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On K3 Surfaces Admitting Finite Non-Symplectic Group Actions

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Abstract. For a pair (X, G) of a complex K3 surface X and its finite automorphism group G, we call the value $I(X, G) := |\operatorname{Im}(G \to \operatorname{Aut}(H^{2,0}(X)))|$ the transcendental value and the Euler number $\varphi(I(X, G))$ of $I(X, G)$ the transcendental index. This paper classifies the pairs (X, G) with the maximal transcendental index 20 and the pair (X, G) with $I(X, G) = 40$ up to isomorphisms. We also determine the set of transcendental values and apply this to determine the set of global canonical indices of complex projective threefolds with only canonical singularities and with numerically trivial canonical Weil divisor.

0. Introduction

Let X be a K3 surface, that is, a simply connected smooth projective complex surface with a nowhere vanishing holomorphic two form. We denote by S_X, T_X and ω_X the Néron Severi lattice, the transcendental lattice and a nowhere vanishing holomorphic two form of X. We denote the multiplicative group of the I-th roots of unity, its specified generator $\exp(\frac{2\pi\sqrt{-1}}{I})$ and the cardinality of its generators by μ_I, ζ_I and $\varphi(I)$.

Let G be a subgroup of $\operatorname{Aut}(X)$ and $\alpha : G \to \mathbb{C}^\times$ the character of the natural representation of G on the space $H^{2,0}(X) = \mathbb{C}\omega_X$. Then, there exists a positive integer $I(X, G)$ which fits in with the following exact sequence ([Ni1, Theorem 0.1], [St, Lemma 2.1]):

$$1 \to G_N := \operatorname{Ker}\alpha \to G \xrightarrow{\alpha} \mu_{I(X,G)} \to 1.$$

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Table 1. The list of all the numbers I with $\varphi(I) \leq 21$.

$\varphi(I)$	20	18	16	12	10	8	6	4	2	1
I	66	54	60	42	22	30	18	12	6	2
	50	38	48	36	11	24	14	10	4	1
	44	27	40	28		20	9	8	3	
	33	19	34	26		16	7	5		
	25		32	21		15				
			17	13						

It is shown by Nikulin [Ni1, ibid.] that $\varphi(I(X,G)) \mid \operatorname{rank} T_X \leq 21$ whence the candidates of the values $I(X,G)$ and $\varphi(I(X,G))$ lie in the Table 1.

We call $I(X,G)$ the transcendental value and $\varphi(I(X,G))$ the transcendental index of (X,G).

Nikulin and Mukai ([Ni1, Section 5], [Mu, Theorem 0.3]) classified the finite groups G for which there exist K3 surfaces X with $G \subset \mathrm{Aut}(X)$ and $I(X,G) = 1$.

In this paper we study a pair (X,G) of a K3 surface X and a *finite* group G such that $G \subset \mathrm{Aut}(X)$ and that $I(X,G) \neq 1$.

First we determine the pairs (X,G) with $\varphi(I(X,G)) = 20$, the maximal possible transcendental index up to isomorphism and calculate the full automorphism groups of such K3 surfaces (Main Theorem 1). We also determine the pairs (X,G) with $I(X,G) = 40$ up to isomorphism and calculate the full automorphism groups of such K3 surfaces (Main Theorem 2), which will answer for Kondo's question to the second author.

Next we determine the set of transcendental values completely (Main Theorem 3, see also Proposition 4). This is one of the important steps towards the complete understanding of finite automorphism groups of K3 surfaces. Then we give some application for threefolds with numerically trivial canonical divisor (Corollary 5).

Employing pairs $(X_I, < g_I >)$ defined in Proposition 4, we can state our main results as follows:

MAIN THEOREM 1. *Assume that* $I = I(X,G)$ *is either* 66*,* 33*,* 44*,* 50*, or* 25*. Then,*

(1) $(X,G) \simeq (X_I, < g_I >)$ *in the case where* I *is even and* $(X,G) \simeq$

$(X_{2I}, < g_{2I}^2 >)$ *in the case where* I *is odd, and*

(2) $\mathrm{Aut}(X) = G \simeq \mathbb{Z}/I$ *in the case where* I *is even and* $\mathrm{Aut}(X) \simeq \mathbb{Z}/2I$ *in the case where* I *is odd.*

This Theorem gives a complete classification of pairs (X, G) with the maximal possible transcendental indices and show, in particular, that such pairs are determined uniquely by their transcendental values.

MAIN THEOREM 2. *Assume that* $I(X, G) = 40$. *Then,*

(1) $(X, G) \simeq (X_{40}, < g_{40} >)$ *and*

(2) $\mathrm{Aut}(X) = G \simeq \mathbb{Z}/40$.

MAIN THEOREM 3. *Set* $\mathbb{TV}_{K3} := \{I(X, G) \mid (X, G)$ *is a pair of a K3 surface* X *and its finite automorphism group* $G\}$. *Then,* $\mathbb{TV}_{K3} = \{I \mid \varphi(I) \leq 20\} - \{60\}$, *or in other words, among the candidates in Table 1, only* 60 *cannot be realised as transcendental values of any pairs* (X, G).

Moreover, for each $I \in \mathbb{TV}_{K3}$, *there exists a K3 surface* X_I *admitting a cyclic group action* $\langle g_I \rangle$ *with* $\langle g_I \rangle \simeq \langle \alpha(g_I) \rangle = \mu_I$.

The main point is to show the nonexistence of pairs (X, G) with $I(X, G) = 60$. The existence part will immediately follows from the Table 1 and the next Proposition:

PROPOSITION 4 (cf. [Ko, Section 7], [Og1, Proposition 2]).

The following pair $(X_I, \langle g_I \rangle)$ *of a K3 surface* X_I *defined by the indicated minimal Weierstrass equation (except for (14) and (15)) and its cyclic automorphism group* $\langle g_I \rangle$ *satisfies* $\langle g_I \rangle \simeq \langle \alpha(g_I) \rangle = \mu_I$:

(1) *([Ko])* $X_{66} : y^2 = x^3 + t(t^{11} - 1)$ *and* $g_{66}^*(x, y, t) = (\zeta_{66}^{40} x, \zeta_{66}^{27} y, \zeta_{66}^{54} t)$;

(2) *([Ko])* $X_{44} : y^2 = x^3 + x + t^{11}$ *and* $g_{44}^*(x, y, t) = (\zeta_{44}^{22} x, \zeta_{44}^{11} y, \zeta_{44}^{34} t)$;

(3) *(cf. [Ko])* $X_{54} : y^2 = x^3 + t(t^9 - 1)$ *and* $g_{54}^*(x, y, t) = (\zeta_{27}^2 x, -\zeta_{27}^3 y, \zeta_{27}^6 t)$;

(4) *(cf. [Ko])* $X_{38} : y^2 = x^3 + t^7 x + t$ *and* $g_{38}^*(x, y, t) = (\zeta_{19}^7 x, -\zeta_{19} y, \zeta_{19}^2 t)$;

(5) $X_{48} : y^2 = x^3 + t(t^8 - 1)$ *and* $g_{48}^*(x, y, t) = (\zeta_{48}^2 x, \zeta_{48}^3 y, \zeta_{48}^6 t)$;

(6) *(cf. [Ko])* $X_{34} : y^2 = x^3 + t^7 x + t^2$ *and* $g_{34}^*(x, y, t) = (\zeta_{17}^7 x, -\zeta_{17}^2 y, \zeta_{17}^2 t)$;

(7) *([Og1])* $X_{32} : y^2 = x^3 + t^2x + t^{11}$ *and* $g^*_{32}(x, y, t) = (\zeta_{32}^{18}x, -\zeta_{32}^{11}y, \zeta_{32}^{2}t)$*;*
(8) *([Ko])* $X_{42} : y^2 = x^3 + t^5(t^7 - 1)$ *and* $g^*_{42}(x, y, t) = (\zeta_{42}^{2}x, \zeta_{42}^{3}y, \zeta_{42}^{18}t)$*;*
(9) *([Ko])* $X_{36} : y^2 = x^3 + t^5(t^6 - 1)$ *and* $g^*_{36}(x, y, t) = (\zeta_{36}^{2}x, \zeta_{36}^{3}y, \zeta_{36}^{30}t)$*;*
(10) *([Ko])* $X_{28} : y^2 = x^3 + tx + t^7$ *and* $g^*_{28}(x, y, t) = (\zeta_{28}^{14}x, \zeta_{28}^{7}y, \zeta_{28}^{2}t)$*;*
(11) *(cf. [Ko])* $X_{26} : y^2 = x^3+t^5x+t$ *and* $g^*_{26}(x, y, t) = (\zeta_{13}^{5}x, -\zeta_{13}y, \zeta_{13}^{2}t)$*;*
(12) $X_{30} : y^2 = x^3 + (t^{10} - 1)$ *and* $g^*_{30}(x, y, t) = (\zeta_{30}^{10}x, y, \zeta_{30}^{3}t)$*;*
(13) $X_{20} : y^2 = x^3 + (t^5 - 1)x$ *and* $g^*_{20}(x, y, t) = (\zeta_{20}^{10}x, \zeta_{20}^{5}y, \zeta_{20}^{4}t)$*;*
(14) *(cf. [Ko])* $X_{50} := (z^2 = x_0^6 + x_0x_1^5 + x_1x_2^5) \subset \mathbb{P}(1, 1, 1, 3)$ *and* $g^*_{50}[x_0 : x_1 : x_2 : z] = [x_0 : \zeta_{25}^{20}x_1 : \zeta_{25}x_2 : -z]$*;*
(15) X_{40}*: the minimal resolution of the surface* $\overline{X_{40}} := (z^2 = x_0(x_0^4x_2 + x_1^5 - x_2^5)) \subset \mathbb{P}(1, 1, 1, 3)$ *having* 5 *ordinary double points* $[0 : 1 : \zeta_5^i : 0]$ *(*$0 \le i \le 4$*) and* $g^*_{40}[x_0 : x_1 : x_2 : z] = [x_0 : \zeta_{20}x_1 : \zeta_4x_2 : \zeta_8z]$.

