J. Math. Sci. Univ. Tokyo
5 (1998), 299–332.

Determination of the Limiting Coefficient for Exponential Functionals of Random Walks with Positive Drift

By Katsuhiro Hirano

Abstract. Let $(S_n, n \geq 1)$ be a random walk satisfying $ES_1 > 0$ and h be a Laplace transform of a non-negative finite measure on $(0, \infty)$. Under additional conditions of S_1 and h, we consider the asymptotic behavior of $Eh(\sum_{i=1}^n e^{S_i})$. In particular we determine the limiting coefficient for asymptotic of this quantity in terms of the unique solution of the certain functional equation with boundary conditions. This solution corresponds to the Green function of $2^{-1}e^{-x}\triangle$ on $\mathbf{R}$. We apply our result to random processes in random media. Moreover we obtain the random walk analogue of Kotani's limit theorem for Brownian motion.

1. Introduction and the statement of theorem

Let $(S_n, n \geq 1)$ be a random walk, i.e., $S_1, S_2 - S_1, \cdots$ are independent and identically distributed random variables. The random variable of the form $\sum_{i=1}^n e^{S_i}$, $n \geq 1$, appears in various contexts. For example we meet this type of variable on random difference equation, hitting probability of a random walk in random media, and non-extinction probability and expected number of particles of a branching process and birth and death process in random media.

In [1] Afanas'ev considered the rate of decay of the tail probability of the maximum of a transient random walk moving in a random environment. To obtain the rate of decay, he analyzed the mean of a functional of the variable $\sum_{i=1}^n e^{S_i}$ where $(S_n, n \geq 1)$ is generated by i.i.d. random variables of the environment. To generalize this problem and clarify his proof, the author considered asymptotic behavior of the mean of exponential functionals of a random walk. In [6] we showed that there exists a limit of the mean of exponential functionals of a random walk and its limiting constant is positive and

1991 *Mathematics Subject Classification.* Primary 60J15; Secondary 60G50.

finite under certain conditions. Applying this to the problem of Afanas'ev, we could fill a gap in his result. But the probabilistic representation of the limiting constant was not given there.

As a continuous version of the result of Afanas'ev, Kawazu and Tanaka [8] considerd the rate of decay of the tail probability of the maximum of a diffusion in a drifted Brownian environment by using Yor's exact formula, in [16], of joint distribution of $\exp(B_t)$ and $\int_0^t \exp(2B_s)ds$ where $(B_t, t \geq 0)$ is a one-dimensional Brownian motion. Roughly speaking, they showed that if $\beta > \alpha > 0$ and $\sup_{x>0} x^\beta |h(x)| < \infty$, then

$$Eh\left(\int_0^t e^{B_s+\alpha s} ds\right) \sim c\, t^{-3/2} e^{-\alpha^2 t/2} \qquad \text{as} \quad t \to \infty, \tag{1.1}$$

where

$$c = \frac{2^{\frac{5}{2}}}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \int_0^\infty y^{2\alpha} h(4/z) e^{-uz} x \sinh x \, dx dy dz, \quad u = (1+y^2)/2 + y \cosh x.$$

It seems difficult to understand the probabilistic meaning of limiting constant c even if we go back to the proof of Yor's formula. On the other hand, if h is the Laplace transform of a non-negative finite measure ν on $(0, \infty)$, i.e.,

$$h(x) = \int_0^\infty e^{-xt} \nu(dt), \tag{1.2}$$

Kotani [10] showed that

$$E\left[f(B_t) h\Big(\int_0^t V(B_s) ds\Big)\right] \sim c_1 \, t^{-3/2} \qquad \text{as} \quad t \to \infty,$$

under additional conditions of f and $V \geq 0$. Here c_1 is explicitely represented as follows:

$$c_1 = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} dx\, f(x) \int_0^\infty \nu(d\lambda) g_\lambda(-\infty, 0)\, g_\lambda(-\infty, x), \tag{1.3}$$

where $g_\lambda(x, y)$ is the Green function of $(2V)^{-1}\triangle$ on $\mathbf{R}$ and $-\infty$ is the entrance boundary of the diffusion with generator $(2V)^{-1}\triangle$. For detail and strict conditions we refer to [10]. We note that the assumption (1.2) is valid when we consider the rate of decay of the maximum of a diffusion in a drifted Brownian environment, and that we can take $V(x) = e^x$ and

$f(x) = e^{\alpha x}$. Therefore, if h has a form (1.2), Kotani's theorem combined with Cameron-Martin transform implies (1.1) and gives another expression of c, that is,

$$c = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} dx\, e^{\alpha x} \int_0^\infty \nu(d\lambda) g_\lambda(-\infty, 0)\, g_\lambda(-\infty, x),$$

where $g_\lambda(-\infty, x)$ is defined by setting $V(x) = e^x$ for $g_\lambda(-\infty, x)$ in (1.3). So we get a probabilistic representation of the limiting constant in this case.

A random walk analogue of (1.1) was already given in [6]. Hence our main object in this paper is to obtain the expression of the limiting constant in the random walk case like the second expression of c. To be more concrete, we find function corresponding to $g_\lambda(-\infty, \cdot)$ and characterize it as the unique solution of the certain probabilistic functional equation with boundary conditions.

Now let us state the conditions of S_1 and our main theorem. To give a condition of S_1 we introduce

$$\psi(\theta) = \log E \exp(-\theta S_1), \qquad \theta \in \mathbf{R}.$$

Our conditions are the following:

Condition (A). *ψ is finite on an open interval of $(0, \infty)$ and $\psi'(\alpha) = 0$ for some $\alpha > 0$ which is contained in that interval.*

Condition (B). *The distribution of S_1 is not supported by any non-centered lattice (i.e., of the form $\{a + bz; z \in \mathbf{Z}\}$ with $0 < a < b$).*

We note that Condition (A) implies $0 < ES_1 \leq +\infty$ and $e^{\psi(\alpha)} < 1$. Conditions (A) and (B) appear in [3], [4], [7], [9] and [15], and the random walk satisfying these conditions has been studied in a number of papers. To state our main theorem we need some preparations. When Condition (A) is satisfied, we introduce an associated random walk $(\zeta_n, n \geq 1)$ where the distribution of ζ_1 is given by

$$P(\zeta_1 \in dy) = e^{-\alpha y - \psi(\alpha)} P(S_1 \in dy). \tag{1.4}$$

Let τ and ρ be the time of the first entry into the open positive half line and the closed negative half line by the walk $(\zeta_n, n \geq 1)$ respectively, i.e.,

$$\begin{aligned} \tau &= \min\{n > 0; \zeta_n > 0\}, \\ \rho &= \min\{n > 0; \zeta_n \leq 0\}. \end{aligned} \tag{1.5}$$

Since $E\zeta_1 = 0$ and $E|\zeta_1|^2 < \infty$, τ and ρ are well defined and the finiteness of $E\zeta_\tau$ and $E|\zeta_\rho|$ follows. Our main theorem is

Theorem. *Let Conditions (A) and (B) be satisfied, and let functions f, h and W satisfy the following:*
(1) f is bounded and continuous on $\mathbf{R}$.
(2) h has an expression

$$h(x) = \int_0^\infty e^{-xt} \nu(dt),$$

where ν is a non-negative finite measure on $(0, \infty)$.
(3) W is non-negative and continuous on $\mathbf{R}$.
(4) There exist positive numbers β, γ and δ such that

$$h(x) = O(x^{-\beta}) \qquad \text{as } x \to +\infty,$$

$$\limsup_{x \to -\infty} W(x) e^{-\gamma x} < \infty, \qquad \liminf_{x \to +\infty} W(x) e^{-\delta x} > 0,$$

with $\beta\delta > \alpha$. Then as $n \to \infty$,

$$E\left[f(S_n) h\left(\sum_{i=1}^n W(S_i) \right) \right] \sim c\, n^{-3/2} e^{\psi(\alpha) n},$$

where

$$c = \frac{1}{\sqrt{2\pi \psi''(\alpha)}} \int_0^\infty \nu(dt) \int_{\mathbf{R}} dx\, e^{\alpha x - tW(x)} f(x) g_t(0) \widehat{g}_t(x)$$

and g_t is the unique solution of the functional equation

$$g_t(x) = E_x\left[e^{-tW(\zeta_1)} g_t(\zeta_1) \right], \qquad x \in \mathbf{R},\ t > 0,$$

with boundary conditions

$$\lim_{x \to -\infty} \frac{g_t(x)}{|x|} = \frac{1}{E\zeta_\tau}, \qquad \lim_{x \to +\infty} g_t(x) = 0.$$

$\widehat{g}_t$ is the unique solution of the above equation with $E\zeta_\tau$ and ζ_1 are replaced by $E|\zeta_\rho|$ and $-\zeta_1$ respectively.

To prove our theorem we need some conditional limit theorems. We investigate limit law of $(S_n, n \geq 1)$ conditioned to stay negative. Especially we will show that $2k$-dimensional distributions $(S_1, \cdots, S_k, S_n, S_{n-1}, \cdots, S_{n-k-1})$ conditioned on $\Lambda_n = (S_1 \leq 0, \cdots, S_n \leq 0)$ converges as $n \to \infty$ weakly to a product of two k-dimensional distributions which are identified as the distributions of harmonic transform of the associated random walk. Limit theorems of random walks conditioned on Λ_n have been treated in many literatures. For example, the convergence of the Laplace transform of S_n conditioned on Λ_n has been shown by Iglehart [7]. The convergence of the law of $(S_1, \cdots, S_k)$ conditioned on Λ_n has been studied by Keener [9] and Bertoin and Doney [3]. But for our purpose we need to improve and extend their results since we have to know the limit law of $(S_1, \cdots, S_k, S_n, S_{n-1}, \cdots, S_{n-k-1})$ conditioned on Λ_n.

This paper is organized as follows. In Section 2 we give notations, fundamental lemmas and some asymptotic results of the associated random walk. Section 3 contains several asymptotic results and conditional limit theorems of the walk $(S_n, n \geq 1)$. In Section 4 we give a solution of the functional equation with boundary conditions in Theorem. The proof of Theorem and counter examples are given in Section 5. In Section 6 we treat three applications of our theorem.

