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# *Elementary Construction of the Sheaf of Small 2-microfunctions and an Estimate of Supports*

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**Abstract.** The aim of this paper is to give our elementary construction of the sheaf of small 2-microfunctions and obtain an estimate of the support of solution complexes with coefficients in this sheaf. As applications we show results of solvability in the framework of microfunctions for a class of differential equations whose characteristic varieties are Lagrangian.

## 1. Introduction

The theory of the second microlocalization along regular involutive submanifolds was initiated by M. Kashiwara and J. M. Bony. M. Kashiwara has constructed the sheaf $\mathcal{C}_V^2$ of 2-microfunctions microlocalizing the sheaf $\mathcal{O}_X$ of holomorphic functions twice. (Refer to Kashiwara–Laurent [3].) Since this sheaf is too large to decompose microfunctions into second microlocal singularities, Kataoka–Tose [7] and Kataoka–Okada–Tose [6] introduced a new sheaf what is called the sheaf of *small 2-microfunctions*. Schapira–Takeuchi [9] constructed later functorially the same sheaf by introducing a bimicrolocalization functor. Here we will also give another elementary construction based on the idea of K. Kataoka, with application to the study of microfunction solutions for a class of differential equations whose characteristic varieties are Lagrangian.

We give the plan of this paper.

Section 2 is preliminaries of subsequent sections. We review the theory of microlocal study of sheaves, 2-microlocal analysis, etc.

In Section 3, we give our construction of the sheaf $\tilde{\mathcal{C}}_V^2$ of small 2-microfunctions in a simple way.

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In Section 4 we give a theorem of supports, that is, when a regular involutive submanifold $V$ is defined by:

$$V = \left\{ (x; \sqrt{-1}\xi \cdot dx) \in \sqrt{-1}\overset{\circ}{T}{}^*\mathbf{R}^n; \xi' = (\xi_1, \ldots, \xi_{n-1}) = 0 \right\}, \tag{1.1}$$

we study a simple sufficient condition under which a solution complex with coefficients in $\tilde{\mathcal{C}}_V^2$ vanishes locally in the derived category. Our construction of the sheaf $\tilde{\mathcal{C}}_V^2$ enable us to estimate the support of solution complexes.

Section 5 is devoted to the application to a class of linear differential equations. Let $P$ be a differential operator with analytic coefficients on $\mathbf{R}^n$ satisfying the following condition. One supposes for some $k \in \mathbf{N}$ and for some $l \in \mathbf{N}$:

$$|\sigma(P)(x; \sqrt{-1}\xi/|\xi|)| \sim (|x_n^k| + |\xi'|/|\xi|)^l, \tag{1.2}$$

in a real neighborhood of any point of

$$\Sigma = \left\{ (x; \sqrt{-1}\xi \cdot dx) \in \sqrt{-1}\overset{\circ}{T}{}^*\mathbf{R}^n; x_n = 0, \, \xi' = 0 \right\}. \tag{1.3}$$

Here, $\sigma(P)$ denotes the principal symbol of $P$. Though this operator $P$ is not partially elliptic along $V$ as in the Bony-Schapira's theory, we will prove that the operator $P$ induces an isomorphism:

$$P : \tilde{\mathcal{C}}_V^2 \quad \xrightarrow{\sim} \quad \tilde{\mathcal{C}}_V^2 \tag{1.4}$$

in a neighborhood of a point of $\{x_n = 0\}$.

Moreover we consider the differential operator $P$ of the form:

$$P = D_{x_1} + a x_2 D_{x_2} \tag{1.5}$$

with $a \in \mathbf{C} \setminus \mathbf{R}$, $D_{x_j} = \dfrac{\partial}{\partial x_j}$ $(j = 1, 2)$. We will prove local solvability of $Pu = f$ on $\Sigma = \{x_2 = \xi_1 = 0\}$ in the space $\mathcal{A}_V^2$ of second analytic functions for any $f \in \mathcal{A}_V^2$. Then, together with the above isomorphism (1.4), we get solvability in the space $\mathcal{C}_M$ of microfunctions for any $f \in \mathcal{C}_M$. As a corollary of this fact, we can also prove existence of microfunction solutions for the second order degenerate elliptic equation:

$$(D_{x_1}^2 + (x_2 D_{x_2})^2)u = f, \tag{1.6}$$

for all $f \in \mathcal{C}_M$.

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## 2. Preliminaries

### 2.1. Microlocal study of sheaves

In this subsection, we recall some notation on microlocal study of sheaves. (Refer to Kashiwara–Schapira [4].)

Let $X$ be a topological space and $A$ a unitary ring. We denote by $\mathbf{D}(X)$ the derived category of the abelian category of sheaves of $A$-modules on $X$ and $\mathbf{D}^+(X)$ denotes the full subcategory of $\mathbf{D}(X)$ consisting of complexes with cohomology bounded from below.

When $X$ is a real $C^\infty$ manifold, $\mathrm{SS}(F)$ denotes the micro-support of an object $F$ of $\mathbf{D}^+(X)$. We recall several functorial important formulae on the micro-support under several operations on sheaves.

Let $Z$ be a locally closed subset of $X$ and $G \in \mathrm{Ob}(\mathbf{D}^+(X))$. Then

$$\mathrm{SS}(R\Gamma_Z(G)) \subset \mathrm{SS}(G)\widehat{+}\mathrm{SS}(\mathbf{Z}_Z)^a. \tag{2.1}$$

Here $\mathrm{SS}(\mathbf{Z}_Z)^a$ is the image of $\mathrm{SS}(\mathbf{Z}_Z)$ by the antipodal map $a : T^*X \to T^*X$, $(x;\xi) \mapsto (x;-\xi)$, and $\mathbf{Z}_Z$ is the zero sheaf on $X \setminus Z$ and the constant sheaf with the stalk $\mathbf{Z}$ on $Z$.

Now we describe the set of the right-hand side of (2.1) with a system of local coordinates. For two conic subsets $A$, $B$ of $T^*X$, the subset $A\widehat{+}B$ is defined. Let $(x_\circ;\xi_\circ) \in T^*X$, $\xi_\circ \neq 0$. Then $(x_\circ;\xi_\circ)$ does not belong to $A\widehat{+}B$ if and only if there exists a positive number $\delta$ such that

$$\begin{aligned}\Big\{\Big(x+\varepsilon y;\frac{x^*}{\varepsilon}+y^*\Big) \in T^*X; (x;x^*) \in B^a,& \\ |x-x_\circ|+|x^*|+|y|+|y^*-\xi_\circ| < \delta, \quad 0<\varepsilon<\delta\Big\} &\cap A = \emptyset.\end{aligned} \tag{2.2}$$

Next let $Y$ and $X$ be real $C^\infty$ manifolds and assume that a $C^\infty$ map $f : Y \to X$ is smooth (i.e. $(f_*)_y : T_yY \to T_{f(y)}X$ is surjective for any $y \in Y$). For $F \in \mathrm{Ob}(\mathbf{D}^+(X))$ one has

$$\mathrm{SS}(f^{-1}F) = \rho(\varpi^{-1}(\mathrm{SS}(F))), \tag{2.3}$$

where $\rho$, $\varpi$ are the natural maps associated to $f$, from $Y \underset{X}{\times} T^*X$ to $T^*Y$ and $T^*X$ respectively:

$$T^*X \xleftarrow{\varpi} Y \underset{X}{\times} T^*X \xrightarrow{\rho} T^*Y. \tag{2.4}$$

Moreover, let $Z$ be a closed subset of $Y$. Then:

$$\mathrm{SS}(\mathbf{Z}_Z) \subset N^*(Z). \tag{2.5}$$

Here $N^*(Z)$ is the conormal cone to $Z$.