This together with Beauville-Kawamata-Morrison's arguments ([Bo, Proposition 8],[Ka1, Theorem 3.2],[Mo, Theorems 1 and 2]) and with the existence of crepant terminalisations of canonical threefolds ([Ka2, Corollary 4.5], [Re, Main Theorem]) gives the following application for threefolds with numerically trivial canonical divisor:

COROLLARY 5 (cf. [Og1, Main Theorem] also [Be, Proposition 8], [Ka1, Theorem 3.2] and [Mo, Theorems 1 and 2]). *Let* X *be a normal projective complex threefold with only canonical singularities and with* $K_X \equiv 0$*. Denote by* $I(X)$ *the global canonical index of* X*;* $I(X) := \min\{n \in \mathbb{Z}_{>0} \mid \mathcal{O}_X(nK_X) \simeq \mathcal{O}_X\}$*. Set:*
$\mathbb{I}_{can} := \{I(X) \mid X \text{ has only canonical singularities}\}$*;*
$\mathbb{I}_{term} := \{I(X) \mid X \text{ has only terminal singularities}\}$*; and*
$\mathbb{I}_{smooth} := \{I(X) \mid X \text{ is non-singular}\}$*.*
Then, $\mathbb{I}_{can} = \mathbb{I}_{term} = \mathbb{I}_{smooth} = \{I \mid \varphi(I) \le 20\} - \{60\}$*. In particular, the so-called Beauville number* $B = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ *is the best possible universal bound for the global canonical indices of canonical threefolds with* $K_X \equiv 0$.

This Corollary gives an answer to Catanese's question to the second author at the Trento International Conference held in June 1994.

We close this paragraph by posing the following interesting open problem related to Corollary 5:

Problem.

(1) Determine the set of the global canonical indices of surfaces with only klt singularities and with numerically trivial canonical Weil divisor (cf. [Zh, Lemma 2.3], [Bl, Theorem C]).

(2) Determine the set of the global canonical indices of threefolds with only klt singularities and with numerically trivial canonical Weil divisor.

This paper is motivated by the following beautiful Theorem due to Kondo:

Kondo's Theorem ([Ko, Main Theorem and Section 3]). *Let X be a K3 surface admitting an automorphism g such that*

(1) *S_X is unimodular,*

(2) *$g^* \mid_{S_X} = id$, and*

(3) *$g^* \mid_{T_X}$ is of order 66 (resp. of order 44).*

Then, X is isomorphic to X_{66} (resp. X_{44}) in Proposition 4.

Besides this Theorem, our basic ingredients are lattice theory, especially, the classification of even $2-$elementary hyperbolic lattices [Ni2, Theorems 4.3.1, 4.3.2], theory of elliptic surfaces [Kd, Theorems 6.2, 9.1], the topological Lefschetz fixed point formula (eg. [Ue, Lemma 1.6]), the holomorphic Lefschetz fixed point formula ([AS1, Page 542] and [AS2, Page 567]) and the following remarkable Theorem on arithemetic [MM, Main Theorem]:

Masley and Montgomery's Theorem. *The ring of cyclotomic integers $\mathbb{Z}[\zeta_I]$ is PID if and only if I belongs to the set $\{I \mid \varphi(I) \le 21\} \cup \{35, 45, 70, 84, 90\}$.*

Indeed, thanks to this Theorem, we can reinterpret Nikulin's Theorem in terms of cyclotomic integers, which will turn out to be very useful:

Lemma (1.1). *Let X be a K3 surface and g an automorphism of X. Set $I(X, \langle g \rangle) = I$,* $\operatorname{rank} T_X = r$ *and regard T_X as a $\mathbb{Z}[\langle g \rangle]-$module via the natural action of g on T_X. Let $\Phi_I(x) \in \mathbb{Z}[x]$ be the $I-$th cyclotomic polymonial. Then,*

(1) *([Ni1, Theorem 0.1]) the eigen values of $g^* \mid T_X$ are the primitive $I-$th roots of unity. In particular,* $\operatorname{ord}(g^* \mid T_X) = I$ *and $I = 1$ if and only if $g^* \mid T_X = id$,*

(2) $\mathrm{Ann}(T_X) = \langle \Phi_I(g) \rangle$ *and* T_X *is then naturally a torsion free* $\mathbb{Z}[\langle g \rangle]/\langle \Phi_I(g) \rangle$*−module, and*

(3) *under the identification* $\mathbb{Z}[\langle g \rangle]/\langle \Phi_I(g) \rangle = \mathbb{Z}[\zeta_I]$ *through the correspondence* $g(\mathrm{mod}\langle \Phi_I(g) \rangle) \leftrightarrow \zeta_I$, $T_X \simeq \mathbb{Z}[\zeta_I]^{\oplus(r/\varphi(I))}$ *as* $\mathbb{Z}[\zeta_I]$*−modules.*

It might be worth mentioning here that Masley and Montgomery's Theorem has been already effectively applied by the second author to determine canonical Calabi-Yau threefolds W with $c_2(W) = 0$ [Og2, Main Theorem]. This article will provide another application of this Theorem and the method here will be also fully applied in [OZ, Main Result].

Some results in this article have been obtained by the first author as her master thesis at University of Tokyo 1996 [Ma]. This article may be regarded as an extended version of her master thesis.

IMPORTANT REMARK. After finishing our preliminary version, Professor S. Kondo kindly informed us that Professor G. Xiao also proved in his preprint, "Non-symplectic involutions of a K3 surface", the uniqueness of pairs (X, G) with $I(X, G) = 66$, 50, 44, 54 and 48 and the nonexistence of pairs (X, G) with $I(X, G) = 60$. However, our method based on cohomological arguments is quite different from his method. Indeed, his method of proof is based on Hurwitz type argument and theory of rational surfaces, especially Hirzebruch surfaces, via the study of appropriate quotients $X \to X/G'$ $(G' \subset G)$. An advantage of our method consists in its applicability to not only even order cases but also odd order cases, which he did not get in touch with, on an equal footing.

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Notation. Besides standard notation in algebraic geometry, we employ the following notation.

We denote by $\pi(C)$ the genus of a smooth complete curve C.

For a finite automorphism g (resp. a finite automorphism group G) of a smooth surface X we write $X^g = \{x \in X \mid g(x) = x\}$ (resp. $X^{[G]} = \cup_{g \in G, g \neq id} X^g$). Note that X^g is a smooth algebraic set of X, while is not irreducible in general. If $P \in X^g$ and $n = \mathrm{ord}(g)$, then there exist local coordinates (x_P, y_P) around P such that $g^*(x_P, y_P) = (\zeta_n^{n_1} x_P, \zeta_n^{n_2} y_P)$. In this case, $P \in X^g$ is called of type $\frac{1}{n}(n_1, n_2)$. Note that this P is an isolated point of X^g if and only if $n_i \not\equiv 0 (\mathrm{mod}\, n)$ for each $i = 1, 2$.

For a given lattice $(L, \langle, \rangle)$, we denote by $L(m)$ the lattice $(L, m \cdot \langle, \rangle)$. By A_l, D_m $(m \geq 4)$ and E_n $(n = 6, 7, 8)$, we denote the negative definite lattices corresponding to the Dynkin's diagrams of the indicated types. By U we denote the lattice defined by the Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We also freely employ the notation and notion fixed in Introduction.

1. Preliminaries

In this section we observe some elementary facts which will be frequently applied in this article.