2. Preliminaries

The purpose of this section is to introduce the notations and to investigate properties of the associated random walk. For every real number x, we denote the law of random walks and Markov processes starting at x by P_x. For simplicity we set $P = P_0$. If ν is a measure on $[0, \infty)$, its Laplace transform will be denoted by $\widetilde{\nu}$, that is, for $\theta > 0$, $\widetilde{\nu}(\theta) = \int_{0-}^{\infty} e^{-\theta x} d\nu(x)$. Let $(\zeta_n, n \geq 1)$ be a random walk and $\mu_n = \max_{0 \leq i \leq n} \zeta_i$. ζ means ζ_1 hereafter. Recalling τ and ρ in (1.5), we introduce the following quantities. For $n \geq 1$,

$$\begin{aligned} u_n(x) &= P(\zeta_n \leq x,\ \rho > n), \\ v_n(x) &= P(-\zeta_n \leq x,\ \tau > n), \\ u_0(x) &= v_0(x) = 1_{(x \geq 0)}. \end{aligned}$$

For $x < 0$, these functions are 0. Set

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \qquad v(x) = \sum_{n=0}^{\infty} v_n(x).$$

The quantities $u(x)$ and $v(x)$ will play crucial roles in this paper. Other but equivalent expressions of $u(x)$ and $v(x)$ are well known. We can find them in Bertoin and Doney [3], Keener [9] and Tanaka [14]. Especially $u(x)$ and $v(x)$ are the renewal functions of ζ_τ and $-\zeta_\rho$ respectively. An important fact for $u(x)$ and $v(x)$ are given by the following general lemma. Essentially the proof was obtained by Tanaka [14]. But our case somewhat differes from his similar lemma. We need careful treatment for (2) below.

LEMMA 1. *Let $x \geq 0$.*
(1) If $\rho < \infty$, a.s., then $Eu(x - \zeta) = u(x)$.
(2) If $\tau < \infty$, a.s., then $Ev(x + \zeta) = v(x)$.

PROOF. We prove only (2). (1) is proved in a similar way and more easily. The proof is devided into two parts. Set $a = \sum_{n=1}^{\infty} v_n(0) = \sum_{n=0}^{\infty} P(\zeta_{n+1} = 0, \tau > n)$.

Step 1. $1 + a = Ev(\zeta)$.

By the assumption $\tau < \infty$, a.s.,

$$\begin{aligned} 1 = P(\tau < \infty) &= P(\tau = 1) + \sum_{n=1}^{\infty} P(\tau = n+1) \\ &= P(\zeta > 0) + \sum_{n=1}^{\infty} P(\tau > n,\ \zeta_{n+1} > 0) \\ &= P(\zeta \geq 0) + \sum_{n=1}^{\infty} P(\tau > n,\ \zeta_{n+1} \geq 0) - a \\ &= P(\zeta \geq 0) + \sum_{n=1}^{\infty} Ev_n(\zeta) - a \\ &= Ev(\zeta) - a. \end{aligned}$$

Thus we have $v(0) = 1 + a = Ev(\zeta)$. That is, the lemma holds if $x = 0$.

Step 2. $1 + a = v(x) - Ev(x+\zeta) + Ev(\zeta),\ x > 0.$

Since $v_0(x) = 1$ if $x > 0$, similar calculations as in Step 1 show

$$\begin{aligned} v(x) - 1 &= v_1(x) + \sum_{n=1}^{\infty} v_{n+1}(x) \\ &= P(0 < -\zeta \le x) + \sum_{n=1}^{\infty} P(0 < -\zeta_{n+1} \le x,\ \tau > n) + a \\ &= P(0 < -\zeta \le x) + \sum_{n=1}^{\infty} E\Big(v_n(x+\zeta) - v_n(\zeta)\Big) + a \\ &= P(0 < -\zeta \le x) + Ev(x+\zeta) - P(x+\zeta \ge 0) \\ &\quad -\Big\{Ev(\zeta) - P(\zeta \ge 0)\Big\} + a \\ &= Ev(x+\zeta) - Ev(\zeta) + a, \end{aligned}$$

which proves Step 2. Combining Steps 1 and 2, we get the desired result. □

This lemma will be used in the next section when we introduce two homogeneous Markov processes. In the rest of this section we investigate some asymptotic properties of $(\zeta_n, n \ge 1)$. To obtain them we use the following two lemmas without proofs.

LEMMA 2 (see [7] and [15]).
(1) Let $\sum_{n=0}^{\infty} a_n t^n = \exp(\sum_{n=1}^{\infty} b_n t^n)$ *for* $|t| < 1$. *If* $b_n \sim bn^{-\frac{3}{2}}$, *then* $a_n \sim (b \exp B) n^{-\frac{3}{2}}$ *with* $B = \sum_{n=1}^{\infty} b_n$.
(2) Let $c_n \ge 0$, $d_n \ge 0$, $c_n \sim cn^{-\frac{3}{2}}$ *and* $d_n \sim dn^{-\frac{3}{2}}$. *If* $a_n = \sum_{j=0}^{n} c_{n-j} d_j$, *then* $a_n \sim (cD + dC) n^{-\frac{3}{2}}$ *with* $C = \sum_{n=0}^{\infty} c_n$ *and* $D = \sum_{n=0}^{\infty} d_n$.

LEMMA 3 (Spitzer-Baxter identity, see [5]). *For any* $\theta \ge 0$, $|t| < 1$,

$$1 + \sum_{n=1}^{\infty} \widetilde{u}_n(\theta) t^n = \exp\left\{ \sum_{n=1}^{\infty} \frac{t^n}{n} E(e^{-\theta\zeta_n}; \zeta_n > 0) \right\}.$$

If we replace $\widetilde{u}_n(\theta)$ *by* $\widetilde{v}_n(\theta)$, *then* $E(e^{-\theta\zeta_n}; \zeta_n > 0)$ *must be replaced by* $E(e^{\theta\zeta_n}; \zeta_n \le 0)$.

We consider the following conditions of ζ.

(a) $E\zeta = 0$, $0 < E|\zeta|^2 = \sigma^2 < \infty$ and $E|\zeta|^3 < \infty$.

(b) The distribution of ζ is not supported by any non-centered lattice (i.e., of the form $\{a + bz; z \in \mathbf{Z}\}$ with $0 < a < b$).

If $(\zeta_n, n \geq 1)$ is the associated random walk of $(S_n, n \geq 1)$, it is easy to see that ζ satisfies these conditions with $\sigma^2 = \psi''(\alpha)$. Lemmas 4, 5 and 6 below can be proved under the conditions (a) and (b). But we prove lemmas only in the case where ζ is non-lattice random variable and satisfies the condition (a). Some points to be paid attention in the case where ζ is centered lattice random variable and satisfies the condition (a) are given by REMARK in the last of this section.

The result below was indirectly shown in [2]. But for the reader's convenience we shall provide its proof.

LEMMA 4. *For every* $\theta > 0$,

$$\lim_{n\to\infty} \sqrt{n} E(e^{-\theta\zeta_n}; \zeta_n > 0) = \frac{1}{\sqrt{2\pi}\sigma\theta}.$$

PROOF. Let $\Phi_n(x) = P(\zeta_n \leq x\sigma\sqrt{n})$ and $\Phi(x)$ be the standard normal distribution. If ζ is non-lattice, under the condition (a), the asymptotic expansion in the central limit theorem

$$\Phi_n(x) = \Phi(x) + \frac{E\zeta^3}{6\sigma^3\sqrt{2\pi n}}(1 - x^2)\, e^{-\frac{1}{2}x^2} + o\left(\frac{1}{\sqrt{n}}\right)$$

holds uniformly on $\mathbf{R}$ (see e.g. [5]). Using this and integration by parts, we have

$$\begin{aligned} E(e^{-\theta\zeta_n}; \zeta_n > 0) &= \int_{0+}^{\infty} e^{-\theta\sigma\sqrt{n}x} d\Phi_n(x) \\ &= \theta\sigma\sqrt{n} \int_0^{\infty} e^{-\theta\sigma\sqrt{n}x}(\Phi_n(x) - \Phi_n(0))dx \\ &= \theta\sigma\sqrt{n} \int_0^{\infty} e^{-\theta\sigma\sqrt{n}x}(\Phi(x) - \Phi(0))dx + o(n^{-\frac{1}{2}}) \\ &= \int_0^{\infty} e^{-\theta\sigma\sqrt{n}x}\Phi'(x)dx + o(n^{-\frac{1}{2}}) \\ &= e^{\frac{1}{2}(\theta\sigma\sqrt{n})^2}(1 - \Phi(\theta\sigma\sqrt{n})) + o(n^{-\frac{1}{2}}). \end{aligned}$$

Moreover we have the following asymptotic formula of $\Phi(x)$

$$e^{\frac{1}{2}x^2}(1-\Phi(x)) \sim (\sqrt{2\pi}x)^{-1} \quad \text{as} \ \ x \to \infty,$$

which shows the lemma. □

Set

$$U(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_0^x u(y)dy, \qquad V(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_0^x v(y)dy.$$

Then we can prove the following lemma.

LEMMA 5. *For every $\theta > 0$ and $x \geq 0$,*
(1) $\lim_{n\to\infty} n^{\frac{3}{2}}\widetilde{u}_n(\theta) = \widetilde{U}(\theta), \quad \lim_{n\to\infty} n^{\frac{3}{2}}\widetilde{v}_n(\theta) = \widetilde{V}(\theta).$
(2) $\lim_{n\to\infty} n^{\frac{3}{2}}u_n(x) = U(x), \quad \lim_{n\to\infty} n^{\frac{3}{2}}v_n(x) = V(x).$

PROOF. Combining (1) of Lemma 2, Lemmas 3 and 4, we have for $\theta > 0$,

$$\lim_{n\to\infty} n^{\frac{3}{2}}\widetilde{u}_n(\theta) = \frac{1}{\sqrt{2\pi}\sigma\theta}\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}E(e^{-\theta\zeta_n};\zeta_n > 0)\right\} = \frac{\widetilde{u}(\theta)}{\sqrt{2\pi}\sigma\theta}.$$

It is easy to see that the last term of the above is $\widetilde{U}(\theta)$. Using the same method, we get the assertion for $\widetilde{V}$. By the extended continuity theorem for Laplace transform (see [5]), (1) yields (2). The proof of the lemma is complete. □

The next lemma is very useful to obtain various asymptotic results in the next section.