### 2.2. 2-microlocal analysis

Let $M$ be an open subset of $\mathbf{R}^n$ with coordinates $x = (x_1, \dots, x_n)$ and $X$ a complex neighborhood of $M$ in $\mathbf{C}^n$ with coordinates $z = (z_1, \dots, z_n)$. Let $(z, \zeta)$ be the associated coordinates on $T^*X$ with $z = x + \sqrt{-1}y$, $\zeta = \xi + \sqrt{-1}\eta$. Then $(x; \sqrt{-1}\xi \cdot dx)$ denotes a point of $T^*_M X (\simeq \sqrt{-1}T^*M)$ with $\xi \in \mathbf{R}^n$. Let $V$ be the following regular involutive submanifold of $\mathring{T}^*_M X (= T^*_M X \setminus M)$:

$$V = \left\{(x; \sqrt{-1}\xi \cdot dx) \in \mathring{T}^*_M X; \xi_1 = \cdots = \xi_d = 0\right\} \qquad (1 \leq d < n). \tag{2.6}$$

We put

$$x = (x', x''), \quad x' = (x_1, \dots, x_d), \quad x'' = (x_{d+1}, \dots, x_n), \tag{2.7}$$

etc. We set, moreover,

$$N = \left\{z \in X; \operatorname{Im} z'' = 0\right\}, \tag{2.8}$$

$$\widetilde{V} = T^*_N X. \tag{2.9}$$

This space $\widetilde{V}$ is called a partial complexification of $V$. It is equipped with the sheaf

$$\mathcal{C}_{\widetilde{V}} = \mu_N(\mathcal{O}_X)[n-d] \tag{2.10}$$

of microfunctions with holomorphic parameters $z'$, where $\mu_N$ denotes the functor of Sato's microlocalization along $N$. Refer to Kashiwara–Schapira [4]. M. Kashiwara constructed the sheaf $\mathcal{C}^2_V$ of 2-microfunctions along $V$ on $T^*_V \widetilde{V}$ from $\mathcal{C}_{\widetilde{V}}$ by

$$\mathcal{C}^2_V = \mu_V(\mathcal{C}_{\widetilde{V}})[d]. \tag{2.11}$$

We also define

$$\mathcal{A}^2_V = \mathcal{C}_{\widetilde{V}}|_V, \tag{2.12}$$

$$\mathcal{B}^2_V = R\Gamma_V(\mathcal{C}_{\widetilde{V}})[d] = \mathcal{C}^2_V|_V. \tag{2.13}$$

We call $\mathcal{A}_V^2$ the sheaf of 2-analytic functions and $\mathcal{B}_V^2$ the sheaf of 2-hyperfunctions along $V$. Note that these complexes $\mathcal{C}_{\widetilde{V}}$, $\mathcal{C}_V^2$ and $\mathcal{B}_V^2$ are concentrated in degree 0.

There are fundamental exact sequences concerning $\mathcal{C}_V^2$. On $V$,

$$0 \longrightarrow \mathcal{A}_V^2 \longrightarrow \mathcal{B}_V^2 \longrightarrow \overset{\circ}{\pi}_{V*}\left(\mathcal{C}_V^2|_{\overset{\circ}{T}{}^*_V\widetilde{V}}\right) \longrightarrow 0, \tag{2.14}$$

$$0 \longrightarrow \mathcal{C}_M|_V \longrightarrow \mathcal{B}_V^2, \tag{2.15}$$

where $\overset{\circ}{\pi}_V$ is the restriction of the projection $\pi_V : T_V^*\widetilde{V} \to V$ to $\overset{\circ}{T}{}^*_V\widetilde{V}$, and $\mathcal{C}_M(= \mu_M(\mathcal{O}_X)[n])$ is the sheaf of Sato microfunctions on $M$. We note that the inclusion $\mathcal{C}_M|_V \hookrightarrow \mathcal{B}_V^2$ is not surjective. Refer to Kataoka–Okada–Tose [6] for a counter-example.

Moreover there exists the canonical spectrum map

$$\mathrm{Sp}_V^2 : \pi_V^{-1}\mathcal{B}_V^2 \quad\longrightarrow\quad \mathcal{C}_V^2 \tag{2.16}$$

on $T_V^*\widetilde{V}$. By using $\mathrm{Sp}_V^2$ we define

$$\mathrm{SS}_V^2(u) = \mathrm{supp}\left(\mathrm{Sp}_V^2(u)\right) \tag{2.17}$$

for $u \in \mathcal{B}_V^2$. This subset $\mathrm{SS}_V^2(u)$ is called the second singular spectrum of $u$ along $V$.

Refer to Kashiwara–Laurent [3] for more details.

The exact sequence (2.14) shows that the sheaf $\mathcal{C}_V^2$ gives second microlocal analytic singularities for sections of $\mathcal{B}_V^2$. This sheaf $\mathcal{C}_V^2$ is, however, too large to study microfunction solutions for a class of differential equations, because the inclusion in (2.15) is not surjective.

For this reason Kataoka–Tose [7] constructed the subsheaf $\overset{\circ}{\mathcal{C}}{}_V^2$ of $\mathcal{C}_V^2|_{\overset{\circ}{T}{}^*_V\widetilde{V}}$ satisfying the exact sequence

$$0 \longrightarrow \mathcal{A}_V^2 \longrightarrow \mathcal{C}_M|_V \longrightarrow \overset{\circ}{\pi}_{V*}(\overset{\circ}{\mathcal{C}}{}_V^2) \longrightarrow 0 \tag{2.18}$$

by using the comonoidal transformation. Kataoka–Okada–Tose [6] also constructed $\tilde{\mathcal{C}}_V^2$ as the image sheaf of the morphism

$$\overset{\circ}{\pi}{}_V^{-1}(\mathcal{C}_M|_V) \longrightarrow \mathcal{C}_V^2|_{\overset{\circ}{T}{}^*_V\widetilde{V}} \tag{2.19}$$

to have the same exact sequence as (2.18). Schapira–Takeuchi [9] constructed later the same sheaf

$$\mathcal{C}_{MN} = \mu_{MN}(\mathcal{O}_X)[n] \tag{2.20}$$

by using the functor $\mu_{MN}$ of Schapira–Takeuchi's bimicrolocalization. Refer also to Takeuchi [10] for details.