LEMMA (1.1). *Let X be a K3 surface and g an automorphism of X. Set $I(X, \langle g \rangle) = I$, $\mathrm{rank}\, T_X = r$ and regard T_X as a $\mathbb{Z}[\langle g \rangle]-$module via the natural action of g on T_X. Let $\Phi_I(x) \in \mathbb{Z}[x]$ be the $I-$th cyclotomic polymonial. Then,*

(1) *([Ni1, Theorem 0.1]) the eigen values of $g^* \mid T_X$ are the primitive $I-$th roots of unity. In particular, $\mathrm{ord}(g^* \mid T_X) = I$ and $I = 1$ if and only if $g^* \mid T_X = id$,*
(2) *$\mathrm{Ann}(T_X) = \langle \Phi_I(g) \rangle$, and T_X is then naturally a torsion free $\mathbb{Z}[\langle g \rangle]/\langle \Phi_I(g) \rangle-$module, and*
(3) *under the identification $\mathbb{Z}[\langle g \rangle]/\langle \Phi_I(g) \rangle = \mathbb{Z}[\zeta_I]$ through the correspondence $g (\mathrm{mod} \langle \Phi_I(g) \rangle) \leftrightarrow \zeta_I$, $T_X \simeq \mathbb{Z}[\zeta_I]^{\oplus (r/\varphi(I))}$ as $\mathbb{Z}[\zeta_I]-$modules.*

PROOF. The statement (1) is shown by Nukulin [Ni1, Section 3]. The satement (2) is a simple reinterpretation of (1) in terms of group algebra. Recall that torsion free modules are in fact free if the coefficient ring is PID.

Now, combining (2) with Table 1 and Masley and Montgometry's Theorem in Introduction, we get the assertion (3). □

Lemma (1.2). *Let X be a K3 surface and G a finite automorphism group of X. Assume that* $\operatorname{rank} T_X \geq 14$. *Then $G_N = \{1\}$, or equivalently, $G \simeq \alpha(G)$.*

Proof. Assume the contrary that G_N contains an element g of prime order p. Then, p is either $2, 3, 5$ or 7 and $|X^g| = 24/(p+1)$ by [Ni1, Section 5]. (See also [Mu, Proposition 1.2]). Note that $g^* \mid T_X = id$ and that $g^* \mid S_X$ has an eigen value 1 corresponding to the pullback of the ample class of $X/\langle g \rangle$. Now writing $r = \operatorname{rank} T_X$ and applying the topological Lefschetz fixed point formula (eg. [Ue, Lemma 1.6]), we get the following contradiction:

$8 \geq 24/(p+1) = \chi_{top}(X^g) = \sum_{i=0}^{4}(-1)^i \operatorname{tr}(g^* \mid H^i(X, \mathbb{Z})) = 2 + \operatorname{tr}(g^* \mid S_X) + \operatorname{tr}(g^* \mid T_X) = 2 + r + \operatorname{tr}(g^* \mid S_X) \geq 2 + r + 1 - (22 - r - 1) = 2r - 18 \geq 10$. □

In what follows, set $S_X^* = \operatorname{Hom}(S_X, \mathbb{Z})$, $T_X^* = \operatorname{Hom}(T_X, \mathbb{Z})$ and regard $S_X \subset S^* \subset S_X \otimes \mathbb{Q}$, $T_X \subset T^* \subset T_X \otimes \mathbb{Q}$. We denote by $l(S_X)$ the minimal number of generators of the finite abelian group S_X^*/S_X. We call S_X $p-$elementary if S_X^*/S_X is a $p-$elementary abelian group (possibly $\{0\}$).

Recall that S_X (resp. T_X) is an even lattice of signature $(1, \operatorname{rank} S_X - 1)$ (resp. of signature $(2, \operatorname{rank} T_X - 2)$) and $\operatorname{rank} S_X + \operatorname{rank} T_X = 22$.

Lemma (1.3) (cf. [Ni2, Theorem 10.1.2], [Ko, Theorem 6.1]). *Let X be a K3 surface. Assume that X admits an automorphism g such that*

(1) *$g^* \mid S_X = id$,*

(2) *$I(\langle g \rangle)$ has at least two distinct prime divisors.*

Then, S_X is isomorphic to either U, $U \oplus E_8$, or $U \oplus E_8^{\oplus 2}$.

Proof. Choose distinct prime divisors p_i and elements h_i $(i = 1, 2)$ such that $\operatorname{ord}(h_i) = p_i$. From the natural isomorphism $S_X^*/S_X \simeq H^2(X, \mathbb{Z})/(S_X \oplus T_X) \simeq T_X^*/T_X$ which commutes with the action of $\operatorname{Aut}(X)$, we get $h_i^* \mid T_X^*/T_X = id$. Combining this with $(\sum_{k=0}^{p_i-1}(h_i^*)^k) \mid T_X = 0$ (1.1)(2), we find that $p_i x \equiv 0 (\operatorname{mod} T_X)$ if $x \in T_X^*$. Thus, T_X^*/T_X is a p_i-elementary abelian group. Hence $S_X^*/S_X \simeq T_X^*/T_X = \{0\}$. This means that S_X is unimodular. Now the result follows from [Se, Chapter 5, Theorem 5]. $\square$

Similarly, we get the following:

LEMMA (1.4) (cf. [Ni2, ibid.], [Ko, ibid.]). *Let X be a K3 surface. Assume that X admits an automorphism g such that*

(1) $g^* \mid S_X = id$, *and*
(2) $I(\langle g \rangle)$ *is a primary, that is,* $I(\langle g \rangle) = p^n$ *for a prime* p.

Then, S_X *is a* $p-$*elementary lattice.*

LEMMA (1.5) (cf. [PS], [Ko], [Se], [Ni3]). *Let X be a K3 surface.*

(1) *If S_X represents zero, then X admits an elliptic fibration. In particular, every K3 surface with $\rho(X) \geq 5$ admits an elliptic fibration.*
(2) *If S_X contains a sublattice isomorphic to U, then X admits a Jacobian fibration. In particular, if* $\operatorname{rank} S_X \geq 3 + l(S_X)$, *then X admits a Jacobian fibration.*

PROOF. The first half part of the assertion (1) is shown by [PS, Section 3, Corollary 3]. The remaining assertion in (1) now follows from [Se, Page 43, Corollary 2]. The first half part of (2) is proved by [Ko, Lemma 2.1]. The last half is then a direct consequence of the so-called splitting Theorem due to [Ni3, Corollary 1.13.5]. $\square$

REMARK. The last assertion of (2) will be fully applied in [OZ].

We close this section by noticing the following:

LEMMA (1.6) (cf. [PS, Section 2, Proposition 2]). *Let X be a K3 surface and g_i $(i = 1, 2)$ automorphisms of X such that $g_1^* \mid S_X = g_2^* \mid S_X$ and that $g_1^* \omega_X = g_2^* \omega_X$. Then $g_1 = g_2$ in* $\operatorname{Aut}(X)$.

PROOF. It follows from the assumption and (1.1) that $g_1^* \mid H^2(X, \mathbb{Z}) = g_1^* \mid H^2(X, \mathbb{Z})$. Now the result follows from the injectivity part of the global Torelli Theorem for algebraic K3 surfaces ([PS, Section 2, Proposition 2], [BPV, Chapter 8, Proposition 11.3]). $\square$

2. Uniqueness of pairs (X, G) with $I(X, G) = 66$, 33 and 44

Let (X, G) be a pair with $I := I(X, G) = 66$, 33 or 44. Note that $\operatorname{rank} T_X = 20$, $\operatorname{rank} S_X = 2$ whence $G \simeq \alpha(G) = \mu_I$ (1.2). Let g be the generator of G with $g^*\omega_X = \zeta_I\omega_X$. Set $h = g^2$.

LEMMA (2.1). *$h^* \mid T_X$ is of order $I/2$ (resp. I) in the case where $I = 44, 66$ (resp. $I = 33$).*

PROOF. This follows from (1.1)(1). □

LEMMA (2.2). *$h^* \mid S_X = id$.*

PROOF. Since g is of finite order, $g^* \mid S_X$ has an eigen value 1 corresponding to the pull back of an ample class of $X/\langle g\rangle$. Combining this with $\operatorname{rank} S_X = 2$, we readily get the result. □

LEMMA (2.3). $S_X \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

PROOF. Since $I(X, \langle h\rangle)$ is either 33 or 22, we may apply (1.3) for (X, h) to get the result. □

Thus X admits a Jacobian fibration $f : X \to \mathbb{P}^1$. Let $C(\simeq \mathbb{P}^1)$ be a section and F a general fiber of f. Then, $(C)^2 = -2$ and $(F)^2 = 0$ whence the classes $[C]$ and $[F]$ lie in the boundary of the effective cone $\overline{NE}(X)(\subset S_X \otimes \mathbb{R})$ of X. (See [KMM, Section 0-1] for definition of several cones and their relations.)

LEMMA (2.4). *$\varphi^* \mid S_X = id$ for each $\varphi \in \operatorname{Aut}(X)$. In particular, $g^* \mid S_X = id$.*

PROOF. The result follows from $\partial\overline{NE}(X) = \mathbb{R}_{\geq 0}[C] \cup \mathbb{R}_{\geq 0}[F]$. □

PROOF OF THEOREM 1 FOR PAIRS (X, G) WITH $I(X, G) = 66$, 33 AND 44.