LEMMA 6. *For every $\theta > 0$ and $x \leq 0$,*

$$\lim_{n\to\infty} n^{\frac{3}{2}}E_x(e^{\theta\zeta_n};\tau > n) = u(-x)\,\widetilde{V}(\theta).$$

PROOF. If $x = 0$, the assertion is just Lemma 5. Therefore we prove the case $x < 0$. Let k_n be the first index k at which μ_n is attained. Then we have

$$
\begin{aligned}
& E_x(e^{\theta\zeta_n};\tau>n) \\
&= e^{\theta x}\sum_{j=0}^{n} E(e^{\theta\zeta_n};\mu_n\le -x, k_n=j) \\
&= e^{\theta x}\sum_{j=0}^{n} E(e^{\theta\zeta_n};\zeta_0,\cdots,\zeta_{j-1}<\zeta_j\le -x,\zeta_{j+1},\cdots,\zeta_n\le\zeta_j) \\
&= e^{\theta x}\sum_{j=0}^{n} E(e^{\theta\zeta_j};\zeta_0,\cdots,\zeta_{j-1}<\zeta_j\le -x)E(e^{\theta\zeta_{n-j}};\tau>n-j) \qquad (2.1)\\
&= e^{\theta x}\sum_{j=0}^{n} \widetilde{v}_{n-j}(\theta)\int_{0-}^{-x} e^{\theta y}du_j(y).
\end{aligned}
$$

In the last equality we use the duality lemma. (see e.g. [5]). In view of Lemma 5, applying (2) of Lemma 2 to the last term, we have

$$\lim_{n\to\infty} n^{\frac{3}{2}}E_x(e^{\theta\zeta_n};\tau>n) = e^{\theta x}\Big\{\widetilde{V}(\theta)\int_{0-}^{-x} e^{\theta y}du(y) + \widetilde{v}(\theta)\int_{0-}^{-x} e^{\theta y}dU(y)\Big\}.$$

Applying integration by parts to the first term in the above bracket, we see

$$\widetilde{V}(\theta)\int_{0-}^{-x} e^{\theta y}du(y) = \widetilde{V}(\theta)\Big\{e^{-\theta x}u(-x) - \theta\int_{0}^{-x} e^{\theta y}u(y)dy\Big\}.$$

The identity $\widetilde{v}(\theta)=\sqrt{2\pi}\sigma\theta\widetilde{V}(\theta)$ and the definition of $U(x)$ imply

$$\widetilde{v}(\theta)\int_{0-}^{-x} e^{\theta y}dU(y) = \theta\widetilde{V}(\theta)\int_{0}^{-x} e^{\theta y}u(y)dy.$$

Combining the above three results, the proof of the lemma is complete. □

Remark. To prove Lemmas 4, 5 and 6 in the case where ζ is centered lattice and satisfies the condition (a) (without $E|\zeta|^3<\infty$), we enumerate the modified points. In this case the support of the distribution of ζ is concentrated on the set $\{dz; z\in\mathbf{Z}\}$ with some $d>0$, and the local limit theorem

$$\lim_{n\to\infty}\left[\sup_{k\in\mathbf{Z}}\left|\sqrt{n}P(\zeta_n=dk) - \frac{d}{\sqrt{2\pi}\sigma}\exp\Big\{-\frac{(dk)^2}{2n\sigma^2}\Big\}\right|\right]=0$$

holds (see e.g. [5]). Using this, we have for $\theta > 0$,

$$\lim_{n\to\infty} \sqrt{n} E(e^{-\theta\zeta_n}; \zeta_n > 0) = \frac{d}{\sqrt{2\pi}\sigma} \sum_{k=1}^{\infty} e^{-\theta dk},$$

which corresponds to Lemma 4. Let U and V be defined by

$$U(x) = \frac{d}{\sqrt{2\pi}\sigma} \sum_{j=0}^{[x/d]-1} u(dj), \qquad V(x) = \frac{d}{\sqrt{2\pi}\sigma} \sum_{j=0}^{[x/d]} v(dj),$$

where $[x/d]$ is the integer part in x/d. Then (1) of Lemma 5 holds, and (2) of Lemma 5 holds at every continuous point. If x takes value on the set $\{dz; z = 0, -1, \cdots\}$, Lemma 6 holds in this case.

3. Conditional limit theorems

Throughout this section, Conditions (A) and (B) are assumed to be satisfied. In this section we study some asymptotic results and conditional limit theorems of $(S_n, n \geq 1)$. In particular Lemma 10 plays a key role in the proof of Theorem. From now on λ stands for $e^{\psi(\alpha)}$. Let f be a function on $\mathbf{R}^n$. By (1.4), we have for $x \in \mathbf{R}$,

$$E_x f(S_1, \cdots, S_n) = \lambda^n e^{-\alpha x} E_x[e^{\alpha\zeta_n} f(\zeta_1, \cdots, \zeta_n)]. \tag{3.1}$$

The relation (3.1) combined with Lemma 6 implies the next two lemmas.

Lemma 7. *Let $K \in \mathbf{R}$ be fixed and $M_n = \max_{0\leq i\leq n} S_i$.*
(1) If $x \leq K$, then as $n \to \infty$,

$$P_x(M_n \leq K) \sim u(K - x) e^{\alpha(K-x)} \widetilde{V}(\alpha)\, n^{-\frac{3}{2}} \lambda^n.$$

(2) If $\theta > -\alpha$ and $x \leq K$, then as $n \to \infty$,

$$E_x(e^{\theta S_n}; M_n \leq K) \sim u(K - x) e^{\alpha(K-x)+\theta K} \widetilde{V}(\theta + \alpha)\, n^{-\frac{3}{2}} \lambda^n.$$

(3) If $\theta > \alpha$, then as $n \to \infty$,

$$E(e^{-\theta M_n}) \sim \widetilde{\eta}(\theta)\, n^{-\frac{3}{2}} \lambda^n,$$

where $\eta(y) = \widetilde{V}(\alpha) e^{\alpha y} u(y)$.

PROOF. Since (1) follows from (2) by setting $\theta = 0$, we prove (2). Using the notations in Section 2 and (3.1), we have

$$E_x(e^{\theta S_n}; M_n \le K) = \lambda^n e^{\theta K + \alpha(K-x)} E_{x-K}(e^{(\theta+\alpha)\zeta_n}; \tau > n).$$

Therefore if $\theta + \alpha > 0$, Lemma 6 implies (2). Using (3.1) and doing a similar calculation as in (2.1), we see

$$\begin{aligned} Ee^{-\theta M_n} &= \lambda^n E e^{-\theta \mu_n + \alpha \zeta_n} \\ &= \lambda^n E e^{-(\theta-\alpha)\mu_n - \alpha(\mu_n - \zeta_n)} \\ &= \lambda^n \sum_{j=0}^{n} \tilde{v}_j(\alpha) \tilde{u}_{n-j}(\theta - \alpha). \end{aligned}$$

Taking account of (2) of Lemma 2 and Lemma 5, we find that $\lim_{n\to\infty} \lambda^{-n} n^{\frac{3}{2}} E e^{-\theta M_n}$ exists if $\theta > \alpha$. Hence (1) combined with the extended continuity theorem for Laplace transform provides (3). The proof of the lemma is complete. □

REMARK. The limits of the left hand side of Lemma 7 have been considered by Afanas'ev [1], Bertoin and Doney [3], Doney [4], Iglehart [7] and Veraverbeke and Teugels [15]. However their results are not given in explicit forms nor in appropriate forms for our purpose.

LEMMA 8. *Let $x \le K$ and f be a bounded continuous function on $(-\infty, K]$. Then*

$$\lim_{n\to\infty} E_x[f(S_n) \mid M_n \le K] = \frac{1}{c_K} \int_{-\infty}^{K} f(y) v(K-y) e^{\alpha y} dy,$$

where $c_K = \int_{-\infty}^{K} v(K-y) e^{\alpha y} dy$.

PROOF. It follows from (1) and (2) of Lemma 7 that if $\theta \ge 0$,

$$\begin{aligned} \lim_{n\to\infty} E_x(e^{\theta S_n} \mid M_n \le K) &= e^{\theta K} \frac{\widetilde{V}(\theta+\alpha)}{\widetilde{V}(\alpha)} \\ &= \frac{\int_{-\infty}^{K} e^{(\theta+\alpha)y}\, v(K-y) dy}{\int_{-\infty}^{K} e^{\alpha y}\, v(K-y) dy}. \end{aligned}$$

In the last equality we used the change of variable. Hence the continuity theorem for Laplace transform implies the lemma. □

We make use of the following lemma in the proof of Lemma 10. This lemma holds for arbitrary random walks.

LEMMA 9. *Let f_i, $0 \le i \le k$, be Borel measurable functions. If the left or right hand side of below exists, then the following equality holds.*

$$\int_{\mathbf{R}} dx f_0(x) E_x[f_1(S_1)\times\cdots\times f_k(S_k)] = \int_{\mathbf{R}} dx f_k(x) E_x[f_{k-1}(S_1')\times\cdots\times f_0(S_k')],$$

where $S_i' = -S_i$.

PROOF. Let $k = 1$ and $F(x) = P(S_1 \le x)$. Using Fubini's theorem repeatedly, we have

$$\begin{aligned}\int dx f(x) E_x g(S_1) &= \int dF(y) \int dx f(x) g(x+y) \\ &= \int dF(y) \int dx f(x-y) g(x) \\ &= \int dx\, g(x) \int dF(y) f(x-y) \\ &= \int dx\, g(x) E_x f(S_1'),\end{aligned}$$

which shows the lemma if $k = 1$. Assume that the lemma holds if $k = 1, \cdots, n$. Set $g(x) = E_x[f_{n+1}(S_1)]$ and $h(x) = E_x[f_{n-1}(S_1') \times \cdots \times f_0(S_n')]$. By the assumption and the Markov property, we have

$$\begin{aligned}&\int dx f_0(x) E_x[f_1(S_1) \times \cdots \times f_{n+1}(S_{n+1})] \\ &= \int dx f_0(x) E_x[f_1(S_1) \times \cdots \times f_n(S_n) g(S_n)] \\ &= \int dx f_n(x) g(x) E_x[f_{n-1}(S_1') \times \cdots \times f_0(S_n')] \\ &= \int dx f_n(x) h(x) E_x[f_{n+1}(S_1)] \\ &= \int dx f_{n+1}(x) E_x[f_n(S_1') h(S_1')] \\ &= \int dx f_{n+1}(x) E_x[f_n(S_1') \times \cdots \times f_0(S_{n+1}')].\end{aligned}$$

By induction the lemma is proved. □

To prove the next lemma that is special interest to us, we must define two homogeneous Markov processes $(Y_{K,n}, n \geq 1)$ and $(Z_{K,n}, n \geq 1)$ on $(-\infty, K]$. Let $(Y_{K,n}, n \geq 1)$ (respectively $(Z_{K,n}, n \geq 1)$) has a transition function p_K (respectively q_K) as follows.

$$\begin{aligned} p_K(x, dy) &= \frac{u(K-y)}{u(K-x)} P(x+\zeta \in dy) 1_{(y \leq K)}, \qquad x \leq K, \\ q_K(x, dy) &= \frac{v(K-y)}{v(K-x)} P(x-\zeta \in dy) 1_{(y \leq K)}, \qquad x \leq K. \end{aligned}$$

We introduce the initial distribution

$$P(Z_{K,0} \in dx) = \frac{1}{c_K} e^{\alpha x} v(K-x) dx \, 1_{(x \leq K)}, \tag{3.2}$$

where c_K is defined in Lemma 8. Since $E\zeta = 0$, it follows from Lemma 1 that p_K and q_K certainly are Markov transition functions on $(-\infty, K]$. The form of p_K means that $u(K - \cdot)$ is a harmonic function for the associated random walk killed as it enters (K, ∞), and that $\{u(K - \zeta_n) 1(\mu_n \leq K); n \geq 0\}$ is a Martingale with respect to P_x, $x \leq K$. The law of $(Y_{K,n}, n \geq 1)$ is connected with that of $(\zeta_n, n \geq 1)$ in the following identity. Let f be a function on $\mathbf{R}^m$ and $x \leq K$. Then

$$\begin{aligned} & E_x f(Y_{K.1}, \cdots, Y_{K.m}) \\ &= \frac{1}{u(K-x)} E_x[f(\zeta_1, \cdots, \zeta_m) u(K-\zeta_m); \mu_m \leq K]. \end{aligned} \tag{3.3}$$

Now we prove the main object of this section.

