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On the Discrepancy of the β-Adic van der Corput Sequence

By Syoiti Ninomiya

Abstract. The β-adic van der Corput sequence is constructed. When β satisfies some conditions, the order of discrepancy of the sequence become $O(\log M/M)$ or $O((\log M)^2/M)$.

1. Introduction

It is well known that low-discrepancy sequences and their discrepancy play essential roles in quasi-Monte Carlo methods [6]. The author constructed a new class of low-discrepancy sequences N_β [7] by using the β-adic transformation [9][11]. Here, β is a real number greater than 1; when β is an integer greater than or equal to 2, N_β becomes the classical van der Corput sequence in base β. Therefore, the class N_β can be regarded as a generalization of the van der Corput sequence. N_β also contains a new construction by Barat and Grabner [1] [7]. The principle of the construction of N_β is that we can consider the van der Corput sequence to be a Kakutani adding machine [10]. Pagès [8] and Hellekalek [4] also considered the van der Corput sequence from this point of view. In [7], it is shown that when β satisfies the following two conditions:

- Markov condition: β is Markov, that is to say, for this β, the β-adic transformation becomes Markov,
- Pisot-Vijayaraghavan condition: All conjugates of β with respect to its characteristic equation belong to $\{z \in \mathbf{C} \mid |z| < 1\}$,

the discrepancy of N_β decreases in the fastest order $O(N^{-1}\log N)$. In this paper, we consider the case in which β is not necessarily Markov. We introduce the function $\phi_\beta(z)$ from Ito and Takahashi [5]. It is shown that when β satisfies the following condition **(PV)**:

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(PV) All zeroes of $1 - \phi_\beta(z)$ except for $z = 1$ belong to $\{z \in \mathbf{C} \mid |z| > \beta\}$,

which is a generalization of the above Pisot-Vijayaraghavan condition, the discrepancy of N_β decreases in the order $O(N^{-1}(\log N)^2)$. We also remark that the condition **(PV)** is considered to be a condition for the second eigenvalue of the Perron-Frobenius operator associated with the β-adic transformation.

2. Low-discrepancy sequence

First, we recall the notions of a uniformly distributed sequence and the discrepancy of points [6]. A sequence $x_1, x_2, \ldots$ in the s-dimensional unit cube $I^s = \prod_{i=1}^s [0, 1)$ is said to be uniformly distributed in I^s when

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_J(x_n) = \lambda_s(J)$$

holds for all subintervals $J \subset I^s$, where c_J is the characteristic function of J and λ_s is the s-dimensional Lebesgue measure. If $x_1, x_2, \ldots \in I^s$ is a uniformly distributed sequence, the formula

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{I^s} f(x)\, dx \tag{2.1}$$

holds for any Riemann integrable function on I^s. The discrepancy of the point set $P = \{x_1, x_2, \ldots, x_N\}$ in I^s is defined as follows:

$$D_N(\mathcal{B}; P) = \sup_{B\in\mathcal{B}} \left| \frac{A(B; P)}{N} - \lambda_s(B) \right| \tag{2.2}$$

where $\mathcal{B} \subset \wp(I^s)$ is a non-empty family of Lebesgue measurable subsets and $A(B; P)$ is the counting function that indicates the number of n, where $1 \le n \le N$, for which $x_n \in B$. When $\mathcal{J}^* = \{\prod_{i=1}^s [0, u_i), 0 \le u_i < 1\}$, the star discrepancy $D_N^*(P)$ is defined by $D_N^*(P) = D_N(J^*; P)$. When $S = \{x_1, x_2, \ldots\}$ is a sequence in I^s, we define $D_N^*(S)$ as $D_N^*(S_N)$, where S_N is the point set $\{x_1, x_2, \ldots, x_N\}$. Let S be a sequence in I^s. It is known that the following two conditions are equivalent:

1. S is uniformly distributed in I^s;

2. $\lim_{N\to\infty} D_N^*(S) = 0$.

The following classical theorem shows the importance of the notion of discrepancy:

THEOREM 2.1 (Koksma-Hlawka [6]). *If f has bounded variation $V(f)$ on $\bar{I}^s$ in the sense of Hardy and Krause, then for any $x_1, x_2, \ldots, x_N \in I^s$, we have*

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_{I^s} f(x)\,dx \right| \leq V(f) D_N^*(x_1, \ldots, x_N).$$

Schmidt [12] showed that, when $s = 1$ or 2, there exists a positive constant C that depends only on s, and the following inequality holds for an arbitrary point set P consisting of N elements:

$$D_N^*(P) \geq C \frac{(\log N)^{s-1}}{N}. \tag{2.3}$$

If (2.3) holds, then there exists a positive constant C that depends only on s, and any sequence $S \subset I^s$ satisfies

$$D_N^*(S) \geq C \frac{(\log N)^{s}}{N} \tag{2.4}$$

for infinitely many N. Taking account of (2.3) and (2.4), we define a low-discrepancy sequence for the one-dimensional case as follows:

DEFINITION 2.1. Let S be an one-dimensional sequence in $[0, 1)$. If $D_N^*(S)$ satisfies

$$D_N^*(S) = O(N^{-1} \log N)$$

then S is called a low-discrepancy sequence.

Hereafter we consider only the case where $s = 1$. We now introduce the classical van der Corput sequence [2] [6].

DEFINITION 2.2. Let $p \geq 2$ be an integer. Every integer $n \geq 0$ has a unique digit expansion

$$n = \sum_{j=0}^{\infty} a_j(n) p^j, \qquad a_j(n) \in \{0, 1, \ldots, p-1\} \text{ for all } j \geq 0,$$

in base p. Let $\tau = \{\tau_j\}_{j\geq 0}$ be a set of permutations τ_j of $\{0, 1, \ldots, p-1\}$. Then the radical-inverse function ϕ_p^τ is defined by

$$\phi_p^\tau(n) = \sum_{j=0}^{\infty} \tau_j(a_j(n))p^{-j-1} \qquad \text{for all integers} \qquad n \geq 0.$$

The van der Corput sequence in base p with digit permutations τ is the sequence $\{\phi_p^\tau(n)\}_{n=0}^{\infty} \subset [0, 1)$.

THEOREM 2.2 ([2][6]). *For an arbitrary integer $p \geq 2$, the van der Corput sequence in base p is a low-discrepancy sequence.*

3. β-adic transformation

In this section we define the fibred system and the β-adic transformation, following [5] [13].

C, **R**, **Z**, and **N** are the sets of all complex numbers, all real numbers, all integers, and all natural numbers, respectively. We also set

$$\begin{array}{rcl} \mathbf{R}_{>a} & = & \{r \in \mathbf{R} \mid r > a\} \\ \mathbf{Z}_{\geq n} & = & \{i \in \mathbf{Z} \mid i \geq n\} \\ & \vdots & \end{array}$$

and so on. For $x \in \mathbf{R}$, $[x]$ denotes the integer part of x.

DEFINITION 3.1. Let B be a set and $T : B \to B$ be a map. The pair (B, T) is called a fibred system if the following conditions are satisfied:

1. There is a finite countable set A.

2. There is a map $k : B \to A$, and the sets

$$B(i) = k^{-1}(\{i\}) = \{x \in B : k(x) = i\}$$

form a partition of B.

3. For an arbitrary $i \in A$, $T|_{B(i)}$ is injective.

Definition 3.2. Let $\Omega = A^{\mathbf{N}}$ and $\sigma : \Omega \to \Omega$ be the one-sided shift operator. Let $k_j(x) = k(T^{j-1}x)$. We derive a canonical map $\varphi : B \to \Omega$ from

$$\varphi(x) = \{k_j(x)\}_{n=1}^{\infty}.$$

φ is called the representation map.

We have the following commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{T} & B \\ \varphi\big\downarrow & & \varphi\big\downarrow \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array}$$

Definition 3.3. If a representation map φ is injective, φ is called a valid representation.

Definition 3.4. Let $\omega \in \Omega$. If $\omega \in \mathrm{Im}(\varphi)$, ω is called an admissible sequence.

Definition 3.5. The cylinder of rank n defined by $a_1, a_2, \ldots, a_n \in A$ is the set

$$B(a_1, a_2, \ldots, a_n) = B(a_1) \cap T^{-1}B(a_2) \cap \ldots \cap T^{-n+1}B(a_n).$$

We define B to be a cylinder of rank 0.

For a sequence $a \in \Omega$, we write the i-th element of a as $a(i)$, that is, $a = (a(0), a(1), a(2), \ldots)$.

Definition 3.6. Let $\beta > 1$ and $\beta \in \mathbf{R}$. Let $f_\beta : [0,1) \to [0,1)$ be the function defined by

$$f_\beta(x) = \beta x - [\beta x].$$

Let $A = \mathbf{Z} \cap [0, \beta)$. Then we have the following fibred system $([0,1), f_\beta)$:

$$\begin{array}{ccc} [0,1) & \xrightarrow{f_\beta} & [0,1) \\ \varphi\big\downarrow & & \varphi\big\downarrow \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array} \tag{3.1}$$

The representation map φ of this fibred system is defined as follows:

$$\varphi(x)(n) = k, \quad \text{if} \ \frac{k}{\beta} \le f_\beta^n(x) < \frac{(k+1)}{\beta}$$

where $f_\beta^0(x) = x$, and $f_\beta^{n+1}(x) = f_\beta(f_\beta^n(x))$. Let X_β be the closure of $\mathrm{Im}(\varphi)$ in the product space Ω with the product topology. The lexicographical order $\prec$ (resp. $\succ$) is defined in Ω as follows: $\omega \prec \omega'$ (resp. $\omega \succ \omega'$) if and only if there exists an integer n such that $\omega(k) = \omega'(k)$ for $k < n$ and $\omega(n) < \omega'(n)$ (resp. $\omega(n) > \omega'(n)$). We also define $\preceq$ (resp. $\succeq$) as $\prec$ (resp. $\succ$) or equal. In this situation, we set

$$f_\beta^n(1) = \lim_{x \nearrow 1} f_\beta^n(x),$$

$$\zeta_\beta = \max\{X_\beta\} = \varphi(1),$$

and

$$\rho_\beta(a) = \sum_{n=0}^{\infty} a(n)\beta^{-n-1}.$$

We have the following diagram:

$$\begin{array}{ccc} [0,1] & \xrightarrow{f_\beta} & [0,1] \\ \varphi\big\downarrow\big\uparrow\rho_\beta & & \varphi\big\downarrow\big\uparrow\rho_\beta \\ X_\beta & \xrightarrow{\sigma} & X_\beta \end{array} \tag{3.2}$$

This diagram is called a β-adic transformation.

We use the following notation for periodic sequences:

$$\begin{aligned} (a_1, a_2, \dots, \dot{a}_n, \dots \dot{a}_{n+m}) = (a_1, a_2, \dots, \ & a_n, a_{n+1}, \dots, a_{n+m}, \\ & a_n, a_{n+1}, \dots, a_{n+m}, \\ & \vdots \\ & a_n, a_{n+1}, \dots, a_{n+m}, \\ & \dots) \end{aligned}$$

We introduce the following proposition from Ito and Takahashi [5].

Proposition 3.1. *For an arbitrary $\beta \in \mathbf{R}_{>1}$ the following statements hold in (3.2).*

1. $\sigma \circ \varphi = \varphi \circ f_\beta$ *on* $[0,1)$.

2. $\varphi : [0,1] \to X_\beta$ *is an injection and is strictly order-preserving, i.e.,* $t < s$ *implies that* $\varphi(t) \prec \varphi(s)$.

3. $\rho_\beta \circ \varphi = \mathrm{id}$ *on* $[0,1]$.

4. $\rho_\beta \circ \sigma = f_\beta \circ \rho_\beta$ *on* $\mathrm{Im}(\varphi)$.

5. $\rho_\beta : X_\beta \to [0,1]$ *is a continuous surjection and is order-preserving, i.e.,* $\omega \prec \omega'$ *implies that* $\rho_\beta(\omega) \le \rho_\beta(\omega')$.

6. *For an arbitrary* $t \in [0,1]$, $\rho_\beta^{-1}(t)$ *consists either of a one point* $\varphi(t)$ *or of two points* $\varphi(t)$ *and* $\sup\{\varphi(s) \mid s < t\}$. *The latter case occurs only when* $f_\beta^n(t) = (\dot{0})$ *for some* $n > 0$.

We also remark that the following proposition holds:

Proposition 3.2.

$$X_\beta = \{\omega \in \Omega \mid \sigma^n \omega \preceq \zeta_\beta, \quad \textit{for all} \quad n \ge 0\}$$

Definition 3.7. Let $u \in X_\beta$. If there exist $n \in \mathbf{Z}_{\ge 1}$ which satisfies $u(i) = u(i+n)$ for any $i \in \mathbf{Z}$, u is called a periodic sequence. When $u \in X_\beta$ is periodic, we define the period of u as $\min\{n \in \mathbf{Z}_{\ge 1} \mid u(i) = u(i+n)$ for any $i \in \mathbf{Z}\}$.

The following definition is from Parry [9].

Definition 3.8. If ζ_β has periodic tail whose period is m, that is, $\sigma^l \zeta_\beta$ is periodic for some non-negative integer l and the period of $\sigma^l \zeta_\beta$ is m, then β and β-adic transformation (3.2) are called Markov. In this case, β is the unique $z > 1$ solution of the following equation:

$$z^{m+l} - \sum_{i=1}^{m+l} a_{i-1} z^{m+l-i} = z^l - \sum_{i=1}^{l} a_{i-1} z^{l-i} \tag{3.3}$$

where

$$\zeta_\beta = (a_0, a_1, \ldots, a_{l-1}, \dot{a}_l, a_{l+1}, \ldots, \dot{a}_{l+m-1})$$

and

$$l = \min\{l \in \mathbf{Z}_{\geq 0} \mid \sigma^l \zeta_\beta \text{ is periodic}\}.$$

This equation is called the characteristic equation of β. When $l = 0$, β is called simple. When β is Markov, $p(\beta)$ denotes the length of the period of ζ_β's periodic tail.

When β is not necessarily Markov, the notion of the characteristic equation is generalized as follows. This function was first studied in Takahashi [14][15] and Ito and Takahashi [5].

Definition 3.9.

$$\phi_\beta(z) = \sum_{n \geq 0} \zeta_\beta(n) \left(\frac{z}{\beta}\right)^{n+1}$$

We also have the following proposition from Ito and Takahashi [5].

Proposition 3.3. *$\phi_\beta(z)$ converges in a neighborhood of the unit disk $\{z \in \mathbf{C} \mid |z| \leq 1\}$ and the equation $1 - \phi_\beta(z) = 0$ has only one simple root at $z = 1$ in a neighborhood of the unit disk.*

Remark 3.1. When β is Markov, $1 - \phi_\beta(\beta/z) = 0$ becomes the characteristic equation of β.

4. Constructing the sequence

In this section, a sequence $N_\beta \subset [0, 1)$ is defined by the use of β-adic transformation, following [7]. Let $\beta \in \mathbf{R}_{>1}$ and let $([0, 1], f_\beta, X_\beta, \sigma, \varphi, \rho_\beta)$ be a β-adic transformation (3.2). Let $B = [0, 1)$, and $A, \Omega, \zeta_\beta, B(a_1, \dots, a_n)$ be the same as in the previous section.

Definition 4.1. Let $n \in \mathbf{Z}_{\geq 0}$. Define

$$\begin{aligned} X_\beta(n) &= \begin{cases} \{(\dot{0})\}, & n = 0 \\ \{\omega \in X_\beta \mid \sigma^{n-1}\omega \neq (\dot{0}) \quad \text{and} \quad \sigma^n \omega = (\dot{0})\}, & n \neq 0 \end{cases}, \\ Y_\beta(n) &= \{(\omega(0), \dots, \omega(n-1)) \mid \omega \in X_\beta\}, \end{aligned}$$

and

$$Y^0_\beta(n) = \{(a_0, \ldots, a_{n-1}) \mid (a_0, \ldots, a_{n-2}, a_{n-1} + 1) \in Y_\beta(n)\}.$$

Let $k \in \mathbf{Z}_{\geq 0}$, $u \in Y_\beta(k)$, and $v \in Y_\beta(l)$. Define $Y_\beta(u;n)$, $Y^0_\beta(u;n)$, $Y_\beta(u;n;v)$, $Y^0_\beta(u;n;v)$, $G_\beta(n)$, $G_\beta(u;n)$, $G^0_\beta(n)$, $G^0_\beta(u;n)$, and $G^0_\beta(u;n;v)$ as follows:

$$\begin{array}{rcl}
Y_\beta(u;n) & = & \{u \cdot \omega \mid u \cdot \omega \in Y_\beta(k+n)\} \\
Y^0_\beta(u;n) & = & \{u \cdot \omega \mid u \cdot \omega \in Y^0_\beta(k+n)\} \\
Y_\beta(u;n;v) & = & \{u \cdot \omega \cdot v \mid u \cdot \omega \cdot v \in Y_\beta(k+n+l)\} \\
Y^0_\beta(u;n;v) & = & \{u \cdot \omega \cdot v \mid u \cdot \omega \cdot v \in Y^0_\beta(k+n+l)\} \\
G_\beta(n) & = & \sharp Y_\beta(n) \\
G^0_\beta(n) & = & \sharp Y^0_\beta(n) \\
G_\beta(u;n) & = & \sharp Y_\beta(u;n) \\
G^0_\beta(u;n) & = & \sharp Y^0_\beta(u;n) \\
G_\beta(u;n;v) & = & \sharp Y_\beta(u;n;v) \\
G^0_\beta(u;n;v) & = & \sharp Y^0_\beta(u;n;v)
\end{array}$$

where $u \cdot v$ means the concatenation of u and v, that is to say,

$$u \cdot v = (u(0), \ldots, u(n-1), v(0), v(1), \ldots).$$

Finally we set $Y_\beta(0) = Y^0_\beta(0) = \{\epsilon\}$ where ϵ is the empty word and satisfies $\epsilon \cdot u = u \cdot \epsilon = u$ for any $u \in Y_\beta(n)$.

Definition 4.2. Define the right-to-left lexicographical order $\overset{r-l}{\prec}$ in $\bigsqcup_{n=0}^\infty X_\beta(n)$ as follows: $\omega \overset{r-l}{\prec} \omega'$ if and only if $(\omega(n-1), \ldots, \omega(0)) \prec (\omega'(m-1), \ldots, \omega'(0))$ where $\omega \in X_\beta(n)$ and $\omega' \in X_\beta(m)$.

Definition 4.3 (N_β [7]). Define $L_\beta = \{\omega_i\}_{i=0}^\infty$ as $\bigsqcup_{n=0}^\infty X_\beta(n)$ ordered in right-to-left lexicographical order, that is, L_β is $\bigsqcup_{n=0}^\infty X_\beta(n)$ as a set and $\omega_i \overset{r-l}{\prec} \omega_j$ holds for all $i < j$. Then, the sequence N_β is defined as follows:

$$N_\beta = \{\rho_\beta(\omega_i)\}_{i=0}^\infty .$$

Example 4.1. If $\beta = \frac{1+\sqrt{5}}{2}$, then $\zeta_\beta = (\dot{1}, \dot{0})$ and elements of N_β are calculated as follows:

$$\begin{aligned}
N_\beta(0) &= \rho_\beta(0) = 0 \\
N_\beta(1) &= \rho_\beta(1) = 0.618033988749895\ldots \\
N_\beta(2) &= \rho_\beta(01) = 0.381966011250106\ldots \\
N_\beta(3) &= \rho_\beta(001) = 0.23606797749979\ldots \\
N_\beta(4) &= \rho_\beta(101) = 0.854101966249686\ldots \\
N_\beta(5) &= \rho_\beta(0001) = 0.145898033750316\ldots \\
N_\beta(6) &= \rho_\beta(1001) = 0.763932022500212\ldots \\
N_\beta(7) &= \rho_\beta(0101) = 0.527864045000422\ldots \\
N_\beta(8) &= \rho_\beta(00001) = 0.0901699437494747\ldots \\
N_\beta(9) &= \rho_\beta(10001) = 0.70820393249937\ldots \\
N_\beta(10) &= \rho_\beta(01001) = 0.472135954999581\ldots \\
N_\beta(11) &= \rho_\beta(00101) = 0.326237921249265\ldots \\
N_\beta(12) &= \rho_\beta(10101) = 0.944271909999161\ldots \\
N_\beta(13) &= \rho_\beta(000001) = 0.0557280900008416\ldots \\
N_\beta(15) &= \rho_\beta(100001) = 0.673762078750737\ldots \\
N_\beta(16) &= \rho_\beta(010001) = 0.437694101250947\ldots \\
&\vdots
\end{aligned}$$

From this definition, we immediately have the following proposition:

PROPOSITION 4.1. *If β is an integer greater than or equal to 2 then N_β is the van der Corput sequence in base β with all digit permutations $\tau_j = \mathrm{id}$.*

From Theorem 2.2 and Proposition 4.1, we see that if $\beta \in \mathbf{Z}_{\geq 2}$ then N_β is a low-discrepancy sequence, that is to say, $D^*_M(N_\beta) = O(M^{-1} \log M)$ holds for all $\beta \in \mathbf{Z}_{\geq 2}$. We also have the following theorem:

THEOREM 4.1. *Let β be a real number greater than 1, and let the following condition* **(PV)** *hold:*

(PV) *All zeroes of* $1 - \phi_\beta(z)$ *except for* $z = 1$ *belong to* $\{z \in \mathbf{C} \mid |z| > \beta\}$.

Then,

$$D^*_M(N_\beta) = O\left(\frac{(\log M)^2}{M}\right)$$

holds. Moreover, if β *is Markov, then*

$$D^*_M(N_\beta) = O\left(\frac{\log M}{M}\right)$$

holds.

Remark 4.1. When β is Markov, the condition **(PV)** is equivalent to the condition that all conjugates of β with respect to its characteristic equation (3.3) belong to $\{z \in \mathbf{C} \mid |z| < 1\}$.

Remark 4.2. In [7], the case in which β is Markov is proved.

To prove this theorem, we provide lemmas and definitions. We use the following notations:

$$\omega[i,j) = \begin{cases} (\omega(i), \ldots, \omega(j-1)), & i < j \\ \epsilon, & i = j \end{cases},$$

where $\omega \in X_\beta$ and $i, j \in \mathbf{Z}_{\geq 0}$. $R_\beta(u) = \lambda(B(u))$ where, λ is the one-dimensional Lebesgue measure, $u \in X_\beta(n)$, and $B(u)$ is the cylinder (3.5). For a sequence S, $S[N]$ denotes the point set consisting of the first N elements of S, and $S[N; M] = S[N + M] \setminus S[N]$.

Definition 4.4. For any $k \geq 0$ and $u \in Y_\beta(k)$, define

$$e(u) = \{i \in \mathbf{Z}_{\geq 0} \mid \zeta_\beta[0, i+1) \cdot u \notin Y_\beta(k + i + 1)\}.$$

Lemma 4.1 ([5]). *For an arbitrary* $k \geq 0$ *and* $u \in Y_\beta(k)$, *we have the following partitioning of* $Y_\beta(u; n)$:

$$Y_\beta(u; n) = \bigsqcup_{j=1}^{n} Y^0_\beta(u; j) \cdot \zeta_\beta[0, n-j) \bigsqcup \max\{Y_\beta(u; n)\}$$

PROOF. It is trivial to show that the left-hand side includes the right-hand side.

If $v = (a_1, \dots, a_{n+k}) \in Y_\beta(u;n) \setminus Y_\beta^0(u;n)$ and $v \neq \max\{Y_\beta(u;n)\}$, then there exists an integer l that satisfies

$$k + 1 \leq l \leq n + k$$

and

$$\min\{w \in Y_\beta(u;n) \mid w \succ v\} = (a_1, \dots, a_l + 1, 0, \dots, 0).$$

This means that

$$(a_{l+1}, \dots, a_{n+k}) = \zeta_\beta[0, n + k - l)$$

and

$$(a_1, \dots, a_{l-1}, a_l + 1) \in Y_\beta^0(u; l - k)$$

hold. □

Taking account of Lemma 4.1, we give the following definition:

DEFINITION 4.5. For an arbitrary $u \in Y_\beta(n)$, define an integer $d(u)$ as follows: $d(u) = k$ if

$$u \in Y_\beta^0(k) \cdot \zeta_\beta[0, n - k)$$

holds. Remark that $\max\{Y_\beta(n)\} = \zeta_\beta[0, n)$.

From Lemma 4.1, Definition 4.4, and Definition4.5 we have the following lemma:

LEMMA 4.2. *For any $k, l, n \geq 0$, $u \in Y_\beta(k)$, and $v \in Y_\beta(l)$, we have the following partitioning of $Y_\beta(u;n;v)$:*

$$Y_\beta(u;n;v) \cong \begin{cases} \displaystyle\bigsqcup_{\substack{1 \leq j \leq n \\ n-j-1 \notin e(v)}} Y_\beta^0(u;j) \cdot \zeta_\beta[0, n-j), & \\ \qquad\qquad\qquad\qquad\qquad \text{if } \; n + k - d(\max\{Y_\beta(u;n)\}) - 1 \in e(v) \\ \displaystyle\bigsqcup_{\substack{1 \leq j \leq n \\ n-j-1 \notin e(v)}} Y_\beta^0(u;j) \cdot \zeta_\beta[0, n-j) \bigsqcup \max\{Y_\beta(u;n)\}, & \text{otherwise.} \end{cases}$$

Lemma 4.3. *For any $n \geq 0$ and $u \in Y_\beta(n)$,*

$$R_\beta(u) = \frac{1}{\beta^{d(u)}} \left(1 - \sum_{i=0}^{n-d(u)-1} \frac{\zeta_\beta(i)}{\beta^{i+1}} \right)$$

holds.

Proof. Let $u = u^0 \cdot \zeta_\beta[0, n - d(u))$ where $u^0 \in Y_\beta^0(d(u))$. From Definition 3.6,

$$\begin{aligned} R_\beta(u^0) &= \rho_\beta((u^0(0), \ldots, u^0(d(u) - 1) + 1) \\ &\quad - \rho_\beta((u^0(0), \ldots, u^0(d(u) - 1)) = \frac{1}{\beta^{d(u)}} \end{aligned}$$

and

$$R_\beta(\zeta_\beta[0, n - d(u))) = 1 - \sum_{i=0}^{n-d(u)-1} \frac{1}{\beta^{i+1}}.$$

When $v \cdot w \in Y_\beta(m)$, it follows that $R_\beta(v \cdot w) = R_\beta(v) R_\beta(w)$. Then, the lemma holds. □

Remark 4.3. From Definition 3.6, it follows that

$$f_\beta^n(x) = \beta^n \left(x - \sum_{i=0}^{n-1} \frac{\varphi(x)(i)}{\beta^{i+1}} \right)$$

for any $x \in [0, 1]$ and $n \geq 0$. Then, we have

$$R_\beta(u) = \frac{1}{\beta^n} f_\beta^{n-d(u)}(1)$$

for any $u \in Y_\beta(n)$ and $n \geq 0$, from Lemma 4.3.

Lemma 4.4 ([5]). *Let r be the absolute value of the second smallest zero of $1 - \phi_\beta(z)$, that is, $r = \min\{|z| \mid z \in \mathbf{C},\ z \neq 1,\ 1 - \phi_\beta(z) = 0\}$. Then for any small $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ and*

$$\left| G_\beta^0(u; n) - \frac{\beta^{n+k} R_\beta(u)}{\phi'_\beta(1)} \right| \leq \frac{C_\varepsilon}{n} \left(\frac{\beta}{r - \varepsilon} \right)^n$$

holds for any $n \geq 0$, $k \geq 0$ and $u \in Y_\beta(k)$.

PROOF. Let $k \geq 0$ and $u \in Y_\beta(k)$. Remark that

$$R_\beta(u) = \sum_{u \cdot v \in Y_\beta(u;n)} R_\beta(u \cdot v) \tag{4.1}$$

holds. From (4.1), Lemma 4.1, and Remark 4.3, we have

$$\beta^{n+k} R_\beta(u) = \sum_{j=0}^{n-1} f_\beta^j(1) G_\beta^0(u; n-j) + f_\beta^{n+l}(1) \tag{4.2}$$

where $l = k - d(\max\{Y_\beta(u;n)\}) \geq 0$. Remark that the formal power series

$$\sum_{n \geq 1} z^n \sum_{j=0}^{n-1} f_\beta^j(1) G_\beta^0(u; n-j) \beta^{-(n+k)}$$

converges for $|z| < 1$. We have the following equality from (4.2):

$$\begin{aligned} \beta^k \sum_{n \geq 1} z^n R_\beta(u) \quad &= \quad \sum_{n \geq 1} \left(\frac{z}{\beta}\right)^n \sum_{j=0}^{n-1} f_\beta^j(1) G_\beta^0(u; n-j) \\ &\quad + \sum_{n \geq 1} \left(\frac{z}{\beta}\right)^n f_\beta^{n+l}(1) \end{aligned} \tag{4.3}$$

We also have

$$\begin{aligned} &\sum_{n \geq 1} \left(\frac{z}{\beta}\right)^n \sum_{j=0}^{n-1} f_\beta^j(1) G_\beta^0(u; n-j) \\ &= \sum_{j \geq 1} \sum_{n \geq j} f_\beta^{j-1}(1) G_\beta^0(u; n-j+1) \left(\frac{z}{\beta}\right)^n \\ &= \sum_{j \geq 0} f_\beta^j(1) \left(\frac{z}{\beta}\right)^j \sum_{n \geq 1} G_\beta^0(u; n) \left(\frac{z}{\beta}\right)^n \end{aligned}$$

and, from Remark 4.3,

$$(1-z) \sum_{n \geq 0} f_\beta^n(1) \left(\frac{z}{\beta}\right)^n$$

$$= (1-z) + (1-z)\sum_{n\geq 1}\left(1-\sum_{i=0}^{n-1}\frac{\zeta_\beta(i)}{\beta^{i+1}}\right)z^n$$
$$= 1 - \sum_{n\geq 0}\zeta_\beta(n)\left(\frac{z}{\beta}\right)^{n+1} = 1-\phi_\beta(z).$$

By using these two equalities, we obtain from (4.3) that

$$\sum_{n\geq 1} G^0_\beta(u;n)\left(\frac{z}{\beta}\right)^n = \frac{z\beta^k R_\beta(u)}{1-\phi_\beta(z)} - \frac{(1-z)\sum_{n\geq 1} f_\beta^{n+l}(1)(z/\beta)^n}{1-\phi_\beta(z)}. \tag{4.4}$$

Consider the function

$$\begin{aligned} h_u(z) &= \sum_{n\geq 1}\left(G^0_\beta(u;n)\left(\frac{z}{\beta}\right)^n - \frac{\beta^k R_\beta(u)}{\phi'_\beta(1)}z^n\right) \\ &= \frac{z\beta^k R_\beta(u)}{1-\phi_\beta(z)} - \frac{(1-z)\sum_{n\geq 1} f_\beta^{n+l}(1)(z/\beta)^n}{1-\phi_\beta(z)} \\ &\quad - \frac{z\beta^k R_\beta(u)}{(1-z)\phi'_\beta(1)}. \end{aligned} \tag{4.5}$$

The second equality comes from (4.4). From Proposition 3.3, we see that $h_u(z)$ is analytic in a neighborhood of $\{z\in\mathbf{C} \mid |z|\leq r-\varepsilon,\ z\neq 1\}$. We also see from (4.5) that $\lim_{z\to 1}(1-z)h_u(z)=0$. Considering the fact that $\beta^k R_\beta(u)\leq 1$ for any $u\in Y_\beta(k)$, $k\geq 1$ and that the second term of the right-hand side of (4.4) and its derivative are bounded uniformly in l, we see that there exists a constant C_ε and

$$\sup_{\substack{k\geq 1,\ u\in Y_\beta(k)\\ |z|=r-\varepsilon}} \left|h'_u(z)\right| < C_\varepsilon \tag{4.6}$$

holds. Then we have

$$\begin{aligned} n!\left|\frac{G^0_\beta(u;n)}{\beta^n} - \frac{\beta^k R_\beta(u)}{\phi'_\beta(1)}\right| &= \left|h_u^{(n)}(0)\right| \\ &= \left|\frac{d^{n-1}h'_u}{dz^{n-1}}(0)\right| \\ &= \left|\frac{(n-1)!}{2\pi(r-\varepsilon)^n}\int_{|z|=r-\varepsilon} h'_u(z)\,dz\right| \end{aligned}$$

$$\leq \quad (n-1)!\frac{C_\varepsilon}{(r-\varepsilon)^n}$$

and the lemma follows. □

LEMMA 4.5. *If $\beta \in \mathbf{R}_{>1}$ is Markov and $\zeta_\beta = (a_0, \ldots, a_{l-1}, \dot{a}_l, \ldots, a_{l+\dot{m}-1})$, where $m = p(\beta)$ and $l = \min\{l \in \mathbf{Z}_{\geq 0} \mid \sigma^l \zeta_\beta$ is periodic$\}$, then we have the following statements:*

1. *For an arbitrary $v \in X_\beta$, $\{G^0_\beta(n)\}_{n=0}^\infty$ and $\{G_\beta(n)\}_{n=0}^\infty$ satisfy the following linear recurrent equation:*

$$\begin{aligned}(4.7)\qquad & G_\beta(\epsilon; n+m+l; v) - \sum_{i=0}^{m+l-1} a_i G_\beta(\epsilon; n+m+l-i-1; v) \\ &= G_\beta(\epsilon; n+l; v) - \sum_{i=0}^{l-1} a_i G_\beta(\epsilon; n+l-i-1; v) \\ &= G_\beta(\zeta_\beta[0,l); n; v).\end{aligned}$$

2. *For arbitrary $u \in Y_\beta(k)$, $k \geq m+l$ and $v \in X_\beta$, the following equations hold for any $n \geq m+l-k+d$:*

$$(4.8)\quad G_\beta(u; n; v) = \begin{cases} \displaystyle\sum_{i=1}^{m+l-k+d} a_{k-d-1+i} G_\beta(\zeta_\beta[0,l); n-i; v) \\ \qquad\qquad \text{when} \quad d > k-m-l \\ G_\beta(\zeta_\beta[0,l); n; v) \\ \qquad\qquad \text{when} \quad d = k-m-l \end{cases}$$

$$\begin{aligned}(4.9)\qquad G_\beta(\zeta_\beta[0,l); n; v) \quad &= \quad \sum_{i=1}^{m} a_{l+i-1} G_\beta(\epsilon; n-i; v) \\ &\quad + G_\beta(\zeta_\beta[0,l); n-m; v)\end{aligned}$$

where $d = d(u[k-m-l, k)) + k - m - l$.

PROOF. First, we remark that $u = u[0,d) \cdot \zeta_\beta[0, k-d)$. From Proposition 3.2, we have the following partitioning:

$$Y_\beta(\epsilon; n+l; v) \setminus \bigsqcup_{j=0}^{l-1} \bigsqcup_{i=0}^{a_j - 1} \zeta_\beta[0,j) \cdot i \cdot Y_\beta(\epsilon; n+l-j-1; v)$$

$$= Y_\beta(\zeta_\beta[0,l);n;v)$$

$$= Y_\beta(\epsilon;n+m+l;v) \setminus \bigsqcup_{j=0}^{m+l-1} \bigsqcup_{i=0}^{a_j-1} \zeta_\beta[0,j) \cdot i \cdot Y_\beta(\epsilon;n+m+l-j-1;v).$$

Then, (4.7) holds. When $d = k-m-l$, it is trivial to obtain (4.8) from Proposition 3.2. When $d > k-m-l$, we obtain the following partitioning:

$$Y_\beta(u;n;v) = \bigsqcup_{j=1}^{m+l-(k-d)} \bigsqcup_{i=0}^{a_{k-d+j-1}-1} u[0,d) \cdot \zeta_\beta[0,k-d+j) \cdot i \cdot w \cdot v$$

where $\zeta_\beta[0,l) \cdot w \cdot v \in Y_\beta(\zeta_\beta[0,l);n-j;v)$. We also have

$$\begin{aligned} Y_\beta(\zeta_\beta[0,l);n;v) &= \bigsqcup_{j=1}^{m} \bigsqcup_{i=0}^{a_{l+j-1}-1} \zeta_\beta[0,l) \cdot \zeta_\beta[l,l+j-1) \cdot i \cdot Y_\beta(\epsilon;n-j;v) \\ &\quad \bigsqcup \zeta_\beta[0,l+m) \cdot Y_\beta(\zeta_\beta[0,l);n-m;v). \end{aligned}$$

The lemma follows from these partitionings. □

Proof of Theorem 4.1. Let $k > 0$, $u \in Y_\beta(k)$. Let $M \in \mathbf{N}$ and $b = (b_0, b_1, \ldots, b_{m-1}) = L_\beta(M)$. We assume M to satisfy $m > k$. Define

$$\Delta(I;P) = A(I;P) - M\lambda(I),$$

where I is an interval in $[0,1)$ and $P = \{x_1, x_2, \ldots, x_M\} \subset [0,1)$. For any finite sets of points P, P' in $[0,1)$ and any intervals $I, I' \subset [0,1)$, $I \cap I' = \emptyset$,

$$\begin{aligned} \Delta(I;P \sqcup P') &= \Delta(I;P) + \Delta(I;P') \\ \Delta(I \sqcup I';P) &= \Delta(I;P) + \Delta(I';P) \end{aligned} \tag{4.10}$$

hold. Here, $P \sqcup P'$ is the disjoint union of P and P' or the union of P and P' with multiplicity. From Definition 4.3 and (4.10), we have

$$\begin{aligned} \Delta(B(u);N_\beta[M]) &= \Delta(B(u); \bigsqcup_{j=0}^{m-1} \bigsqcup_{i=0}^{b_j-1} Y_\beta(\epsilon;j;v_{ij})) \\ &= \sum_{j=0}^{m-1} \sum_{i=0}^{b_j-1} \Delta(B(u); Y_\beta(\epsilon;j;v_{ij})) \end{aligned} \tag{4.11}$$

where $v_{ij} = i \cdot b[j+1, m)$. Consider the $0 \leq j \leq k$ part of the right hand side of (4.11).

$$\sum_{j=0}^{k} \sum_{i=0}^{b_j - 1} |\Delta(B(u); Y_\beta(\epsilon; j; v_{ij}))| \leq \sum_{j=0}^{k} ([\beta]+1) G_\beta(j) R_\beta(u) \tag{4.12}$$

holds from the definition of Δ. From Lemma 4.1 and Lemma 4.4, there exists a constant C' and $G_\beta(j) \leq C' \beta^j$ holds for any j. From this and $R_\beta(u) \leq \beta^{-k}$, there exists a constant C_0, and

$$\sum_{j=0}^{k} ([\beta]+1) G_\beta(j) R_\beta(u) < C_0$$

is satisfied for any k. Then, from (4.11) and (4.12), we have

$$\Delta(B(u); N_\beta[M]) \leq C_0 + \sum_{j=k+1}^{m-1} \sum_{i=0}^{b_j - 1} |\Delta(B(u); Y_\beta(\epsilon; j; v_{ij}))| . \tag{4.13}$$

Define

$$\begin{aligned} \delta(u; n) &= G_\beta^0(u; n) - \frac{\beta^{n+k} R_\beta(u)}{\phi'_\beta(1)} \\ \delta(n) &= G_\beta^0(n) - \frac{\beta^n}{\phi'_\beta(1)} \end{aligned}$$

for $u \in Y_\beta(k)$ and $k, n \geq 0$. From this definition,

$$\begin{aligned} |\Delta(B(u); Y_\beta^0(n))| &= |G_\beta^0(u; n) - R_\beta(u) G_\beta^0(k+n)| \\ &= |\delta(u; n) - R_\beta(u) \delta(k+n)| \end{aligned} \tag{4.14}$$

holds. Then, from Lemma 4.2 we have

$$\begin{aligned} &\sum_{j=k+1}^{m-1} \sum_{i=0}^{b_j - 1} |\Delta(B(u); Y_\beta(\epsilon; j; v_{ij}))| \\ &\leq \sum_{j=k+1}^{m-1} \sum_{i=0}^{b_j - 1} \left(\sum_{\substack{l=1,\ldots,j \\ j-l-1 \notin e(v_{ij})}} \left| \Delta(B(u); Y_\beta^0(l) \cdot \zeta_\beta[0, j-l)) \right| + 1 \right) \\ &\leq \sum_{j=k+1}^{m-1} \sum_{i=0}^{b_j - 1} \left(\sum_{l=1}^{j} \left| \Delta(B(u); Y_\beta^0(l)) \right| + 1 \right) . \end{aligned} \tag{4.15}$$

From the **(PV)** condition and Lemma 4.4, there exist $r > \beta$ and a constant C_r that satisfy

$$|\delta(u;n)| \leq \frac{C_r}{n}\left(\frac{\beta}{r}\right)^n \tag{4.16}$$

for any $n, k > 0$ and $u \in Y_\beta(k)$. From (4.13), (4.14), (4.15), (4.16), and $r > \beta$, we see that

$$\begin{aligned}
&\Delta(B(u); N_\beta[M]) \\
&\leq C_0 + C_r([\beta]+1) \\
&\quad \cdot \sum_{j=k+1}^{m-1}\left(\sum_{l=1}^{j}\left(\frac{1}{l}\left(\frac{\beta}{r}\right)^l + \frac{1}{k+l}\left(\frac{\beta}{r}\right)^{k+l} R_\beta(u)\right)+1\right) \\
&= O(m) = O(\log M)
\end{aligned} \tag{4.17}$$

holds.

Choose an arbitrary $t \in [0,1)$. Let $M \in \mathbf{N}$ and $L_\beta(M) = (b_0, \ldots, b_{m-1})$. Let $B(t_0, \ldots, t_{m-1})$ be a cylinder of rank m that satisfies $t \in B(t_0, \ldots, t_{m-1})$. Then we have

$$[0,t) = B_{s_1} \sqcup B_{s_2} \sqcup \ldots \sqcup B_{s_k} \sqcup R,$$

where $0 \leq s_1 < s_2 < \ldots < s_k = m-1$, B_{s_i} is a disjoint union of up to $[\beta]+1$ cylinders of rank s_i and $\lambda(R) < \beta^{-m+1}$. Then from (4.10) and (4.17), we have

$$|\Delta([0,t); N_\beta[M])| = O((\log M)^2),$$

and therefore

$$D_M^*(N_\beta) = O\left(\frac{(\log M)^2}{M}\right).$$

In the following part, we consider the case in which β is Markov. Let $\zeta_\beta = (a_0, \ldots, a_{l'-1}, \dot{a_{l'}}, \ldots, \dot{a_{l-1}})$ and $l - l' = p(\beta)$. Then, β is the unique $z > 1$ solution of

$$z^l - \sum_{i=0}^{l-1} a_i z^{l-1-i} = z^{l'} - \sum_{i=0}^{l'-1} a_i z^{l'-1-i}. \tag{4.18}$$

Let $\alpha_1, \ldots, \alpha_q$ be the conjugates of β with respect to the equation (4.18), that is,

$$z^l - \sum_{i=0}^{l-1} a_i z^{l-1-i} - z^{l'} + \sum_{i=0}^{l'-1} a_i z^{l'-1-i} = (z-\beta)\prod_{i=1}^{q}(z-\alpha_i)^{l_i}$$

where $l_i \geq 1$, $\alpha_i \neq \alpha_j$ for all $i \neq j$ and $\sum_{i=1}^{q} l_i = l - 1$. We also have

$$|\alpha_i| < 1, \quad \text{for all } i \in \{1, \ldots, q\} \tag{4.19}$$

from the **(PV)** condition. Let $v \in X_\beta$. From Lemma 4.5, there exist complex numbers c, c_{ij} ($i = 1, \ldots, q$, $j = 0, \ldots l_i - 1$) that satisfy the following equation:

$$G_\beta(\epsilon; n; v) = c\beta^n + \sum_{i=1}^{r} \sum_{j=0}^{l_i-1} c_{ij} n^j \alpha_i^n \quad \text{for all} \quad n \in \mathbf{N}. \tag{4.20}$$

From Lemma 4.3, Lemma 4.5, and (4.20), we have

$$\begin{aligned} &\Delta(B(u); N_\beta[G_\beta(\epsilon; k+n; v)]) \\ &= \begin{cases} \displaystyle\sum_{h=1}^{q} \sum_{j=0}^{l_h-1} c_{hj} \left(n^j \alpha_h^n - \frac{1}{\beta^k}(k+n)^j \alpha_h^{k+n} \right), \\ \qquad\qquad \text{when} \quad d = k - l \\ \displaystyle\sum_{i=k-d}^{l-1} a_i \sum_{h=1}^{q} \sum_{j=0}^{l_h-1} c_{hj} \\ \quad \cdot \left((k+n-d)^j \alpha_h^{k+n-d-i} - \tfrac{1}{\beta^{d+i}}(k+n)^j \alpha_h^{k+n} \right), \\ \qquad\qquad \text{when} \quad d > k - l \end{cases} \end{aligned} \tag{4.21}$$

where $u \in Y_\beta(k)$, $n \in \mathbf{N}$, and $d = d(u[\max\{0, k-l+1\}, k+1)) + k - l$. From (4.10), (4.13), (4.15), (4.19), and (4.21), there exists a constant C that satisfies the following inequality (4.22) for any cylinder $B(u)$ of any rank k and $M > G_\beta(l+d)$.

$$|\Delta(B(u); N_\beta[M])| < C \tag{4.22}$$

Then, we obtain

$$D_M^*(N_\beta) = O\left(\frac{\log M}{M}\right)$$

by the above reasoning. □

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IBM Research, Tokyo Research Laboratory
1623-14 Shimotsuruma
Yamato-shi, Kanagawa-ken 242
Japan
E-mail: ninomiya@trl.ibm.co.jp

Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba Meguro-ku, Tokyo 153
Japan