Assume first that I is either 66 or 44. Then, by (2.1), (2.3) and (2.4), (X, g) satisfies the condition (1), (2) and (3) in Kondo's Theorem quoted in Introduction. Thus there exists a biregular map $\varphi_I : X \simeq X_I$ for each I.

Since $(\varphi_I^{-1} \circ g_I \circ \varphi_I)^* \omega_X = \zeta_I \omega_I = g^* \omega_X$ and $(\varphi_I^{-1} \circ g_I \circ \varphi_I)^* \mid S_X = id = g^* \mid S_X$, it follows from (1.6) that $\varphi_I^{-1} \circ g_I \circ \varphi_I = g$. Thus $(X, \langle g \rangle) \simeq (X_I, \langle g_I \rangle)$.

We show that $\mathrm{Aut}(X) = G$. Let a be an element of $\mathrm{Aut}(X)$ and set $I(X, \langle a \rangle) = J$. Then $I(X, \langle a, g \rangle) = LCM(J, I)$. Combining this with Table 1, we readily see that $J \mid I$. Using this, we find an integer n such that $a^* \omega_X = (g^n)^* \omega_X$. Combining this with (2.4) and (1.6), we get $a = g^n$. This implies $\mathrm{Aut}(X) = G$. Now we are done for the case where $I = 66$ and 44.

Next assume that $I = 33$. Let us regard C as a zero section of the Jacobian fibration $f : X \to \mathbb{P}^1$ and denote by ι the automorphism of X induced by the inversion of the generic fiber E_η with respect to C. Then $I(X, \langle \iota \circ g \rangle) = 66$, and $\langle \iota, g \rangle$ is finite. Indeed $\iota \circ g = g \circ \iota$ by (1.6) and (2.4). This implies $(X, \langle \iota \circ g \rangle) \simeq (X_{66}, \langle g_{66} \rangle)$, whence $(X, \langle g \rangle) \simeq (X_{66}, \langle g_{66}^2 \rangle)$.

Now we are done. □

3. Uniqueness of pairs (X, G) with $I(X, G) = 50$ and 25

In this section we prove Theorem 1 in the case where $I(X, G) = 50$ and 25. First we treat the case where $I(X, G) = 25$. The same argument as in Section 2 shows the following:

Lemma (3.1). $\operatorname{rank} T_X = 20$, $\operatorname{rank} S_X = 2$, *and there exists an elment* $g \in G$ *such that* $G = \langle g \rangle$, g *is of order* 25 *and that* $g^* \mid S_X = id$.

The next Lemma is crucial to determine (X, G).

Lemma (3.2). $S_X \simeq \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$.

Proof. By (3.1) and (1.4), there exists a non-negative integer l with $S_X^*/S_X \simeq T_X^*/T_X \simeq (\mathbb{Z}/5)^{\oplus l}$. First we determine the value l.

Claim (3.3). $l \neq 0$.

Proof. Assume the contrary that S_X is unimodular. Since $g^* \mid S_X = id$, it follows from main Theorem of [Ko, Introduction] that $\mathrm{ord}(g)$ is a divisor of either $66, 44, 42, 36, 28$, or 12, a contradiction. □

Claim (3.4). $l = 1$.

PROOF. Note by (1.1) that $T_X \simeq \mathbb{Z}[\zeta_{25}]$ as $\mathbb{Z}[\zeta_{25}]-$modules. Let b_i $(1 \leq i \leq 20)$ be the $\mathbb{Z}-$basis of T_X corresponding to the $\mathbb{Z}-$basis ζ_{25}^{i-1} of $\mathbb{Z}[\zeta_{25}]$ under this isomorphism. Translating the relations, $\zeta_{25} \cdot \zeta_{25}^{i-1} = \zeta_{25}^{i}$ $(1 \leq i \leq 19)$ and $\zeta_{25} \cdot \zeta_{25}^{19} = \zeta_{25}^{20} = -(1 + \zeta_{25}^{5} + \zeta_{25}^{10} + \zeta_{25}^{15})$ in $\mathbb{Z}[\zeta_{25}]$ into T_X by the above isomorphism, we get $g^*(b_i) = b_{i+1}$ $(1 \leq i \leq 19)$ and $g^*(b_{20}) = -b_1 - b_6 - b_{11} - b_{16}$. Let $y \in T_X^*(\in T_X \otimes \mathbb{Q})$ be any element. Then there exist integers y_i such that $y = \frac{1}{5}(\sum_{i=1}^{20} y_i b_i)$. Since $g^* \mid T_X^*/T_X = id$, we calculate modulo T_X that $0 \equiv g^*(y) - y = \frac{1}{5}(\sum_{i=1}^{19} y_i b_{i+1} + y_{20}(-b_1 - b_6 - b_{11} - b_{16}) - \sum_{i=1}^{20} y_i b_i) = \frac{1}{5}(-(y_1 + y_{20})b_1 + (y_5 - y_{20} - y_6)b_6 + (y_{10} - y_{20} - y_{11})b_{11} + (y_{15} - y_{20} - y_{16})b_{16} + \sum_{i \neq 1,6,11,16}(y_{i-1} - y_i)b_i)$. This readily implies that $y_i = ky_1 (\bmod 5)$ $(5k-4 \leq i \leq 5k)$. Combining this with (3.3), we get $T_X^*/T_X = \langle \frac{1}{5}(\sum_{k=1}^{4} k(\sum_{i=5k-4}^{5k} b_i))\rangle \simeq \mathbb{Z}/5$. □

We return back to the proof of (3.2).

Let e_i $(i = 1, 2)$ be $\mathbb{Z}-$basis of S_X and a, b, c integers with $\langle e_1, e_1\rangle = 2a$, $\langle e_2, e_2\rangle = 2c$, $\langle e_1, e_2\rangle = b$. Then, $4ac - b^2 = -5$ by (3.3). Now we may assume by the reduction theory of quadratic forms (eg. [BS, Chapter 2, Section 7, Problem 12]) that a and b satisfy $a > 0$, $b < 0$ and $-b + 5^{1/2} > 2a > b + 5^{1/2} > 0$. Thus $a = 1$, $b = -1$, $c = -1$. This completes the proof of (3.2). □

We now translate (3.2) into more geometrical terms to determine X.

Let $W(X)$ be the reflection group on $S_X \otimes \mathbb{R}$ generated by $r_b : x \mapsto x + (x \cdot b)b$ where $b \in S_X$ satisfies $b^2 = -2$. Since the nef and big cone is a fundamental domain for this action on the closure of the positive cone ([PS, Section 7], [BPV, Chapter 8, Proposition 3.9]), we may assume that $e_1 = [H]$ for a nef and big divisor H with $(H)^2 = 2$. Set $e_2 = [C_2]$ and $e_1 - e_2 = [C_1]$. Since $(e_i)^2 = -2$ and $(H.e_i) = 1$, C_i may be chosen to be effective. Combining this with the semi-ampleness of H and with the equality $\operatorname{rank} S_X = 2$, we easily find that C_i are then smooth rational curves and $\partial\overline{NE}(X) = \mathbb{R}_{\geq 0}[C_1] \cup \mathbb{R}_{\geq 0}[C_2]$. In particular, X contains neither smooth rational curves nor smooth ellptic curves other than C_i $(i = 1, 2)$.

LEMMA (3.5). *H is ample and free.*

PROOF. Since $(H.C_i) = 1$, the ampleness of H follows from Kleiman's criterion. Applying the Riemann-Roch Theorem and the vanishing Theorem, we calculate that $h^0(\mathcal{O}_X(H)) = 3$, while $h^0(\mathcal{O}_X(H - C_i)) =$

$h^0(\mathcal{O}_X(C_{2-i})) = 1$. Thus $|H|$ has no fixed components whence a general element C of $|H|$ is an irreducible reduced curve with $p_a(C) = 2$ (cf. [SD, Proposition 2.6]). Since $|K_C|$ is free, the freeness of H now follows from the exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(H) \to \mathcal{O}_C(K_C) \to 0$ and the equality $h^1(\mathcal{O}_X) = 0$ (cf. [SD, Section 3]). $\square$

Let $f : X \to \mathbb{P}^2$ be the finite double cover given by $|H|$ and $B \subset \mathbb{P}^2$ the ramification curve. Note that B is a smooth sextic curve. Using the last assertion in (3.1), we find an element $h \in \mathrm{Aut}(\mathbb{P}^2)$ such that $f \circ g = h \circ f$. Note that h is of order 25 and satisfies $h(B) = B$. Let $[x_0 : x_1 : x_2]$ be homogeneous coordinates of $\mathbb{P}^2$ under which the co-action of h is diagonalized as $h^* = \mathrm{diag}(a, b, c)$.

Claim (3.6). a, b, c are mutually distinct.