LEMMA 10. *Let $m \in \mathbf{N}$ and $x \leq K$. If g is a bounded function on $\mathbf{R}^m$ and f is a bounded continuous function on $\mathbf{R}^{m+1}$, then as $n \to \infty$,*

$$\begin{aligned} & E_x\Big[g(S_1, \cdots, S_m) f(S_n, \cdots, S_{n-m}) \mid M_n \leq K\Big] \\ & \to E_x g(Y_{K,1}, \cdots, Y_{K,m}) \; E f(Z_{K,0}, \cdots, Z_{K,m}). \end{aligned}$$

PROOF. Let $K \in \mathbf{R}$ be fixed. To avoid complexities, we omit K in $Y_{K,i}$ and $Z_{K,i}$ and set $\Lambda_n = (M_n \leq K)$. Without loss of generality, we assume

$0 \le f,\ g \le 1$ and $f(x_0, \cdots, x_m) = \prod_{i=0}^m f_i(x_i)$ with f_i bounded continuous on $\mathbf{R}$. Denote $g(S_1, \cdots, S_m)$ by $g(S)$. By the Markov property,

$$\begin{aligned} & E_x\Big[g(S) f(S_n, \cdots, S_{n-m}) \mid \Lambda_n\Big] \\ & = \frac{1}{P_x(\Lambda_n)} E_x\Big[g(S); \Lambda_m, E_{S_m}\Big[\prod_{i=0}^m f_i(S_{n-m-i}); \Lambda_{n-m}\Big]\Big] \\ & = E_x\left[g(S); \Lambda_m, \frac{P_{S_m}(\Lambda_{n-m})}{P_x(\Lambda_n)} h_{n-m}(S_m)\right], \end{aligned} \tag{3.4}$$

where $h_n(y) = E_y[\prod_{i=0}^m f_i(S_{n-i}) \mid \Lambda_n]$. Using (1) of Lemma 7, we have for $y \le K$,

$$\frac{P_y(\Lambda_{n-m})}{P_x(\Lambda_n)} \to \lambda^{-m} \frac{u(K-y)}{u(K-x)} e^{\alpha(x-y)} \qquad \text{as } n \to \infty. \tag{3.5}$$

Set $h(y) = E_y[\prod_{i=0}^{m-1} f_i(S_{m-i}); \Lambda_m]$. Then

$$\begin{aligned} h_n(y) &= \frac{1}{P_y(\Lambda_n)} E_y[f_m(S_{n-m}); \Lambda_{n-m}, h(S_{n-m})] \\ &= \frac{P_y(\Lambda_{n-m})}{P_y(\Lambda_n)} E_y[f_m(S_{n-m}) h(S_{n-m}) \mid \Lambda_{n-m}]. \end{aligned}$$

Therefore we have by (1) of Lemma 7 and Lemma 8,

$$\lim_{n\to\infty} h_n(y) = \frac{\lambda^{-m}}{c_K} \int_{-\infty}^K dz\, e^{\alpha z} v(K-z) f_m(z) h(z). \tag{3.6}$$

Collecting (3.4)–(3.6) and then using Fatou's lemma, we have

$$\begin{aligned} & \liminf_{n\to\infty} E_x\Big[g(S) \prod_{i=0}^m f_i(S_{n-m}) \mid \Lambda_n\Big] \\ & \ge E_x\Big[g(S), \Lambda_m, \lambda^{-m} \frac{u(K-S_m)}{u(K-x)} e^{\alpha(x-S_m)}\Big] \\ & \quad \times \frac{\lambda^{-m}}{c_K} \int_{-\infty}^K dy\, e^{\alpha y} v(K-y) f_m(y) h(y). \end{aligned} \tag{3.7}$$

Using (3.1) and then Lemma 9, we rewrite the last term in (3.7) with obvious notations as follows.

$$\frac{\lambda^{-m}}{c_K} \int_{-\infty}^K dy\, e^{\alpha y} v(K-y) f_m(y) h(y)$$

$$
\begin{aligned}
&= \frac{\lambda^{-m}}{c_K}\int_{-\infty}^{K} dy\, e^{\alpha y} v(K-y) f_m(y) E_y\Big[\prod_{i=0}^{m-1} f_i(S_{m-i});\Lambda_m\Big] \\
&= \frac{1}{c_K}\int_{-\infty}^{K} dy\, v(K-y) f_m(y) E_y\Big[\prod_{i=0}^{m-1} f_i(\zeta_{m-i}) e^{\alpha\zeta_m};\mu_m\le K\Big] \\
&= \frac{1}{c_K}\int_{-\infty}^{K} dy\, e^{\alpha y} f_0(y) E_y\Big[\prod_{i=1}^{m} f_i(\zeta_i') v(K-\zeta_m');\mu_m'\le K\Big] \qquad (3.8)\\
&= \frac{1}{c_K}\int_{-\infty}^{K} dy\, e^{\alpha y} v(K-y) f_0(y) E_y\Big[\prod_{i=1}^{m} f_i(Z_i)\Big] \\
&= E\Big[\prod_{i=0}^{m} f_i(Z_i)\Big].
\end{aligned}
$$

In the last equality we have used (3.2). Similar calculations show that

$$
\begin{aligned}
&E_x\left[g(S),\Lambda_m,\lambda^{-m}\frac{u(K-S_m)}{u(K-x)}e^{\alpha(x-S_m)}\right] \\
&= \frac{1}{u(K-x)}E_x[g(\zeta_1,\cdots,\zeta_m)\,u(K-\zeta_m),\mu_m\le K] \qquad (3.9)\\
&= E_x g(Y_1,\cdots,Y_m).
\end{aligned}
$$

Combining (3.7)–(3.9), we see

$$
\begin{aligned}
&\liminf_{n\to\infty} E_x[g(S)f(S_n,\cdots,S_{n-m})\mid\Lambda_n] \qquad (3.10)\\
&\quad\ge E_x g(Y_1,\cdots,Y_m)\,Ef(Z_0,\cdots,Z_m).
\end{aligned}
$$

Set $f=1$ on (3.10). Then we get

$$\liminf_{n\to\infty} E_x[g(S)\mid\Lambda_n]\ge E_x g(Y_1,\cdots,Y_m).$$

Replacing g by $1-g$ in the above, we have

$$
\begin{aligned}
E_x g(Y_1,\cdots,Y_m) &\ge 1-\liminf_{n\to\infty}E_x[(1-g)(S)\mid\Lambda_n] \\
&= \limsup_{n\to\infty}E_x[g(S)\mid\Lambda_n].
\end{aligned}
$$

Thus we have

$$\lim_{n\to\infty}E_x[g(S)\mid\Lambda_n]=E_x g(Y_1,\cdots,Y_m).$$

Replacing f by $1-f$ in (3.10) and using the above result, we get

$$E_x g(Y_1,\cdots,Y_m)\,Ef(Z_0,\cdots,Z_m)\geq \limsup_{n\to\infty} E_x[g(S)f(S_n,\cdots,S_{n-m})\mid \Lambda_n].$$

Thus our lemma is established. □

Remark. (i) If $f\equiv 1$, Lemma 10 was obtained by Keener [9] for integer valued random walks and by Bertoin and Doney [3] for general case with concise proof.

(ii) In [1] Afanas'ev claimed that the limit of the left hand side of Lemma 10 exists, and used it without proof. However he did not show what the limit is.

(iii) Lemma 10 corresponds to the following fact for Brownian motion with drift. Let $\Omega = C([0,\infty)\to \mathbf{R})$ and P_x be Wiener measure on Ω satisfying $P_x(\omega(0)=x)=1$. Set $\tau=\inf\{t>0;\omega(t)-\alpha t=0\}$, $\alpha>0$. If A and B are Borel sets of $C([0,t_0]\to(0,\infty))$, then for $x>0$,

$$\begin{aligned}&\lim_{t\to\infty} P_x(\{\omega(s)-\alpha s, s\leq t_0\}\in A,\ \{\omega(t-s)-\alpha(t-s), s\leq t_0\}\in B\mid \tau>t)\\ &\quad = Q_x(A)\times\int_0^\infty dy\,\alpha^2 y e^{-\alpha y}Q_y(B),\end{aligned}$$

where Q_z, $z>0$, is the law of 3-dimensional Bessel process starting at z on Ω.

4. Solution of the functional equation

In this section we assume that W satisfies the conditions in Theorem. To abbreviate the notations, let us put, for $x\leq K$ and $t>0$,

$$\begin{aligned} g_{K,t}(x) &= u(K-x)E_x\left[\exp\Big\{-t\sum_{i=1}^\infty W(Y_{K,i})\Big\}\right],\\ \widehat{g}_{K,t}(x) &= v(K-x)E_x\left[\exp\Big\{-t\sum_{i=1}^\infty W(Z_{K,i})\Big\}\right].\end{aligned}$$

These functions appear naturally in the proof of Theorem in the next section. To prove our theorem, we need to investigate the properties of these functions. We prove lemmas only for $g_{K,t}$. It is trivial that similar results hold for $\widehat{g}_{K,t}$.