We take the following regular involutive submanifold of $\overset{\circ}{T}{}^*X$:

$$V^{\mathbf{C}} = \left\{(z;\zeta\cdot dz) \in \overset{\circ}{T}{}^*X; \zeta' = 0\right\}, \tag{2.21}$$

which is a complexification of $V$. For this space, one sets the complex submanifold $\widetilde{V}^{\mathbf{C}}$ of $\overset{\circ}{T}{}^*(X\times X)$ as in Laurent [8]. We remark that $T^*_{V^{\mathbf{C}}}\widetilde{V}^{\mathbf{C}}$ is a complexification of $T^*_V\widetilde{V}$.

We denote by $\mathcal{E}_X$ the sheaf of rings of microdifferential operators and by $\mathcal{E}^2_{V^{\mathbf{C}}}$ the sheaf of rings of 2-microdifferential operators. We denote, moreover, by $\sigma(P)$ (resp. $\sigma_{V^{\mathbf{C}}}(P)$) the principal symbol of a microdifferential (resp. 2-microdifferential) operator $P$. We can regard a microdifferential operator $P$ as a 2-microdifferential operator in a neighborhood of a point of $V^{\mathbf{C}}$:

$$\mathcal{E}_X|_{V^{\mathbf{C}}} \xrightarrow{\sim} \mathcal{D}^2_{V^{\mathbf{C}}} := \mathcal{E}^2_{V^{\mathbf{C}}}|_{V^{\mathbf{C}}}. \tag{2.22}$$

For this operator $P$, $\sigma_{V^{\mathbf{C}}}(P)$ is the lowest degree term of the Taylor expansion of $\sigma(P)$ along $V^{\mathbf{C}}$:

$$\begin{cases} \sigma(P)(z,\zeta) = \displaystyle\sum_{|\alpha|\ge m} a_\alpha(z,\zeta'')\zeta'^{\alpha}, \\ \sigma_{V^{\mathbf{C}}}(P)(z,\zeta'',z'^*) = \displaystyle\sum_{|\alpha|=m} a_\alpha(z,\zeta'')z'^{*\alpha}. \end{cases} \tag{2.23}$$

Let $U$ be an open subset of $T^*_{V^{\mathbf{C}}}\widetilde{V}^{\mathbf{C}}$. Then, for a 2-microdifferential operator $P \in \mathcal{E}^2_{V^{\mathbf{C}}}(U)$, $P$ is invertible on $U$ if and only if $\sigma_{V^{\mathbf{C}}}(P) \neq 0$ on $U$.

Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module defined in a neighborhood of a point of $V$. One says that $\mathcal{M}$ is partially elliptic along $V$ if:

$$\mathrm{Ch}^2_{V^{\mathbf{C}}}(\mathcal{M}) \cap \overset{\circ}{T}{}^*_V\widetilde{V} = \emptyset. \tag{2.24}$$

Here the subset $\mathrm{Ch}^2_{V^{\mathbf{C}}}(\mathcal{M})$ of $T^*_{V^{\mathbf{C}}}\widetilde{V}^{\mathbf{C}}$ is the microcharacteristic variety of $\mathcal{M}$ along $V^{\mathbf{C}}$. Next let $P$ be a microdifferential operator defined in a neighborhood of a point of $V$, which is partially elliptic along $V$. Since this operator $P$ induces an isomorphism $P : \mathcal{C}_V^2 \xrightarrow{\sim} \mathcal{C}_V^2$, any microfunction (or any 2-hyperfunction) solution for the equation $Pu = 0$ always belongs to $\mathcal{A}_V^2$.

Refer to Laurent [8] and Bony–Schapira [1] for more details.

## 3. Elementary Construction of the Sheaf $\tilde{\mathcal{C}}_V^2$

In this section we give our elementary construction of the sheaf $\tilde{\mathcal{C}}_V^2$ in order to estimate the support of solution complexes with coefficients in this sheaf. Our construction makes it possible to give this estimate. This construction is based on the idea of K. Kataoka. We follow the notation of Subsection 2.2 in this, and all subsequent sections unless otherwise specified.

First of all, we set $M = \mathbf{R}^n$ with coordinates $x = (x_1, \ldots, x_n)$, $X = M^{\mathbf{C}} = \mathbf{C}^n$ with coordinates $z = (z_1, \ldots, z_n)$, and

$$\widetilde{X} = X \times (\mathbf{R}^d \setminus \{0\}), \tag{3.1}$$

$$H_c = \left\{(z, \xi') \in \widetilde{X}; \langle \operatorname{Im} z', \xi' \rangle \leq c|\operatorname{Im} z''|\right\}, \tag{3.2}$$

$$G = \left\{(z, \xi') \in \widetilde{X}; \operatorname{Im} z'' = 0\right\} \tag{3.3}$$

for $c > 0$. We identify

$$\left\{(z', x'', \xi'; \sqrt{-1}\xi'' \cdot dx'') \in \overset{\circ}{T}{}^*_G \widetilde{X}; \operatorname{Im} z' = 0\right\} \tag{3.4}$$

with $\overset{\circ}{T}{}^*_V \widetilde{V}$ through the correspondence

$$(x, \xi'; \sqrt{-1}\xi'' \cdot dx'') \longleftrightarrow (x; \sqrt{-1}\xi'' \cdot dx''; \sqrt{-1}\xi' \cdot dx'). \tag{3.5}$$

Definition 3.1 (small 2-microfunction). One sets

$$\tilde{\mathcal{C}}_V^2 = \varinjlim_c H^n\left(\left(\mu_G R\Gamma_{H_c}(p^{-1}\mathcal{O}_X)\right)\Big|_{\overset{\circ}{T}{}^*_V \widetilde{V}}\right) \tag{3.6}$$

on $\overset{\circ}{T}{}^*_V \widetilde{V}$, where $p : \widetilde{X} \to X$ is the projection and $c$ tends to $+\infty$. One calls $\tilde{\mathcal{C}}_V^2$ the sheaf of small 2-microfunctions along $V$.

We can find from the next theorem that this sheaf $\tilde{\mathcal{C}}_V^2$ coincides with that of Kataoka–Tose and Kataoka–Okada–Tose, and also with $\mathcal{C}_{MN}$ of Schapira–Takeuchi. Therefore we get the exact sequence:

$$0 \longrightarrow \mathcal{A}_V^2 \longrightarrow \mathcal{C}_M|_V \longrightarrow \overset{\circ}{\pi}_{V*}(\tilde{\mathcal{C}}_V^2) \longrightarrow 0 \tag{3.7}$$

in this case too.