Proof. Using the topological Lefschetz formula and (1.1), (3.1), we calculate that $\chi_{top}(X^g) = 2 + \mathrm{tr}(g^* \mid S_X) + \mathrm{tr}(g^* \mid T_X) = 4$. Assume the contrary that $a = b$. Then, $(\mathbb{P}^2)^h = (x_2 = 0) \coprod \{[0 : 0 : 1]\}$ and $X^g = f^{-1}((x_2 = 0)) \coprod f^{-1}([0 : 0 : 1])$. Note that the smoothness of X^g implies that $f^{-1}((x_2 = 0))$ is a smooth curve of genus 2. Now combining these all together with $|f^{-1}([0 : 0 : 1])| \le 2$, we get the following contradiction: $4 = \chi_{top}(X^g) = (2 - 2 \times 2) + |f^{-1}([0 : 0 : 1])| \le -2 + 2 = 0$. $\square$

Set $P_0 = [1 : 0 : 0]$, $P_1 = [0 : 1 : 0]$, $P_2 = [0 : 0 : 1]$. By (3.6), we have $(\mathbb{P}^2)^h = \{P_0\} \coprod \{P_1\} \coprod \{P_2\}$ and $X^g = f^{-1}(P_0) \coprod f^{-1}(P_1) \coprod f^{-1}(P_2)$. Since $\chi_{top}(X^g) = 4$ and $|f^{-1}(P_i)|$ is either 1 or 2 for each i, the topological Lefschetz formula shows that exactly two of P_i's lie in B (and the other one does not).

Claim (3.7). Eaxctly two of a^5, b^5 and c^5 coincide.

Proof. Assume the contrary that the statement is false. Since h^5 is of order 5, a^5, b^5 and c^5 are then mutually distinct, whence $X^{g^5} = f^{-1}((\mathbb{P}^2)^{h^5}) = f^{-1}(P_0) \coprod f^{-1}(P_1) \coprod f^{-1}(P_2)$. This gives $\chi_{top}(X^{g^5}) = 4$. On the other hand, using the topological Lefschetz formula, we calculate that $\chi_{top}(X^{g^5}) = 2 + \mathrm{tr}(g^* \mid S_X) + \mathrm{tr}(g^* \mid T_X) = 2 + 2 + 5 \times (-1) = -1$, a contradiction. $\square$

Now, adjusting the order of the coordinates and replacing g by another generator of G if necessary, we may write the co-action of h as $h^* = \mathrm{diag}(1, \zeta_5^n, \zeta_{25})$, where n denotes some integer with $1 \le n \le 4$. Set $L = (x_2 = 0)$ in $\mathbb{P}^2$. Then $(\mathbb{P}^2)^{h^5} = L \coprod \{P_2\}$ and $X^{g^5} = f^{-1}(L) \coprod f^{-1}(P_2)$. Using the fact that $f^{-1}(L)$ is a smooth curve of genus 2, and applying the topological Lefschetz formula, we calculate $-1 = \chi_{top}(X^{g^5}) = (2 - 2 \times 2) + |f^{-1}(P_2)|$, whence $|f^{-1}(P_2)| = 1$. This means $P_2 \in B$. Thus either $P_0 \notin B$ or $P_1 \notin B$. Since $h^* = \mathrm{diag}(\zeta_5^{5-n}, 1, \zeta_{25}^{1+5(5-n)})$, there exist integers l and m such that $(l, 5) = (m, 5) = 1$ and that $(h^*)^l = \mathrm{diag}(\zeta_5^m, 1, \zeta_{25})$, we may assume without loss of generality that $P_0 \notin B$. Let $F := \sum_{i+j+k=6} a_{ijk} x_0^i x_1^j x_2^k$ be a defining polynomial of B. We determine F and n. For simplicity of notation, we write $x_0^i x_1^j x_2^k \in F$ if $a_{ijk} \neq 0$. Using $P_0 \notin B$, we see that $x_0^6 \in F$ and $h^*(F) = F$. This implies that $x_0^i x_1^j x_2^k \in F$ only if either $1 \le n \le 3$ and $(i, j, k) \in \{(6, 0, 0), (1, 5, 0)\}$ or $n = 4$ and $(i, j, k) \in \{(6, 0, 0), (1, 5, 0), (0, 1, 5)\}$. Combining this with the smoothness of B, we readily see that $n = 4$ and $F = \alpha x_0^6 + \beta x_0 x_1^5 + \gamma x_1 x_2^5$, where α, β and γ denote some constant with $\alpha\beta\gamma \neq 0$. Now, multiplying each coordinate x_i by a suitable non-zero constant if necessary, we may normalise the polynomial F as $F = x_0^6 + x_0 x_1^5 + x_1 x_2^5$. This means that the defining equation of X in $\mathbb{P}(1, 1, 1, 3)$ is $z^2 = x_0^6 + x_0 x_1^5 + x_1 x_2^5$ and the co-action of g is $g^* = \mathrm{diag}(1, \zeta_5^4, \zeta_{25}, 1)$ under appropriate coordinates. This implies $(X, G) \simeq (X_{50}, \langle g_{50}^2 \rangle)$.

LEMMA (3.8). $\mathrm{Aut}(X_{50}) = \langle g_{50} \rangle$.

PROOF. Let θ be an element of $\mathrm{Aut}(X_{50})$. Then $I(X_{50}, \langle \theta, g_{50} \rangle) = 50$ by the Table 1. Thus, there exists an integer n such that $(\theta \circ g_{50}^{-n})^* \omega_{X_{50}} = \omega_{X_{50}}$. On the other hand, using the description of the effective cone, we see that $(\theta \circ g_{50}^{-n})^* \mid S_X$ is at most of order 2. Then it follows from (1.6) that $(\theta \circ g_{50}^{-n})$ is of finite order. Combing this with (1.2) and (1.6), we get $(\theta \circ g_{50}^{-n}) = id$. This implies the result. $\square$

This completes the proof of Theorem 1 in the case where $I(X, G) = 25$.

Next consider the case where $I(X, G) = 50$. Then G is of order 50 and there exists an element g of G with $I(X, \langle g \rangle) = 25$. Thus $(X, \langle g \rangle) \simeq (X_{50}, \langle g_{50}^2 \rangle)$ by the previous argument. Combining this with (3.8), we get $(X, G) \simeq (X_{50}, \langle g_{50} \rangle)$. Now we are done.

4. Uniqueness of pairs (X, G) with $I(X, G) = 40$

In this section, we show main Theorem 2. Let (X, G) be a pair with $I(X, G) = 40$. Using (1.1) and (1.2), we readily find

Lemma (4.1).
(1) $\operatorname{rank} T_X = 16$ *and* $\operatorname{rank} S_X = 6$.
(2) *There exists an element* $g \in G$ *such that* $G = \langle g \rangle$, $\operatorname{ord}(g) = 40$ *and that* $g^*\omega_X = \zeta_{40}\omega_X$.

Set $h := g^8$ and $f := g^{10}$.

Lemma (4.2).
(1) $h^* \mid S_X \otimes \mathbb{C}$ *is diagonalised as* $\operatorname{diag}(1, 1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$, *and*
(2) $f^* \mid S_X = id$.

Proof. The assertin (2) readily follows from (1). We show the assertion (1). Assuming the contrary that $h^* \mid S_X = id$, we shall derive a contradiction. Since $\operatorname{rank} S_X = 6$, X admits an elliptic fibration $\Phi : X \to \mathbb{P}^1$. Since $h^* \mid S_X = id$, there exists $\overline{h} \in \operatorname{Aut}(\mathbb{P}^1)$ such that $\Phi \circ h = \overline{h} \circ \Phi$. Using $\operatorname{ord}(h) = 5$, $h^*\omega_X = \zeta_5\omega_X$, and the fact that no elliptic curve admits complex multiplication of order 5, we find that $\overline{h}$ is also of order 5. Thus we may choose an inhomogeneous coordinate t of $\mathbb{P}^1$ under which the co-action of h is written as $(\overline{h})^*t = \zeta_5^a t$ where a is an integer with $(a, 5) = 1$. Then $(\mathbb{P}^1)^{\overline{h}} = \{0, \infty\}$. Since $h^* \mid S_X = id$, every singular fiber of Φ other than X_0 and X_∞ must be irreducible, namely, either of type I_1 or of type II. In addition, if X_t $(t \neq 0, \infty)$ is a singular fiber, then $X_{\zeta_5^n t}$ $(1 \leq n \leq 5)$ are also the singular fibers of the same type as X_t. Indeed, they are permuted by h.

Claim (4.3). X^h is either
(1) $\{P_1, P_2, P_3\} \coprod \{Q\}$ or
(2) $\{P_1, P_2, P_3\} \coprod \{Q\} \coprod E$,

where $P_i \in X^h$ are of type $\frac{1}{5}(2, 4)$, $Q \in X^h$ is of type $\frac{1}{5}(3, 3)$, and E is a smooth fiber of Φ.