Lemma 11. *Let $x < N < K$ and $\tau_N^{(K)} = \min\{n > 0; Y_{K,n} > N\}$. Then*

$$P_x\Big(\tau_N^{(K)} < \infty\Big) \le \frac{u(K-N)}{u(K-x)}.$$

Proof. By (3.3), we have

$$\begin{aligned}
&P_x\Big(\tau_N^{(K)} < \infty\Big)\\
&= \sum_{j=1}^{\infty} P_x\Big(\tau_N^{(K)} = j\Big)\\
&= \frac{1}{u(K-x)} \sum_{j=1}^{\infty} E_x\Big[u(K-\zeta_j); \zeta_1, \cdots, \zeta_{j-1} \le N < \zeta_j \le K\Big]\\
&\le \frac{u(K-N)}{u(K-x)} \sum_{j=1}^{\infty} P_x(\zeta_1, \cdots, \zeta_{j-1} \le N < \zeta_j)\\
&= \frac{u(K-N)}{u(K-x)}.
\end{aligned}$$

The proof of the lemma is complete. □

Lemma 12. *Let $K \in \mathbf{R}$ and $t > 0$ be fixed.*
(1) For all $x \le K$, $g_{K,t}(x) = E_x[e^{-tW(\zeta)} g_{K,t}(\zeta); \zeta \le K]$.
(2) If $K' \ge K$, then for all $x \le K$, $0 < g_{K,t}(x) \le g_{K',t}(x)$.
(3) $\lim_{x \to -\infty} g_{K,t}(x)/(K-x) = (E\zeta_\tau)^{-1}$.

Proof. (1) follows from the Markov property and (3.3). We prove (2). By Jensen's inequality,

$$g_{K,t}(x) \ge \exp\Big\{-t \sum_{n=1}^{\infty} E_x W(Y_{K,n})\Big\}.$$

Therefore if we show that $\sum_{n=1}^{\infty} E_x W(Y_{K,n}) < \infty$, the positivity of $g_{K,t}(x)$ follows. Set $c_1 = \sup_{y \le K} W(y) e^{-\gamma y}$. Using this and (3.3), we have

$$\begin{aligned}
E_x W(Y_{K,n}) &\le c_1 E_x(e^{\gamma Y_{K,n}})\\
&= c_1 \frac{1}{u(K-x)} E_x[e^{\gamma \zeta_n} u(K-\zeta_n); \mu_n \le K].
\end{aligned}$$

However Chebyshev's inequality shows $u(K-x) \le \widetilde{u}(\theta)e^{\theta(K-x)}$ for $\theta > 0$. Hence, setting $\theta = \gamma/2$ and $c_2 = c_1\widetilde{u}(\theta)$ and calculating as in (2.1), we see

$$
\begin{aligned}
E_x W(Y_{K,n}) &\le c_2 \frac{1}{u(K-x)} e^{\gamma K} E_{x-K}(e^{\theta\zeta_n}; \tau > n) \\
&= c_2 \frac{1}{u(K-x)} e^{\theta(x+K)} \sum_{j=0}^{n} \widetilde{v}_{n-j}(\theta) \int_{0-}^{K-x} e^{\theta y} du_j(y) \qquad (4.1)\\
&\le c_2 \frac{1}{u(K-x)} e^{\gamma K} \sum_{j=0}^{n} \widetilde{v}_{n-j}(\theta) u_j(K-x).
\end{aligned}
$$

Therefore we have

$$
\sum_{n=1}^{\infty} E_x W(Y_{K,n}) \le c_3\, e^{\gamma K} < \infty,
$$

where $c_3 = c_1\widetilde{u}(\theta)\widetilde{v}(\theta)$ does not depend on x. Thus the positivity of $g_{K,t}(x)$ is obtained. Let $K' \ge K \ge x$. From the definition of $g_{K,t}$,

$$
\begin{aligned}
g_{K,t}(x) &= \lim_{n\to\infty} u(K-x) E_x\Big[e^{-t\sum_1^n W(Y_{K,i})}\Big] \\
&= \lim_{n\to\infty} E_x\Big[e^{-t\sum_1^n W(\zeta_i)} u(K-\zeta_n); \mu_n \le K\Big] \\
&\le \lim_{n\to\infty} E_x\Big[e^{-t\sum_1^n W(\zeta_i)} u(K'-\zeta_n); \mu_n \le K'\Big] \\
&= g_{K',t}(x),
\end{aligned}
$$

which proves (2). We turn to the proof of (3). By the renewal theory, $u(y) \sim (E\zeta_\tau)^{-1} y$ as $y \to \infty$ (see e.g. [5]), it is enough to prove that $\lim_{x\to-\infty} g_{K,t}(x)/u(K-x) = 1$. Let $N < 0$ be fixed and $x < N$. Then

$$
\begin{aligned}
0 \le 1 - \frac{g_{K,t}(x)}{u(K-x)} &= E_x\Big[1 - e^{-t\sum_1^\infty W(Y_{K,i})}\Big] \\
&\le P_x\Big(\tau_N^{(K)} < \infty\Big) + t \sum_{n=1}^{\infty} E_x\Big(W(Y_{K,n}); \tau_N^{(K)} = \infty\Big). \qquad (4.2)
\end{aligned}
$$

Using the renewal theory and Lemma 11, we see

$$
P_x\Big(\tau_N^{(K)} < \infty\Big) \le \frac{u(K-N)}{u(K-x)} \to 0 \qquad \text{as } x \to -\infty. \qquad (4.3)
$$

Applying the method in (4.1), we have

$$\begin{aligned} E_x\Big(W(Y_{K,n});\tau_N^{(K)}=\infty\Big) &\le c_1E_x\Big(e^{\theta Y_{K,n}};\tau_N^{(K)}>n\Big)\\ &\le c_2\frac{1}{u(K-x)}e^{\theta(K+N)}\sum_{j=0}^{n}\widetilde{v}_{n-j}(\theta)u_j(N-x). \end{aligned}$$

Therefore we have

$$\sum_{n=1}^{\infty}E_x\Big(W(Y_{K,n});\tau_N^{(K)}=\infty\Big)\le c_3\,e^{\theta(K+N)}=c_4\,e^{\theta N}, \tag{4.4}$$

where c_4 does not depend on N and x. Combining (4.2)–(4.4) and letting $N\to-\infty$, we get (3). The proof of the lemma is complete. □

To investigate the properties of $\lim_{K\to\infty} g_{K,t}(x)$, we use the following lemma.

LEMMA 13. *Set* $f_{k,t}(x)=g_{k+1,t}(x)-g_{k,t}(x)$ *for* $x\le k$. *Then*

$$\sup_{x\le k} f_{k,t}(x)\le u(1)\,\exp\Big\{-t\min_{s\ge k}W(s)\Big\}.$$

PROOF. Using the definition of $g_{k,t}$ and then (3.3), we have

$$\begin{aligned} f_{k,t}(x) &= \lim_{n\to\infty}\Big\{u(k+1-x)E_x\Big[e^{-t\sum_1^n W(Y_{k+1,i})}\Big]\\ &\qquad -u(k-x)E_x\Big[e^{-t\sum_1^n W(Y_{k,i})}\Big]\Big\}\\ &= \lim_{n\to\infty}\Big\{E_x\Big[e^{-t\sum_1^n W(\zeta_i)}u(k+1-\zeta_n);\mu_n\le k+1\Big]\\ &\qquad -E_x\Big[e^{-t\sum_1^n W(\zeta_i)}u(k-\zeta_n);\mu_n\le k\Big]\Big\}\\ &= \lim_{n\to\infty}\Big\{E_x\Big[e^{-t\sum_1^n W(\zeta_i)}\{u(k+1-\zeta_n)-u(k-\zeta_n)\};\mu_n\le k\Big]\\ &\qquad +E_x\Big[e^{-t\sum_1^n W(\zeta_i)}u(k+1-\zeta_n);k<\mu_n\le k+1\Big]\Big\}\\ &= \lim_{n\to\infty}\{J_n+K_n\}. \end{aligned}$$

In the third equality we used the fact $(\mu_n \le k+1) = (\mu_n \le k) + (k < \mu_n \le k+1)$. We estimate J_n and K_n. Using the property $u(y+z) \le u(y) + u(z)$ (see e.g. [5]), we have

$$J_n \le u(1)\, P_x(\mu_n \le k) \to 0 \qquad \text{as } n \to \infty.$$

Set $T = \tau_k^{(k+1)}$. Then by (3.3),

$$\begin{aligned} K_n &= u(k+1-x)E_x\Big[e^{-t\sum_1^n W(Y_{k+1,i})}; T \le n\Big] \\ &\le u(k+1-x)E_x\Big[e^{-tW(Y_{k+1,T})}; T \le n\Big] \\ &\le u(k+1-x)P_x(T<\infty)\exp\Big\{-t\min_{k\le s\le k+1} W(s)\Big\} \\ &\le u(1)\exp\Big\{-t\min_{s\ge k} W(s)\Big\}. \end{aligned}$$

The last inequality comes from Lemma 11. Collecting the above results, the proof of the lemma is complete. □

From (2) of Lemma 12, we can define $g_t(x) := \lim_{n\to\infty} g_{n,t}(x)$ for all $x \in \mathbf{R}$. If $x \le n$, we express $g_t(x)$ as

$$g_t(x) = \sum_{j=n}^{\infty} f_{j,t}(x) + g_{n,t}(x). \tag{4.5}$$

By Lemma 13, $g_t(x)$ is finite for all x. The following lemma shows that g_t is a solution of the functional equation with boundary conditions in Theorem.

LEMMA 14. *Let $t > 0$ be fixed.*
(1) For all $x \in \mathbf{R}$, $g_t(x) = E_x[e^{-tW(\zeta)}g_t(\zeta)]$.
(2) $\lim_{x\to-\infty} g_t(x)/|x| = (E\zeta_\tau)^{-1}$.
(3) $\lim_{x\to+\infty} g_t(x) = 0$.