THEOREM 3.2. *Let $q_\circ = (x_\circ; \sqrt{-1}\xi''_\circ \cdot dx''; \sqrt{-1}\xi'_\circ \cdot dx') \in \overset{\circ}{T}{}^*_V \widetilde{V}$. Then we have*

$$\tilde{\mathcal{C}}^2_{V, q_\circ} = \varinjlim_{Z} H^n_Z(\mathcal{O}_X)_{x_\circ}. \tag{3.8}$$

*Here $Z$ ranges through the family of closed subsets of $X$ such that*

$$Z = M + \sqrt{-1}(\Gamma + (\Gamma' \times \{0\})) \tag{3.9}$$

*and $\Gamma$ (resp. $\Gamma'$) is a closed convex cone with the vertex $0$ in $\mathbf{R}^n$ (resp. $\mathbf{R}^d$) satisfying the properties*

$$\Gamma \subset \{(y', y'') \in \mathbf{R}^n; \langle y'', \xi''_\circ \rangle < 0\} \cup \{0\}, \tag{3.10}$$

$$\Gamma' \subset \{y' \in \mathbf{R}^d; \langle y', \xi'_\circ \rangle < 0\} \cup \{0\}. \tag{3.11}$$

We prepare some propositions which is needed in the proof of Theorem 3.2. The following lemma is fundamental.

LEMMA 3.3 (Vietoris-Begle theorem). *Let $Y$ and $X$ be topological spaces and let $f : Y \to X$ be a continuous map. Assume $f$ is proper with contractible fibers and surjective. Then for any $F \in \mathrm{Ob}(\mathbf{D}^+(X))$, $F \xrightarrow{\sim} Rf_* f^{-1}F$ is an isomorphism. In particular, for any $q \in \mathbf{Z}$ we have*

$$H^q(Y, f^{-1}F) \simeq H^q(X, F). \tag{3.12}$$

In the same manner in Kataoka [5, Lemma 4.1.10], we get the following proposition.

PROPOSITION 3.4. *In the preceding situation of Definition 3.1, let $V$ be an open subset of $\widetilde{X}$ with convex fibers. Assume that there exists a compact* *subset $K$ of $\mathbf{R}^d \setminus \{0\}$ such that $V \subset X \times K$. Then for any sheaf $F$ on $X$ and for any $q \in \mathbf{Z}$ one has*

$$H^q(V, p^{-1}F) \simeq H^q(p(V), F). \tag{3.13}$$

Proof. We may assume that $F$ is a flabby sheaf by the general theory of cohomology groups with coefficients in sheaves. It is clear that there exists a family $\{K_m\}_{m\in\mathbf{N}}$ of closed subsets of $\widetilde{X}$, satisfying:

$$\begin{cases} V = \bigcup_m K_m, \quad K_m \subset \mathrm{Int}(K_{m+1}) \quad \text{for all } m, \\ \text{any fiber of } K_m \text{ is convex for all } m. \end{cases} \tag{3.14}$$

Since $p|_{X\times K}$ is proper, the restriction map of $p$ to $K_m$, $p|_{K_m} : K_m \to p(K_m)$ is also proper with convex fibers. Therefore, applying Lemma 3.3, we are able to get

$$H^q(K_m, p^{-1}F) \simeq H^q(p(K_m), F) \tag{3.15}$$

for any $m \in \mathbf{N}$.

Note first that each $p(K_m)$ is closed in $X$ and $F$ is flabby. Hence, we have

$$\begin{aligned} H^q(p(K_m), F) &\overset{\sim}{\leftarrow} \varinjlim_{p(K_m)\subset W} H^q(W, F) \\ &= \begin{cases} \varinjlim\limits_{p(K_m)\subset W} H^0(W, F), & \text{if } q = 0, \\ 0, & \text{if } q \neq 0, \end{cases} \end{aligned} \tag{3.16}$$

where $W$ ranges through the family of open neighborhoods of $p(K_m)$ in $X$.

Note moreover that the projective system $\{H^{q-1}(K_m, p^{-1}F)\}_{m\in\mathbf{N}}$ satisfies the Mittag-Leffler condition for a given $q \in \mathbf{Z}$ from the flabbiness of $F$. Therefore we have

$$H^q(V, p^{-1}F) \overset{\sim}{\to} \varprojlim_m H^q(K_m, p^{-1}F), \tag{3.17}$$

and hence, taking the projective limit on both sides of (3.15), we get

$$H^q(V, p^{-1}F) = \begin{cases} H^0(p(V), F), & \text{if } q = 0, \\ 0, & \text{if } q \neq 0. \end{cases} \tag{3.18}$$

This completes the proof of Proposition 3.4. □

PROOF OF THEOREM 3.2. We assume that $q_\circ = (x_\circ; \sqrt{-1}dx_n; \sqrt{-1}dx_1)$ for the sake of simplicity. Note that we identify $q_\circ \in \overset{\circ}{T}{}^*_V \widetilde{V}$ with a point $(x_\circ, 1, 0, \dots, 0; \sqrt{-1}dx_n) \in T^*_G \widetilde{X}$. The stalk of $\tilde{\mathcal{C}}^2_V$ at $q_\circ$ is expressed by

$$\begin{aligned} (3.19) \qquad \tilde{\mathcal{C}}^2_{V,q_\circ} &\simeq \varinjlim_{c} H^n \left(\mu_G R\Gamma_{H_c}(p^{-1}\mathcal{O}_X)\right)_{q_\circ} \\ &= \varinjlim_{c,\widetilde{Z},\pi(q_\circ)\in\widetilde{U}} H^n_{\widetilde{Z}\cap H_c\cap\widetilde{U}} \left(\widetilde{U}, p^{-1}\mathcal{O}_X\right). \end{aligned}$$

Here $\pi$ denotes the projection $\pi : T^*_G\widetilde{X} \to G$ and $\widetilde{Z}$ ranges through the family of closed subsets of $\widetilde{X}$ such that

$$(3.20) \qquad C_G\left(\widetilde{Z}\right)_{\pi(q_\circ)} \subset \left\{v \in T_G\widetilde{X}; \langle v, q_\circ\rangle > 0\right\} \cup \{0\},$$

and $\widetilde{U}$ through the family of open neighborhoods of $\pi(q_\circ) = (x_\circ, 1, 0, \dots, 0)$ in $\widetilde{X}$. Refer to Kashiwara–Schapira [4] for the notion of normal cones.

By a system of local coordinates on $X$, we set

$$(3.21) \quad \widetilde{U} = \widetilde{U}_\varepsilon = \left\{(z, \xi') \in \widetilde{X}; |z - x_\circ| < \varepsilon,\ |\xi' - (1, 0, \dots, 0)| < \varepsilon\right\},$$

and

$$\begin{aligned} (3.22) \quad \widetilde{Z} &= \widetilde{Z}_\varepsilon \\ &= \begin{cases} \left\{(z, \xi') \in \widetilde{X}; -y_n \geq \varepsilon|(y_{d+1}, \dots, y_{n-1})|\right\}, & \text{if } d+1 \neq n, \\ \left\{(z, \xi') \in \widetilde{X}; -y_n \geq 0\right\}, & \text{if } d+1 = n, \end{cases} \end{aligned}$$

for a sufficiently small $\varepsilon > 0$.

