rangle$ of $\sum^{(2)}$ corresponding to the components of 1-dimensional fixed subvariety of f_{γ_α} (then take an element u of $\sum^{(1)}$ such that $\langle u,v,w\rangle \in \sum^{(3)}$), and the sum $\sum_{(3)}$ runs over the elements w of $\sum^{(1)}$ corresponding to the components of 2-dimensional fixed subvariety of f_{γ_α} (then take an element $\langle u,v\rangle$ of $\sum^{(2)}$ such that $\langle u,v,w\rangle \in \sum^{(3)}$).

REMARK 4.3. (1) As one sees in Definition 4.2, a 3-simplex $\langle u, v, w\rangle \in \Sigma^{(3)}$ is chosen for an element $w \in \Sigma^{(1)}$ corresponding to a component of 2-dimensional fixed subvariety of f_{γ_α}. The value of $\mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]$ is independent of a choice of such 3-simplices. Indeed, let $\{\sigma_1, \cdots, \sigma_s\}$ be the set of all 2-simplices in Σ with the property $\langle \sigma_i, w\rangle \in \Sigma^{(3)}$. There exist elements $u_1, \cdots, u_s \in \Sigma^{(1)}$ such that

$$\sigma_i = \langle u_i, u_{i+1}\rangle \qquad (1 \le i \le s), \qquad u_{s+1} = u_1.$$

Then we have

$$u_{i+1} + u_{i-1} = c_i w + d_i u_i \qquad (1 \le i \le s)$$

for certain integers $c_i,\ d_i \in \mathbb{Z}$. Using it,

$$\begin{aligned} d(u_i, u_{i+1}, \alpha/\mu) &= -c_i \cdot d(w, u_i, \alpha/\mu) + d(u_{i-1}, u_i, \alpha/\mu),\\ d(u_i, u_{i+1}, w) &= d(u_{i-1}, u_i, w). \end{aligned}$$

Since $\mathbf{e}\left[\frac{d(u_j,\alpha/\mu,w)}{d(u_j,u_{j+1},w)}\right] = 1 \quad (1 \le j \le s)$ (cf. 5.4.), we have

$$\begin{aligned} \mathbf{e}\left[\frac{d(u_i, u_{i+1}, \alpha/\mu)}{d(u_i, u_{i+1}, w)}\right] &= \mathbf{e}\left[\frac{d(u_i, -u_{i-1}, \alpha/\mu)}{d(u_i, u_{i+1}, w)}\right] \cdot \mathbf{e}\left[\frac{d(u_i, w, \alpha/\mu)}{d(u_i, u_{i+1}, w)}\right]^{c_i}\\ &= \mathbf{e}\left[\frac{d(u_{i-1}, u_i, \alpha/\mu)}{d(u_i, u_{i+1}, w)}\right]\\ &= \mathbf{e}\left[\frac{d(u_{i-1}, u_i, \alpha/\mu)}{d(u_{i-1}, u_i, w)}\right]. \end{aligned}$$

This proves the claim. Also, in Definition 4.2, for any element $\langle v, w\rangle \in \Sigma^{(2)}$ corresponding to a component of 1-dimensional fixed subvariety of f_{γ_α}, an element $u \in \Sigma^{(1)}$ such that $\langle u, v, w\rangle \in \Sigma^{(3)}$ is chosen. The values of $\mathbf{e}\left[\frac{d(u,\alpha/\mu,w)}{d(u,v,w)}\right]$ and $\mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]$ are independent of choice of u. Indeed, let u' be another 1-simplex such that $\langle u', v, w\rangle \in \Sigma^{(3)}$. There exist $c,\ d \in \mathbb{Z}$

such that $u+u' = cv+dw$ as above. Hence, $d(u',v,w) = -d(u,v,w)$ holds. Since $\mathbf{e}\left[\frac{d(\alpha/\mu,v,w)}{d(u,v,w)}\right] = 1$ (cf. 5.4.), we have

$$\mathbf{e}\left[\frac{d(u',v,\alpha/\mu)}{d(u',v,w)}\right] = \mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right]\cdot\mathbf{e}\left[\frac{d(w,v,\alpha/\mu)}{d(u,v,w)}\right]^{-d} = \mathbf{e}\left[\frac{d(u,v,\alpha/\mu)}{d(u,v,w)}\right].$$

This proves the second claim.

(2) If $\alpha \equiv \beta \pmod{\mathfrak{p}}$ for $\alpha,\ \beta \in O_K$, then $\nu(\alpha) = \nu(\beta)$. Indeed, $\alpha-\beta \in (\mu)$ implies that $(\alpha-\beta)/\mu$ is a linear combination of u,v,w over $\mathbb{Z}$. Hence, $d((\alpha-\beta)/\mu,v,w)/d(u,v,w)$, $d(u,(\alpha-\beta)/\mu,w)/d(u,v,w)$, and $d(u,v,(\alpha-\beta)/\mu)/d(u,v,w)$ are rational integers.

(3) Though in our case we define $\nu(\alpha)$ with the use of the cubic determinant, Saito [11] defines $\nu(\alpha)$ without the use of the determinant in the real quadratic field case. However, one can easily see that $\nu(\alpha)$ in Saito [11] is rewritten with the use of the quadratic determinant $d(a,b) = \begin{vmatrix} \sigma_1(a) & \sigma_1(b) \\ \sigma_2(a) & \sigma_2(b) \end{vmatrix}$.

We now state the main theorem:

THEOREM 4.4. *Let K be a totally real cubic field whose class number is 1, and $\mathfrak{p}$ a prime ideal of K, which lies over an odd prime number, generated by a totally positive element μ, and is prime to $6\cdot d_K$. On the space $S_{2m}(\Gamma(\mathfrak{p}))$, we have*

$$y_1 - y_2 = \frac{1}{\sqrt{(-1)^{(q-1)/2}q}}\cdot\frac{2}{[U:U(\mathfrak{p})]}\sum_{\alpha\in(O_K/\mathfrak{p})^\times}\left(\frac{\alpha}{\mathfrak{p}}\right)\nu(\alpha),$$

Here we explain the notation appeared above. Let $q = \#(O_K/\mathfrak{p})$. Let U be the unit group for K, and $U(\mathfrak{p})$ the group of elements of U congruent to 1 modulo $\mathfrak{p}$. The sum $\sum$ runs over a complete system of the representatives of $(O_K/\mathfrak{p})^\times$. Let $\left(\frac{\ }{\mathfrak{p}}\right)$ be the quadratic residue symbol modulo $\mathfrak{p}$.