PROOF. (1) and (2) of Lemma 12 combined with the monotone convergence theorem imply (1). (2) and (3) of Lemma 12 yield

$$\frac{1}{E\zeta_\tau} = \lim_{x\to-\infty} \frac{g_{0,t}(x)}{|x|} \le \liminf_{x\to-\infty} \frac{g_t(x)}{|x|}.$$

By virtue of Lemma 13 and (4.5), (3) of Lemma 12 shows

$$\limsup_{x\to-\infty}\frac{g_t(x)}{|x|}\le\limsup_{x\to-\infty}\frac{1}{|x|}\Big\{\sum_{n=0}^{\infty}f_{n,t}(x)+g_{0,t}(x)\Big\}=\lim_{x\to-\infty}\frac{g_{0,t}(x)}{|x|}=\frac{1}{E\zeta_\tau}.$$

Thus (2) is proved. Let $x\in[n-1,n]$. Then we have

$$\begin{aligned}g_{n,t}(x)&=u(n-x)E_x\Big[e^{-t\sum_1^\infty W(Y_{n,i})}\Big]\\&\le u(n-x)E_x\Big[e^{-tW(Y_{n,1})}\Big]\\&=E_x\Big[e^{-tW(\zeta)}u(n-\zeta);\zeta\le n\Big]\\&\le E\Big[e^{-tW(x+\zeta)}u(1-\zeta)\Big].\end{aligned}$$

Hence we have

$$\sup_{n-1\le x\le n}g_{n,t}(x)\le E\Big[u(1-\zeta)\exp\Big\{-t\min_{x\ge n-1}W(x+\zeta)\Big\}\Big].$$

Applying the dominated convergence theorem to the right hand side, we get

$$\limsup_{n\to\infty}\sup_{n-1\le x\le n}g_{n,t}(x)=0.$$

In addition we have by Lemma 13,

$$\limsup_{n\to\infty}\sup_{n-1\le x\le n}\sum_{j=n}^{\infty}f_{j,t}(x)=0.$$

Two estimates above and (4.5) provide

$$\limsup_{n\to\infty}\sup_{n-1\le x\le n}g_t(x)=0,$$

which shows (3). The proof of the lemma is complete. □

In the next section we show that g_t is uniquely determined by the conditions in Lemma 14.

5. Proof of Theorem

Firstly we prove the convergence part of Theorem. Secondly we show the uniqueness of solution of the functional equation in Theorem. We devide the proof into five parts.

Step 1. Let $K > 0$ and $t > 0$. Then as $n \to \infty$,

$$\lambda^{-n} n^{\frac{3}{2}} E\Big[f(S_n) e^{-t\sum_1^n W(S_i)}; M_n \le K\Big] \to \frac{g_{K,t}(0)}{\sqrt{2\pi\psi''(\alpha)}} \int_{-\infty}^{K} dx\, e^{\alpha x - tW(x)} f(x) \widehat{g}_{K,t}(x).$$

Proof. Taking account of (3.2), (1) of Lemma 7 and the identity $\sqrt{2\pi\psi''(\alpha)}\widetilde{V}(\alpha)e^{\alpha K} = c_K$, Step 1 is equivalent to

$$\lim_{n\to\infty} E\Big[f(S_n)e^{-t\sum_1^n W(S_i)} \mid M_n \le K\Big] = E\Big[e^{-t\sum_1^\infty W(Y_i)}\Big] E\Big[f(Z_0)e^{-t\sum_0^\infty W(Z_i)}\Big],$$

where $Y_i = Y_{K.i}$ and $Z_i = Z_{K,i}$. With no loss of generality we assume $0 \le f \le 1$. For $m \le n$,

$$0 \le f(S_n)e^{-t\sum_1^n W(S_i)} \le f(S_n)e^{-t\sum_1^m W(S_i) - t\sum_{n-m}^n W(S_i)} \le 1.$$

By virtue of Lemma 10, letting $n \to \infty$ and then $m \to \infty$, we get

$$\limsup_{n\to\infty} E\left[f(S_n)e^{-t\sum_1^n W(S_i)} \mid M_n \le K\right] \le E\left[e^{-t\sum_1^\infty W(Y_i)}\right] E\left[f(Z_0)e^{-t\sum_0^\infty W(Z_i)}\right].$$

On the other hand

$$0 \le f(S_n)e^{-t\sum_1^m W(S_i) - t\sum_{n-m}^n W(S_i)} - f(S_n)e^{-t\sum_1^n W(S_i)} \le t\sum_{i=m+1}^{n-m-1} W(S_i).$$

Thus to obtain the liminf estimate, it suffices to show that

$$\limsup_{m\to\infty}\limsup_{n\to\infty} \sum_{i=m+1}^{n-m-1} E\Big(W(S_i) \mid M_n \le K\Big) = 0. \tag{5.1}$$

From the assumption of $W, W(x) \le c_1 e^{\gamma x}$ for $x \le K$. Using this and the Markov property, we have

$$\begin{aligned} E\Big(W(S_i); M_n \le K\Big) &\le c_1\, E(e^{\gamma S_i}; M_n \le K) \\ &= c_1\, E\Big[e^{\gamma S_i} P_{S_i}(M_{n-i} \le K); M_i \le K\Big]. \end{aligned}$$

Let $\theta \in (\alpha, \alpha+\gamma)$. By Chebyshev's inequality and (3) of Lemma 7,

$$P_x(M_n \le K) \le e^{\theta(K-x)} E e^{-\theta M_n} \le c_2\, e^{-\theta x} n^{-\frac{3}{2}} \lambda^n.$$

Combining the two estimates above and (2) of Lemma 7, we have

$$\begin{aligned} E\Big(W(S_i)\,;\, M_n \le K\Big) &\le c_3 (n-i)^{-\frac{3}{2}} \lambda^{n-i} E(e^{(\gamma-\theta)S_i}; M_i \le K) \\ &\le c_4 (n-i)^{-\frac{3}{2}}\, i^{-\frac{3}{2}}\, \lambda^n. \end{aligned}$$

Therefore we have

$$\sum_{i=m+1}^{n-m-1} E\Big(W(S_i) \mid M_n \le K\Big) \le \frac{c_4}{P(M_n \le K)}\, \lambda^n \sum_{i=m+1}^{n-m-1} (n-i)^{-\frac{3}{2}}\, i^{-\frac{3}{2}}.$$

Applying (2) of Lemma 2 and (1) of Lemma 7 to the right hand side, we see

$$\limsup_{n\to\infty} \sum_{i=m+1}^{n-m-1} E\Big(W(S_i) \mid M_n \le K\Big) \le c_5 \sum_{i=m+1}^{\infty} i^{-\frac{3}{2}},$$

which shows (5.1). Hence Step 1 is proved. □

Step 2. Let $K > 0$. Then as $n \to \infty$,

$$\begin{aligned} &\lambda^{-n} n^{\frac{3}{2}} E\left[f(S_n)\, h\Big(\sum_{i=1}^{n} W(S_i)\Big); M_n \le K\right] \\ &\quad \to \frac{1}{\sqrt{2\pi\psi''(\alpha)}} \int_0^\infty \nu(dt) \int_{-\infty}^{K} dx\, e^{\alpha x - tW(x)} f(x) g_{K,t}(0) \widehat{g}_{K,t}(x). \end{aligned}$$

PROOF. By (1) of Lemma 7, there exists l_K such that $P(M_n \le K) \le l_K\, n^{-\frac{3}{2}} \lambda^n$, $n \ge 1$. Using this, we have

$$\Big| E\Big[f(S_n) e^{-t\sum_1^n W(S_i)}; M_n \le K\Big]\Big| \le \|f\|_\infty\, l_K\, n^{-\frac{3}{2}} \lambda^n, \qquad n \ge 1.$$

From the assumption of h,

$$E\left[f(S_n)h\Big(\sum_{i=1}^{n} W(S_i)\Big); M_n \le K\right]$$
$$= \int_0^{\infty} \nu(dt)\, E\left[f(S_n)e^{-t\sum_1^n W(S_i)}; M_n \le K\right].$$

Therefore the dominated convergence theorem and Step 1 yeild Step 2. □

Step 3. The following relation holds.

$$\limsup_{K\to\infty}\limsup_{n\to\infty} \lambda^{-n} n^{\frac{3}{2}} E\left[\left|f(S_n)\right| h\Big(\sum_{i=1}^{n} W(S_i)\Big); M_n > K\right] = 0.$$

PROOF. By the assumptions of h and W, there exist positive constants A, B and K such that $h(x) \le Ax^{-\beta}$ for $x > 0$ and $W(x) \ge Be^{\delta x}$ for $x > K$. Hence, on $(M_n > K)$, we have

$$h\Big(\sum_{j=1}^{n} W(S_j)\Big) \le AW(M_n)^{-\beta} \le AB^{-\beta}\exp\{-\beta\delta M_n\}.$$

Using this inequality, we see

$$E\left[\left|f(S_n)\right| h\Big(\sum_{i=1}^{n} W(S_i)\Big); M_n > K\right] \le c\, E(e^{-\beta\delta M_n}; M_n > K),$$

where $c = AB^{-\beta}\|f\|_\infty$. Assuming $\beta\delta > \alpha$, we can apply (3) of Lemma 7 to the right hand side. So we get

$$\limsup_{n\to\infty} \lambda^{-n} n^{\frac{3}{2}} E\left[\left|f(S_n)\right| h\Big(\sum_{i=1}^{n} W(S_i)\Big); M_n > K\right] \le c\int_K^{\infty} e^{-\beta\delta x}\eta(dx),$$

which shows Step 3. □

Step 4. As $n \to \infty$,

$$\lambda^{-n} n^{\frac{3}{2}} E\left[f(S_n)\, h\Big(\sum_{i=1}^{n} W(S_i)\Big)\right]$$
$$\to \frac{1}{\sqrt{2\pi\psi''(\alpha)}}\int_0^{\infty} \nu(dt)\int_{\mathbf{R}} dx\, e^{\alpha x - tW(x)} f(x) g_t(0)\widehat{g}_t(x).$$

PROOF. Steps 2 and 3 show that

$$\begin{aligned}
&\lim_{n\to\infty} \lambda^{-n} n^{\frac{3}{2}} E\left[f(S_n)\, h\Big(\sum_{i=1}^{n} W(S_i)\Big)\right] \\
&= \lim_{K\to\infty}\lim_{n\to\infty} \lambda^{-n} n^{\frac{3}{2}} E\left[f(S_n)\, h\Big(\sum_{i=1}^{n} W(S_i)\Big); M_n \le K\right] \\
&= \lim_{K\to\infty} \frac{1}{\sqrt{2\pi\psi''(\alpha)}} \int_0^\infty \nu(dt) \int_{-\infty}^{K} dx\, e^{\alpha x - tW(x)} f(x) g_{K,t}(0) \widehat{g}_{K,t}(x)
\end{aligned}$$

and this limit is finite. Without loss of generality we assume $f \ge 0$. By (2) of Lemma 12, for fixed $x \in \mathbf{R}$ and $t > 0$, $g_{K,t}(0) \nearrow g_t(0)$ and $\widehat{g}_{K,t}(x) \nearrow \widehat{g}_t(x)$ as $K \to \infty$. Thus the monotone convergence theorem implies Step 4. Therefore the first half part of Theorem is proved. □

Step 5. g_t is the unique solution of the functional equation in Theorem.