Now, consider the following exact sequences,

$$\begin{aligned} (3.23) \quad 0 \longrightarrow \mathcal{O}_{X,x_\circ} &\longrightarrow \varinjlim_{\varepsilon} H^0\left(\widetilde{U}_\varepsilon \setminus \left(\widetilde{Z}_\varepsilon \cap H_c\right), p^{-1}\mathcal{O}_X\right) \\ &\longrightarrow \varinjlim_{\varepsilon} H^1_{\widetilde{Z}_\varepsilon\cap H_c\cap\widetilde{U}_\varepsilon}\left(\widetilde{U}_\varepsilon, p^{-1}\mathcal{O}_X\right) \longrightarrow 0, \end{aligned}$$

$$(3.24)\qquad 0 \longrightarrow \varinjlim_{\varepsilon} H^{j-1}\left(\widetilde{U}_\varepsilon \setminus \left(\widetilde{Z}_\varepsilon \cap H_c\right), p^{-1}\mathcal{O}_X\right) \longrightarrow \varinjlim_{\varepsilon} H^j_{\widetilde{Z}_\varepsilon \cap H_c \cap \widetilde{U}_\varepsilon}\left(\widetilde{U}_\varepsilon, p^{-1}\mathcal{O}_X\right) \longrightarrow 0,$$

for $j \geq 2$. We can easily ascertain that any fiber of $\widetilde{U}_\varepsilon \setminus \left(\widetilde{Z}_\varepsilon \cap H_c\right)$ is a convex subset of $\mathbf{R}^d \setminus \{0\}$. Therefore we get by Proposition 3.4,

$$(3.25)\qquad H^{j-1}\left(\widetilde{U}_\varepsilon \setminus \left(\widetilde{Z}_\varepsilon \cap H_c\right), p^{-1}\mathcal{O}_X\right) \simeq H^{j-1}\left(p\left(\widetilde{U}_\varepsilon \setminus \left(\widetilde{Z}_\varepsilon \cap H_c\right)\right), \mathcal{O}_X\right),$$

for any $j \in \mathbf{Z}$.

Set, moreover, $U_\varepsilon := p\left(\widetilde{U}_\varepsilon\right)$, $Z_\varepsilon := p\left(\widetilde{Z}_\varepsilon\right)$. We define a closed subset $K_{(c,\varepsilon)}$ in $X$ by

$$(3.26)\qquad K_{(c,\varepsilon)} = Z_\varepsilon \cap \bigcap_{|\xi'^* - (1,0,\dots,0)| < \varepsilon} p\left(\{(z,\xi') \in H_c; \xi' = \xi'^*\}\right).$$

Then it is easy to see that $p\left(\widetilde{U}_\varepsilon \setminus \left(\widetilde{Z}_\varepsilon \cap H_c\right)\right) = U_\varepsilon \setminus K_{(c,\varepsilon)}$. Therefore there exist the following exact sequences:

$$(3.27)\qquad 0 \longrightarrow \mathcal{O}_{X,x_\circ} \longrightarrow \varinjlim_{\varepsilon} H^0\left(p\left(\widetilde{U}_\varepsilon \setminus \left(\widetilde{Z}_\varepsilon \cap H_c\right)\right), \mathcal{O}_X\right) \longrightarrow \varinjlim_{\varepsilon} H^1_{K_{(c,\varepsilon)} \cap U_\varepsilon}\left(U_\varepsilon, \mathcal{O}_X\right) \longrightarrow 0,$$

$$(3.28)\qquad 0 \longrightarrow \varinjlim_{\varepsilon} H^{j-1}\left(p\left(\widetilde{U}_\varepsilon \setminus \left(\widetilde{Z}_\varepsilon \cap H_c\right)\right), \mathcal{O}_X\right) \longrightarrow \varinjlim_{\varepsilon} H^j_{K_{(c,\varepsilon)} \cap U_\varepsilon}\left(U_\varepsilon, \mathcal{O}_X\right) \longrightarrow 0,$$

for $j \geq 2$.

Thus for every $j \in \mathbf{Z}$, we have

$$(3.29)\qquad H^j\left(\left(\mu_G R\Gamma_{H_c}(p^{-1}\mathcal{O}_X)\right)\Big|_{\overset{\circ}{T^*_V}\widetilde{V}}\right)_{q_\circ} \simeq \varinjlim_{\varepsilon} H^j_{K_{(c,\varepsilon)}}(\mathcal{O}_X)_{x_\circ}.$$

In particular we obtain

$$\tilde{\mathcal{C}}^2_{V, q_\circ} \simeq \varinjlim_{c,\varepsilon} H^n_{K_{(c,\varepsilon)}}(\mathcal{O}_X)_{x_\circ}. \tag{3.30}$$

One easily checks that the right-hand side of (3.30) is equal to that of (3.8). □

REMARK 3.5. Assume $d+1=n$. In this case, $K_{(c,\varepsilon)}$ is a closed convex subset of $X$ for all $c$ and for all $\varepsilon$. Hence one has the vanishing of cohomology groups:

$$H^j\left(\left(\mu_G R\Gamma_{H_c}(p^{-1}\mathcal{O}_X)\right)\Big|_{\overset{\circ}{T}{}^*_V\widetilde{V}}\right) = 0 \qquad (j \neq n) \tag{3.31}$$

from the edge of the wedge theorem.

## 4. An Estimate of the Support of Solution Complexes with Coefficients in $\tilde{\mathcal{C}}^2_V$

Hereafter unless otherwise specified, we assume that $d+1=n$, that is to say, a regular involutive submanifold $V$ is defined by $\xi_1 = \cdots = \xi_{n-1} = 0$.

In order to study microfunction solutions for linear differential equations, we reduce the problem to the small second microfunction solutions through the morphism (3.7). Here using the our construction of the sheaf $\tilde{\mathcal{C}}^2_V$ in Section 3, we have an estimate of the support of solution complexes with coefficients in $\tilde{\mathcal{C}}^2_V$. We prove this theorem by means of the micro-support.

We denote by $\mathcal{D}_X$ the sheaf of rings of finite-order holomorphic differential operators on $X$. Let $\mathcal{M}$ be an arbitrary coherent $\mathcal{D}_X$-module, and we denote by $\mathrm{char}(\mathcal{M})$ the characteristic variety of $\mathcal{M}$.