Proof. Since $X^h = (X_0)^h \coprod (X_\infty)^h$ and $h^*\omega_X = \zeta_5\omega_X$, X^h consists of l isolated points of type $\frac{1}{5}(2, 4)$, say, P_i $(i = 1, \ldots, l)$, m isolated points of type $\frac{1}{5}(3,3)$, say, Q_j $(j = 1, \dots, m)$, n smooth rational curves in $X_0 \cup X_\infty$, say, C_k $(k = 1, \dots, n)$, and p smooth elliptic curves in $X_0 \cup X_\infty$, say, E_q $(q = 1, \dots, p)$, where l, m, n and p are some non-negative integers. Then using the topological Lefschetz fixed point formula, we calculate $l+m+2n = \chi_{top}(X^h) = 2 + \mathrm{tr}(h^* \mid S_X) + \mathrm{tr}(h^* \mid T_X) = 4$. On the other hand, using the holomorphic Lefschetz fixed point formula ([AS1, Page 542] and [AS2, Page 567]), we calculate $1 + \zeta_5^{-1} = \sum_{i=0}^{2} (-1)^i \,\mathrm{tr}(h^* \mid H^i(\mathcal{O}_X)) = \sum_{i=1}^{l} a(P_i) + \sum_{j=1}^{m} a(Q_j) + \sum_{k=1}^{n} b(C_k) + \sum_{q=1}^{p} b(E_q)$, where $a(P_i) = 1/(1-\zeta_5^2)(1-\zeta_5^4)$, $a(Q_j) = 1/(1-\zeta_5^3)(1-\zeta_5^3)$, $b(C_k) = (1-\pi(C_k))/(1-\zeta_5) - \zeta_5(C_k)^2/(1-\zeta_5)^2 = (1+\zeta_5)/(1-\zeta_5)^2$, and $b(E_q) = (1-\pi(E_q))/(1-\zeta_5) - \zeta_5(E_q)^2/(1-\zeta_5)^2 = 0$. From this, we readily get that $(-2l+m+3n+5)\zeta_5 + (-l-2m+4n+5)\zeta_5^2 + (-2l+m+3n+5)\zeta_5^3 = 0$, whence $l = 3 + 2n$ and $m = 1 + n$. Combining this with $l + m + 2n = 4$, we get $n = 0$, $l = 3$ and $m = 1$. This also implies that p is at most one. Now we are done. □

Now the next two claims, which contradict each other, completes the proof of (4.2)(1).

CLAIM (4.4). The case (2) in (4.3) does not occur.

CLAIM (4.5). The case (1) in (4.3) does not occur.

PROOF OF (4.4). Assuming the contrary that this occurs, we derive a contradiction. Since $g(X^h) = X^h$, we get $g(E) = E$. Thus, there exists $\overline{g} \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\Phi \circ g = \overline{g} \circ \Phi$. We may adjust an inhomogeneous coordinate t of $\mathbb{P}^1$ as $E = X_0$ and $\overline{g}^* t = \zeta_I^k t$ where k is an integer. Let $5r$ (resp. $5s$) be the number of singular fibers of Φ of type I_1 (resp. of type II) lying over $\mathbb{P}^1 - \{\infty\}$. Note that $r+s \neq 0$. Indeed, Φ has at least two singular fibers by the monodromy reason. Using $24 = \chi_{top}(X) = 5r + 10s + \chi_{top}(X_\infty)$, we see that (r,s) is either $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, $(2,0)$, $(2,1)$, $(3,0)$ or $(4,0)$. This with $\mathrm{ord}(g) = 40$ implies $\overline{g}^{20} = id$. Let ω_E be a nowhere vanishing holomorphic $1-$form of E and set $(g \mid E)^* \omega_E = \alpha \omega_E$. Since $(g^*)^{20} \omega_X = -\omega_X$ and $\overline{g}^{20} = id$, we have $(g^{20} \mid E)^* \omega_E = -\omega_E$. Thus $\alpha^{20} = -1$, whence $\alpha \notin \mu_4 \cup \mu_6$, a contradiction. □

PROOF OF (4.5). Assuming the contrary that this occurs, we derive a contradiction. Set $k = g^4$. Then $k^2 = h (= g^8)$ and k is of order 10.

Since $g(\{P_1, P_2, P_3\}) = \{P_1, P_2, P_3\}$ and $g(Q) = Q$, we see that $k(P_i) = P_i$ for each i and $k(Q) = Q$. Combining this with $X^k \subset X^h$, we get $X^k = \{P_1, P_2, P_3, Q\}$. Since $k^*\omega_X = \zeta_{10}\omega_X$, the type of a point P in X^k is of the form $\frac{1}{10}(n_1, 11 - n_1)$. In addition if $P = P_i$ then $\frac{1}{5}(n_1, 11 - n_1)$ is same as $\frac{1}{5}(2, 4)$ and if $P = Q$ then $\frac{1}{5}(n_1, 11 - n_1)$ is same as $\frac{1}{5}(3, 3)$. This implies that $P_i \in X^k$ is either of type $\frac{1}{10}(2, 9)$ or of type $\frac{1}{10}(7, 4)$ and $Q \in X^k$ is of type $\frac{1}{10}(3, 8)$. Let a and b be the numbers of points P_i of type $\frac{1}{10}(2, 9)$ and of type $\frac{1}{10}(7, 4)$. Then $a + b = 3$. On the other hand, by apply the holomorphic Lefschetz fixed point formula for k, we get:

$$\begin{aligned} 1 + \zeta_{10}^{-1} &= \sum_{i=0}^{2} (-1)^i \operatorname{tr}(f^* \mid H^i(\mathcal{O}_X)) \\ &= a/(1 - \zeta_{10}^2)(1 - \zeta_{10}^9) + b/(1 - \zeta_{10}^7)(1 - \zeta_{10}^4) + 1/(1 - \zeta_{10}^3)(-\zeta_{10}^8). \end{aligned}$$

This equation is readily simplified as $(2a - 4b - 2) - (a + 3b - 1)\zeta_5^2 - (a + 3b - 1)\zeta_5^3 = 0$, whence $b = 0$ and $a = 1$. However, this contradicts the previous equality $a + b = 3$. This completes the proof of (4.5). □

Now we have completed the proof of (4.2). □

Lemma (4.6).

(1) *$S_X \simeq U(2) \oplus D_4$.*
(2) *$X^{g^{20}} = R \coprod C$, where R is a smooth rational curve and C is a smooth curve with $\pi(C) = 6$. In particular, $g(R) = R$ and $g(C) = C$, and*
(3) *$f \mid C \neq id$.*

Proof. Since $(g^{20})^* \mid S_X = id$ and $(g^{20})^*\omega_X = -\omega_X$ by (4.2), it follows from (1.4) that S_X is 2−elementary. Now we may apply Nikulin's classification of even 2−elementary hyperbolic lattices ([Ni2, Theorems 4.3.1, 4.3.2], see also [Ko, Section 6]) to find that S_X is isomorphic to either (i) $U \oplus A_1^{\oplus 4}$, (ii) $U(2) \oplus A_1^{\oplus 4}$, (iii) $U \oplus D_4$, or (iv) $U(2) \oplus D_4$. We eliminate the cases (i), (ii) and (iii). In the cases (i) and (ii), X admits an elliptic fibration $\alpha : X \to \mathbb{P}^1$ whose reducible singular fibers are $a_1 I_2 + a_2 III$ ($a_1 + a_2 = 4$) ([Ko, Lemma 2.2]). Since $f^* \mid S_X = id$, there exists $\overline{f} \in \operatorname{Aut}(\mathbb{P}^1)$ such that $f \circ \alpha = \alpha \circ \overline{f}$. Again by $f^* \mid S_X = id$, each smooth rational curve on X is $f-$stable. Thus, we have $\overline{f} = id$. Let E be any smooth fiber of α and $\omega_E \neq 0$ a holomorphic 1-form of E. Then $\omega_E \wedge \alpha^* dt$ gives a nowhere vanishing holomorphic 2-form of X around E whence $(g \mid E)^* \omega_E = \zeta_4 \omega_E$. In particular, the $J-$invariant map $J : \mathbb{P}^1 \to \mathbb{P}^1$ of α is constant, or more pricisely, $J \equiv j(\mathbb{C}/\mathbb{Z} + \mathbb{Z}\zeta_4)$. Thus, possible singular fibers of α are of Type III, of Type III^* or of Type I_0^* by Kodaira's classification of singular fibers ([Kd, Theorem 9.1], [BPV, Page 159, Table 6]). In particular, these are all reducible. Combining this with previous observation, we see that α has either exactly 4 singular fibers of Type III. Then $24 = \chi_{top}(X) = 4 \times 3 = 12$, a contradiction. This eliminates the cases (i) and (ii). Next we eliminate the case (iii). In the case (iii), again by Nikulin's classification of the fixed locus of an involution ι with $\iota^* \mid S_X = id$ and with $\iota^* \omega_X = -\omega_X$ ([Ni2, Theorem 4.2.2], see also [Ko, Section 6]) we see that $X^{g^{20}} = C \coprod R_1 \coprod R_2$, where C is a smooth curve of genus 7 and R_i are smooth rational curves. Since $g(X^{g^{20}}) = X^{g^{20}}$, this implies $h(C) = C$, $h(R_1) = R_1$, and $h(R_2) = R_2$. Since C, R_1 and R_2 are independent in S_X, this contradicts $\dim S_X^{h^*} = 2$. Thus we get the assertion (1). Then $X^{g^{20}} = C \coprod R$, where C is a smooth curve of genus 6 and R is a smooth rational curve by Nikulin's classification quoted above. This readily implies the assertion (2). Finally we show (3). Since $S_X \simeq U(2) \oplus D_4$, X admits an elliptic fibration $\beta : X \to \mathbb{P}^1$ having exactly one reducible singular fiber of Type I_0^*. Since $f^* \mid S_X = id$, there exists $\overline{f} \in \mathrm{Aut}(\mathbb{P}^1)$ such that $f \circ \beta = \beta \circ \overline{f}$. If $\overline{f} = id$, then the same argument as before implies the $J-$invariant map of β takes a constant value $j(\mathbb{C}/\mathbb{Z} + \mathbb{Z}\zeta_4)$, whence β has exactly one singular fiber. However, this is impossible. Thus $\overline{f} \neq id$ whence each irreducible component of X^f is either a smooth elliptic curve, a smooth rational curve or an isolated point. This implies $C \not\subset X^f$, that is, $f \mid C \neq id$. $\square$

LEMMA (4.7). *$g^* \mid S_X \otimes \mathbb{C}$ is diagonalised as* $\mathrm{diag}(1, 1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$.