5. Proof of Theorem 4.4

In this section we prove Theorem 4.4. From 5.1 till 5.6, we are engaged in the proof in the case $m=1$. In 5.7, we prove in the case $m \geq 2$.

5.1. From now on, we assume that the norm $N(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ is a power of some odd prime. Let π be the representation of $SL_2(\mathbb{F}_q) \cong SL_2(O_K)/\Gamma(\mathfrak{p})$ on the space $S_2(\Gamma(\mathfrak{p}))$. Take an element η of O_K such that $\left(\frac{\eta}{\mathfrak{p}}\right) = -1$ (here $\left(\frac{\ }{\mathfrak{p}}\right)$ is the quadratic residue symbol modulo $\mathfrak{p}$). Put $\epsilon = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\epsilon' = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$. Then the difference $y_1 - y_2$ (cf. Section 2) of multiplicities of two irreducible characters in tr π was expressed as

$$y_1 - y_2 = \frac{1}{\sqrt{(-1)^{(q-1)/2}q}}(\mathrm{tr}\ \pi(\epsilon) - \mathrm{tr}\ \pi(\epsilon')). \tag{7}$$

Hence we shall compute tr $\pi(\epsilon)$ and tr $\pi(\epsilon')$ in order to study $y_1 - y_2$.

The matrix ϵ (resp. ϵ') induces the automorphism f_ϵ (resp. $f_{\epsilon'}$) of $\overline{\Gamma(\mathfrak{p}) \setminus \mathfrak{H}^3}$. By Proposition 3.12, we can extend the automorphism f_ϵ (resp. $f_{\epsilon'}$) to the biholomorphic automorphism $\widetilde{f_\epsilon}$ (resp. $\widetilde{f_{\epsilon'}}$) of $X(\mathfrak{p})$, respectively. Let Ω^3 be the sheaf of germs of holomorphic 3-forms on $X(\mathfrak{p})$. It is known that the space $S_2(\Gamma(\mathfrak{p}))$ is isomorphic to $H^0(X(\mathfrak{p}), \Omega^3)$. Let $\mathrm{tr}(\widetilde{f_\epsilon}|H^0(X(\mathfrak{p}), \Omega^3))$ (resp. $\mathrm{tr}(\widetilde{f_{\epsilon'}}|H^0(X(\mathfrak{p}), \Omega^3))$) be the trace of the linear transformation of $H^0(X(\mathfrak{p}), \Omega^3)$ induced by $\widetilde{f_\epsilon}$ (resp. $\widetilde{f_{\epsilon'}}$). Then we see that tr $\pi(\epsilon)$ (resp. tr $\pi(\epsilon')$) is equal to $\mathrm{tr}(\widetilde{f_\epsilon}|H^0(X(\mathfrak{p}), \Omega^3))$ (resp. $\mathrm{tr}(\widetilde{f_{\epsilon'}}|H^0(X(\mathfrak{p}), \Omega^3))$).

5.2. By the holomorphic Lefschetz formula (Theorem 2.8), we have

$$\sum_{i=0}^{3}(-1)^i \mathrm{tr}(\widetilde{f_\epsilon}|H^i(X(\mathfrak{p}), \Omega^3)) = \tau(\epsilon),$$
$$\sum_{i=0}^{3}(-1)^i \mathrm{tr}(\widetilde{f_{\epsilon'}}|H^i(X(\mathfrak{p}), \Omega^3)) = \tau(\epsilon').$$

Let $\mathcal{O}_{X(\mathfrak{p})}$ be the structure sheaf of $X(\mathfrak{p})$. By the Serre duality theorem, we have

$$H^i(X(\mathfrak{p}), \Omega^3) = H^{3-i}(X(\mathfrak{p}), \mathcal{O}_{X(\mathfrak{p})}) \qquad (i = 1, 2, 3).$$

We know that $H^0(X(\mathfrak{p}), \mathcal{O}_{X(\mathfrak{p})}) = \mathbb{C}$. By Theorem 7.1 in Freitag [5], we have $H^1(X(\mathfrak{p}), \mathcal{O}_{X(\mathfrak{p})}) = H^2(X(\mathfrak{p}), \mathcal{O}_{X(\mathfrak{p})}) = 0$. Therefore, we conclude that

$$\mathrm{tr}(\widetilde{f_\epsilon}|H^0(X(\mathfrak{p}), \Omega^3)) - \mathrm{tr}(\widetilde{f_{\epsilon'}}|H^0(X(\mathfrak{p}), \Omega^3)) = \tau(\epsilon) - \tau(\epsilon').$$

By (7), the difference $y_1 - y_2$ is expressed as

$$y_1 - y_2 = \frac{1}{\sqrt{(-1)^{(q-1)/2}q}} \left(\tau(\epsilon) - \tau(\epsilon')\right). \tag{8}$$

5.3. By Lemma 3.3, the fixed subvariety of $\widetilde{f_\epsilon}$ is contained in the surfaces arising from the resolution of the cusps which are $\Gamma(\mathfrak{p})$-equivalent to the cusps of the form $\alpha/\mu \quad (\alpha \in O_K,\ O_K\alpha + O_K\mu = O_K)$. The same thing holds for $\widetilde{f_{\epsilon'}}$.

Take a fixed point $\lambda = \alpha/\mu$ of f_ϵ (resp. $f_{\epsilon'}$). Then the stabilizer $\Gamma(\mathfrak{p})_\lambda$ of λ in $\Gamma(\mathfrak{p})$ is isomorphic to $\left\{\begin{pmatrix} e & m \\ 0 & e^{-1}\end{pmatrix} \mid e \in U(\mathfrak{p}),\ m \in \mathfrak{p}\right\}$ by Lemma 2.2. Hence λ is of type $(\mathfrak{p}, U(\mathfrak{p})^2)$ (here we set $U(\mathfrak{p})^2 = \{u^2 \mid u \in U(\mathfrak{p})\ \}$). Hence $\mathcal{O}(\lambda)$ is isomorphic to $\mathcal{O}(\mathfrak{p}, U(\mathfrak{p})^2)$ (here $\mathcal{O}(\lambda)$ denotes the ring of holomorphic functions at a neighborhood of λ). Then the automorphism of $\mathcal{O}(\lambda)$ given by $\widetilde{f_\epsilon}$ (resp. $\widetilde{f_{\epsilon'}}$) is transformed to that of $\mathcal{O}(\mathfrak{p}, U(\mathfrak{p})^2))$ given by

$$(\mathbf{e}[z_1], \mathbf{e}[z_2], \mathbf{e}[z_3]) \mapsto \left(\mathbf{e}\left[z_1 + \sigma_1\left(\left(\frac{1}{\alpha^2}\right)\right)\right], \mathbf{e}\left[z_2 + \sigma_2\left(\left(\frac{1}{\alpha^2}\right)\right)\right], \mathbf{e}\left[z_3 + \sigma_3\left(\left(\frac{1}{\alpha^2}\right)\right)\right]\right)$$

(resp.

$$(\mathbf{e}[z_1], \mathbf{e}[z_2], \mathbf{e}[z_3]) \mapsto \left(\mathbf{e}\left[z_1 + \sigma_1\left(\left(\tfrac{\eta}{\alpha^2}\right)\right)\right], \mathbf{e}\left[z_2 + \sigma_2\left(\left(\tfrac{\eta}{\alpha^2}\right)\right)\right], \mathbf{e}\left[z_3 + \sigma_3\left(\left(\tfrac{\eta}{\alpha^2}\right)\right)\right]\right)).$$

Here $\left(\frac{1}{\alpha^2}\right)$ (resp. $\left(\frac{\eta}{\alpha^2}\right)$) is an element of O_K such that $\alpha^2\left(\frac{1}{\alpha^2}\right) \equiv 1 \pmod{\mathfrak{p}}$ (resp. $\alpha^2\left(\frac{\eta}{\alpha^2}\right) \equiv \eta \pmod{\mathfrak{p}}$). By the isomorphism $\mathcal{O}(\mathfrak{p}, U(\mathfrak{p})^2) \xrightarrow{\sim} \mathcal{O}(O_K, U(\mathfrak{p})^2)$ induced by

$$(\mathbf{e}[z_1], \mathbf{e}[z_2], \mathbf{e}[z_3]) \mapsto \left(\mathbf{e}\left[\frac{z_1}{\sigma_1(\mu)}\right], \mathbf{e}\left[\frac{z_2}{\sigma_2(\mu)}\right], \mathbf{e}\left[\frac{z_3}{\sigma_3(\mu)}\right]\right),$$

the automorphisms of $\mathcal{O}(\mathfrak{p}, U(\mathfrak{p})^2)$ described above are transformed to those of $\mathcal{O}(O_K, U(\mathfrak{p})^2)$ given by

$$\begin{aligned}&(\mathbf{e}[z_1], \mathbf{e}[z_2], \mathbf{e}[z_3]) \mapsto \\ &\left(\mathbf{e}\left[z_1 + \sigma_1\left(\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\right)\right], \mathbf{e}\left[z_2 + \sigma_2\left(\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\right)\right],\right. \\ &\qquad\qquad\left.\mathbf{e}\left[z_3 + \sigma_3\left(\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)\right)\right]\right)\end{aligned}$$

(resp.

$$\begin{aligned}&(\mathbf{e}[z_1], \mathbf{e}[z_2], \mathbf{e}[z_3]) \mapsto \\ &\left(\mathbf{e}\left[z_1 + \sigma_1\left(\frac{1}{\mu}\left(\frac{\eta}{\alpha^2}\right)\right)\right], \mathbf{e}\left[z_2 + \sigma_2\left(\frac{1}{\mu}\left(\frac{\eta}{\alpha^2}\right)\right)\right],\right. \\ &\qquad\qquad\left.\mathbf{e}\left[z_3 + \sigma_3\left(\frac{1}{\mu}\left(\frac{\eta}{\alpha^2}\right)\right)\right]\right).\end{aligned}$$

5.4. Let m be an element of K such that $(e-1)m \in O_K$ for any element e of $U(\mathfrak{p})^2$. For example, $\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right)$ and $\frac{1}{\mu}\left(\frac{\eta}{\alpha^2}\right)$ have such property. By the proof of Proposition 3.10, the extended automorphism $\widetilde{g_m}$ of g_m to the cusp resolution is given by

$$\widetilde{g_m} : (t_1, t_2, t_3) \mapsto \left(\mathbf{e}\left[\frac{d(m,v,w)}{d(u,v,w)}\right] t_1, \mathbf{e}\left[\frac{d(u,m,w)}{d(u,v,w)}\right] t_2, \mathbf{e}\left[\frac{d(u,v,m)}{d(u,v,w)}\right] t_3\right)$$

using coordinates t_1, t_2, t_3 in $(\mathbb{C}^3)_\sigma$ $\quad(\sigma = \langle u, v, w\rangle \in \Sigma^{(3)})$. We here consider the fixed subvariety of $\widetilde{g_m}$ for $m = \frac{1}{\mu}\left(\frac{1}{\alpha^2}\right), \frac{1}{\mu}\left(\frac{\eta}{\alpha^2}\right)$. Put $\epsilon = \widetilde{g_1}$, $\epsilon' = \widetilde{g_\eta}$. For simplicity, put

$$\begin{aligned}&e_1 := \mathbf{e}\left[\frac{d(\frac{1}{\mu}\left(\frac{1}{\alpha^2}\right), v, w)}{d(u,v,w)}\right], \quad e_2 := \mathbf{e}\left[\frac{d(u, \frac{1}{\mu}\left(\frac{1}{\alpha^2}\right), w)}{d(u,v,w)}\right], \\ &e_3 := \mathbf{e}\left[\frac{d(u, v, \frac{1}{\mu}\left(\frac{1}{\alpha^2}\right))}{d(u,v,w)}\right].\end{aligned}$$

If $e_1 \neq 1, e_2 \neq 1$, and $e_3 \neq 1$, then ϵ has only a fixed point $(0,0,0)$ in $(\mathbb{C}^3)_\sigma$. If exactly one of e_1, e_2, e_3 equals to 1, then ϵ has a 1-dimensional fixed subvariety in $(\mathbb{C}^3)_\sigma$. If exactly two of e_1, e_2, e_3 equal to 1, then ϵ has a 2-dimensional fixed subvariety in $(\mathbb{C}^3)_\sigma$. The map ϵ has no 3-dimensional fixed subvariety because of the relation (5) in the proof of Proposition 3.10. Put

$$e_1' := \mathbf{e}\left[\frac{d(\frac{1}{\mu}\left(\frac{\eta}{\alpha^2}\right), v, w)}{d(u,v,w)}\right], \quad e_2' := \mathbf{e}\left[\frac{d(u, \frac{1}{\mu}\left(\frac{\eta}{\alpha^2}\right), w)}{d(u,v,w)}\right],$$
$$e_3' := \mathbf{e}\left[\frac{d(u, v, \frac{1}{\mu}\left(\frac{\eta}{\alpha^2}\right))}{d(u,v,w)}\right].$$

Then the same thing holds for ϵ'.

5.5. In this subsection, we compute the contribution $\tau(\epsilon, X_\alpha^\epsilon)$ from the fixed subvariety X_α^ϵ of ϵ. The same thing holds for ϵ'. We suppose $X_\alpha^\epsilon \cap (\mathbb{C}^3)_\sigma \neq \phi$, and use the notation in the preceding subsection. Let $\mathcal{K}_{X(\mathfrak{p})}$ be the canonical bundle of $X(\mathfrak{p})$. Let $c_1(\bullet)$ (resp. $c_2(\bullet)$) be the first (resp. second) Chern class of $\bullet$.

(i) *The case of* $\dim X_\alpha^\epsilon = 0$.

In this case, we have $e_1 \neq 1$, $e_2 \neq 1$, and $e_3 \neq 1$. We find that $X_\alpha^\epsilon = \{(0,0,0)\}$. Since $\mathcal{K}_{X(\mathfrak{p})}|X_\alpha^\epsilon$ and $\mathcal{N}_{X_\alpha^\epsilon}$ are trivial, we have

$$\operatorname{ch}(\mathcal{K}_{X(\mathfrak{p})}|X_\alpha^\epsilon)(\epsilon) = 1,$$
$$\prod_\theta \mathcal{U}^\theta(\mathcal{N}_\alpha^\epsilon(\theta)) = \mathcal{T}(X_\alpha^\epsilon) = 1,$$
$$\det(1 - \epsilon|(\mathcal{N}_\alpha^\epsilon)^*) = (1 - e_1^{-1})(1 - e_2^{-1})(1 - e_3^{-1}).$$

Therefore, we obtain

$$\tau(\epsilon, X_\alpha^\epsilon) = -\frac{e_1 e_2 e_3}{(1-e_1)(1-e_2)(1-e_3)}.$$

(ii) *The case of* $\dim X_\alpha^\epsilon = 1$.

Assume $e_1 \neq 1$, $e_2 \neq 1$, and $e_3 = 1$. Then X_α^ϵ is t_3-axis. We find that $X_\alpha^\epsilon = F_{\langle u,v\rangle}$. Put $d = c_1(\mathcal{N}_\alpha^\epsilon) = d_1 + d_2$, $d_i = c_1(\mathcal{N}_\alpha^\epsilon(\theta_i))$ $(i = 1, 2)$, and $c_1 = c_1(X_\alpha^\epsilon)$. Here we put

$$\theta_1 = 2\pi \cdot \frac{d(\frac{1}{\mu}(\frac{1}{\alpha^2}), v, w)}{d(u,v,w)}, \quad \theta_2 = 2\pi \cdot \frac{d(u, \frac{1}{\mu}(\frac{1}{\alpha^2}), w)}{d(u,v,w)}.$$

Then we have

$$\begin{aligned}
\mathrm{ch}(\mathcal{K}_{X(\mathfrak{p})}|X^{\epsilon}_{\alpha})(\epsilon) &= e_1e_2(1-c_1-d),\\
\mathcal{U}^{\theta_1}(\mathcal{N}^{\epsilon}_{\alpha}(\theta_1)) &= \frac{1-e_1^{-1}}{1-e_1^{-1}\exp(-d_1)},\\
\mathcal{U}^{\theta_2}(\mathcal{N}^{\epsilon}_{\alpha}(\theta_2)) &= \frac{1-e_2^{-1}}{1-e_2^{-1}\exp(-d_2)},\\
\mathcal{T}(X^{\epsilon}_{\alpha}) &= 1+\frac{c_1}{2},\\
\det(1-\epsilon|(\mathcal{N}^{\epsilon}_{\alpha})^*) &= (1-e_1^{-1})(1-e_2^{-1}).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\tau(\epsilon, X^{\epsilon}_{\alpha})\\
&= \frac{e_1e_2(1-c_1-d)}{(1-e_1^{-1})(1-e_2^{-1})}\left(\frac{1-e_1^{-1}}{1-e_1^{-1}\exp(-d_1)}\right)\\
&\quad\cdot\left(\frac{1-e_2^{-1}}{1-e_2^{-1}\exp(-d_2)}\right)\left(1+\frac{c_1}{2}\right)[X^{\epsilon}_{\alpha}]\\
&= \frac{e_1e_2(1-c_1-d)\left(1+\frac{c_1}{2}\right)}{(1-e_1^{-1}(1-d_1))(1-e_2^{-1}(1-d_2))}[X^{\epsilon}_{\alpha}]\\
&= \frac{e_1e_2}{(1-e_1^{-1})(1-e_2^{-1})}\left\{\frac{(1-c_1-d)\left(1+\frac{c_1}{2}\right)}{\left(1+\frac{e_1^{-1}}{1-e_1^{-1}}d_1\right)\left(1+\frac{e_2^{-1}}{1-e_2^{-1}}d_2\right)}\right\}[X^{\epsilon}_{\alpha}]\\
&= \frac{e_1e_2}{(1-e_1^{-1})(1-e_2^{-1})}\left\{\left(1-c_1-d+\frac{c_1}{2}\right)\left(1-\frac{e_1^{-1}}{1-e_1^{-1}}d_1\right)\right.\\
&\qquad\qquad\left.\cdot\left(1-\frac{e_2^{-1}}{1-e_2^{-1}}d_1\right)\right\}[X^{\epsilon}_{\alpha}]\\
&= \frac{e_1e_2}{(1-e_1^{-1})(1-e_2^{-1})}\left(-\frac{c_1}{2}-\frac{1}{1-e_1^{-1}}d_1-\frac{1}{1-e_2^{-1}}d_2\right)[X^{\epsilon}_{\alpha}]\\
&= \frac{e_1e_2}{(1-e_1^{-1})(1-e_2^{-1})}\left(-\frac{c_1}{2}[X^{\epsilon}_{\alpha}]-\frac{1}{1-e_1^{-1}}d_1[X^{\epsilon}_{\alpha}]-\frac{1}{1-e_2^{-1}}d_2[X^{\epsilon}_{\alpha}]\right).
\end{aligned}$$

Here $c_1[X^{\epsilon}_{\alpha}] = 2-g(X^{\epsilon}_{\alpha})$ ($g(X^{\epsilon}_{\alpha})$ is the genus of X^{ϵ}_{α}). Since X^{ϵ}_{α} is rational, $g(X^{\epsilon}_{\alpha})=0$. By Tsushima [16], section 2, we have $d_1[X^{\epsilon}_{\alpha}] = F_{\langle u\rangle}\cdot {F_{\langle v\rangle}}^2$, and $d_2[X^{\epsilon}_{\alpha}] = {F_{\langle u\rangle}}^2\cdot F_{\langle v\rangle}$.

(iii) *The case of* $\dim X_\alpha^\epsilon = 2$.

Assume $e_1 \neq 1$, and $e_2 = e_3 = 1$. Then X_α^ϵ is a plane defined by $t_1 = 0$. We find that $X_\alpha^\epsilon = F_{\langle u\rangle}$. Put $c_1 = c_1(X_\alpha^\epsilon)$, $c_2 = c_2(X_\alpha^\epsilon)$, and $d = c_1(\mathcal{N}_\alpha^\epsilon)$. Then we have

$$\begin{aligned}
\mathrm{ch}(\mathcal{K}_{X(\mathfrak{p})}|X_\alpha^\epsilon)(\epsilon) &= e_1(1 - c_1 - d),\\
\prod_\theta \mathcal{U}^\theta(\mathcal{N}_\alpha^\epsilon(\theta)) &= 1 - \frac{e_1^{-1}d}{1-e_1^{-1}},\\
\mathcal{T}(X_\alpha^\epsilon) &= 1 + \frac{1}{12}c_2 + c_1^2,\\
\det(1-\epsilon|(\mathcal{N}_\alpha^\epsilon)^*) &= 1 - e_1^{-1}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\tau(\epsilon, X_\alpha^\epsilon)\\
&= \frac{e_1}{1-e_1^{-1}}\left\{(1-c_1-d)\left(1+\frac{1}{12}c_2+\frac{1}{12}c_1^2\right)\left(1-\frac{e_1^{-1}d}{1-e_1^{-1}}\right)\right\}[X_\alpha^\epsilon]\\
&= \frac{e_1}{1-e_1^{-1}}\left(1-c_1-d-\frac{e_1^{-1}d}{1-e_1^{-1}}+\frac{e_1^{-1}}{1-e_1^{-1}}c_1d\right.\\
&\qquad\qquad\left.+\frac{e_1^{-1}}{1-e_1^{-1}}d^2+\frac{1}{12}c_2+\frac{1}{12}c_1^2\right)[X_\alpha^\epsilon]\\
&= \frac{e_1}{1-e_1^{-1}}\left(\frac{e_1^{-1}}{1-e_1^{-1}}c_1d+\frac{e_1^{-1}}{1-e_1^{-1}}d^2+\frac{1}{12}c_2+\frac{1}{12}c_1^2\right)[X_\alpha^\epsilon]\\
&= \frac{e_1}{1-e_1^{-1}}\left(-\frac{c_1d[X_\alpha^\epsilon]+d^2[X_\alpha^\epsilon]}{1-e_1}+\frac{1}{12}\left(c_2[X_\alpha^\epsilon]+c_1^2[X_\alpha^\epsilon]\right)\right).
\end{aligned}$$

Since X_α^ϵ is rational, we have $c_2[X_\alpha^\epsilon] + c_1^2[X_\alpha^\epsilon] = 12$ by the formula of Noether. We find that $d^2[X_\alpha^\epsilon] = (X_\alpha^\epsilon)^3 = d(u)$ by Tsushima [16], section 2. Let $\{D_i\}_{i\in I}$ be the set of all irreducible divisors arising from the cusp resolutions of $\overline{\Gamma(\mathfrak{p})\setminus\mathfrak{H}^3}$. Then the total Chern class $c(X_\alpha^\epsilon)$ of X_α^ϵ is expressed as

$$c(X_\alpha^\epsilon) = \prod_{D_i \neq X_\alpha^\epsilon}(1 + D_i|X_\alpha^\epsilon)$$

(cf. Satake [12], Tsushima [16]). From this, $c_1(X_\alpha^\epsilon) = \sum_{D_i\neq X_\alpha^\epsilon} D_i|X_\alpha^\epsilon$. Hence we have $c_1d[X_\alpha^\epsilon] = c(u)$.

5.6. We return to the equation (8). Now let us calculate the difference $\tau(\epsilon)-\tau(\epsilon')$. For any element $x \in O_K$, let $\nu(x)$ be as in Definition 4.2. The contribution to $\tau(\epsilon)$ (resp. $\tau(\epsilon')$) from the fixed subvarieties in the resolution of the cusp α/μ is $\nu((1/\alpha^2))$ (resp. $\nu((\eta/\alpha^2))$) by 5.4 and 5.5. By Lemma 3.4, we have

$$\begin{aligned}\tau(\epsilon) =& \sum_{\alpha\in(O_K/\mathfrak{p})^\times/\widetilde{U}} \nu\left(\left(\frac{1}{\alpha^2}\right)\right)\\ =&\frac{1}{[U:U(\mathfrak{p})]}\sum_{\alpha\in(O_K/\mathfrak{p})^\times} \nu\left(\left(\frac{1}{\alpha^2}\right)\right)\\ =&\frac{1}{[U:U(\mathfrak{p})]}\sum_{\alpha\in(O_K/\mathfrak{p})^\times} \left(1+\left(\frac{\alpha}{\mathfrak{p}}\right)\right)\nu(\alpha),\end{aligned}$$

$$\begin{aligned}\tau(\epsilon') =& \sum_{\alpha\in(O_K/\mathfrak{p})^\times/\widetilde{U}} \nu\left(\left(\frac{\eta}{\alpha^2}\right)\right)\\ =&\frac{1}{[U:U(\mathfrak{p})]}\sum_{\alpha\in(O_K/\mathfrak{p})^\times} \nu\left(\left(\frac{\eta}{\alpha^2}\right)\right)\\ =&\frac{1}{[U:U(\mathfrak{p})]}\sum_{\alpha\in(O_K/\mathfrak{p})^\times} \left(1-\left(\frac{\alpha}{\mathfrak{p}}\right)\right)\nu(\alpha).\end{aligned}$$

We thus get the formula in Theorem 4.4. for the case $m=1$.

5.7. Let $m \geq 2$. Put $D := X(\mathfrak{p}) - \Gamma(\mathfrak{p}) \setminus \mathfrak{H}^3$. We denote by $\mathcal{L} := \Omega^3(\log D)$ be the sheaf of germs of 3-forms with logarithmic poles along D on $X(\mathfrak{p})$. Then we have $S_{2m}(\Gamma(\mathfrak{p})) = H^0(X(\mathfrak{p}), \mathcal{L}^{\otimes(m-1)} \otimes \Omega^3)$ for any positive integer m. If $\mathrm{tr}(\widetilde{f}_\epsilon | H^0(X(\mathfrak{p}), \mathcal{L}^{\otimes(m-1)}\otimes\Omega^3)$ is the trace of the linear transformation of $H^0(X(\mathfrak{p}), \mathcal{L}^{\otimes(m-1)} \otimes \Omega^3)$ induced by $\widetilde{f}_\epsilon$, then $\mathrm{tr}\ \pi(\epsilon) = \mathrm{tr}(\widetilde{f}_\epsilon | H^0(X(\mathfrak{p}), \mathcal{L}^{\otimes(m-1)} \otimes \Omega^3))$. The same thing holds for $\mathrm{tr}\ \pi(\epsilon')$. Since $\mathcal{L}^{\otimes(m-1)}$ is the pull-back of an ample sheaf under the morphism $X(\mathfrak{p}) \to \overline{\Gamma(\mathfrak{p}) \setminus \mathfrak{H}^3}$, we have

$$H^i(X(\mathfrak{p}), \mathcal{L}^{\otimes(m-1)} \otimes \Omega^3) = 0 \qquad (i \geq 1)$$

by the Kodaira vanishing theorem. Hence we have

$$\begin{aligned}
&\mathrm{tr}\ \pi(\epsilon) - \mathrm{tr}\ \pi(\epsilon') \\
&= \sum_{i=0}^{3} (-1)^i \mathrm{tr}(\widetilde{f_\epsilon} | H^i(X(\mathfrak{p}), \mathcal{L}^{\otimes(m-1)} \otimes \Omega^3) \\
&\quad - \sum_{i=0}^{3} (-1)^i \mathrm{tr}(\widetilde{f_{\epsilon'}} | H^i(X(\mathfrak{p}), \mathcal{L}^{\otimes(m-1)} \otimes \Omega^3).
\end{aligned}$$

Since $\mathcal{L}^{\otimes(m-1)}$ is trivial around D by Lemma 5.8 below, $\mathrm{ch}((\mathcal{L}^{\otimes(m-1)} \otimes \Omega^3)|X_\alpha^\epsilon)(\epsilon) = \mathrm{ch}(\Omega^3|X_\alpha^\epsilon)(\epsilon)$ holds in 2.7. The same thing holds for ϵ'. Thus we have

$$\mathrm{tr}\ \pi(\epsilon) - \mathrm{tr}\ \pi(\epsilon') = \tau(\epsilon) - \tau(\epsilon')$$

by using $\tau(\epsilon)$, $\tau(\epsilon')$ in 5.2 and the holomorphic Lefschetz formula. We get the equation (8) on $S_{2m}(\Gamma(\mathfrak{p}))$. In other words, the case $m \geq 2$ is reduced to the case $m = 1$. □

LEMMA 5.8. *The notation being as in 5.7, $\mathcal{L}$ is trivial around D.*

PROOF. It suffices to prove the claim for $Y(M,V)$ in Proposition 3.10. In the coordinate system (t_1, t_2, t_3) of the resolution $Y(M,V)$, we have

$$(2\pi\sqrt{-1})^3 dz_1 \wedge dz_2 \wedge dz_3 = d(u,v,w) \cdot \frac{dt_1 \wedge dt_2 \wedge dt_3}{t_1 t_2 t_3}.$$

Hence the holomorphic 3-form $dz_1 \wedge dz_2 \wedge dz_3$ on $G(M,V) \setminus \mathfrak{H}^3$ extends to a nowhere vanishing section of ${\Omega_{Y(M,V)}}^3(\log\ D)$. Here we put $D := Y(M,V) - G(M,V) \setminus \mathfrak{H}^3$. This proves the Lemma. □

6. An example

In this section, we give an example to Theorem 4.4.

6.1. Let K be the field $\mathbb{Q}(w)$ defined by

$$w^3 + 2w^2 - w - 1 = 0.$$

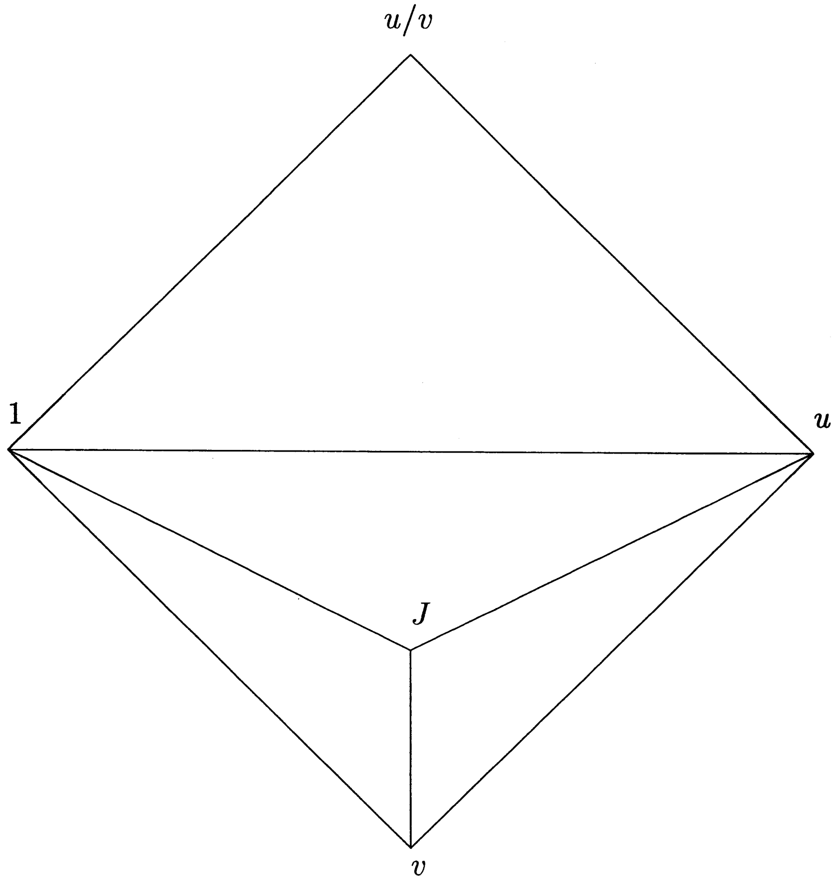

Fig. 2

The discriminant of this equation is 7^2, and K is a totally real Galois cubic field with class number 1. It is known that $O_K = \mathbb{Z} + \mathbb{Z}w + \mathbb{Z}w^2$. Thomas-Vasquez [13] shows that $U/\{\pm 1\}$ is freely generated by w^{-1} and $(1+w)w^{-1}$, and that U^+ is freely generated by $u := w^2$ and $v := (w+1)^2$. Moreover, they shows that $\langle 1, u, u/v \rangle$ and $\langle 1, u, v \rangle$ form a fundamental domain for the action of U^+ on $\mathbb{R}_+^3$, where $\mathbb{R}_+ := \{r \in \mathbb{R} \mid r > 0\}$. Put $J := 1 + w + w^2$. Then each of the triples $(1, u, u/v)$, $(1, u, J)$, $(1, v, J)$, and (u, v, J) is a basis of O_K. Therefore, the diagram in Fig. 2 gives a cusp resolution for the cusp of type (O_K, U^+) :

One can see that 13 is completely decomposed in K. We may assume that $w = \sigma_1(w) > \sigma_2(w) > \sigma_3(w)$. We have

$$-3 < \sigma_3(w) < -2, \quad -1 < \sigma_2(w) < -1/2, \quad 0 < w < 1.$$

Hence if we put $\mu = 2 - w$, then μ is totally positive. We find that $\mathfrak{p} := (\mu)$ is a prime ideal of K lying over 13. A simple computation shows that $U^+/U(\mathfrak{p})^2$ is a cyclic group with order 6 generated by the image of u. Hence if we denote by Σ the complex consisting of

$$\langle u^i, u^{i+1}, u^{i+1}/v\rangle,\ \langle u^i, u^{i+1}, u^iJ\rangle,\ \langle u^i, u^iv, u^iJ\rangle,\ \langle u^{i+1}, u^iv, u^iJ\rangle \qquad (0 \le i \le 5),$$

and their faces, then Σ describe the cusp resolution for the cusp of type $(O_K, U(\mathfrak{p})^2)$. We shall compute $y_1 - y_2$ for this prime ideal $\mathfrak{p}$ below.

6.2. Let Σ be the complex as above. Let m be any positive integer such that $1 \le m \le 12$. For any element $\langle u', v', w'\rangle \in \Sigma^{(3)}$, we have

$$\mathbf{e}\left[\frac{d(m/\mu, v', w')}{d(u', v', w')}\right] \ne 1, \quad \mathbf{e}\left[\frac{d(u', m/\mu, w')}{d(u', v', w')}\right] \ne 1, \quad \mathbf{e}\left[\frac{d(u', v', m/\mu)}{d(u', v', w')}\right] \ne 1.$$

Hence the fixed subvariety is 0-dimensional. Put

$$\nu(m; \langle u', v', w'\rangle) := \frac{\mathbf{e}\left[\frac{d(m/\mu, v', w')}{d(u', v', w')}\right]}{1 - \mathbf{e}\left[\frac{d(m/\mu, v', w')}{d(u', v', w')}\right]} \cdot \frac{\mathbf{e}\left[\frac{d(u', m/\mu, w')}{d(u', v', w')}\right]}{1 - \mathbf{e}\left[\frac{d(u', m/\mu, w')}{d(u', v', w')}\right]} \cdot \frac{\mathbf{e}\left[\frac{d(u', v', m/\mu)}{d(u', v', w')}\right]}{1 - \mathbf{e}\left[\frac{d(u', v', m/\mu)}{d(u', v', w')}\right]}.$$

Then we get

$$\nu(m) = -\sum_{\sigma \in \Sigma^{(3)}} \nu(m; \sigma).$$

We put $\zeta := \exp(2\pi i/13)$. The values of $\nu(m;\sigma)$ $(\sigma \in \Sigma^{(3)})$ are as follows :

$$\begin{aligned}
\nu(m;\langle 1,u,u/v\rangle) &= \frac{\zeta^{m}}{(1-\zeta^{m})(1-\zeta^{2m})(1-\zeta^{4m})},\\
\nu(m;\langle u,u^2,u^2/v\rangle) &= \frac{\zeta^{6m}}{(1-\zeta^{m})(1-\zeta^{3m})(1-\zeta^{6m})},\\
\nu(m;\langle u^2,u^3,u^3/v\rangle) &= \frac{-\zeta^{11m}}{(1-\zeta^{3m})(1-\zeta^{4m})(1-\zeta^{8m})},\\
\nu(m;\langle u^3,u^4,u^4/v\rangle) &= \frac{-\zeta^{6m}}{(1-\zeta^{m})(1-\zeta^{2m})(1-\zeta^{4m})},\\
\nu(m;\langle u^4,u^5,u^5/v\rangle) &= \frac{-\zeta^{4m}}{(1-\zeta^{m})(1-\zeta^{3m})(1-\zeta^{6m})},\\
\nu(m;\langle u^5,u^6,u^6/v\rangle) &= \frac{\zeta^{4m}}{(1-\zeta^{3m})(1-\zeta^{4m})(1-\zeta^{8m})},
\end{aligned}$$

$$\begin{aligned}
\nu(m;\langle 1,u,J\rangle) &= \frac{-\zeta^{7m}}{(1-\zeta^{3m})^2(1-\zeta^{4m})},\\
\nu(m;\langle u,u^2,uJ\rangle) &= \frac{-\zeta^{5m}}{(1-\zeta^{4m})^2(1-\zeta^{m})},\\
\nu(m;\langle u^2,u^3,u^2J\rangle) &= \frac{\zeta^{m}}{(1-\zeta^{m})^2(1-\zeta^{3m})},\\
\nu(m;\langle u^3,u^4,u^3J\rangle) &= \frac{\zeta^{3m}}{(1-\zeta^{3m})^2(1-\zeta^{4m})},\\
\nu(m;\langle u^4,u^5,u^4J\rangle) &= \frac{\zeta^{4m}}{(1-\zeta^{4m})^2(1-\zeta^{m})},\\
\nu(m;\langle u^5,u^6,u^5J\rangle) &= \frac{-\zeta^{4m}}{(1-\zeta^{m})^2(1-\zeta^{3m})},
\end{aligned}$$

$$\begin{aligned}
\nu(m;\langle 1,v,J\rangle) &= \frac{-\zeta^{9m}}{(1-\zeta^{2m})(1-\zeta^{3m})(1-\zeta^{6m})},\\
\nu(m;\langle u,uv,uJ\rangle) &= \frac{\zeta^{5m}}{(1-\zeta^{4m})(1-\zeta^{6m})(1-\zeta^{8m})},
\end{aligned}$$

$$\begin{aligned}
\nu(m;\langle u^2,u^2v,u^2J\rangle)&=\frac{\zeta^{11m}}{(1-\zeta^{m})(1-\zeta^{2m})(1-\zeta^{8m})},\\
\nu(m;\langle u^3,u^3v,u^3J\rangle)&=\frac{\zeta^{2m}}{(1-\zeta^{2m})(1-\zeta^{3m})(1-\zeta^{6m})},\\
\nu(m;\langle u^4,u^4v,u^4J\rangle)&=\frac{-1}{(1-\zeta^{4m})(1-\zeta^{6m})(1-\zeta^{8m})},\\
\nu(m;\langle u^5,u^5v,u^5J\rangle)&=\frac{-1}{(1-\zeta^{m})(1-\zeta^{2m})(1-\zeta^{8m})},
\end{aligned}$$

$$\begin{aligned}
\nu(m;\langle u,v,J\rangle)&=\frac{-1}{(1-\zeta^{3m})^2(1-\zeta^{6m})},\\
\nu(m;\langle u^2,uv,uJ\rangle)&=\frac{-1}{(1-\zeta^{4m})^2(1-\zeta^{8m})},\\
\nu(m;\langle u^3,u^2v,u^2J\rangle)&=\frac{-1}{(1-\zeta^{m})^2(1-\zeta^{2m})},\\
\nu(m;\langle u^4,u^3v,u^3J\rangle)&=\frac{\zeta^{12m}}{(1-\zeta^{3m})^2(1-\zeta^{6m})},\\
\nu(m;\langle u^5,u^4v,u^4J\rangle)&=\frac{\zeta^{3m}}{(1-\zeta^{4m})^2(1-\zeta^{8m})},\\
\nu(m;\langle u^6,u^5v,u^5J\rangle)&=\frac{\zeta^{4m}}{(1-\zeta^{m})^2(1-\zeta^{2m})}.
\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
&\sum_{i=0}^{5}\bigl\{\nu(m;\langle u^i,u^{i+1},u^{i+1}/v\rangle)+\nu(m;\langle u^i,u^{i+1},u^iJ\rangle)\\
&\qquad\qquad\qquad+\nu(m;\langle u^{i+1},u^{i+1},u^iJ\rangle)\bigr\}\\
&=\sum_{i=0}^{5}\nu(m;\langle u^i,u^iv,u^iJ\rangle)=0.
\end{aligned}$$

Thus we have $\nu(m)=0\ \ (1\le m\le 12)$. Using these, we obtain

$$\begin{aligned}
y_1-y_2&=\frac{1}{\sqrt{13}}\cdot\frac{2}{[U:U(\mathfrak{p})]}\cdot\sum_{m=1}^{12}\left(\frac{m}{\mathfrak{p}}\right)\nu(m)\\
&=0.
\end{aligned}$$

REMARK 6.3. The above result agrees with the fact that there does not exist a totally imaginary quadratic extension of K with the relative discriminant $\mathfrak{p}$. One can verify this fact as follows: $\{w^{-1}, w^{-1}+1\}$ is a fundamental system of units for K (cf. 6.1). One can see that -1 is quadratic residue, but w^{-1} and $w^{-1}+1$ are quadratic nonresidue modulo $\mathfrak{p}$. Consequently, the above fact is verified by the following Lemma.

LEMMA 6.4 (Naito). *Let K be as above, and $\mathfrak{p}$ a prime ideal of K. We assume that the prime p which divided by $\mathfrak{p}$ is odd and that p is totally decomposed in K. Moreover, we suppose that -1 is quadratic residue, and w^{-1} is quadratic nonresidue modulo $\mathfrak{p}$. Then there does not exist a totally imaginary quadratic extension of K with the relative discriminant $\mathfrak{p}$.*

PROOF. Let ∞_i be the real infinite prime of K corresponding to σ_i $(i = 1, 2, 3)$. Put $\mathfrak{f} = \mathfrak{p}\infty_1\infty_2\infty_3$. Let $H_\mathfrak{f}$ be the ideal class group for $\mathfrak{f}$. Since we here consider quadratic extensions, it suffices to study $H_\mathfrak{f}/{H_\mathfrak{f}}^2$. Let U be the unit group of K, and $\overline{U}$ the image of U by the map $U \to (O_K/\mathfrak{p})^\times \times \{\pm 1\}^3$, $u \mapsto (u \bmod \mathfrak{p}, \operatorname{sgn}\sigma_1(u), \operatorname{sgn}\sigma_2(u), \operatorname{sgn}\sigma_3(u))$. Here $\operatorname{sgn}\sigma_i(u)$ denotes the signature of $\sigma_i(u)$. Then we have $H_\mathfrak{f} \cong \left((O_K/\mathfrak{p})^\times \times \{\pm 1\}^3\right)/\overline{U}$. Since we deal with $H_\mathfrak{f}/{H_\mathfrak{f}}^2$, it suffices to consider the image of U in $\left((O_K/\mathfrak{p})^\times/(O_K/\mathfrak{p})^{\times 2}\right) \times \{\pm 1\}^3$. Let $\widehat{U}$ be its image, and $\widehat{u} \in \widehat{U}$ the image of $u \in U$.

As we saw in 6.1, we have $w^{-1} > 0, \sigma_2(w^{-1}) < 0$, and $\sigma_3(w^{-1}) < 0$. Since $w^{-1}\sigma_2(w^{-1})\sigma_3(w^{-1}) = 1$, $\sigma_2(w^{-1})$ or $\sigma_3(w^{-1})$ is quadratic residue modulo $\mathfrak{p}$. We may assume that $\sigma_2(w^{-1})$ is quadratic residue modulo $\mathfrak{p}$ by exchange σ_2 for σ_3 if necessary. We denote the image of $u \in U$ in $(O_K/\mathfrak{p})^\times/(O_K/\mathfrak{p})^{\times 2}$ by $\left(\frac{u}{\mathfrak{p}}\right)$. Then we have

$$\begin{aligned} \widehat{w^{-1}} &= (-1, 1, -1, -1), \\ \widehat{\sigma_2(w^{-1})} &= (1, -1, -1, 1), \\ \widehat{-1} &= (1, -1, -1, -1). \end{aligned} \tag{9}$$

Hence the rank of $\widehat{U}$ is 3. Thus the elementary 2-extension for $\mathfrak{f}$ is a quadratic extension. However, if we put $\mathfrak{f}' = \mathfrak{p}\infty_1\infty_3$, then the elementary

2-extension for $\mathfrak{f}'$is a quadratic extension by (9). Hence the former extension agrees with the latter one. Therefore this extension is not totally imaginary. □

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(Received December 24, 1997)

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