THEOREM 4.1. *Let $q_\circ = (x_\circ; \pm\sqrt{-1}dx_n; \sqrt{-1}\eta'_\circ \cdot dx') \in \overset{\circ}{T}{}^*_V\widetilde{V}$. Then*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}^2_V)_{q_\circ} = 0, \tag{4.1}$$

*if there exists a positive number $\delta$ such that*

$$\Big\{(z; (\xi' + \sqrt{-1}\varepsilon\eta') \cdot dz' \pm (\xi_n + \sqrt{-1}) \cdot dz_n) \in T^*X; \quad |z - x_\circ| + |\eta' - \eta'_\circ| < \delta, \quad |\mathrm{Im}\, z_n| + |\xi| < \varepsilon\delta\Big\} \cap \mathrm{char}(\mathcal{M}) = \emptyset \tag{4.2}$$

*for any* $\varepsilon$ *with* $0 < \varepsilon < \delta$*.*

REMARK 4.2. In the situation of Theorem 4.1, one gets not only (4.1) but also

$$q_\circ \notin \operatorname{supp}\left(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_V^2)\right) \tag{4.3}$$

if the same condition (4.2) holds.

PROOF OF THEOREM 4.1. We may assume that $q_\circ = (x_\circ; \sqrt{-1}dx_n; \sqrt{-1}\eta'_\circ \cdot dx') \in \overset{\circ}{T}{}^*_V\widetilde{V}$ by a coordinate transformation. Using the fact of Remark 3.5, one has:

$$\begin{aligned}
(4.4)\qquad & \mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_V^2)_{q_\circ} \\
&= \varinjlim_c H^j R\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \left(\mu_G R\varGamma_{H_c}(p^{-1}\mathcal{O}_X)\right)\Big|_{\overset{\circ}{T}{}^*_V\widetilde{V}}[n]\right)_{q_\circ} \\
&= \varinjlim_c H^{j+n}\mu_G R\varGamma_{H_c}(p^{-1}F)_{q_\circ} \\
&\simeq \varinjlim_c H^{j+n} R\varGamma_{\widetilde{Z}\cap H_c}(p^{-1}F)_{(x_\circ,\eta'_\circ)},
\end{aligned}$$

similarly as we did in Proof of Theorem 3.2. Here we set $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ and $\widetilde{Z} = \left\{(z, \xi') \in \widetilde{X}; y_n \le 0\right\}$.

Hence the $j$-th cohomology group (4.4) on $\mathcal{M}$ vanishes at $q_\circ$ for all $j \in \mathbf{Z}$ provided that there exists a positive number $c_0 > 1$ such that

$$R\varGamma_{\widetilde{Z}\cap H_c}(p^{-1}F)_{(x_\circ,\eta'_\circ)} \simeq 0 \tag{4.5}$$

for any $c > c_0$. Therefore it suffices to study a sufficient condition in order that we have (4.5).

On the other hand, we define a real analytic function $f_c$ on $\widetilde{X} = X \times (\mathbf{R}^d \setminus \{0\})$ by

$$f_c(z, \xi') = -c \cdot y_n - \langle y', \xi' \rangle. \tag{4.6}$$

Assume that

$$(x_\circ, \eta'_\circ; df_c(x_\circ, \eta'_\circ)) \notin \mathrm{SS}\left(R\varGamma_{\widetilde{Z}}(p^{-1}F)\right), \tag{4.7}$$

and we find that the vanishing (4.5) holds by the definition of the micro-support and the fact that $f_c(x_\circ, \eta'_\circ) = 0$.

In this way we have been able to reduce the condition on the vanishing of the cohomology groups to that on the micro-support (4.7). It suffices to estimate the micro-support SS $\left(R\Gamma_{\widetilde{Z}}(p^{-1}F)\right)$.

Applying the estimates on the micro-support in Subsection 2.1, we find immediately that the condition (4.7) holds if there exists a positive number $\delta$ such that

$$\begin{aligned}(4.8)\quad &\Big\{(z', z_n + \sqrt{-1}\epsilon t_n; z'^* \cdot dz' + (z_n^* + \sqrt{-1}\frac{t_n^*}{\epsilon}) \cdot dz_n) \in T^*X; \\ &\quad -\delta < t_n < \delta, \quad \text{“}\operatorname{Im} z_n < 0,\ t_n^* = 0\text{”} \quad \text{or} \quad \text{“}\operatorname{Im} z_n = 0,\ 0 \le t_n^* < \delta\text{”}, \\ &\quad |z - x_\circ| + |z'^* - \sqrt{-1}\eta'_\circ| + |z_n^* - \sqrt{-1}c| < \delta\Big\} \cap \operatorname{char}(\mathcal{M}) = \emptyset\end{aligned}$$

for any $\epsilon$ with $0 < \epsilon < \delta$. Note that one can take $\delta$ according to $c$. Here we put $\varepsilon = (\operatorname{Im} z_n^* + \dfrac{t_n^*}{\epsilon})^{-1}$ and $\delta = c^{-2}$. Then by the formula:

$$\text{(4.9)} \qquad \mathrm{SS}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) = \operatorname{char}(\mathcal{M}),$$

we can easily obtain the expression (4.2) as required. $\square$

REMARK 4.3. K. Takeuchi also got the same result as Theorem 4.1 in a different way at the same time.

## 5. Application

In this section, applying the theorem of supports in Section 4 to a class of linear differential equations whose characteristic varieties are Lagrangian, we argue the structure of small 2-microfunction solutions. After having given its solvability in the framework of 2-analytic functions, we also study the solvability in the framework of microfunctions. We follow the notation of Sections 3 and 4.

We set $V$ as in Section 4. Let $\Sigma \subset V$ be the following Lagrangian manifold of $\overset{\circ}{T}{}^*_M X$:

$$\text{(5.1)} \qquad \Sigma := \left\{(x; \sqrt{-1}\xi \cdot dx) \in \overset{\circ}{T}{}^*_M X; x_n = 0,\ \xi' = 0\right\}.$$

We consider a differential operator defined on $M$ with $\sigma(P)(x_\circ^*) = 0$ for a point $x_\circ^* \in \Sigma$. We assume that for some $k \in \mathbf{N}$ and for some $l \in \mathbf{N}$,

$$|\sigma(P)(x;\sqrt{-1}\xi/|\xi|)| \sim (|x_n^k| + |\xi'|/|\xi|)^l, \tag{5.2}$$

in a real neighborhood of a point $x_\circ^* \in \Sigma$.

In a neighborhood of a point $x_\circ^*$ of $\Sigma$, the equation $Pu = f$ is uniquely solvable in the space of microfunctions for any $f \in \mathcal{C}_{M,x^*}$ at any point $x^* \notin \Sigma$, because the principal symbol of $P$ never vanishes there. One cannot apply the theory of Subsection 2.2 to this operator $P$, since $P$ is not partially elliptic along $V$ as in the Bony-Schapira's theory. Hence we consider this operator on the Lagrangian manifold $\Sigma$.

We note that if $k = 1$, Grigis–Schapira–Sjöstrand [2] has given a theorem on the propagation of analytic singularities for the operator which satisfies the property (5.2). This operator is called *transversally elliptic* in a neighborhood of $x_\circ^*$.

First, we will prove the following theorem on small 2-microfunction solutions by using the theorem of supports in Section 4.

THEOREM 5.1. *We take $V$ and $\Sigma$ as above. Let $P$ be a differential operator with analytic coefficients defined on $M$ and set $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$. Assume* (5.2) *for some $k \in \mathbf{N}$ and for some $l \in \mathbf{N}$ in a real neighborhood of $x_\circ^* \in \Sigma$. Then for any $q_\circ \in \overset{\circ}{\pi}{}_V^{-1}(x_\circ^*)$, one has*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_V^2)_{q_\circ} = 0. \tag{5.3}$$

PROOF. Set $x_\circ^* = (x_\circ', 0; \pm\sqrt{-1}dx_n)$ and $q_\circ = (x_\circ', 0; \pm\sqrt{-1}dx_n; \sqrt{-1}\eta_\circ' \cdot dx') \in \overset{\circ}{T}{}_V^*\widetilde{V}$ with $\eta_\circ' \in \mathbf{R}^{n-1} \setminus \{0\}$. Let $m$ be the order of $P$. By the assumption (5.2) it is easy to see that the principal symbol of $P$ is of the form:

$$\sigma(P)(x;\xi) = \sum_{|\alpha|=l} a_\alpha(x;\xi)(\xi')^{\alpha'}(x_n^k)^{\alpha_n} \tag{5.4}$$

in a neighborhood of $(x_\circ', 0) \in M$. Here $\alpha = (\alpha', \alpha_n)$, $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and $a_\alpha(x;\xi)$ is a real analytic function and homogeneous in $\xi$ of degree $m - |\alpha'|$ for $\alpha$.

On the other hand, there exists a positive constant $C$ such that

$$C(|\eta'|+|t|)^l \le \Big|\sum_{|\alpha|=l} a_\alpha(x'_\circ,0;0,\pm1)(\eta')^{\alpha'}t^{\alpha_n}\Big| \tag{5.5}$$

for all $(\eta',t)\in \mathbf{R}^n$ by (5.2).

It suffices to show that

$$\sigma(P)(z;\xi'+\sqrt{-1}\varepsilon\eta',\pm(\xi_n+\sqrt{-1}))\neq 0 \tag{5.6}$$

where

$$|z'-x'_\circ|+|z_n|+|\eta'-\eta'_\circ|<\delta, \quad |\operatorname{Im} z_n|+|\xi|<\varepsilon\delta, \quad 0<\varepsilon<\delta \tag{5.7}$$

for a good small $\delta>0$.

Here note that $\dfrac{z_n^k}{\varepsilon}=\dfrac{x_n^k}{\varepsilon}+o(1)$ as $\delta\to 0$ under the condition (5.7). Then one has:

$$\begin{aligned}
&\frac{1}{\sqrt{-1}^m\varepsilon^l}\,\sigma(P)(z;\xi'+\sqrt{-1}\varepsilon\eta',\pm(\xi_n+\sqrt{-1})) \\
&= \sum_{|\alpha|=l} a_\alpha(z;\varepsilon\eta'-\sqrt{-1}\xi',\pm(1-\sqrt{-1}\xi_n))\Big(\eta'-\sqrt{-1}\frac{\xi'}{\varepsilon}\Big)^{\alpha'}\Big(\frac{z_n^k}{\varepsilon}\Big)^{\alpha_n} \\
&= \sum_{|\alpha|=l}\Big\{a_\alpha(z;\varepsilon\eta'-\sqrt{-1}\xi',\pm(1-\sqrt{-1}\xi_n)) \\
&\qquad\qquad \times\Big(\eta'-\sqrt{-1}\frac{\xi'}{\varepsilon}\Big)^{\alpha'}+o(1)\Big\}\Big(\frac{x_n^k}{\varepsilon}\Big)^{\alpha_n}
\end{aligned} \tag{5.8}$$

as $\delta\to 0$. Setting $t=\dfrac{x_n^k}{\varepsilon}$, we get the inequalities:

$$\begin{aligned}
&\Big|\frac{1}{\sqrt{-1}^m\varepsilon^l}\,\sigma(P)(z;\xi'+\sqrt{-1}\varepsilon\eta',\pm(\xi_n+\sqrt{-1}))\Big| \\
&\ge \Big|\sum_{|\alpha|=l} a_\alpha(x'_\circ,0;0,\pm1)(\eta'_\circ)^{\alpha'}t^{\alpha_n}\Big| - \Big|\sum_{|\alpha|=l} o(1)t^{\alpha_n}\Big| \\
&\ge C(|\eta'_\circ|+|t|)^l-\sum_{|\alpha|=l}|o(1)||t|^{\alpha_n} \\
&> 0,
\end{aligned} \tag{5.9}$$

because of the inequality (5.5) and the fact that

$$\tag{5.10} \begin{cases} a_\alpha(z;\varepsilon\eta' - \sqrt{-1}\xi', \pm(1-\sqrt{-1}\xi_n)) \longrightarrow a_\alpha(x'_\circ, 0; 0, \pm 1), \\ \left(\eta' - \sqrt{-1}\dfrac{\xi'}{\varepsilon}\right)^{\alpha'} \longrightarrow (\eta'_\circ)^{\alpha'}, \end{cases}$$

as $\delta \to 0$. This completes the proof. □

In the situation of Theorem 5.1, we get the following isomorphism in a neighborhood of a point $x^*_\circ \in \Sigma$ in $V$ by the fundamental exact sequence (3.7).

$$\tag{5.11} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}^2_V) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M|_V).$$

This shows that the structure of solutions for $Pu = f$ in $\mathcal{C}_M|_V$ has been reduced to that in $\mathcal{A}^2_V$.

We will show, as another application, the results of solvability in the framework of 2-analytic functions and microfunctions. Hereafter we assume that $n = 2$.

Theorem 5.2. *Let $P$ be the linear differential operator of the form:*

$$\tag{5.12} P = D_{x_1} + a x_2 D_{x_2}$$

*on $M$ with $a \in \mathbf{C} \setminus \mathbf{R}$, $D_{x_j} = \dfrac{\partial}{\partial x_j}$ $(j = 1, 2)$. Then for any microfunction $f \in \mathcal{C}_{M,x^*}$ the following equation*

$$\tag{5.13} Pu = f, \quad u \in \mathcal{C}_{M,x^*}$$

*is solvable at any point $x^* \in \overset{\circ}{T}{}^*_M X$.*

Proof. We may assume that $x^* \in \Sigma = \{x_2 = 0, \xi_1 = 0\}$. In fact, the structure of solutions for the equation (5.13) is trivial outside $\Sigma$. It is enough to prove the next lemma by applying Theorem 5.1 to this case and by the isomorphism (5.11).

Lemma 5.3. *Let $x^*$ be any point of $\Sigma$ and let $P$ be the differential operator in Theorem 5.2. Then for any $f \in \mathcal{A}^2_{V,x^*}$, there exists $u \in \mathcal{A}^2_{V,x^*}$ such that $Pu = f$.*

Proof of Lemma 5.3. We assume that $x^* = x^*_\circ = (x_{\circ,1}, 0; \sqrt{-1}dx_2) \in \Sigma$ for the sake of simplicity. Recall first that $\mathcal{A}^2_V = \mathcal{C}_{\widetilde{V}}|_V$ and that any germ $f(z_1, x_2) \in \mathcal{C}_{\widetilde{V}, x^*_\circ}$ is represented as boundary value of a holomorphic function:

$$f(z_1, x_2) = b_{D_\varepsilon \times U_\varepsilon}(F(z_1, z_2))_{x^*_\circ}. \tag{5.14}$$

Here $D_\varepsilon$ and $U_\varepsilon$ are domains defined by:

$$D_\varepsilon = \{z_1 \in \mathbf{C}; |z_1 - x_{\circ,1}| < \varepsilon\}, \tag{5.15}$$
$$U_\varepsilon = \{z_2 \in \mathbf{C}; |z_2| < \varepsilon, \operatorname{Im} z_2 > 0\}, \tag{5.16}$$

for $\varepsilon > 0$, and $F(z_1, z_2)$ is a holomorphic function in $D_\varepsilon \times U_\varepsilon$.

Secondly, we define domains $U_{\varepsilon,1}$ and $U_{\varepsilon,2}$ in $\mathbf{C}$ by:

$$U_{\varepsilon,1} = \left\{z_2 \in \mathbf{C}; |z_2| < \varepsilon, 0 < \arg z_2 < \frac{3}{2}\pi\right\}, \tag{5.17}$$
$$U_{\varepsilon,2} = \left\{z_2 \in \mathbf{C}; |z_2| < \varepsilon, -\frac{1}{2}\pi < \arg z_2 < \pi\right\}. \tag{5.18}$$

Since $\bigcup_{j=1,2} (D_\varepsilon \times U_{\varepsilon,j})$ is a Stein manifold, one can find $F_j(z_1, z_2) \in \mathcal{O}(D_\varepsilon \times U_{\varepsilon,j})$ for $j = 1, 2$ such that

$$F(z_1, z_2) = F_1(z_1, z_2) + F_2(z_1, z_2) \tag{5.19}$$

in $D_\varepsilon \times U_\varepsilon$ by the solvability of the first Cousin problem. Therefore, when we set $f_j(z_1, x_2) := b_{D_\varepsilon \times U_\varepsilon}(F_j(z_1, z_2))_{x^*_\circ}$ $(j = 1, 2)$, we have:

$$f(z_1, x_2) = f_1(z_1, x_2) + f_2(z_1, x_2). \tag{5.20}$$

Moreover, we define:

$$U_j(z_1, z_2) = \int_{[\alpha_j, z_1]} F_j(\zeta_1, z_2 e^{a(\zeta_1 - z_1)}) d\zeta_1, \tag{5.21}$$

for $j = 1, 2$. Here we choose a point $\alpha_j \in D_\varepsilon$ satisfying

$$\begin{cases} 0 < \operatorname{Im}(a(\alpha_1 - x_{\circ,1})) < \dfrac{1}{2}\pi, \\ -\dfrac{1}{2}\pi < \operatorname{Im}(a(\alpha_2 - x_{\circ,1})) < 0, \end{cases} \tag{5.22}$$

and $[\alpha_j, z_1]$ denotes the integration path

$$(5.23) \qquad [\alpha_j, z_1] : \zeta_1 = \alpha_j + t(z_1 - \alpha_j), \qquad 0 \le t \le 1.$$

Then, this definition (5.21) is well-defined for a sufficiently small $\varepsilon' > 0$ and for any point $(z_1, z_2) \in D_{\varepsilon'} \times U_{\varepsilon'}$, because we have

$$(5.24) \qquad |z_2 e^{a(\zeta_1 - z_1)}| \le |z_2| e^{|a||\alpha_j - z_1|} \le \varepsilon' e^{2|a|\varepsilon} < \varepsilon,$$

and

$$(5.25) \quad \begin{aligned} \arg\{z_2 e^{a(\zeta_1 - z_1)}\} &= \arg z_2 + \operatorname{Im}\{a(\zeta_1 - z_1)\} \\ &= \arg z_2 + (1-t)\operatorname{Im}\{a(\alpha_j - x_{\circ,1}) + a(x_{\circ,1} - z_1)\}. \end{aligned}$$

We can easily ascertain that $U_j(z_1, z_2) \in \mathcal{O}(D_{\varepsilon'} \times U_{\varepsilon'})$ and that $U_j(z_1, z_2)$ satisfies the following differential equation:

$$(5.26) \qquad (D_{z_1} + a z_2 D_{z_2})\, U_j(z_1, z_2) = F_j(z_1, z_2)$$

for $j = 1,\ 2$.

Finally, we define $u_j(z_1, x_2) := b_{D_{\varepsilon'} \times U_{\varepsilon'}}(U_j(z_1, z_2))_{x_\circ^*} \in \mathcal{C}_{\widetilde{V}, x_\circ^*}$ $(j = 1,\ 2)$, and $u(z_1, x_2) = u_1(z_1, x_2) + u_2(z_1, x_2)$. Then one obtains:

$$(5.27) \qquad Pu = Pu_1 + Pu_2 = f_1 + f_2 = f.$$

This completes the proof of Theorem 5.2. □

Corollary 5.4. *For the second order degenerate elliptic operator* $P = D_{x_1}^2 + (x_2 D_{x_2})^2$, *one also obtains the same result of solvability as Theorem 5.2.*

Remark 5.5. In the situation of Theorem 5.2 or Corollary 5.4, we can claim further that

$$(5.28) \qquad \operatorname{Ker}(\mathcal{A}_V^2 \xrightarrow[P]{} \mathcal{A}_V^2) \simeq \operatorname{Ker}(\mathcal{C}_M|_V \xrightarrow[P]{} \mathcal{C}_M|_V)$$

by the isomorphism (5.11). This fact is also familiar by means of an estimate of the support of solution complexes with coefficients in $\mathcal{C}_V^2$.

By this assertion, we can get the following exact sequence for the operator in Theorem 5.2 or Corollary 5.4.

$$0 \longrightarrow \mathcal{A}_V^{2\,P} \longrightarrow \mathcal{C}_M|_V \underset{P}{\longrightarrow} \mathcal{C}_M|_V \longrightarrow 0. \tag{5.29}$$

Here we set $\mathcal{A}_V^{2\,P} = \mathrm{Ker}(\mathcal{A}_V^2 \underset{P}{\longrightarrow} \mathcal{A}_V^2)$.

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