PROOF. Using the fact that R and C are independant in $S_X^{g^*}$ and (4.2)(1), we see that $g^* \mid S_X \otimes \mathbb{C}$ is diagonalised as either $\mathrm{diag}(1, 1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$ or $\mathrm{diag}(1, 1, \zeta_{10}, \zeta_{10}^3, \zeta_{10}^7, \zeta_{10}^9)$. Suppose the last case occurs. Then $(g^5)^* \mid S_X \otimes \mathbb{C}$ is diagonalised as $\mathrm{diag}(1, 1, -1, -1, -1, -1)$ whence $\chi_{top}(X^{g^5}) = 0$ by the topological Lefschetz fixed point formula. On the other hand, we have $X^{g^5} = C^{g^5} \coprod R^{g^5}$ by $X^{g^5} \subset X^{g^{20}}$, whence by (4.6)(3), $\chi_{top}(X^{g^5}) = \chi_{top}(C^{g^5}) + 2 \geq 2$, a contradiction. Now we are done. $\square$

Lemma (4.8). *$X^{g^5} = R \coprod S$ where R is same as in (4.6)(2) and S is a finte set of points.*

Proof. Since $S_X \simeq U(2) \oplus D_4$, X admits an elliptic fibration $\Phi := \Phi_{|E|} : X \to \mathbb{P}^1$ whose reducible singular fibers are exactly one I_0^*, say $2C_0 + \sum_{i=1}^4 C_i$ ([Ko, Lemma 2.2]). Since $(g^5)^* \mid S_X = id$, there exists $\overline{g^5} \in \mathrm{Aut}(\mathbb{P}^1)$ such that $g^5 \circ \Phi = \Phi \circ \overline{g^5}$. Since each smooth rational curve on X is g^5-stable, $\overline{g^5} \neq id$ by the same argument as in (4.6). Then $X^{g^5} (\subset X^{g^{20}})$ consists of one smooth rational curve $C_0 = R$ and a finite set of points. □

Lemma (4.9). $\det S_X^{g^*} = -20$ *or* -5.

Proof. Since the lattice $M := \langle [R], [C] \rangle$ is of finite index, say r, in $S_X^{g^*}$, we have $r^2 = |\det M| / |\det S_X^{g^*}|$. Now the result follows from $\det M = (R)^2 \cdot (C)^2 = -20$. □

Lemma (4.10). *$X^{g^8} = D \coprod \{P\}$, where D is a smooth curve of genus 2. In particular, $g(D) = D$.*

Proof. By (4.2) and the topological Lefschetz fixed point formula, we have $\chi_{top}(X^{g^8}) = -1 < 0$. This means that X^{g^8} contains a smooth curve D with $\pi(D) \geq 2$. Since D is nef and big and the multiplicity of the eigen value 1 of $(g^8)^* \mid S_X \otimes \mathbb{C}$ is 2, the remaining one-dimensional component of X^{g^8} is a smooth rational curve, say E, if exists. Assuming the contrary that this is the case, we set $X^{g^8} = D \coprod E \coprod S_1 \coprod S_2$, where S_1 (resp. S_2) is a finite set of points of type $\frac{1}{5}(2,4)$ (resp. of type $\frac{1}{5}(3,3)$). Then, applying the holomorphic Lefschetz fixed point formula, we get $1 + \zeta_5^{-1} = (-(D)^2/2 + 1)(1 + \zeta_5)/(1 - \zeta_5)^2 + |S_1|/(1 - \zeta_5^2)(1 - \zeta_5^4) + |S_2|/(1 - \zeta_5^3)^2$, whence $|S_1| = 2(-(D)^2/2 + 1) + 3$ and $|S_2| = (-(D)^2/2 + 1) + 1$. On the other hand, by the topological Lefschetz fixed point formula, we have $(-(D)^2 + 2) + |S_1| + |S_2| = -1$. Combining these all, we get $(D)^2 = 4$, $|S_1| = 1$ and $|S_2| = 0$. This gives $\det(\langle [D], [E] \rangle) = -8$. However, since $\langle [D], [E] \rangle$ is of finite index in $S_X^{g^*}$, this contradicts the fact that $\det \langle [D], [E] \rangle / \det S_X^{g^*}$ is an integer. Thus $X^{g^8} = D \coprod S_1 \coprod S_2$. Now the same calculation as before implies $(D)^2 = 2$, $|S_1| = 1$ and $|S_2| = 0$. This completes the proof. □

Let us considr the generically $2:1-$map $\varphi := \Phi_{|D|} : X \to \mathbb{P}^2$ and take the Stein factorisation $X \xrightarrow{\nu} \overline{X} \xrightarrow{\overline{\varphi}} \mathbb{P}^2$. Let $B (\subset \mathbb{P}^2)$ be a ramification curve of $\overline{\varphi}$. Then B is a sextic curve. Note also that g descends to the action on $\overline{X}$ and there exists $\overline{g} \in \mathrm{Aut}(\mathbb{P}^2)$ such that $\overline{\varphi} \circ g = \overline{g} \circ \varphi$.

LEMMA (4.11). *$(D.R) = 1$ and $\varphi_*(R)$ is a line in $\mathbb{P}^2$. Moreover $S_X^{g^*} = \langle [D], [R] \rangle$.*

PROOF. Since $X^{g^8} \cap X^{g^5} = X^g$, X^g is a finite set of points by (4.8) and (4.10). Combining this with the topological Lefschetz fixed point formula, we find $|X^g| = 3$. Thus $(0 \le) m := |D \cap R| \le 3$. Assume that $\mathrm{mult}(D, R; P) \ge 2$ for some $P \in D \cap R$. Then $T_{D,P} = T_{R,P}$ in $T_{X,P}$. Since $(g^8)^* \mid T_{D,P} = id$ and $(g^5)^* \mid T_{R,P} = id$, this implies $g^* \mid T_{R,P} = id$ whence $g \mid R = id$, a contradiction. Thus $m = (D \cdot R)$ and then $\det(\langle [D], [R] \rangle) = -4 - m^2$. Moreover, since $\langle [D], [R] \rangle$ is of finite index in S^{g^*}, there exists an integer r such that either $r^2 = (4 + m^2)/20$ or $r^2 = (4 + m^2)/5$. Combining this with $0 \le m \le 3$, we get $m = 1$, $|\det S_X^{g^*}| = 5$ and $r = 1$. Since $D = \varphi^* l$ for some line l in $\mathbb{P}^2$, we get from $m = 1$ that $1 = (D \cdot R) = (l \cdot \varphi_*(R))$. This means that $\varphi_*(R)$ is a line in $\mathbb{P}^2$. □

Set $\overline{R} = \varphi_*(R)$.

LEMMA (4.12). *$B = \overline{R} \cup \overline{B}$, where $\overline{B}$ is a smooth quintic curve intersecting $\overline{R}$ transversally (at 5 points).*

PROOF. Since $(\varphi^* \overline{R} \cdot \varphi^* l) = 2$ and $(R \cdot \varphi^* l) = 1$ by (4.11), $\varphi^* \overline{R} = R + \iota(R) + E$ where E is an effective divisor supported in $\mathrm{Exc}(\nu)$ and ι is the covering involution of φ. Since $\iota \circ g = g \circ \iota$, we have $\iota(R) \subset X^{g^5}$ whence $\iota(R) = R$ by the description of X^{g^5}. Thus $\varphi^* \overline{R} = 2R + E$, and $B = \overline{R} + \overline{B}$ for some quintic curve $\overline{B}$. Now it is enough to show the next Lemma to complete (4.12): □

LEMMA (4.13). *$\mathrm{Exc}(\nu)$ consists of 5 disjoint smooth rational curves, say, E_i $(1 \le i \le 5)$ permuted by g. In particular $\mathrm{Sing}(\overline{X})$ consists of 5 ordinary double points.*

PROOF. Since $(g^5)^* \mid S_X = id$, $g^5(R') = R'$ for each smooth rational curve R'. Let E_i be a connected component of $\mathrm{Exc}(\nu)$. If $g(E_i) = E_i$, then $E_i \simeq \mathbb{P}^1$, because $\chi_{top}(X^g) = 3$, $\chi_{top}(D^g) \ge 1$ by $(D \cdot R) = 1$, and $D \cap E_i = \phi$. Then there exist integers a and b such that $E_i = aD + bR$

in S_X. Using $(D \cdot E_i) = 0$ and $(E_i)^2 = -2$, we then get $2a + b = 0$ and $2a^2 + 2ab - 2b^2 = -2$ whence $a^2 = 1/5$, a contradiction. Thus $g(E_i) \neq E_i$ for each connected component of $\mathrm{Exc}(\nu)$. In particular, $\mathrm{Exc}(\nu)$ contains at least 5 connected components. Combining this with $\mathrm{rank}\, S_X = 6$, we get the assertion. □

LEMMA (4.14). *$\overline{g} \in \mathrm{Aut}(\mathbb{P}^2)$ is of order* 20.

PROOF. By (4.6)(2), we have $\overline{g}^{20} \mid \varphi(C) = id$. Since $\varphi(C)$ is not a line, this implies $\overline{g}^{20} = id$. On the other hand, if $\overline{g}^n = id$ for some n with $n < 20$, then $g^{2n} = id$, a contradiction. This implies the result. □

PROOF OF THEOREM 2.

Since $\overline{g}(\overline{R}) = \overline{R}$, we may take homogeneous coordinates $[x_0 : x_1 : x_2]$ of $\mathbb{P}^2$ such that $\overline{R} = (x_0 = 0)$ and that the co-action of $\overline{g}$ is diagonalised as $\overline{g}^* = \mathrm{diag}(1, \zeta_{20}, \zeta_{20}^{4k+1})$ for some integer k with $0 \leq k \leq 4$ (after replacing g by appropriate generator of G if necessary). For the last statement, we use $\overline{g}$ is of order 20 and $\overline{g}^5 \mid \overline{R} = id$. Since $\overline{g}(\overline{B}) = \overline{B}$ and $\overline{R} \cap \overline{B} = \{P_1, \dots, P_5\}$ are permuted by $\overline{g}$, we may set $P_i = [0 : 1 : \zeta_5^i]$ by changing x_1 and x_2 by their suitable constant multiples. Then the equation of $\overline{B}$ is of the form $F(x_0, x_1, x_2) = x_0 f(x_0, x_1, x_2) + (x_1^5 - x_2^5) = 0$. Since $\overline{g}^*(x_1^5 - x_2^5) = \zeta_4(x_1^5 - x_2^5)$, we have $\overline{g}^* f = \zeta_4 f$. This readily implies that $f = \alpha x_0^3 x_2$ if $k = 1$ and $f = 0$ if $k \neq 1$. Combining this with the smoothness of $\overline{B}$, we get $k = 1$, $\alpha \neq 0$ and $F(x_0, x_1, x_2) = \alpha x_0^4 x_2 + x_1^5 - x_2^5$. Now changing x_0 by its suitable constant multiple if necessary, we may normalise the equation of $\overline{B}$ as $x_0^4 x_2 + x_1^5 - x_2^5 = 0$. Thus $\overline{X}$ is isomorphic to the hypersurface $(z^2 = x_0(x_0^4 x_2 + x_1^5 - x_2^5))$ in $\mathbb{P}(1,1,1,3)$ and $g^* = \mathrm{diag}(1, \zeta_{20}, \zeta_4, (-)\zeta_8)$ under this identification. This implies $(X, G) \simeq (X_{40}, \langle g_{40} \rangle)$. Finally we show that $\mathrm{Aut}(X) = G$. Since $(g^{20})^* \mid S_X = id$ and $(g^{20})^* \omega_X = -\omega_X$, we see by (1.6) that g^{20} is in the center of $\mathrm{Aut}(X)$. Thus $\mathrm{Aut}(X)$ stabilises C where C is a curve found in (4.6). Since C is big and semi-ample, this implies that $\mathrm{Aut}(X)$ is finite. Now combining this with $\mathrm{rank}(T_X) = 16$ and (1.2), we find that $\mathrm{Aut}_N(X) = id$. Now Table 1 implies $\mathrm{Aut}(X) = G$. □

REMARK. The referee kindly indicated another proof of Theorem 2 based on (4.6), $(g^{20})^* \mid S_X = id$ (cf. (4.7)) and the following observation: Besides R, there exist exactly 5 smooth rational curves, namely C_i ($i =$

$1, 2, \ldots, 5)$, on X and that they satisfy $(C_i.C_j) = 0$ $(i \neq j)$ and $(C_i.R) = (C_i.C) = 1$, where R and C are the curves found in (4.6).

5. Determination of transendental values

In this section, we prove Theorem 3 and Corollary 5.

The core of this section is to show the following:

THEOREM (5.1). $60 \notin \mathbb{TV}_{K3}$.

PROOF. Assuming the contrary that there exists a pair (X, G) of a K3 surface and it finite automorphism group G with $I = I(X, G) = 60$, we shall derive a contradiction.

First we notice the following:

CLAIM (5.2).

(1) $\operatorname{rank} T_X = 16$ and $\operatorname{rank} S_X = 6$.

(2) There exists an element $g \in G$ such that $G = \langle g \rangle$, $\operatorname{ord}(g) = I$ and that $g^*\omega_X = \zeta_I \omega_X$.

PROOF. This follows from the same argument as in (4.1). $\square$

Set $h = g^{12}$. Note that h is of order 5.

CLAIM (5.3). $h^* \mid S_X = id$.

PROOF. Assume the contrary that $h^* \mid S_X \neq id$. Since h is of order 5, $h^* \mid S_X \otimes \mathbb{C}$ is then diagonalised as $h^* \mid S_X \otimes \mathbb{C} = \operatorname{diag}(1, 1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$. Combining this with the fact that $g^* \mid S_X$ has at least one fixed element (coming from an ample class of $X/\langle g \rangle$) and considering the Euler function, we readily get $(g^{10})^* \mid S_X = id$. Then g^{10} is of order 6 whence S_X is unimodular by (1.3). However, this is impossible, because $\operatorname{rank} S_X = 6$. $\square$

Since $\operatorname{rank} S_X = 6$ and $h^* \mid S_X = id$, we get in the same manner as in the proof of (4.2) that X admits an elliptic fibration $\Phi : X \to \mathbb{P}^1$ such that there exists an element $\overline{h} \in \operatorname{Aut}(\mathbb{P}^1)$ of order 5 with $\Phi \circ h = \overline{h} \circ \Phi$. Again as before, we may then choose an inhomogeneous coordinate t of $\mathbb{P}^1$ under which the co-action of h is written as $(\overline{h})^*t = \zeta_5^a t$ where a is an integer with $(a, 5) = 1$. Then again as before $(\mathbb{P}^1)^{\overline{h}} = \{0, \infty\}$ and every singular fiber of Φ other than X_0 and X_∞ must be of type I_1 or of type II. In addition, if X_t $(t \neq 0, \infty)$ is a singular fiber, then $X_{\zeta_5^n t}$ $(1 \leq n \leq 5)$ are also the singular fibers of the same type as X_t and are permuted by h. Then the same argument as in (4.3) implies

Claim (5.4). X^h is either
(1) $\{P_1, P_2, P_3\} \coprod \{Q\}$ or
(2) $\{P_1, P_2, P_3\} \coprod \{Q\} \coprod E$,
where P_i are of type $\frac{1}{5}(2,4)$, Q is of type $\frac{1}{5}(3,3)$, and E is a smooth fiber of Φ.

Now again the next two claims completes the proof of (5.1). The verifications of these two claims are quite similar to those of (4.4) and (4.5) and are then left to the readers.

Claim (5.5). The case (2) in (5.4) does not occur.

Claim (5.6). The case (1) in (5.5) does not occur.

Now we are done. $\square$

Proof of Theorem 3 and Corollary 5.
Combining (5.1) together with Proposition 4 and Table 1 in Introduction, we get Theorem 3. Details for Proposition 4 are left to the readers as an exercise.

We show Corollary 5 in Introduction. By the existence of crepant terminalisation of canonical threefolds ([Ka2, Corollary 4.5], [Re, Main Theorem]), we have $\mathbb{I}_{can} = \mathbb{I}_{term}$. On the other hand by [Mo, Theorems 1 and 2] based on the argument of [Ka1, Theorem 3.2], we see that $I(X)$ lies certainly in $\{I \mid \varphi(I) \leq 20\} - \{60\}$ if X has only terminal singularities and is not smooth. On the other hand it is shown by [Be, Proposition 8] that $\mathbb{I}_{smooth} = \mathbb{TV}_{K3}$. Now combining these together with Theorem 3, we get Corollary 5. $\square$

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