PROOF. In Section 4, we have already seen that $g_t(x) = \lim_{K\to\infty} g_{K,t}(x)$ is a solution of that equation. Therefore we prove the uniqueness of solution. It is enough to prove the case $t = 1$. We assume that g is another solution of that equation if $t = 1$. Let $\varepsilon > 0$ and $x \in \mathbf{R}$ be fixed. Thanks to the renewal theory and the boundary condition at $-\infty$, there exists $L < x$ such that $|g(y) - g_1(y)|/u(-y) \le \varepsilon$ for $y < L$. On the other hand, by the boundary condition at $+\infty$, we take $K > x$ which satisfies that $|g(y) - g_1(y)| \le \varepsilon$ for $y > K$. Put $f = |g - g_1|$. Using these estimates, we have

$$\begin{aligned}
f(x) &\le E_x\Big[e^{-W(\zeta)} f(\zeta); L \le \zeta \le K\Big] + E_x\Big[e^{-W(\zeta)} f(\zeta); \zeta < L\Big] \\
&\quad + E_x\Big[e^{-W(\zeta)} f(\zeta); \zeta > K\Big] \\
&\le E_x\Big[e^{-W(\zeta)} f(\zeta); L \le \zeta \le K\Big] + \varepsilon E_x\Big[e^{-W(\zeta)} u(-\zeta); \zeta < L\Big] \\
&\quad + \varepsilon E_x\Big[e^{-W(\zeta)}; \zeta > K\Big] \\
&= E_x\Big[e^{-W(\zeta)} f(\zeta); L \le \zeta \le K\Big] + a(x) + b(x).
\end{aligned}$$

Iterating the above inequality, we have for all $n \in \mathbf{N}$,

$$f(x) \le E_x\Big[e^{-\sum_1^n W(\zeta_i)} f(\zeta_n); L \le \zeta_1, \cdots, \zeta_n \le K\Big]$$

$$+\sum_{j=0}^{n-1} E_x\Big[e^{-\sum_1^j W(\zeta_i)}\{a(\zeta_j)+b(\zeta_j)\}; L \le \zeta_1,\cdots,\zeta_j \le K\Big].$$

The first term of the right hand side is less than $E_x(f(\zeta_n); L \le \zeta_n \le K)$. Since $f(y) \le C = C_{K,L}$ if $L \le y \le K$,

$$E_x(f(\zeta_n); L \le \zeta_n \le K) \le C\, P_x(L \le \zeta_n \le K) \to 0 \quad \text{as } n \to \infty.$$

Hence we have

$$\begin{aligned} f(x) \quad \le \quad & \sum_{n=0}^{\infty} E_x\Big[e^{-\sum_1^n W(\zeta_i)} a(\zeta_n); L \le \zeta_1,\cdots,\zeta_n \le K\Big] \\ & + \sum_{n=0}^{\infty} E_x(b(\zeta_n); \mu_n \le K) = I_K + J_K. \end{aligned} \tag{5.2}$$

Set $T = T_L = \min\{n > 0; Y_{K,n} < L\}$. Then we have

$$\begin{aligned} & E_x\Big[e^{-\sum_1^n W(\zeta_i)} a(\zeta_n); L \le \zeta_1,\cdots,\zeta_n \le K\Big] \\ & = \varepsilon E_x\Big[e^{-\sum_1^{n+1} W(\zeta_i)} u(-\zeta_{n+1}); \zeta_{n+1} < L \le \zeta_1,\cdots,\zeta_n \le K\Big] \\ & = \varepsilon u(K-x) E_x\left[e^{-\sum_1^{n+1} W(Y_{K,i})} \frac{u(-Y_{K,n+1})}{u(K-Y_{K,n+1})}; T = n+1\right] \\ & \le \varepsilon u(K-x) E_x\Big[e^{-\sum_1^{T} W(Y_{K,i})}; T = n+1\Big]. \end{aligned}$$

Using this inequality, we see

$$I_K \le \varepsilon u(K-x) E_x\Big[e^{-\sum_1^{T} W(Y_{K,i})}\Big] \to \varepsilon\, g_{K,1}(x) \quad \text{as } L \to -\infty. \tag{5.3}$$

By the definition of $b(y)$,

$$\begin{aligned} E_x(b(\zeta_n); \mu_n \le K) \quad & = \quad \varepsilon E_x\Big[e^{-W(\zeta_{n+1})}; \zeta_1,\cdots,\zeta_n \le K < \zeta_{n+1}\Big] \\ & \le \quad \varepsilon \exp\Big\{-\min_{t \ge K} W(t)\Big\} P_x(\zeta_1,\cdots,\zeta_n \le K < \zeta_{n+1}). \end{aligned}$$

It easily follows from this inequality that

$$J_K \le \varepsilon \exp\Big\{-\min_{t \ge K} W(t)\Big\}. \tag{5.4}$$

Collecting (5.2)–(5.4), we have

$$|g(x) - g_1(x)| \leq \limsup_{K\to\infty} I_K \leq \varepsilon\, g_1(x).$$

Since $\varepsilon > 0$ and $x \in \mathbf{R}$ are arbitrary, this inequality shows $g \equiv g_1$. The proof of Theorem is established. □

From (2) of Lemma 12 we have the positivity of $g_t(0)$ and $\widehat{g}_t(x)$ for each $t > 0$ and $x \in \mathbf{R}$. Therefore we get the following corollary which is used in the next section.

COROLLARY. *Let f be non-negative and $f \not\equiv 0$ on $\mathbf{R}$. Then the limiting constant c in Theorem is positive.*

Before ending this section, we point out that there are counter examples for our theorem if $\beta\delta > \alpha$ does not hold. Let Condition (A) be satisfied. Set $f = 1$, $W(x) = e^x$ and $h(x) \sim ax^{-\alpha}$ for some $a > 0$. In this case $\beta\delta \leq \alpha$. By (3.1),

$$Eh\Big(\sum_{i=1}^{n} e^{S_i}\Big) = \lambda^n E\Big[e^{\alpha\zeta_n} h\Big(\sum_{i=1}^{n} e^{\zeta_i}\Big)\Big].$$

Applying the methods in Kozlov [11], we see

$$E\Big[e^{\alpha\zeta_n} h\Big(\sum_{i=1}^{n} e^{\zeta_i}\Big)\Big] \sim ac\,n^{-\frac{1}{2}} \qquad \text{as } n \to \infty, \tag{5.5}$$

where $c = \lim_{n\to\infty} \sqrt{n} E\{\sum_{i=0}^{n} e^{-\zeta_i}\}^{-\alpha} \in (0,\infty)$. Hence we get

$$Eh\Big(\sum_{i=1}^{n} e^{S_i}\Big) \sim ac\,n^{-\frac{1}{2}}\lambda^n \qquad \text{as } n \to \infty.$$

In this caes Theorem does not hold. Next we assume that $f(+\infty)$ exists, $W(x) = e^x$ and $h(x) \sim ax^{-\beta}$ for some $a > 0$ and $0 < \beta < \alpha$. Notice $\beta\delta < \alpha$. Define the random walk $(\zeta_n^*, n \geq 1)$ where the distribution of ζ_1^* is given by

$$P(\zeta_1^* \in dy) = e^{-\beta y - \psi(\beta)} P(S_1 \in dy).$$

It is easy to see that $E\zeta_1^* > 0$. Set $\theta = e^{\psi(\beta)} \in (\lambda, 1)$. Using $(\zeta_n^*, n \geq 1)$ and the duality lemma, we have

$$\begin{aligned}
\theta^{-n} E\Big[f(S_n) h\Big(\sum_{i=1}^{n} e^{S_i}\Big)\Big] &= E\Big[f(\zeta_n^*) e^{\beta \zeta_n^*} h\Big(\sum_{i=1}^{n} e^{\zeta_i^*}\Big)\Big] \\
&= E\left[f(\zeta_n^*) h\Big(e^{\zeta_n^*} \sum_{i=0}^{n-1} e^{-\zeta_i^*}\Big) \frac{\Big\{e^{\zeta_n^*} \sum_{i=0}^{n-1} e^{-\zeta_i^*}\Big\}^{\beta}}{\Big\{\sum_{i=0}^{n-1} e^{-\zeta_i^*}\Big\}^{\beta}} \right].
\end{aligned}$$

By our assumptions, the dominated convergence theorem implies

$$\lim_{n \to \infty} \theta^{-n} E\Big[f(S_n) h\Big(\sum_{i=1}^{n} e^{S_i}\Big)\Big] = a f(+\infty) E\Big\{\sum_{i=0}^{\infty} e^{-\zeta_i^*}\Big\}^{-\beta}.$$

This is another counter example.

6. Applications

In this section we give three applications to which our theorem and corollary can be used. Second one has been treated in [6] with a different manner.

Application 1. Let $(f_n(s), n \geq 0)$ be a sequence of i.i.d. random generating functions, i.e.,

$$f_n(s) = \sum_{j=0}^{\infty} \pi_n^{(j)} s^j, \qquad n \geq 0, \ |s| < 1,$$

where $\pi_n = (\pi_n^{(j)}, j \geq 0)$, $n \geq 0$ are i.i.d. random vectors satisfying $\pi_n^{(j)} \geq 0$, $\sum_{j=0}^{\infty} \pi_n^{(j)} = 1$. Let $X_0 = 1, X_1, \cdots$ be a branching process in a random medium $\{\pi_n\}$. When $X_0, \cdots, X_n$ and $\pi_0, \cdots, \pi_n$ are given, X_{n+1} is the sum of X_n random variables which take value k with probability $\pi_n^{(k)}$. In terms of $(f_n(s), n \geq 0)$,

$$E\left[s^{X_{n+1}} \mid X_0, \cdots, X_n, \pi_0, \cdots, \pi_n\right] = [f_n(s)]^{X_n}, \quad n \geq 0.$$

Set $T = \min\{n > 0; X_n = 0\}$. It is well known that $E \log f_0'(1) \le 0$ yields $P(T < \infty) = 1$. When $E \log f_0'(1) = 0$ and $f_n(s)$ has an expression

$$f_n(s) = 1 - \frac{\alpha_n}{1 - \beta_n} + \frac{\alpha_n s}{1 - \beta_n s},$$

where $0 \le \alpha_n$, $0 \le \beta_n < 1$ and $\alpha_n + \beta_n \le 1$, Kozlov [11] showed that for some $0 < c < \infty$,

$$P(T > n) \sim c/\sqrt{n} \qquad \text{as} \;\; n \to \infty.$$

For the special case of $f_n(s)$, we consider the rate of decay of $P(T > n)$ when $E \log f_0'(1) < 0$. Let $(p_i, i \ge 0)$ be i.i.d. random variables with values in $[0, 1]$ and

$$f_n(s) = \sum_{j=0}^{\infty} p_n \, q_n^j s^j = \frac{p_n}{1 - q_n s},$$

where $q_n = 1 - p_n$. That is, $(X_n, n \ge 0)$ has a geometric offspring distributions. This process is closely related to random walks in random media. In this case $P(T > n)$ is expressed as

$$P(T > n) = E\Big(1 + \sum_{i=1}^{n} e^{S_i}\Big)^{-1},$$

where $S_n = \sum_{i=1}^{n} \log(p_{i-1}/q_{i-1})$. Set $h(x) = \int_0^\infty e^{-(x+1)t} dt = (1 + x)^{-1}$. Then

$$P(T > n) = Eh\Big(\sum_{i=1}^{n} e^{S_i}\Big).$$

Suppose that $\log(p_1/q_1)$ satisfies Condition (A). If $\alpha \ge 1$, there is no $\beta > \alpha$ such that $h(x) = O(x^{-\beta})$. Thus our theorem can not be applied if $\alpha \ge 1$. In this case the rate of decay of $P(T > n)$ is deduced from the results of [1] and [11] (or counter examples in the last section). Let $0 < \alpha < 1$. If we suppose that $\log(p_1/q_1)$ satisfies Condition (B), the rate of decay of $P(T > n)$ follows from our theorem. Taking $W(x) = e^x, \beta = \gamma = \delta = 1$ and $f = 1$ in Theorem, we have

$$P(T > n) \sim c\, n^{-\frac{3}{2}} \lambda^n \qquad \text{as} \;\; n \to \infty,$$

where $\lambda = \min_{t>0} E(q_1/p_1)^t$ and $0 < c < \infty$. The positivity of c is given by Corollary. In particular, c has a following form.

$$c = \frac{1}{\sqrt{2\pi v}} \int_0^\infty dt \int_{\mathbf{R}} dx\, e^{\alpha x - t(1+\exp x)} g_t(0) \widehat{g}_t(x),$$

where $v = \lambda^{-1} E[(q_1/p_1)^\alpha \log^2(q_1/p_1)]$.

Application 2. Let $(p_i, i \in \mathbf{Z})$ be a doubly infinite sequence of i.i.d. random variables with values in $[0,1]$ and $\mathcal{F}$ be the σ-field generated by $\{p_i\}$. Let $X_0 = 0, X_1, \cdots$ be a random walk in a random medium $\{p_i\}$, i.e.,

$$\begin{aligned} P(X_{t+1} = X_t + 1 \mid \mathcal{F}, X_t = i) &= p_i, \\ P(X_{t+1} = X_t - 1 \mid \mathcal{F}, X_t = i) &= 1 - p_i. \end{aligned}$$

Alternatively one can describe $(X_n, n \geq 0)$ as the sequence of states of a birth and death process in a randm medium with birth parameter p_t and death parameter $q_t = 1 - p_t$. In [13] it was shown that if $E \log(q_0/p_0) > 0$, $\lim_{t\to\infty} X_t = -\infty$ a.s. In this situation $\max_{t\geq 0} X_t < \infty$ a.s. Let $T_n = \min\{t > 0; X_t = n\}$. We consider the rate of decay of $P(T_n < \infty)$ as $n \to \infty$. Set $\xi_i = \log(q_{i-1}/p_{i-1})$. $P(T_n < \infty)$ is expressed as follows. (see [1]).

$$P(T_n < \infty) = EA\Big(A + \sum_{i=1}^n e^{S_i}\Big)^{-1},$$

where $A = 1 + \sum_{n=1}^\infty \exp\{-(\xi_0 + \cdots + \xi_{1-n})\}$ and $S_n = \sum_{i=1}^n \xi_i$. Put $h(x) = \int_0^\infty e^{-xt} \nu(dt)$ where $\nu(t) = 1 - E(e^{-tA})$. Then $h(x) = EA(A+x)^{-1}$. Since A and $(S_n, n \geq 0)$ are independent, we have

$$P(T_n < \infty) = Eh\Big(\sum_{i=1}^n e^{S_i}\Big).$$

We assume that ξ_1 satisfies Condition (A). If $\alpha \geq 1$, it is easy to see that $EA < \infty$ and $h(x) \sim (EA)x^{-1}$. To our regret Theorem can not be applied in this case. The rate of decay of $P(T_n < \infty)$ is given in [1] in this case. Let $0 < \alpha < 1$ and ξ_1 satisfy Condition (B). Choosing $\beta > 0$ such that $\alpha < \beta < 1$ and $E(e^{-\beta\xi_1}) < 1$, we have $b := EA^\beta \leq \sum_{n=0}^\infty (Ee^{-\beta\xi_1})^n < \infty$.

By Chebyshev's inequality, $P(A > y) \leq by^{-\beta}$. Using this and change of variable, we see

$$\begin{aligned}\nu(t) = 1 - E(e^{-tA}) &= t\int_0^\infty e^{-ty} P(A > y)dy \\ &\leq b\,\Gamma(1-\beta)\,t^\beta.\end{aligned}$$

Applying similar calculations to $h(x)$, we have

$$h(x) = x\int_0^\infty e^{-xt}\nu(t)dt \leq b\,\frac{\pi\beta}{\sin\pi\beta}\,x^{-\beta}.$$

Therefore if $0 < \alpha < 1$ and ξ_1 satisfies Condition (B), the rate of decay of $P(T_n < \infty)$ follows from our theorem. Indeed taking $W(x) = e^x$, $\gamma = \delta = 1$ and $f = 1$, Theorem and Corollary imply

$$P(T_n < \infty) \sim c\,n^{-\frac{3}{2}}\lambda^n \qquad \text{as } n \to \infty,$$

where $\lambda = \min_{t>0} E(p_1/q_1)^t$ and $0 < c < \infty$. In this case c is expressed as

$$c = \frac{1}{\sqrt{2\pi v}}\int_0^\infty dF(y)\int_0^\infty dt\int_0^\infty dz\, yz^{\alpha-1}e^{-t(y+z)}g_t(0)\widehat{g}_t(\log z),$$

where $v = \lambda^{-1}E[(q_1/p_1)^\alpha \log^2(q_1/p_1)]$ and $F(y) = P(A \leq y)$.

APPLICATION 3. Now we state the last application. Let $(\zeta_n, n \geq 1)$ be a random walk satisfying the conditions (a), (b) and $Ee^{\alpha\zeta_1} < \infty$ for some $\alpha > 0$. We consider the asymptotic behavior of $E[f(\zeta_n)h(\sum_{i=1}^n W(\zeta_i))]$ for suitable f, h and W. Set $\lambda^{-1} = Ee^{\alpha\zeta_1}$. We define a random walk $(S_n, n \geq 1)$ where the distribution of S_1 is given by

$$P(S_1 \in dy) = \lambda\,e^{\alpha y}P(\zeta_1 \in dy).$$

Let us define the following:

CONDITION (A′). *For some $\alpha > 0$, $E(e^{-\alpha S_1}) < \infty$, $E(S_1 e^{-\alpha S_1}) = 0$ and $E(|S_1|^3 e^{-\alpha S_1}) < \infty$.*

It is trivial that S_1 satisfies Conditions (A′) and (B) and the following identity holds.

$$E\left[f(\zeta_n)h\Big(\sum_{i=1}^n W(\zeta_i)\Big)\right] = \lambda^{-n}E\left[e^{-\alpha S_n}f(S_n)h\Big(\sum_{i=1}^n W(S_i)\Big)\right].$$

We remark that Theorem holds even if we replace Condition (A) by Condition (A$'$). We assume that f is a continuous function such that $\sup_{x\in\mathbf{R}} |f(x)|e^{-\alpha x} < \infty$ and h and W satisfy the conditions in Theorem. Applying Theorem to the right hand side, we get

$$\lim_{n\to\infty} n^{\frac{3}{2}} E\left[f(\zeta_n)h\left(\sum_{i=1}^{n} W(\zeta_i)\right)\right] = \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty \nu(dt)\int_{\mathbf{R}} dx\, e^{-tW(x)} f(x) g_t(0)\widehat{g}_t(x),$$

where g_t and $\widehat{g}_t$ are unique solutions of functional equations in Theorem. Comparing (1.3) and the above, we find that this is the random walk analogue of Kotani's limit theorem for Brownian motion in [10].

Acknowledgement. I would like to express my gratitude to Professor S. Kotani for his helpful suggestions and insightful comments.

References

[1] Afanas'ev, V. I., On a maximum of a transient random walk in random environment, Theory Probab. Appl. **35** (1990), 205–215.

[2] Bahadur, R. R. and R. R. Rao, On deviation of the sample mean, Ann. Math. Statist. **31** (1960), 1015–1027.

[3] Bertoin, J. and R. A. Doney, On conditioning a random walk to stay nonnegative, Ann. Probab. **22** (1994), 2152–2167.

[4] Doney, R. A., On the asymptotic behaviour of first passage times for transient random walks, Probab. Theory Related Fields. **81** (1989), 239–246

[5] Feller, W., An Introduction to Probability Theory and Its Applications, volume II, second edition, John Wiley & Sons, 1971.

[6] Hirano, K., An asymptotic behavior of the mean of some exponential functionals of a random walk, Osaka J. Math. **34** (1997), 953–968.

[7] Iglehart, D. L., Random walks with negative drift conditioned to stay positive, J. Appl. Probab. **11** (1974), 742–751.

[8] Kawazu, K. and H. Tanaka, On the maximum of a diffusion process in a drifted Brownian environment, Seminaire de probabilités XXVII, 1993 (L. N. M. vol. 1557, 78–85), Springer-Verlag, Berlin, Heidelberg, 1993.

[9] Keener, R. W., Limit theorems for random walks conditioned to stay positive, Ann. Probab. **20** (1992), 801–824.

[10] Kotani, S., Analytic approach to Yor's formula of exponential additive functionals of Brownian motion, Ito's stochastic calculus and Probability Theory, Springer-Verlag, Tokyo, 1996.

[11] Kozlov, M. V., On the asymptotic behavior of the probability of non-extinction for critical branching processes in a random environment, Theory Probab. Appl. **21** (1976), 791–803.
[12] Rosén, B., On the asymptotic distribution of sums of independent identically distributed random variables, Ark Mat. **4** (1961), 323–332.
[13] Solomon, F., Random walks in a random environment, Ann. Probab. **3** (1975), 1–31.
[14] Tanaka, H., Time reversal of random walks in one-dimension, Tokyo J. Math. **12** (1989), 159–174.
[15] Veraverbeke, N. and J. L. Teugels, The exponential rate of convergence of the distribution of the maximum of a random walk, part II, J. Appl. Probab. **13** (1976), 733–740.
[16] Yor, M., On some exponential functionals of Brownian motion, Adv. Appl. Probab. **24** (1992), 509–531.

(Received July 25, 1997)

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan