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Averages of Green Functions of Classical Groups

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Abstract. In this paper, we compare the Green functions of $Sp(2n,q)$ and $SO(2n+1,q)$ with those of $GL(n,q^2)$ and find an interesting connection between them. Let $G = Sp_{2n}(\bar{\mathbf{F}}_q)$ or $SO_{2n+1}(\bar{\mathbf{F}}_q)$ and $\bar{G} = GL_n(\bar{\mathbf{F}}_q)$ with Frobenuius map F. The Weyl group W of G is written as $W = DS_n$, where D is an elementary abelian 2-group and S_n is the symmetric group of degree n, which is identified with the Weyl group of $\bar{G}$. Let $Q^G_{T_w}$ be a Green function of G^F where T_w is an F-stable maximal torus of G corresponding to $w \in W$. For $w \in S_n$, we define an average of Green functions $Q^G_{w,D}$ on G^F by $Q^G_{w,D} = |D|^{-1}\sum_{x\in D} Q^G_{T_{wx}}$. Then there exists a natural injection $u_0 \mapsto u$ from the set of unipotent classes of $\bar{G}$ to the set of unipotent classes of G such that the function $Q^G_{w,D}(u)$ on G^F coincides with the Green function $Q^{\bar{G}}_{\bar{T}_w}(u_0)$ on $\bar{G}^{F^2}$.

0. Introduction

Let G be a connected reductive algebraic group defined over $\mathbf{F}_q$, $F : G \to G$ a Frobenius morphism and G^F the finite group of F-fixed points of G. Let T be an F-stable maximal torus of G. Let θ be a character of T over $\bar{\mathbf{Q}}_l$, where l is a prime not dividing q. Deligne and Lusztig have defined a virtual character $R^G_T(\theta)$ of G^F. The character value of $R^G_T(\theta)$ at a unipotent element $u \in G^F$ is independent of θ and thus we can define a Green function Q^G_T on the unipotent elements of G^F by $Q^G_T(u) = R^G_T(\theta)(u)$. The Green functions form an important part of the character table of G^F.

If $G = GL_n(\bar{\mathbf{F}}_q)$ and $G^F = GL(n,q)$ then Green gave a combinatorial method of computing the Green functions of G^F. If $G^F = U(n,q)$ then

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the Green functions are obtained from those of $GL(n,q)$ by the so-called Ennola conjecture, by a simple recipe of changing q to $-q$. If G^F is a symplectic or orthogonal group the Green functions can be computed in principle by an algorithm given originally in [Sh2] and later modified by Lusztig. However, this is a cumbersome method which does not give any insight into the structure of the Green functions.

In this paper we compare the Green functions of $Sp(2n,q)$ and $SO(2n+1,q)$ (q : odd) with those of $GL(n,q^2)$ and find an interesting connection between them. We first define a surjective map f from the set of unipotent classes of $G = Sp_{2n}(\bar{\mathbf{F}}_q)$ or $SO_{2n+1}(\bar{\mathbf{F}}_q)$ onto the set of unipotent classes of $\bar{G} = GL_n(\bar{\mathbf{F}}_q)$. (More precisely, we define this map for the corresponding groups over $\mathbb{C}$.) Let u be an F-stable unipotent element of G, and $C(u) = Z_G(u)/Z_G^0(u)$. We can assume that u is a distinguished element in the sense of [Sh2], so that the classes in G^F which are contained in the class of u in G are parameterized by the elements of $C(u)$. Furthermore, the G^F-conjugacy classes of maximal tori in G are parameterized by elements of the Weyl group W, so that we can denote a set of representatives of these classes by $\{T_w \mid w \in W\}$, where T_1 is a maximally F-split torus of G. We note that we can write $W = DS_n$, where D is an elementary abelian 2-group and S_n, the symmetric group, is the Weyl group of $\bar{G}$. We now fix $u \in G^F$ as above and $w \in S_n$. We then consider the sum

$$|C(u)|^{-1}|D|^{-1}\sum_{x\in D}\sum_{v} Q^G_{T_{wx}}(v)$$

where v runs over the F-stable unipotent elements in the conjugacy class of u in G. In other words, we average the Green functions over the F-fixed points of a unipotent class in G and over the tori T_y such that y maps to a fixed element $w \in S_n$ under the natural map $W \to S_n$. We compare this polynomial in q with the Green function $Q^{\bar{G}}_{T_w}(f(u))$, but considered as a polynomial in q^2, i.e. as a Green function on $\bar{G}^{F^2} = GL(n,q^2)$. Our main result is that for certain good unipotent elements u, these two polynomials are equal. In general, the average Green function on G^F is equal to the Green function on $\bar{G}^{F^2}$ together with some extra terms (which are not computed here). We also remark that in each coset wD of D in W as above, exactly one corresponding torus T_{wx} is anisotropic, and thus exactly one $Q^G_{T_{wx}}$ in our sum is cuspidal. Thus, in principle, assuming we know Harish-

Chandra induction on Green functions, our averages give information on cuspidal functions.

The result is proved by interpreting the Green function as arising from the Springer representation of W on the cohomology of the variety of Borel subgroups of G containing a unipotent element. In fact, we work with the corresponding groups over $\mathbb{C}$ and the varieties of Borel subgroups whose Lie algebras contain a fixed nilpotent element. Then the problem is reduced to showing a connection between the D-fixed points of the cohomology groups of such varieties for the groups $Sp_{2n}(\mathbb{C})$ or $SO_{2n+1}(\mathbb{C})$ and the cohomology groups of corresponding varieties for $GL_n(\mathbb{C})$, where both are regarded as S_n-modules (see Theorem 1.9 and Theorem 1.13).

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1. The Statement of the Results

1.1. Let G be a connected reductive algebraic group defined over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Let $\mathcal{B}$ be the variety of Borel subgroups of G, and for each nilpotent element $A \in \mathfrak{g}$, let $\mathcal{B}_A$ denote the subvariety of $\mathcal{B}$ consisting of all Borel subgroups whose Lie algebra contains A. Let W be the Weyl group of G. We consider the Springer representation of W on the cohomology group $H^i(\mathcal{B}_A) = H^i(\mathcal{B}_A, \mathbb{C})$, which was first constructed by Springer [Sp1], [Sp2], by passing to the groups over $\overline{\mathbb{F}}_q$. Later Lusztig [L1] gave a construction available for both of $\overline{\mathbb{F}}_q$ and $\mathbb{C}$ by making use of the intersection cohomology theory. Let $C(A) = Z_G(A)/Z_G^0(A)$ be the component group of A. Then $C(A)$ acts naturally on $H^i(\mathcal{B}_A)$ and this action of $C(A)$ commutes with that of W. For each $\varphi \in C(A)^\wedge$, we denote by $H^i(\mathcal{B}_A)_\varphi$ the φ-isotypic subspace of $H^i(\mathcal{B}_A)$. Put $d_A = \dim \mathcal{B}_A$. We denote by $\varphi \otimes \chi_{A,\varphi}$ the character of the $C(A) \times W$-module $H^{2d_A}(\mathcal{B}_A)_\varphi$. Then by the Springer correspondence the following holds: $\chi_{A,\varphi}$ is irreducible, and any irreducible character χ of W is expressed as $\chi = \chi_{A,\varphi}$ for a unique pair (A, φ), where A runs over the nilpotent orbits in $\mathfrak{g}$, and $\varphi \in C(A)^\wedge$ is such that $H^{2d_A}(\mathcal{B}_A)_\varphi \neq 0$.

It is known that $H^i(\mathcal{B}_A) = 0$ if i is odd. It is also known that $H^{2i}(\mathcal{B}_A)_\varphi = 0$ for any $i \geq 0$ if $H^{2d_A}(\mathcal{B}_A)_\varphi = 0$, (see for example, [Sh3]).

1.2. From now on we assume that $G = Sp_{2n}$ or $G = SO_{2n+1}$. Then W is the Weyl group of type C_n, and is isomorphic to $S_n \ltimes D$, where S_n is the symmetric group of degree n and $D \simeq (\mathbb{Z}/2\mathbb{Z})^n$. Note that $W = W_n$ is realized as the group of signed permutations of n letters $\{1, 2, \dots, n\}$. Let r_i be the reflection of W which permutes i and $-i$ and leaves the other letters invariant. Then D is the subgroup of W generated by $r_1, \dots, r_n$. We consider the subspace $H^i(\mathcal{B}_A)^D$ of $H^i(\mathcal{B}_A)$ consisting of D-invariant vectors. Then $H^i(\mathcal{B}_A)^D$ has a structure of an S_n-module. We also consider the subspace of $H^i(\mathcal{B}_A)$ consisting of $C(A)$-invariant vectors, which we denote by $H^i(\mathcal{B}_A)_1$ as in 1.1. We put $H^i(\mathcal{B}_A)_1^D = H^i(\mathcal{B}_A)^D \cap H^i(\mathcal{B}_A)_1$. Then $H^i(\mathcal{B}_A)_1^D$ also has a structure of an S_n-module.

Let $W' = W_{n-1}$ be the parabolic subgroup of W of type C_{n-1}. We write $W' = S_{n-1} \ltimes D'$, where D' is the subgroup of D generated by $r_1, \dots, r_{n-1}$. First we show the following lemma.

Lemma 1.3.

$$H^{2i}(\mathcal{B}_A)_1^D = \begin{cases} H^{2i}(\mathcal{B}_A)_1^{D'} & \text{if } i : \textit{even}, \\ 0 & \text{if } i : \textit{odd}. \end{cases}$$

Proof. For each $\chi \in W^\wedge$, we define the parity $p(\chi) = \pm 1$ by the condition that $p(\chi) = 0$ (resp. $p(\chi) = 1$) if $\chi(-1) = \chi(1)$ (resp. $\chi(-1) = -\chi(1)$). For each $\chi \in W^\wedge$ and $\varphi \in C(A)^\wedge$, we consider the $\varphi \otimes \chi$-isotypic subspace $H^{2i}(\mathcal{B}_A)_{\varphi\otimes\chi}$ of $H^{2i}(\mathcal{B}_A)$. Then by Spaltenstein [S1], the following formula holds.

(1.3.1) Assume that $H^{2i}(\mathcal{B}_A)_{\varphi\otimes\chi} \neq 0$. Then we have

$$i \equiv d_A + p(\chi) + p(\chi_{A,\varphi}) \pmod 2.$$

If we take $\varphi = 1 \in C(A)^\wedge$ and $\chi = 1_W \in W^\wedge$, then $H^0(\mathcal{B}_A)_{\varphi\otimes\chi} \neq 0$ since $H^0(\mathcal{B}_A) \simeq \mathbb{C}$ is the trivial $C(A) \times W$-module. This implies, by (1.3.1), that $p(\chi_{A,1}) \equiv d_A \pmod 2$. Hence for any $\chi \in W^\wedge$ such that $H^{2i}(\mathcal{B}_A)_{1\otimes\chi} \neq 0$, we see that $p(\chi) \equiv i \pmod 2$. We now consider $H^{2i}(\mathcal{B}_A)_1$. First assume that i is even. Then for any $\chi \in W^\wedge$ such that $H^{2i}(\mathcal{B}_A)_{1\otimes\chi} \neq 0$, we have $p(\chi) = 0$ and so $\chi(1) = \chi(-1)$. This means that the central element -1 acts trivially on $H^{2i}(\mathcal{B}_A)_1$. Under the realization of W given in 1.2,

$-1 = r_1 r_2 \cdots r_n \in D$. Since $r_1 r_2 \cdots r_{n-1} \in D'$, we see that r_n acts trivially on $H^{2i}(\mathcal{B}_A)_1^{D'}$. This proves the first assertion of the lemma. Next assume that i is odd. Then the similar argument as before shows that -1 acts as a scalar multiplication by -1 on $H^{2i}(\mathcal{B}_A)_1$ if it is non zero. Since $-1 \in D$, this implies that $H^{2i}(\mathcal{B}_A)_1^D = 0$. The lemma follows from this. □

1.4. Let $\bar{G} = GL_n$. We denote objects associated with $\bar{G}$ as $\overline{\mathcal{B}}, \bar{\mathfrak{g}}$, etc. For any nilpotent element $A' \in \bar{\mathfrak{g}}$, $H^i(\overline{\mathcal{B}}_{A'})$ has a structure of S_n-module. In what follows, we shall compare the S_n-module structures for suitable $H^{4i}(\mathcal{B}_A)_1^D$ and $H^{2i}(\overline{\mathcal{B}}_{A'})$. We perform this by defining a map $f : A \mapsto A'$ from the set of nilpotent orbits of $\mathfrak{g}$ to the set of nilpotent orbits in $\bar{\mathfrak{g}}$.

Let $\mathcal{N}_{\mathfrak{g}}$ be the set of nilpotent orbits in $\mathfrak{g}$. We shall describe the set $\mathcal{N}_{\mathfrak{g}}$. First assume that $G = Sp_{2n}$. Then via the Jordan normal form, $\mathcal{N}_{\mathfrak{g}}$ is in bijection with the set $\mathcal{P} = \mathcal{P}_{2n}$ of partitions $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ of $2n$ (i.e., $\sum i \cdot m_i = 2n$) such that m_i is even for odd i. Next assume that $G = SO_{2n+1}$. Then $\mathcal{N}_{\mathfrak{g}}$ is in bijection with the set $\mathcal{P}' = \mathcal{P}'_{2n+1}$ of partitions $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ of $2n+1$ such that m_i is even for even i. Finally, in the case where $\bar{G} = GL_n$, the set $\mathcal{N}_{\bar{\mathfrak{g}}}$ of nilpotent orbits in $\bar{\mathfrak{g}}$ is in bijective correspondence with the set $\overline{\mathcal{P}} = \overline{\mathcal{P}}_n$ of partitions λ of n.

1.5. By making use of the Springer correspondence, we define a map $f : \mathcal{N}_{\mathfrak{g}} \to \mathcal{N}_{\bar{\mathfrak{g}}}$ as follows. First note that the irreducible characters of W are parameterized by the pairs of partitions $(\alpha; \beta)$, where $\alpha : \alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_r \ge 0$ and $\beta : \beta_1 \ge \beta_2 \ge \cdots \ge \beta_s \ge 0$ with $\sum \alpha_i + \sum \beta_j = n$. We denote this set by $\mathcal{P}^\natural = \mathcal{P}^\natural_n$. By adding 0 to the sequence α or β, we may assume that $r = s$. Let us denote by $\chi_{(\alpha;\beta)}$ the irreducible character of W corresponding to $(\alpha; \beta)$. Note that under this correspondence, $(n; -)$ corresponds to the unit character 1_W and $(-; 1^n)$ corresponds to the sign character ε of W.

Now the Springer correspondence gives an injective map $\mathcal{N}_{\mathfrak{g}} \to W^\wedge$ by $A \mapsto \chi_{A,1}$. Then $\chi_{A,1}$ is expressed as $\chi_{(\alpha;\beta)}$ for some $(\alpha; \beta) \in \mathcal{P}^\natural$. We define a sequence of integers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r$ by $\lambda_1 = \alpha_1 + \beta_1$, $\lambda_2 = \alpha_2 + \beta_2, \dots$. Then $\lambda : \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r$ gives rise to a partition of n. We put $f(A) = A'$ where $A' \in \mathcal{N}_{\bar{\mathfrak{g}}}$ is the nilpotent orbit corresponding to the partition λ. Thus the map $f : \mathcal{N}_{\mathfrak{g}} \to \mathcal{N}_{\bar{\mathfrak{g}}}$ is defined. By abuse of notation, we regard the map f as the corresponding map $\mathcal{P} \to \overline{\mathcal{P}}$ or $\mathcal{P}' \to \overline{\mathcal{P}}$ induced from f.

We shall describe the map f more explicitly using the description of the Springer correspondence for classical groups given in [Sh2], (see also [L2]).

(a) The case $G = Sp_{2n}$.

Assume that $\lambda = (1^{m_1}, 2^{m_2}, \dots) \in \mathcal{P}$. We express the sequence λ in the decreasing order as $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots$. Let us define a sequence $\{a_1, a_2, \dots ,\}$ by the following rule. If $\lambda_i = 2k$, we put $a_i = k$. If $\lambda_i = 2k+1$, and if it is expressed as

$$\lambda_{i-1} > \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+2r-1} > \lambda_{i+2r}$$

for some $r > 0$, we put $a_i = a_{i+2} = \cdots = a_{i+2r-2} = k$ and $a_{i+1} = a_{i+3} = \cdots = a_{i+2r-1} = k+1$. Then we have $a_1 \geq a_3 \geq \cdots$ and $a_2 \geq a_4 \geq \cdots$. We now define a pair of partitions $(\alpha;\beta) \in \mathcal{P}^\natural$ by

$$\alpha : a_1 \geq a_3 \geq \cdots , \quad \beta : a_2 \geq a_4 \geq \cdots .$$

Then we have $\chi_{A,1} = \chi_{(\alpha;\beta)}$. Hence the partition $f(\lambda) \in \overline{\mathcal{P}}$ is given by

$$f(\lambda) = (a_1 + a_2, a_3 + a_4, \dots),$$

by adding 0 on the end of the sequence of a_i, if necessary.

(b) The case $G = SO_{2n+1}$.

As in the case (a), we express $\lambda = (1^{m_1}, 2^{m_2}, \dots) \in \mathcal{P}'$ in the decreasing order as $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots$. We define a sequence $\{a_1, a_2, \dots\}$ by the following rule. Assume that $\lambda_i = 2k+1$ for some $k \geq 0$. We put

$$a_i = \begin{cases} k & \text{if } i : \text{odd}, \\ k+1 & \text{if } i : \text{even}. \end{cases}$$

Assume that $\lambda_i = 2k$ for some $k \geq 0$, and that it is expressed as

$$\lambda_{i-1} > \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+2r-1} > \lambda_{i+2r}$$

for some $r > 0$. We put $a_i = a_{i+2} = \cdots = a_{i+2r-2} = k-1$, and $a_{i+1} = a_{i+3} = \cdots = a_{i+2r-1} = k+1$ if i is odd, and put $a_i = a_{i+1} = \cdots = a_{i+2r-1} =$

k if i is even. Then we have $a_1 \ge a_3 \ge \cdots$, and $a_2 \ge a_4 \ge \cdots$. We define a pair of partitions $(\alpha;\beta) \in \mathcal{P}^\natural$ by

$$\alpha : a_1 \ge a_3 \ge \cdots, \quad \beta : a_2 \ge a_4 \ge \cdots .$$

Then we have $\chi_{A,1} = \chi_{(\alpha:\beta)}$. Hence the partition $f(\lambda) \in \overline{\mathcal{P}}$ is given by

$$f(\lambda) = (a_1 + a_2, a_3 + a_4, \dots).$$

1.6. We define subsets $\mathcal{P}_{\mathrm{ev}} \subset \mathcal{P}$ and $\mathcal{P}'_{\mathrm{ev}} \subset \mathcal{P}'$ as follows.

$$\begin{aligned}\mathcal{P}_{\mathrm{ev}} &= \{\lambda = (1^{m_1}, 2^{m_2}, \dots) \in \mathcal{P} \mid m_i : \text{ even for } i \ge 1\},\\ \mathcal{P}'_{\mathrm{ev}} &= \{\lambda = (1^{m_1}, 2^{m_2}, \dots) \in \mathcal{P}' \mid m_i : \text{ even for } i > 1\}.\end{aligned}$$

Note that m_1 is always odd for $\lambda \in \mathcal{P}'_{\mathrm{ev}}$. We denote by $(\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$ the set of $\mathcal{N}_{\mathfrak{g}}$ corresponding to $\mathcal{P}_{\mathrm{ev}}$ or $\mathcal{P}'_{\mathrm{ev}}$, respectively. From the description of the map f given in 1.5, it is then easy to see the following.

(1.6.1) For each $\lambda = (1^{m_1}, 2^{m_2}, \dots) \in \mathcal{P}_{\mathrm{ev}}$ (resp. $\lambda = (1^{m_1+1}, 2^{m_2}, \dots) \in \mathcal{P}'_{\mathrm{ev}}$), $f(\lambda) \in \overline{\mathcal{P}}$ is given by $f(\lambda) = (1^{m_1/2}, 2^{m_2/2}, \dots)$. Hence the restriction of the map f on $(\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$ gives a bijection $(\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}} \simeq \mathcal{N}_{\overline{\mathfrak{g}}}$. In particular, $f : \mathcal{N}_{\mathfrak{g}} \to \mathcal{N}_{\overline{\mathfrak{g}}}$ is surjective.

Here we give some examples of the map f for small rank cases. In the following tables, the first column denotes the elements in $\mathcal{N}_{\mathfrak{g}}$, where the asterisk indicates the elements in $(\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$.

Table 1. $G = Sp_6, \quad \bar{G} = GL_3$.

A		$\chi_{A,1}$	$f(A)$
1^6	*	$(-;1^3)$	1^3
21^4		$(1^3;-)$	1^3
2^21^2	*	$(1;1^2)$	21
41^2		$(21;-)$	21
2^3		$(1^2;1)$	21
3^2	*	$(1;2)$	3
42		$(2;1)$	3
6		$(3;-)$	3

Table 2. $G = SO_7$, $\bar{G} = GL_3$.

A		$\chi_{A,1}$	$f(A)$
1^7	$*$	$(-;1^3)$	1^3
2^21^3	$*$	$(-;21)$	21
31^4		$(1;1^2)$	21
32^2		$(1^2;1)$	21
3^21	$*$	$(1;2)$	3
51^2		$(2;1)$	3
7		$(3;-)$	3

1.7. We are interested in comparing the S_n-module structures of $H^{4i}(\mathcal{B}_A)_1^D$ and $H^{2i}(\overline{\mathcal{B}}_{f(A)})$. Note that $H^j(\mathcal{B}_A)_1^D = 0$ unless $j \equiv 0 \pmod 4$ by Lemma 1.3. First we consider the special case where $A = 0$, and show that there exists a natural isomorphism of S_n-modules $\theta_0 : H^{2i}(\overline{\mathcal{B}}) \xrightarrow{\sim} H^{4i}(\mathcal{B})^D$. We consider a polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ on which W acts as $w(x_i) = \pm x_j$ if $w(i) = \pm j$ as a signed permutation of $\{1, 2, \dots, n\}$. Then we have a surjective W-equivariant homomorphism $\alpha : \mathbb{C}[x_1, \dots, x_n] \to H^*(\mathcal{B})$ where the kernel J is the ideal generated by non-constant homogeneous W-invariant polynomials. We claim that the image of $\mathbb{C}[x_1^2, \dots, x_n^2]$ under the map α coincides with $H^*(\mathcal{B})^D$. In fact, let $J_0 = \mathbb{C}[x_1^2, \dots, x_n^2] \cap J$. By applying the average operator $|D|^{-1}\sum_{w \in D} w$ on J_0, we see that J_0 is the ideal of $\mathbb{C}[x_1^2, \dots, x_n^2]$ generated by non-constant homogeneous S_n-invariant polynomials. Hence we have

$$\alpha(\mathbb{C}[x_1^2, \dots, x_n^2]) \simeq \mathbb{C}[x_1^2, \dots, x_n^2]/J_0 \simeq H^*(\overline{\mathcal{B}}).$$

But since $H^*(\mathcal{B})$ is a regular W-module, we see that $\dim H^*(\mathcal{B})^D = |S_n| = \dim H^*(\overline{\mathcal{B}})$. The claim follows from this.

We now consider a similar S_n-equivariant surjective map $\overline{\alpha} : \mathbb{C}[x_1, \dots, x_n] \to H^*(\overline{\mathcal{B}})$. Thanks to the above claim, one can construct an isomorphism $\theta_0 : H^*(\overline{\mathcal{B}}) \xrightarrow{\sim} H^*(\mathcal{B})^D$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{C}[x_1, \dots, x_n] & \xrightarrow{\widetilde{\theta}} & \mathbb{C}[x_1^2, \dots, x_n^2] \\ {\scriptstyle\overline{\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ H^*(\overline{\mathcal{B}}) & \xrightarrow{\theta_0} & H^*(\mathcal{B})^D, \end{array} \tag{1.7.1}$$

where $\widetilde{\theta}$ is the isomorphism defined by $x_i \mapsto x_i^2$.

1.8. The natural inclusion $\mathcal{B}_A \hookrightarrow \mathcal{B}$ induces a graded algebra homomorphism $\phi = \phi_A : H^*(\mathcal{B}) \to H^*(\mathcal{B}_A)$. Clearly this map is $C(A)$-equivariant, where $C(A)$ acts trivially on $H^*(\mathcal{B})$. Also it is known (e.g. Spaltenstein [S2, Lemma 2.5]) that ϕ is W-equivariant. Hence ϕ induces an S_n-equivariant map $H^i(\mathcal{B})^D \to H^i(\mathcal{B}_A)_1^D$, which we denote by ϕ_D. We denote by $\overline{\phi}$ the similar map for $\bar{G}$ induced from the inclusion $\overline{\mathcal{B}}_{A'} \hookrightarrow \overline{\mathcal{B}}$ for $A' \in \mathcal{N}_{\bar{\mathfrak{g}}}$. We can now state our main result.

Theorem 1.9. *Let $G = Sp_{2n}$ or SO_{2n+1}, and put $\bar{G} = GL_n$. Let $f : \mathcal{N}_{\mathfrak{g}} \to \mathcal{N}_{\bar{\mathfrak{g}}}$ be the map defined in 1.5. Then for each $A \in \mathcal{N}_{\mathfrak{g}}$, there exists a unique S_n-equivariant map $\theta : H^{2i}(\overline{\mathcal{B}}_{f(A)}) \to H^{4i}(\mathcal{B}_A)_1^D$ such that the following diagram commutes.*

$$\begin{array}{ccc} H^{2i}(\overline{\mathcal{B}}) & \xrightarrow{\theta_0} & H^{4i}(\mathcal{B})^D \\ {\scriptstyle\overline{\phi}}\big\downarrow & & \big\downarrow{\scriptstyle\phi_D} \\ H^{2i}(\overline{\mathcal{B}}_{f(A)}) & \xrightarrow{\theta} & H^{4i}(\mathcal{B}_A)_1^D \end{array} \tag{1.9.1}$$

Moreover, the map θ is injective. Hence, θ gives an isomorphism $H^{2i}(\overline{\mathcal{B}}_{f(A)}) \xrightarrow{\sim} \operatorname{Im}\phi_D$ as S_n-modules.

The proof of the theorem will be given in Sections 2 and 3. The following special case would be worth mentioning.

Corollary 1.10. *Assume that $A \in (\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$. Then we have*

$$H^{2i}(\mathcal{B}_A)^D \simeq \begin{cases} H^i(\overline{\mathcal{B}}_{f(A)}) & \text{if } i : \text{even}, \\ 0 & \text{if } i : \text{odd} \end{cases}$$

as S_n-modules. Moreover, $C(A)$ acts trivially on $H^{4i}(\mathcal{B}_A)^D$ and the map $H^{4i}(\mathcal{B})^D \to H^{4i}(\mathcal{B}_A)^D$ induced from ϕ is surjective.

Proof. Let θ be the map given in the theorem. We shall show that $\operatorname{Im}\theta$ coincides with $H^{4i}(\mathcal{B}_A)^D$. Since θ is injective, we have the following inequalities,

$$\begin{aligned} \sum_{i\ge 0} \dim H^{2i}(\overline{\mathcal{B}}_{f(A)}) &\le \sum_{i\ge 0} \dim H^{4i}(\mathcal{B}_A)_1^D \\ &\le \sum_{j\ge 0} \dim H^{2j}(\mathcal{B}_A)^D. \end{aligned} \tag{1.10.1}$$

Since $A \in (\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$, A is a regular nilpotent element in the Lie algebra of a Levi subgroup L of a parabolic subgroup of G such that the corresponding Weyl group W_L is given as $W_L \simeq S_{\lambda_1} \times \cdots \times S_{\lambda_k}$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a partition of n corresponding to $f(A)$ in $\mathcal{N}_{\overline{\mathfrak{g}}}$. Now by Alvis-Lusztig [AL], we see that the cohomology algebra $H^*(\mathcal{B}_A)$ is isomorphic to $\mathrm{Ind}_{W_L}^W 1$ as W-modules. Hence

$$\dim H^*(\mathcal{B}_A)^D = \langle \mathrm{Ind}_{W_L}^W 1, \mathrm{Ind}_D^W 1 \rangle_W = |S_n|/|W_L|.$$

But by [HS], we see also that $H^*(\overline{\mathcal{B}}_{f(A)})$ is isomorphic to $\mathrm{Ind}_{W_L}^{S_n} 1$ as S_n-modules. This implies that

$$\dim H^*(\mathcal{B}_A)^D = \dim H^*(\overline{\mathcal{B}}_{f(A)}),$$

and so the inequalities in (1.10.1) are actually equalities. Hence we have $\operatorname{Im}\theta = H^{4i}(\mathcal{B}_A)^D$ as asserted.

The above argument also shows that $H^{4i+2}(\mathcal{B}_A)^D = 0$ and $H^{4i}(\mathcal{B}_A)^D = H^{4i}(\mathcal{B}_A)^D_1$. The first statement of the lemma follows from this. Now it is known by Spaltenstein (see [HS]) that the map $\overline{\phi}$ is always surjective. The second statement follows from this by using the theorem. $\square$

Remark 1.11. As remarked in the proof of Corollary 1.10, the map $\overline{\phi} : H^*(\overline{\mathcal{B}}) \to H^*(\overline{\mathcal{B}}_A)$ is surjective for any $A \in \mathcal{N}_{\overline{\mathfrak{g}}}$ in the case of GL_n. According to de Concini-Procesi [CP] and Tanisaki [T], this map is interpreted as follows; let $\mathcal{O}$ be the nilpotent orbit in $\overline{\mathfrak{g}}$ containing an element $A^\vee$ which corresponds to the dual partition of A, and let $\overline{\mathcal{O}}$ be its closure in $\overline{\mathfrak{g}}$. We consider the coordinate ring $\mathbb{C}[\mathfrak{t} \cap \overline{\mathcal{O}}]$ of the scheme theoretic intersection of $\overline{\mathcal{O}}$ with a Cartan subalgebra $\mathfrak{t}$ in $\overline{\mathfrak{g}}$. Then $\mathbb{C}[\mathfrak{t} \cap \overline{\mathcal{O}}]$ affords a structure of graded S_n-module which is isomorphic to $H^*(\overline{\mathcal{B}}_A)$. Moreover, the map $\overline{\phi}$ coincides with the natural surjection $\mathbb{C}[\mathfrak{t} \cap \overline{\mathfrak{g}}_{\mathrm{nil}}] \to \mathbb{C}[\mathfrak{t} \cap \overline{\mathcal{O}}]$, where $\overline{\mathfrak{g}}_{\mathrm{nil}}$ denotes the nilpotent variety of $\overline{\mathfrak{g}}$.

The similar construction of graded W-module $\mathbb{C}[\mathfrak{t} \cap \overline{\mathcal{O}}]$ is also available for other cases. In fact, Tanisaki [T] showed, in the case of Sp_{2n}, that for any $A \in (\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$, the W-module $\mathbb{C}[\mathfrak{t} \cap \overline{\mathcal{O}}]$ is isomorphic to $\mathrm{Ind}_{W_L}^W 1$. Here $\mathfrak{t}$ is a Cartan subalgebra in $\mathfrak{g}$ and $\overline{\mathcal{O}}$ is the closure of the nilpotent orbit in $\mathfrak{g}$ containing $A^\vee$, where $A^\vee$ is the element corresponding to the dual partition of A (regarded as an element in $\mathfrak{gl}_{2n}$). Note, for any $A \in (\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$, that $A^\vee$ also belongs to $\mathcal{N}_{\mathfrak{g}}$. W_L is the parabolic subgroup of W as in the proof of Corollary 1.10. We also have $\mathbb{C}[\mathfrak{t} \cap \mathfrak{g}_{\mathrm{nil}}] \simeq H^*(\mathcal{B})$ as graded W-modules. ($\mathfrak{g}_{\mathrm{nil}}$ denotes the nilpotent variety in $\mathfrak{g}$.) On the other hand, we have $H^*(\mathcal{B}_A) \simeq \mathrm{Ind}_{W_L}^W 1$ by [AL] as before. However, the natural map $H^*(\mathcal{B}) \to H^*(\mathcal{B}_A)$ is in general not surjective. In fact, the smallest example is that of $G = Sp_{12}$ and $A = (4^2, 2^2) \in (\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$. In this case, the W-module $H^{2d_A}(\mathcal{B}_A)$ is equal to $\chi_{(21;21)} + \chi_{(1^2;31)}$, and the latter component corresponds to a non-trivial character of $C(A)$. Hence $C(A)$ acts non-trivially on $H^*(\mathcal{B}_A)$. The corollary suggests, even in this case, that one will recover a natural isomorphism $\mathbb{C}[\mathfrak{t} \cap \overline{\mathcal{O}}]^D \simeq H^*(\mathcal{B}_A)^D$ once we restrict to the D-fixed point subspaces.

1.12. We now pass to the setting in the Introduction, namely we consider the groups defined over a finite field $\mathbb{F}_q$. We assume that q is odd. Then the set $\mathcal{N}_G$ of unipotent classes in G has the identical parameterization with $\mathcal{N}_{\mathfrak{g}_\mathbb{C}}$, where $\mathfrak{g}_\mathbb{C}$ is the corresponding Lie algebra over $\mathbb{C}$. For each unipotent class $C \in \mathcal{N}_G$, we fix a split representative $u_1 \in C^F$ (see [Sh2], where it is called a distinguished element). Then the representatives in the G^F-conjugacy classes in C^F are in one to one correspondence with $C(u_1) \simeq C(A)$. (Here A is a nilpotent element corresponding to u_1. Note that $C(A)$ is abelian in our case.) We denote by u_c the representative corresponding to $c \in C(A)$. For each $w \in W$, let T_w be an F-stable maximal torus of G obtained from the split maximal torus T by twisting by w. We consider the Green functions $Q_{T_w}^G$ associated to T_w. It is known by [L3], [Sh3], that the values at u_c of Green functions can be interpreted as

$$Q_{T_w}^G(u_c) = \sum_{i \ge 0} \mathrm{Tr}((w,c), H^{2i}(\mathcal{B}_A))q^i, \tag{1.12.1}$$

where $H^{2i}(\mathcal{B}_A)$ is regarded as a $W \times C(A)$-module.

We take $A \in (\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$ and let C be the corresponding unipotent class in G. For each $u \in C^F$, we consider the average of Green functions over D as follows; for each $w \in S_n$, let

$$Q_{u,D}^G(w) = |D|^{-1} \sum_{x \in D} Q_{T_{wx}}^G(u).$$

We fix $u \in C^F$, and regard $Q_{u,D}^G$ as a class function on S_n. Then by Corollary 1.10, we have

$$Q_{u,D}^G(w) = \sum_{i \ge 0} \mathrm{Tr}(w, H^{4i}(\mathcal{B}_A)^D)q^{2i}.$$

In particular, $Q^G_{u,D}$ does not depend on the choice of $u \in C^F$. By this formula $Q^G_{u,D}$ may be regarded as a polynomial in q for each $w \in S_n$, which we denote by $Q^G_{u,D}(q)(w)$.

Let $f(u)$ be a unipotent class in $\bar{G}$ corresponding to a nilpotent element $f(A) \in \mathcal{N}_{\bar{\mathfrak{g}}_{\mathbf{C}}}$. The similar formula as (1.12.1) holds for the Green functions of $\bar{G}$, and one can write, for each $w \in S_n$,

$$Q^{\bar{G}}_{T_w}(f(u)) = \sum_{i \geq 0} \mathrm{Tr}(w, H^{2i}(\overline{\mathcal{B}}_{f(A)}))q^i.$$

We regard $Q^{\bar{G}}_{T_w}(f(u))$ as a class function on S_n by fixing $f(u)$, and denote it as $Q^{\bar{G}}_{f(u)}(q)(w)$, as a polynomial in q. Then the following theorem is an immediate consequence of Corollary 1.10.

THEOREM 1.13. *Assume that $A \in (\mathcal{N}_{\mathfrak{g}_{\mathbf{C}}})_{\mathrm{ev}}$. Then we have*

$$Q^G_{u,D}(q) = Q^{\bar{G}}_{f(u)}(q^2).$$

2. The Construction of θ

In this section, we shall construct the map θ, i.e., we prove the following proposition.

PROPOSITION 2.1. *Under the assumption of Theorem 1.9, there exists a unique S_n-equivariant map $\theta : H^{2i}(\overline{\mathcal{B}}_{f(A)}) \to H^{4i}(\mathcal{B}_A)^D$ satisfying the commutative diagram* (1.9.1).

2.2. The injectivity of θ will be proved in Section 3. Note, since the map $\overline{\phi}$ is surjective, the uniqueness of the map θ will follow once we construct θ. So the remaining part of this section is devoted to the construction of θ. As was discussed in Remark 1.11, de Concini-Procesi [CP] and Tanisaki [T] showed that the cohomology ring $H^*(\overline{\mathcal{B}}_{A'})$ is isomorphic to $\mathbb{C}[\mathfrak{t} \cap \overline{\mathcal{O}}]$ as graded S_n-modules in the case of $\bar{G} = GL_n$. The essential step in their proof is to construct an S_n-equivariant map from $\mathbb{C}[\mathfrak{t} \cap \overline{\mathcal{O}}]$ to $H^*(\overline{\mathcal{B}}_{A'})$ commuting with the isomorphism $\mathbb{C}[\mathfrak{t} \cap \bar{\mathfrak{g}}_{\mathrm{nil}}] \simeq H^*(\overline{\mathcal{B}})$. Our strategy is quite similar to theirs, in particular, to that of [CP]. In our discussion, the role of $\mathbb{C}[\mathfrak{t} \cap \overline{\mathcal{O}}]$ is replaced by $H^*(\overline{\mathcal{B}}_{f(A)})$. Following [CP], we reduce the problem of the construction to the case where A is of special type. First we show the following lemma.

LEMMA 2.3. *Let $A, A' \in \mathcal{N}_{\mathfrak{g}}$ and assume that A is contained in the closure $\overline{\mathcal{O}}_{A'}$ of the G-orbit $\mathcal{O}_{A'}$ of A'. Then there exists a W-equivariant map $\phi_{A,A'} : H^i(\mathcal{B}_A) \to H^i(\mathcal{B}_{A'})$ such that the following diagram commutes.*

$$\begin{tikzcd} H^i(\mathcal{B}) \arrow[r, "\phi_{A'}"] \arrow[d, "\phi_A"'] & H^i(\mathcal{B}_{A'}) \\ H^i(\mathcal{B}_A) \arrow[ur, "\phi_{A,A'}"'] & \end{tikzcd}$$

PROOF. We may assume that A and A' are adjacent with respect to the closure relations, i.e., there exists no A'' such that $\overline{\mathcal{O}}_A \subsetneq \overline{\mathcal{O}}_{A''} \subsetneq \overline{\mathcal{O}}_{A'}$. Let A^- be a nilpotent element in $\mathfrak{g}$ such that the triple $\{A, H, A^-\}$ satisfies the relation $[H, A] = 2A, [H, A^-] = -2A^-$ and $[A, A^-] = H$. By Slodowy [Slo], $S = A + Z_{\mathfrak{g}}(A^-)$ is a transversal slice in $\mathfrak{g}$ to $\mathcal{O}_A$. Hence $S \cap \overline{\mathcal{O}}_{A'}$ is also a transversal slice in $\overline{\mathcal{O}}_{A'}$ to $\mathcal{O}_A$. Thus the natural map $\varphi : G \times (S \cap \overline{\mathcal{O}}_{A'}) \to \overline{\mathcal{O}}_{A'}$ is a smooth map. Since $\overline{\mathcal{O}}_{A'}$ is irreducible, φ is dominant. Now assume that $S \cap \mathcal{O}_{A'} = \emptyset$. Since any nilpotent element $A'' \in S$ has the property that $A \in \overline{\mathcal{O}}_{A''}$, our assumption implies that $S \cap \overline{\mathcal{O}}_A = \{A\}$. Then the image of φ is contained in $\overline{\mathcal{O}}_A$. But this contradicts the fact that φ is dominant.

Now we may assume that $A' \in S$. We recall the construction of Springer representations due to Lusztig [L1]. Consider the Grothendieck map $\rho : \widetilde{\mathfrak{g}} \to \mathfrak{g}$, where

$$\widetilde{\mathfrak{g}} = \{(x, gB) \in \mathfrak{g} \times G/B \mid \mathrm{Ad}(g^{-1})x \in LieB\}, \qquad \rho(x, gB) = x.$$

Then the complex $\mathbb{R}\rho_*\mathbb{C}$ is a perverse sheaf on $\mathfrak{g}$ (up to shift) which admits a W-action. Hence for each $x \in \mathfrak{g}$, the stalk at x of i-th cohomology sheaf $\mathcal{H}^i_x(\mathbb{R}\rho_*\mathbb{C})$, which is isomorphic to $H^i(\mathcal{B}_x)$, turns out to be a W-module. Now since $\mathbb{R}\rho_*\mathbb{C}$ is W-equivariant, we have a commutative diagram of hypercohomology,

(2.3.1)
$$\begin{tikzcd} \mathbb{H}^i(\mathfrak{g}, \mathbb{R}\rho_*\mathbb{C}) \arrow[r] \arrow[d] & \mathbb{H}^i(\{A'\}, \mathbb{R}\rho_*\mathbb{C}) \\ \mathbb{H}^i(S, \mathbb{R}\rho_*\mathbb{C}) \arrow[ur] & \end{tikzcd}$$

induced from the inclusions $\{A'\} \subset S \subset \mathfrak{g}$. Note that all the maps are W-equivariant. On the one hand, we have

$$\mathbb{H}^i(\mathfrak{g}, \mathbb{R}\rho_*\mathbb{C}) \simeq H^i(\widetilde{\mathfrak{g}}, \mathbb{C}) \simeq H^i(\mathcal{B})$$

as W-modules, since $\widetilde{\mathfrak{g}}$ is a vector bundle over $\mathcal{B}$. On the other hand, by considering the $\mathbb{G}_m$-action on S, we see that

$$\mathbb{H}^i(S, \mathbb{R}\rho_*\mathbb{C}) \simeq \mathcal{H}^i_A(\mathbb{R}\rho_*\mathbb{C}) \simeq H^i(\mathcal{B}_A)$$

as W-modules (cf. Kazhdan-Lusztig [KL], Lemma 4.5). Now the lemma is immediate from (2.3.1) if we notice that $\mathbb{H}^i(\{A'\}, \mathbb{R}\rho_*\mathbb{C}) \simeq H^i(\mathcal{B}_{A'})$ as W-modules. □

For a given $A_0 \in \mathcal{N}_{\overline{\mathfrak{g}}}$, we denote by $A_0^\natural$ the unique element in $(\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$ such that $f(A_0^\natural) = A_0$. Hence if $A_0 = (1^{m_1}, 2^{m_2}, \dots)$, $A_0^\natural$ is given by $A_0^\natural = (1^{2m_1}, 2^{2m_2}, \dots)$ (resp. $A_0^\natural = (1^{2m_1+1}, 2^{2m_2}, \dots)$) in the case where $G = Sp_{2n}$ (resp. $G = SO_{2n+1}$), respectively. Then we have the following lemma.

Lemma 2.4. *For $A \in \mathcal{N}_{\mathfrak{g}}$, put $A_0 = f(A)$. Then $A_0^\natural \in \overline{\mathcal{O}}_A$. In other words, $\mathcal{O}_{A_0^\natural}$ is the unique minimal orbit (with respect to the closure relations) among the orbits contained in $f^{-1}(A_0)$.*

Proof. We consider the partition $\lambda \in \mathcal{P}$ or $\mathcal{P}'$ corresponding to $A \in \mathcal{N}_{\mathfrak{g}}$, and denote it as $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots$. We also denote the partition $\eta \in \overline{\mathcal{P}}$ corresponding to $A_0 = f(A)$ by $\eta : \eta_1 \geq \eta_2 \geq \cdots$. Then the following formula is easily verified from the definition of the map f given in 1.5.

$$\lambda_{2i-1} + \lambda_{2i} = \begin{cases} 2\eta_i & \text{if } \lambda_{2i-1} - \lambda_{2i} : \text{even}, \\ 2\eta_i + \delta & \text{if } \lambda_{2i-1} : \text{even}, \lambda_{2i} : \text{odd}, \\ 2\eta_i - \delta & \text{if } \lambda_{2i-1} : \text{odd}, \lambda_{2i} : \text{even}, \end{cases}$$

for each $i \geq 1$, where $\delta = 1$ (resp. $\delta = -1$) if $G = Sp_{2n}$ (resp. $G = SO_{2n+1}$), respectively. Now it is easy to see from the above formula that the Young diagram corresponding to $A_0^\natural$ is obtained from that of A by moving several nodes in the edge to lower positions. Hence $A_0^\natural$ is contained in $\overline{\mathcal{O}}_A$, and the lemma is proved. □

2.5. Thanks to Lemma 2.3 and Lemma 2.4, the construction of θ is reduced to the special case where $A = f(A)^\natural$, i.e., $A \in (\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$. We now assume that $A \in (\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$, and put $A_0 = f(A)$. Let us consider the homomorphism $\overline{\phi} : H^*(\overline{\mathcal{B}}) \to H^*(\overline{\mathcal{B}}_{A_0})$. The kernel $\operatorname{Ker}\overline{\phi}$ of the map $\overline{\phi}$ is described by the main result of de Concini-Procesi [CP], which is given as follows; let $S_h(x_1, x_2, \dots, x_k)$ be the total symmetric function of degree h with k variables, i.e., it is defined to be the sum of all monomials in $x_1, x_2, \dots, x_k$ of degree h. Put

$$S_{h,t,k}(x_1, \dots, x_t) = S_h(x_1, \dots, x_t)(x_1 \cdots x_t)^k.$$

Then it follows from Theorem 2.2 and Theorem 4.2 in [CP], that we have

Theorem 2.6 (de Concini-Procesi [CP]). *The ideal* $\operatorname{Ker}\overline{\phi}$ *in* $H^*(\overline{\mathcal{B}})$ *is generated by* $\overline{\alpha}(S_{j,t,k}(x_{i_1}, \dots, x_{i_t}))$ *for any* $i_1, \dots, i_t \in [1, n]$, *subject to the condition that* $j + k = n_k + 1$, *where* n_k *is the rank of* $(A_0^\vee)^k$ *(*$A_0^\vee$ *is the element corresponding to the dual partition of* A_0. *The map* $\overline{\alpha}$ *is given in 1.7.)*

2.7. In view of Theorem 2.6, together with (1.7.1), in order to prove the proposition, we have only to show that the image of $\alpha(S_{j,t,k}(x_{i_1}^2, \dots, x_{i_t}^2))$ vanishes on $H^*(\mathcal{B}_A)^D$ under the map ϕ_D for $A \in (\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$. Furthermore, since ϕ_D is S_n-equivariant, it is enough to check this only for $S_{j,t,k}(x_1^2, \dots, x_t^2)$. We put $T_{j,t,k}(x_1, \dots, x_t) = S_{j,t,k}(x_1^2, \dots, x_t^2)$. We shall show the vanishing of $\phi \circ \alpha(T_{j,t,k})$ by induction on the partial ordering with respect to the closure relations of $\mathcal{N}_{\overline{\mathfrak{g}}}$. Let $\eta \in \overline{\mathcal{P}}$ be the partition corresponding to A_0, and let $\eta^\vee$ be the dual partition of η. We write $n_k = n_k(\eta^\vee)$. In view of Lemma 2.3, if one can find $\sigma \in \overline{\mathcal{P}}$ such that $\sigma < \eta$ and that $n_k(\eta^\vee) = n_k(\sigma^\vee)$, then our assertion is satisfied. Then by further restriction as discussed in [CP, p. 216–217], the proof of the proposition is reduced to verifying the following statement.

(2.7.1) Let $A_0 \in \mathcal{N}_{\overline{\mathfrak{g}}}$ be of type η, where $\eta = ((c+1)^{d_1}, c^{d_2}, 1^{d_3})$, with $c \geq 2, d_2 > 0, d_3 > 0$. (Hence $A = A_0^\natural \in \mathcal{N}_{\mathfrak{g}}$ of type $((c+1)^{2d_1}, c^{2d_2}, 1^{2d_3})$ (resp. $((c+1)^{2d_1}, c^{2d_2}, 1^{2d_3+1})$ if $G = Sp_{2n}$ (resp. $G = SO_{2n+1}$), respectively). Let k be an integer such that $j + t = n_k + 1$ satisfying one of the following conditions;

(i) $k = d_2 + d_3$ with $d_1 = 0, c = 2$,
(ii) $k = d_1 + d_2$ with $d_1 > 0$ or $c > 2$.

Then $\phi \circ \alpha(T_{j,t,k})$ vanishes on $H^*(\mathcal{B}_A)_1^D$. (Note that in the case (i), we have $j+t=1$, and so $T_{j,t,k} = x_1^{2k}$, while in the case (ii), we have $n_k = d_3$.)

2.8. The case (i) will be discussed later in 2.21. First we concentrate on the case (ii). Let V be a vector space with $\dim V = 2n$ (resp. $\dim V = 2n+1$) endowed with a non-degenerate symplectic form (resp. non-degenerate symmetric bilinear form) $(\ ,\ )$ if $G = Sp_{2n}$ (resp. $G = SO_{2n+1}$), respectively. Let $\mathcal{F}(V)$ be the set of total isotropic flags in V. We consider a nilpotent element $A \in (\mathcal{N}_{\mathfrak{g}})_{\mathrm{ev}}$ of the following type;

$$A = \begin{cases} ((c+1)^{2d_1}, c^{2d_2}, 1^{2d_3}) & \text{if } G = Sp_{2n}, \\ ((c+1)^{2d_1}, c^{2d_2}, 1^{2d_3+1}) & \text{if } G = SO_{2n+1}, \end{cases}$$

for some integers $d_2, d_3 > 0$, and c, d_1 as in (ii) of (2.7.1). In the following, we identify $\mathcal{B}$ with $\mathcal{F}(V)$, and $\mathcal{B}_A$ with the set $\mathcal{F}_A(V)$ of A-stable flags in $\mathcal{F}(V)$. Let

$$\begin{aligned} V_0 &= \operatorname{Ker} A, \\ V_1 &= \operatorname{Ker} A \cap \operatorname{Im} A, \\ V_2 &= \operatorname{Ker} A \cap \operatorname{Im} A^c. \end{aligned}$$

We have $\dim V_1 = 2(d_1 + d_2)$ and $\dim V_2 = 2d_1$. Put $k = d_1 + d_2$, and $g = 2k + d_3$. For a given A, there exists a basis of V on which A acts as in the formula in IV 2.19 in Springer-Steinberg [SS]. In particular, we see that V_1 and V_2 are isotropic subspaces in V. But V_0 is not isotropic, nor is the restriction to V_0 of the form $(\ ,\ )$ non-degenerate. We now define a subspace $\widetilde{V}_0$ of V as a smallest subspace containing V_0 such that the form is non-degenerate on $\widetilde{V}_0$. Hence $\dim \widetilde{V}_0 = 2g$ (resp. $\dim \widetilde{V}_0 = 2g+1$) if $G = Sp_{2n}$ (resp. $G = SO_{2n+1}$), respectively. For any integer $i > 0$, we denote by $\mathcal{G}_i(V)$ or $\mathcal{G}_i(\widetilde{V}_0)$ the isotropic Grassmann variety of V or $\widetilde{V}_0$ of degree i, i.e., $\mathcal{G}_i(V)$ is the set of isotropic subspaces of dimension i in V. We have a natural map $\pi_i : \mathcal{F}(V) \to \mathcal{G}_i(V)$. We may regard $\mathcal{G}_i(\widetilde{V}_0)$ as a closed subset of $\mathcal{G}_i(V)$.

Now we fix an integer t such that $1 \le t \le g$, and define a closed subset $\mathcal{X}$ of $\mathcal{F}(V)$ by $\mathcal{X} = \pi_t^{-1}(\mathcal{G}_t(\widetilde{V}_0))$. Furthermore, for each s such that $1 \le s < t$ and that $s \le d_3$, put

$$\mathcal{Y}_s = \{Z \in \mathcal{G}_s(\widetilde{V}_0) \mid Z \in \operatorname{Ker} A, \dim(Z \cap \operatorname{Im} A) \ge 1\}.$$

Then $\mathcal{Y}_s$ is a closed subset of $\mathcal{G}_s(\widetilde{V}_0)$, and we define a subset $\mathcal{X}_s$ of $\mathcal{F}(V)$ by $\mathcal{X}_s = \pi_s^{-1}(\mathcal{Y}_s)$. We have the following lemma.

LEMMA 2.9. *Let $\mathcal{X}^\sharp = \mathcal{X} \cup (\bigcup_{1 \le s < t} \mathcal{X}_s)$. Then we have $\mathcal{F}_A(V) \subset \mathcal{X}^\sharp$.*

PROOF. The proof is similar to the arguments in p. 218 of [CP]. It is enough to show that each irreducible component of $\mathcal{F}_A(V) \simeq \mathcal{B}_A$ is contained in $\mathcal{X}^\sharp$. Note that the irreducible components of $\mathcal{B}_A$ are parameterized by tableaux with $2n$ boxes, and this parameterization is compatible with the locally trivial fibration of $\mathcal{B}_A$ (see Remark 3.7 for a more precise discussion). We denote by $\mathcal{F}_T$ the irreducible component corresponding to a tableau T. But, note that the tableau T does not characterize the irreducible components. In general, it happens that more than two irreducible components correspond to the same tableau T. Anyway, we consider an irreducible component $\mathcal{F}_T$ of $\mathcal{B}_A$. Its open part $\mathcal{F}_T^0$ is described completely by locally trivial fibrations associated to $\mathcal{F}_T$. Since $\mathcal{X}^\sharp$ is a closed subset of $\mathcal{B}$, we have only to show that $\mathcal{F}_T^0 \subset \mathcal{X}^\sharp$.

Now assume that the last t numbers $n, n-1, \dots, n-t+1$ appear in the first column of the tableau T. Then any flag $F \in \mathcal{F}_T^0$ has the form

$$F : Z_1 \subset Z_2 \subset \cdots \subset Z_n$$

with $Z_i \subset \operatorname{Ker} A$ for $i = 1, \dots, t$. (Here Z_i is an isotropic subspace of V of dimension i.) This implies that $\mathcal{F}_T^0 \subset \mathcal{X}$. So we may assume that not all of the integers $n, n-1, \dots, n-t+1$ appear in the first column of T. Let s be the smallest number such that $n, n-1, \dots, n-s+1$ appear in the first column, but $n-s$ does not. (This implies that $s \le d_3$.) Then for any $F = (Z_i) \in \mathcal{F}_T^0$, we see that $Z_1, Z_2, \dots, Z_{s-1} \in \operatorname{Ker} A$. Furthermore, we have $Z_s \in \operatorname{Ker} A$ and $Z_s \cap \operatorname{Im} A \ne \{0\}$. It follows that $\mathcal{F}_T^0 \subset \mathcal{X}_s$. This proves the lemma. □

2.10. We fix a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ (resp. $\{e_1, \dots, e_n, f_1, \dots, f_n, h\}$ of V if $G = Sp_{2n}$ (resp. $G = SO_{2n+1}$) as follows; for any i, j,

$$(e_i, f_j) = \delta_{ij}, \quad (e_i, e_j) = (f_i, f_j) = 0$$

and $(h, h) = 1, (h, e_i) = (h, f_j) = 0$. We assume that the basis is chosen so that $\{e_1, \dots, e_g, f_1, \dots, f_g\}$ (resp. $\{e_1, \dots, e_g, f_1, \dots, f_g, h\}$) gives rise to a basis of $\widetilde{V}_0$ if $G = Sp_{2n}$ (resp. $G = SO_{2n+1}$), and

$$V_2 = \langle e_1, \dots, e_{2d_1} \rangle, \quad V_1 = \langle e_1, \dots, e_{2k} \rangle,$$

$$V_0 = \begin{cases} \langle e_1, \dots, e_g, f_{2k+1}, \dots, f_g \rangle & \text{if } G = Sp_{2n}, \\ \langle e_1, \dots, e_g, f_{2k+1}, \dots, f_g, h \rangle & \text{if } G = SO_{2n+1}. \end{cases}$$

Let B be the Borel subgroup of G defined as the stabilizer of the total flag

$$\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, \dots, e_n\rangle$$

in G. We define a maximal torus T of B by the condition that the opposite Borel subgroup of B with respect to T is the stabilizer of the total flag

$$\langle f_1\rangle \subset \langle f_1, f_2\rangle \subset \cdots \subset \langle f_1, \dots, f_n\rangle.$$

We identify W with $N_G(T)/T$.

Let $H = Sp(\widetilde{V}_0)$ or $SO(\widetilde{V}_0)$ according to the case where $G = Sp_{2n}$ or SO_{2n+1}. We identify H with a subgroup of G consisting of elements which fix any basis element outside of $\widetilde{V}_0$. We put $B_H = B \cap H$ and $T_H = T \cap H$. Then B_H is a Borel subgroup of H which is the stabilizer of the total flag

$$\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, \dots, e_g\rangle$$

in H, and T_H is a maximal torus in H. Let $W_H = N_H(T_H)/T_H$. Then W_H is realized as the group of signed permutations of $\{1, \dots, g\}$.

We now consider the structure of the Grassmann variety $\mathcal{G}_s(\widetilde{V}_0)$. Let us define a sequence

$$U_1 \subset U_2 \subset \cdots \subset U_g \subset U_{g+1} \subset \cdots \subset U_{2g} = \widetilde{V}_0 \tag{2.10.1}$$

of subspaces of $\widetilde{V}_0$ by $U_i = \langle e_1, \dots, e_i\rangle$ if $1 \le i \le g$, and by U_{g+i} the orthogonal complement of U_{g-i} in $\widetilde{V}_0$, for $1 \le i \le g$. (Here we put $U_0 = \{0\}$.) Hence

$$U_{g+i} = \begin{cases} \langle e_1, \dots, e_g, f_g, \dots, f_{g-i+1}\rangle & \text{if } G = Sp_{2n} \\ \langle e_1, \dots, e_g, h, f_g, \dots, f_{g-i+1}\rangle & \text{if } G = SO_{2n+1}. \end{cases}$$

We define a total order on the set $\{\pm 1, \dots, \pm g\}$ by

$$1 \prec 2 \prec \cdots \prec g \prec -g \prec -g+1 \prec \cdots \prec -2 \prec -1.$$

Assume that $s \le g$, and let $\varGamma$ be the set of s-tuples $\gamma = (\gamma_1, \dots, \gamma_s)$ of integers γ_i $(-g \le \gamma_i \le g)$ such that $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_s$ and that the absolute values $|\gamma_i|$ are all distinct. We define an ordering on $\varGamma$ by $\gamma \ge \gamma'$ if

$\gamma_i \succeq \gamma_i'$ for $i = 1, \dots, s$. We put $\overline{\gamma}_i = \gamma_i$ if $\gamma_i > 0$ and $\overline{\gamma}_i = 2g + \gamma_i$ if $\gamma_i < 0$. For each $\gamma \in \varGamma$, let us define subsets $Y_\gamma, \overline{Y}_\gamma$ in $\mathcal{G}_s(\widetilde{V}_0)$ by

$$\begin{aligned} Y_\gamma &= \{Z \in \mathcal{G}_s(\widetilde{V}_0) \mid \dim(Z \cap U_k) = i \text{ for } \overline{\gamma}_i \le k < \overline{\gamma}_{i+1}\}, \\ \overline{Y}_\gamma &= \{Z \in \mathcal{G}_s(\widetilde{V}_0) \mid \dim(Z \cap U_{\overline{\gamma}_i}) \ge i\}. \end{aligned} \tag{2.10.2}$$

Then we have the following lemma.

Lemma 2.11.

(i) $\mathcal{G}_s(\widetilde{V}_0) = \coprod_{\gamma \in \varGamma} Y_\gamma$, *and* Y_γ *is a Schubert cell in* $\mathcal{G}_s(\widetilde{V}_0)$.

(ii) $\overline{Y}_\gamma$ *is the closure of* Y_γ. *Hence* $\overline{Y}_\gamma$ *is a Schubert variety. Furthermore* $\overline{Y}_\gamma \subset \overline{Y}_\beta$ *if and only if* $\gamma \le \beta$.

Proof. Let $Q^{(s)}$ be the stabilizer of U_s in H. Then $Q^{(s)}$ is the maximal parabolic subgroup of H containing B_H. We denote by $W_H^{(s)}$ the corresponding Weyl subgroup of W_H, which is isomorphic to $S_s \times W_{g-s}$, where S_s is the symmetric group on s letters $\{1, \dots, s\}$ and W_{g-s} is the Weyl group of type C_{g-s} with $g - s$ letters $\{s+1, \dots, g\}$. Then the set $\mathcal{D}$ of distinguished representatives for $W_H/W_H^{(s)}$ is given as

$$\mathcal{D} = \{w_\gamma = \begin{pmatrix} 1 & 2 & \cdots & t & t+1 & \cdots & g \\ \gamma_1 & \gamma_2 & \cdots & \gamma_t & \beta_1 & \cdots & \beta_{g-t} \end{pmatrix} \mid \gamma \in \varGamma\}$$

where $0 < \beta_1 < \beta_2 < \cdots < \beta_{g-t}$ are the complement of $\{|\gamma_1|, |\gamma_2|, \dots, |\gamma_t|\}$ in $\{1, 2, \dots, g\}$. Now $\mathcal{G}_s(\widetilde{V}_0)$ is naturally identified with $H/Q^{(s)}$, and the cell $Y_\gamma \subset \mathcal{G}_s(\widetilde{V}_0)$ corresponding to $\gamma \in \varGamma$ is given by

$$Y_\gamma = \{u w_\gamma(U_s) \mid u \in U_H\},$$

where U_H is the maximal unipotent subgroup of B_H. Since $w_\gamma(U_s) = \langle g_1, g_2, \dots, g_s \rangle$, where $g_i = e_{\gamma_i}$ if $\gamma_i > 0$ and $g_i = f_{-\gamma_i}$ if $\gamma_i < 0$, it is easy to see that Y_γ has the form given in (2.10.2). This shows (i). The second statement also follows easily from this. $\square$

2.12. We now consider the variety $\mathcal{X} = \pi_t^{-1}(\mathcal{G}_t(\widetilde{V}_0))$ as in 2.8. Let Q_i be the stabilizer in H of the flag

$$U_1 \subset U_2 \subset \cdots \subset U_i$$

for $1 \le i \le t$. Put $V_i^* = U_i^\perp/U_i$ for $1 \le i \le t$, where $U_i^\perp$ is the orthogonal complement of U_i in V. Then the form $(\ ,\ )$ induces a non-degenerate symplectic or symmetric bilinear form on V_i^*. We consider the flag variety $\mathcal{F}(V_i^*)$. Then the group Q_i acts naturally on $\mathcal{F}(V_i^*)$. The structure of the variety $\mathcal{X}$ is given in the following lemma.

Lemma 2.13. *$\mathcal{X} \simeq H \times^{Q_t} \mathcal{F}(V_t^*)$.*

Proof. Put

$$\mathcal{X}_t^1 = \{F = (Z_i) \in \mathcal{F}(V) \mid Z_i = U_i \text{ for } i = 1, \dots, t\}.$$

Then $\mathcal{X}_t^1$ is a closed subset of $\mathcal{X}$ which is isomorphic to $\mathcal{F}(V_t^*)$. On the other hand, for each $F = (Z_i) \in \mathcal{X}$, we have $Z_t \in \widetilde{V}_0$, and so $Z_1 \subset Z_2 \subset \cdots \subset Z_t$ is a partial flag in $\widetilde{V}_0$. Hence there exists $g \in H$ such that

$$g^{-1}F = (U_1 \subset U_2 \subset \cdots \subset U_t \subset Z'_{t+1} \subset \cdots \subset Z'_n).$$

Hence $g^{-1}F \in \mathcal{X}_t^1$, and so we see that $\mathcal{X} = H \cdot \mathcal{X}_t^1$. Now it is easy to see that the map $\rho : \mathcal{X} \to H/Q_t$, $F = (Z_i) \mapsto (Z_1 \subset Z_2 \subset \cdots \subset Z_t)$ gives rise to a locally trivial fibration with fibre isomorphic to $\mathcal{F}(V_t^*)$. Hence the lemma follows. $\square$

Next, we consider the varieties $\mathcal{X}_s$ for $1 \le s < t$. We define Q_s, V_s, etc. similar to Q_t, V_t, etc. by replacing t by s. Then we have the following lemma.

Lemma 2.14.

(i) *$\mathcal{Y}_s$ is a union of Schubert varieties $\overline{Y}_\gamma$ in $\mathcal{G}_s(\widetilde{V}_0)$ of the form $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_s)$ with $\gamma_1 = 2k, \overline{\gamma}_s \le g + d_3$.*

(ii) *Let $Q^{(s)} \supset Q_s$ be the maximal parabolic subgroup of H stabilizing U_s, and let $f_s : H/Q_s \to H/Q^{(s)} \simeq \mathcal{G}_s(\widetilde{V}_0)$ be the natural projection. Then we have*

$$\mathcal{X}_s \simeq f_s^{-1}(\mathcal{Y}_s) \times^{Q_s} \mathcal{F}(V_s^*),$$

where $f_s^{-1}(\mathcal{Y}_s)/Q_s$ is a union of Schubert varieties in H/Q_s.

PROOF. Under the notation in (2.10.1), we have $V_0 = U_{g+d_3}$, $V_1 = U_{2k}$. Hence, for a given $Z \in \mathcal{G}_s(\widetilde{V}_0)$, the condition that $Z \in \mathcal{Y}_s$ is equivalent to the condition that $\dim(Z \cap U_{2k}) \geq 1$ and $\dim(Z \cap U_{g+d_3}) \geq s$. In other words, Z is contained in $\overline{Y}_\gamma$ with $\overline{\gamma}_1 = 2k$ and $\overline{\gamma}_s \leq g + d_3$. Since $2k - 2g < -g$, we have $\gamma_1 = 2k$. The statement (i) follows from this. Next we consider (ii). It is clear from (i) that $f_s^{-1}(\mathcal{Y}_s)/Q_s$ is a union of Schubert varieties in H/Q_s. Let $\mathcal{X}_s^1$ be the subvariety of $\mathcal{F}(V)$ defined similar to $\mathcal{X}_t^1$ in the proof of Lemma 2.12. Since $U_s \in \mathcal{Y}_s$, $\mathcal{X}_s^1$ is contained in $\mathcal{X}_s$, and we see easily that $\mathcal{X}_s$ coincides with the translation $f_s^{-1}(\mathcal{Y}_s)\mathcal{X}_s^1$ of $\mathcal{X}_s^1$ under the action of $f_s^{-1}(\mathcal{Y}_s)$. On the other hand, the similar argument as in (i) shows that $H\mathcal{X}_s^1$ is a locally trivial fibration $H \times^{Q_s} \mathcal{X}_s^1$ over H/Q_s. Hence $f_s^{-1}(\mathcal{Y}_s)\mathcal{X}_s^1$ is also a locally trivial fibration over $f_s^{-1}(\mathcal{Y}_s)/Q_s$. This proves (ii), and the lemma follows. □

Now we show the following.

PROPOSITION 2.15. *The natural map*

$$H^{2i}(\mathcal{X}^\sharp) \to \Big(\bigoplus_{s=1}^{t-1} H^{2i}(\mathcal{X}_s)\Big) \oplus H^{2i}(\mathcal{X})$$

induced from the closed immersions such as $\mathcal{X}_s \hookrightarrow \mathcal{X}^\sharp$ is injective for each i.

PROOF. We set $\mathcal{X}_k^\sharp = \bigcup_{s=1}^k \mathcal{X}_s$ for $k = 1, 2, \dots, t-1$. We show, by induction on k, that the natural map

$$H^{2i}(\mathcal{X}_k^\sharp) \to \bigoplus_{s=1}^{k} H^{2i}(\mathcal{X}_s)$$

is injective.

First we note that

$$\mathcal{X}_{k-1}^\sharp \cap \mathcal{X}_k \simeq \mathcal{T}_k \times^{Q_k} \mathcal{F}(V_k^*), \tag{2.15.1}$$

for a subset $\mathcal{T}_k$ of H stable under the right multiplication of Q_k such that $\mathcal{T}_k/Q_k$ is a union of Schubert varieties in H/Q_k.

In fact, if $s < k$, we have $\pi_k^{-1}(\mathcal{G}_k(\widetilde{V}_0) \subset \pi_s^{-1}(\mathcal{G}_s(\widetilde{V}_0))$. The similar argument as in the proof of Lemma 2.13 shows that $\pi_k^{-1}(\mathcal{G}_k(\widetilde{V}_0)) \simeq H \times^{Q_k} \mathcal{F}(V_k^*)$. Since $Q_k \subset Q_s$, the map f_s factors through f_k, and we have

$$\pi_k^{-1}(\mathcal{G}_k(\widetilde{V}_0)) \cap \mathcal{X}_s \simeq f_s^{-1}(\mathcal{Y}_s) \times^{Q_k} \mathcal{F}(V_k^*).$$

In particular, $f_s^{-1}(\mathcal{Y}_s)/Q_k$ is a union of Schubert varieties in H/Q_k. Now using Lemma 2.14, we see that

$$\mathcal{X}_s \cap \mathcal{X}_k \simeq (f_s^{-1}(\mathcal{Y}_s) \cap f_k^{-1}(\mathcal{Y}_k)) \times^{Q_k} \mathcal{F}(V_k^*).$$

Here $(f_s^{-1}(\mathcal{Y}_s) \cap f_k^{-1}(\mathcal{Y}_k))/Q_k$ is a union of Schubert varieties in H/Q_k since both are so. The variety $\mathcal{X}_{k-1}^\sharp$ is also described in a similar way. The assertion (2.15.1) now follows easily from this.

It follows from (2.15.1) that we have

$$\mathcal{X}_k - \mathcal{X}_{k-1}^\sharp \simeq \mathcal{T}_k^0 \times^{Q_k} \mathcal{F}(V_k^*), \tag{2.15.2}$$

where $\mathcal{T}_k^0/Q_k$ is a union of Schubert cell in H/Q_k. Now (2.15.2) implies that $\mathcal{X}_k - \mathcal{X}_{k-1}^\sharp$ has a pavement by affine spaces. In particular, we see that $H_c^{\mathrm{odd}}(\mathcal{X}_k - \mathcal{X}_{k-1}^\sharp) = 0$. We also have $H^{\mathrm{odd}}(\mathcal{X}_k) = 0$ by Lemma 2.14. Hence, by using the cohomology long exact sequence for $\mathcal{X}_{k-1}^\sharp \hookrightarrow \mathcal{X}_k^\sharp$, and by induction, we have

$$H^{\mathrm{odd}}(\mathcal{X}_k^\sharp) = 0 \quad \text{for } k = 1, 2, \dots, t-1. \tag{2.15.3}$$

Then we have the following commutative diagram.

$$\begin{tikzcd}
 & & \bigoplus_{s=1}^{k-1} H^{2i}(\mathcal{X}_s) \arrow[r, "\mathrm{id}"] & \bigoplus_{s=1}^{k-1} H^{2i}(\mathcal{X}_s) & \\
0 \arrow[r] & H_c^{2i}(\mathcal{X}_k^\sharp - \mathcal{X}_{k-1}^\sharp) \arrow[r] \arrow[d, "\mathrm{id}"'] & H^{2i}(\mathcal{X}_k^\sharp) \arrow[r] \arrow[u, "\varphi_1"] \arrow[d, "\varphi_2"'] & H^{2i}(\mathcal{X}_{k-1}^\sharp) \arrow[r] \arrow[u, "\varphi_3"] \arrow[d] & 0 \\
0 \arrow[r] & H_c^{2i}(\mathcal{X}_k - \mathcal{X}_{k-1}^\sharp) \arrow[r, "\varphi_4"] & H^{2i}(\mathcal{X}_k) \arrow[r] & H^{2i}(\mathcal{X}_k \cap \mathcal{X}_{k-1}^\sharp) \arrow[r] & 0
\end{tikzcd}$$

Here φ_1, φ_2 and φ_3 are the natural maps induced from closed immersions such as $\mathcal{X}_s \hookrightarrow \mathcal{X}_k^\sharp$. Now the middle horizontal row is exact by (2.15.2) and (2.15.3). The lower horizontal row is also exact by (2.15.1) and (2.15.2).

Now assume that the image of $x \in H^{2i}(\mathcal{X}_k^\sharp)$ is zero under the natural map $\varphi : H^{2i}(\mathcal{X}_k^\sharp) \to \bigoplus_{s=1}^k H^{2i}(\mathcal{X}_s)$. Then $\varphi_1(x) = 0$, $\varphi_2(x) = 0$. Since φ_3 is injective by the assumption, x lies in $H_c^{2i}(\mathcal{X}_k^\sharp - \mathcal{X}_{k-1}^\sharp)$. Since φ_4 is injective, this implies that $x = 0$. So we have shown that φ is injective.

The completely similar argument works also for the last step, i.e., for $\mathcal{X}_{t-1}^\sharp$ and $\mathcal{X}^\sharp$. (In fact, the required property for $\mathcal{X}_s$ is that $\mathcal{Y}_s$ is a union of Schubert varieties. So it can be applied to $\mathcal{X}$ also.) This proves the proposition. □

2.16. We now consider the polynomial $T_{j,t,k}$ in the case (ii) of (2.7.1). We shall show that the image of $\alpha(T_{j,t,k})$ vanishes on $H^*(\mathcal{B}_A)$. By Lemma 2.9, the map ϕ_A factors through the natural map $H^*(\mathcal{B}) \to H^*(\mathcal{X}^\sharp)$. So, in view of Proposition 2.15, in order to show the statement (2.7.1) it is enough to see that the image of $\alpha(T_{j,t,k})$ vanishes under the maps $H^*(\mathcal{B}) \to H^*(\mathcal{X}_s)$ for $1 \le s < t$ and under the map $H^*(\mathcal{B}) \to H^*(\mathcal{X})$. First we show

Lemma 2.17. *The image of $\alpha(T_{j,t,k})$ is zero on $H^*(\mathcal{X})$.*

Proof. Let $P^{(t)}$ be the stabilizer of U_t in G, and $Q^{(t)}$ the stabilizer of U_t in H. Hence $P^{(t)}$ (resp. $Q^{(t)}$) is a maximal parabolic subgroup of G (resp. H) and we have a natural isomorphism $G/P^{(t)} \simeq \mathcal{G}_t(V)$ (resp. $H/Q^{(t)} \simeq \mathcal{G}_t(\widetilde{V}_0)$), respectively. We have a closed immersion $H/Q^{(t)} \hookrightarrow G/P^{(t)}$ corresponding to the inclusion $\mathcal{G}_t(\widetilde{V}_0) \hookrightarrow \mathcal{G}_t(V)$. Let $\pi_t : \mathcal{B} \to G/P^{(t)}$ be the map as before. Since $\mathcal{X} = \pi_t^{-1}(\mathcal{G}_t(\widetilde{V}_0))$, we have a commutative diagram

$$\begin{array}{ccc} H^*(G/P^{(t)}) & \xrightarrow{\pi_t^*} & H^*(\mathcal{B}) \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow \\ H^*(H/Q^{(t)}) & \longrightarrow & H^*(\mathcal{X}). \end{array}$$

Let $W^{(t)}$ be the Weyl subgroup of W corresponding to $P^{(t)}$. Hence $W^{(t)} \simeq S_t \times W_{n-t}$, where S_t is the symmetric group of t letters $\{1, \dots, t\}$ and W_{n-t} is the Weyl group of type C_{n-t} of $n-t$ letters $\{t+1, \dots, n\}$. Now it is known that π_t^* is injective and its image coincides with $H^*(\mathcal{B})^{W^{(t)}}$, ([BGG, 5.5]. See also (3.7) in [LS]. This is the special case of Borho-MacPherson's theorem there). Note that $\alpha(T_{j,t,k})$ is $W^{(t)}$-invariant. It follows that we may assume that $\alpha(T_{j,t,k})$ lies in $H^*(G/P^{(t)})$. Hence in order to show the lemma, it is enough to see the following.

(2.17.1) The image of $\alpha(T_{j,t,k})$ under the map ψ is zero on $H^*(H/Q^{(t)})$.

We show (2.17.1). First we note that the map $\alpha : \mathbb{C}[x_1, \dots, x_n] \to H^*(\mathcal{B})$ is obtained by attaching $\lambda \in X(T)$ to the first Chern class in $H^2(\mathcal{B})$ corresponding to the line bundle $G \times^B \lambda \to G/B$. Here $X(T)$ denotes the character group of T and we identify $\mathbb{C}[x_1, \dots, x_n]$ with the symmetric algebra on $\mathbb{C} \otimes X(T)$. The map $\alpha_H : \mathbb{C}[x_1, \dots, x_g] \to H^*(\mathcal{B}^H)$ is defined similarly, where $\mathcal{B}^H = H/B_H \simeq \mathcal{F}(\widetilde{V}_0)$. Then, by the property of Chern classes we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{C}[x_1, \dots, x_n] & \xrightarrow{\alpha} & H^*(\mathcal{B}) \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}[x_1, \dots, x_g] & \xrightarrow{\alpha_H} & H^*(\mathcal{B}^H),
\end{array}$$

where the right vertical map is the map induced from the natural map $\mathcal{B}^H \to \mathcal{B}$, and the left vertical map is the projection on $x_1, \dots, x_g$ variables, which is obtained from the restriction $X(T) \to X(T_H)$.

Taking D-invariant part and D_H-invariant part, we have

$$(2.17.2) \qquad \begin{array}{ccc}
\mathbb{C}[x_1^2, \dots, x_n^2] & \xrightarrow{\alpha} & H^*(\mathcal{B})^D \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}[x_1^2, \dots, x_g^2] & \xrightarrow{\alpha_H} & H^*(\mathcal{B}^H)^{D_H},
\end{array}$$

where D_H is the subgroup of W_H corresponding to D in W.

Let $\overline{G} = GL(\overline{V})$ and $\overline{H} = GL(\overline{V}_0)$ for $\overline{V} = < e_1, \dots, e_n >$ and $\overline{V}_0 = < e_1, \dots, e_g >$. Then a similar argument works also for $\overline{G}$ and $\overline{H}$, i.e., we have a commutative diagram

$$(2.17.3) \qquad \begin{array}{ccc}
\mathbb{C}[x_1, \dots, x_n] & \xrightarrow{\alpha} & H^*(\overline{\mathcal{B}}) \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}[x_1, \dots, x_g] & \xrightarrow{\alpha_H} & H^*(\overline{\mathcal{B}}^{\overline{H}}).
\end{array}$$

Now (2.17.2) and (2.17.3) implies the following commutative diagram

$$(2.17.4) \qquad \begin{array}{ccc}
H^*(\overline{\mathcal{B}}) & \xrightarrow{\theta_0} & H^*(\mathcal{B})^D \\
\big\downarrow & & \big\downarrow \\
H^*(\overline{\mathcal{B}}^{\overline{H}}) & \xrightarrow{\theta_0^H} & H^*(\mathcal{B}^H)^{D_H}.
\end{array}$$

Let $\overline{P}^{(t)}$ (resp. $\overline{Q}^{(t)}$) be the maximal parabolic subgroup of $\overline{G}$ (resp. $\overline{H}$), which is the stabilizer of the subspace $< e_1, \dots, e_t >$ in $\overline{G}$ (resp. $\overline{H}$), and let $\overline{W}^{(t)}$ (resp. $\overline{W}_H^{(t)}$) be the corresponding Weyl subgroup. Then $\overline{W}^{(t)} \simeq S_t \times S_{n-t}$, $\overline{W}_H^{(t)} \simeq S_t \times S_{g-t}$. Since θ_0 is S_n-equivariant and θ_0^H is S_g-equivariant, we obtain the following commutative diagram, (using [BGG, 5.5] for $\overline{G}$ and $\overline{H}$),

$$\begin{array}{ccccc}
H^*(\bar{\mathcal{G}}_t(\overline{V})) & \xrightarrow{\sim} & H^*(\overline{G}/\overline{P}^{(t)}) & \xrightarrow{\sim} & H^*(\mathcal{B})^{D\cdot\overline{W}^{(t)}} \\
\downarrow & & \downarrow & & \downarrow \\
H^*(\bar{\mathcal{G}}_t(\overline{V}_0)) & \xrightarrow{\sim} & H^*(\overline{H}/\overline{Q}^{(t)}) & \xrightarrow{\sim} & H^*(\mathcal{B}^H)^{D_H\overline{W}_H^{(t)}},
\end{array}$$

where $\bar{\mathcal{G}}_t(\overline{V})$ denotes the Grassmann variety of degree t for $\overline{V}$.

Note, since $D \cdot \overline{W}^{(t)} \supset W^{(t)}$, and $D_H \cdot \overline{W}_H^{(t)} \supset W_H^{(t)}$, that we have a natural injection

$$H^*(\mathcal{B})^{D\cdot\overline{W}^{(t)}} \hookrightarrow H^*(\mathcal{B})^{W^{(t)}} \simeq H^*(G/P^{(t)})$$

and similarly for $H^*(\mathcal{B}^H)$. It follows that we obtain

$$\begin{array}{ccccc}
H^*(\bar{\mathcal{G}}_t(\overline{V}) & \xrightarrow{\sim} & H^*(\mathcal{B})^{D\cdot\overline{W}^{(t)}} & \longrightarrow & H^*(G/P^{(t)}) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \psi} \\
H^*(\bar{\mathcal{G}}_t(\overline{V}_0)) & \xrightarrow{\sim} & H^*(\mathcal{B}^H)^{D_H\cdot\overline{W}_H^{(t)}} & \longrightarrow & H^*(H/Q^{(t)}).
\end{array}$$

Note that $\alpha(T_{j,t,k})$ is $P^{(t)}$-invariant and D-invariant. Hence it is invariant under $D \cdot \overline{W}^{(t)}$. So we may assume that $\alpha(T_{j,t,k})$ lies in $H^*(\mathcal{B})^{D\cdot\overline{W}^{(t)}}$. On the other hand, under the above isomorphism, $\alpha(T_{j,t,k})$ is mapped to an element $\bar{\alpha}(S_{j,t,k}) \in H^*(\bar{\mathcal{G}}_t(\overline{V}_0))$. Now it is known, by Lemma 4.10 in [CP], that the image of $\bar{\alpha}(S_{j,t,k})$ vanishes on $H^*(\bar{\mathcal{G}}_t(\overline{V}_0))$. It follows that the image of $\alpha(T_{j,t,k})$ under the map ψ vanishes on $H^*(H/Q^{(t)})$. This proves (2.17.1), and so the lemma follows. □

Next we show

LEMMA 2.18. *The image of $\alpha(T_{j,t,k})$ is zero on $H^*(\mathcal{X}_s)$ for each s, $(1 \le s < t)$.*

PROOF. Put $R_{k,s}(x_1,\dots,x_s) = (x_1\cdots x_s)^{2k}$. Since $R_{k,s}$ is a factor of $T_{j,t,k}$, to prove the lemma it is enough to show that the image of $\alpha(R_{k,s})$ is zero on $H^*(\mathcal{X}_s)$ for each s. Let $P^{(s)}$ be the maximal parabolic subgroup of G, which is the stabilizer of U_s. As in the proof of Lemma 2.17, we have the following commutative diagram

$$\begin{array}{ccc} H^*(\mathcal{B}) & \xleftarrow{\pi_s^*} & H^*(G/P^{(s)}) \\ \big\downarrow & & \psi\big\downarrow \\ H^*(\mathcal{X}_s) & \longleftarrow & H^*(\mathcal{Y}_s). \end{array}$$

Since $\alpha(R_{k,s})$ is $W^{(s)}$-invariant, we may assume that $\alpha(R_{k,s})$ lies in $H^*(G/P^{(s)})$. So, in order to prove the lemma, it is enough to show that

(2.18.1) The image of $\alpha(R_{k,s})$ under ψ is zero on $H^*(\mathcal{Y}_s)$.

Now the map ψ factors as

$$\psi : H^*(G/P^{(s)}) \xrightarrow{\psi_1} H^*(H/Q^{(s)}) \xrightarrow{\psi_2} H^*(\mathcal{Y}_s),$$

where ψ_1, ψ_2 are natural maps induced from the closed immersions, $\mathcal{Y}_s \hookrightarrow H/Q^{(s)} \hookrightarrow G/P^{(s)}$. But by the similar argument as in the proof of Lemma 2.17 (cf. (2.17.2)), we see that $\psi_1(\alpha(R_{k,s}))$ coincides with $\alpha_H(R_{k,s})$ under the isomorphism $H^*(\mathcal{B}^H)^{W_H^{(s)}} \simeq H^*(H/Q^{(s)})$.

Let X_w ($w \in W_H$) be the basis of $H^*(\mathcal{B}^H)$ dual to the basis of $H_*(\mathcal{B}^H)$ consisting of Schubert classes. Let $\mathcal{D}$ be the distinguished representatives of $W_H/W_H^{(s)}$ as in the proof of Lemma 2.11. Then it is known by Theorem 5.5 in Bernstein-Gelfand-Gelfand [BGG] that the set X_w ($w \in \mathcal{D}$) gives rise to a basis of $H^*(\mathcal{B}^H)^{W_H^{(s)}}$, and that, under the isomorphism $H^*(H/Q^{(s)}) \xrightarrow{\sim} H^*(\mathcal{B}^H)^{W_H^{(s)}}$, the basis X_w ($w \in \mathcal{D}$) coincides with the basis of $H^*(H/Q^{(s)})$ dual to the basis of $H_*(H/Q^{(s)})$ consisting of Schubert classes.

Now since $\alpha_H(R_{k,s})$ is $W_H^{(s)}$-invariant, $\alpha_H(R_{k,s})$ can be written as a linear combination of X_w with $w \in \mathcal{D}$. As in the proof of Lemma 2.11, the set $\mathcal{D}$ is described by the set Γ of s-tuples $\gamma = (\gamma_1,\dots,\gamma_s)$. We have the following lemma.

LEMMA 2.19. *$\alpha_H(R_{k,s}) = X_{w_\beta}$ for $\beta = (2k+1, 2k+2, \dots, 2k+s) \in \Gamma$. (Note that since $s \le d_3$, we have $2k+s \le g$.)*

Assuming Lemma 2.19, we continue the proof of Lemma 2.18. By Lemma 2.14 (i), $\mathcal{Y}_s$ is a union of Schubert varieties $\overline{Y}_\gamma$. So by the similar argument as in Proposition 2.15, (2.18.1) is reduced to showing that the image of X_{w_β} is zero on $H^*(\overline{Y}_\gamma)$ for each γ as in Lemma 2.14. But such γ has the form $\gamma = (2k, \dots)$ and so $\overline{Y}_\beta \not\subset \overline{Y}_\gamma$ since $\beta = (2k+1, \dots)$. This implies that the image of X_{w_β} is zero on $H^*(\overline{Y}_\gamma)$, and (2.18.1) holds. Now Lemma 2.18 follows, and so (2.17.1) under the condition (ii) is verified, modulo the proof of Lemma 2.19. □

2.20. We prove Lemma 2.19. Let $S = \{s_1, \dots, s_g\}$ be the set of simple reflections of W_H, where s_i is a permutation $x_i \leftrightarrow x_{i+1}$ for $i = 1, \dots, g-1$, and s_g is a sign change $x_g \leftrightarrow -x_g$. For each $s_i \in S$, we define an operator Δ_i on $\mathbb{C}[x_1, \dots, x_g]$ by $\Delta_i(f) = (f - s_i(f))/\alpha_i$, where α_i is a simple root with respect to s_i, which is realized as a linear form on $x_1, \dots, x_g$. Then we define an operator Δ_w for each $w \in W_H$ by $\Delta_w = \Delta_{i_1}\Delta_{i_2}\cdots\Delta_{i_r}$ according to the reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$. Note that Δ_w is independent of the choice of a reduced expression of w. The operators Δ_i satisfy the following relations.

$$\begin{aligned}
&\Delta_i^2 = 0, \\
&\Delta_i\Delta_j = \Delta_j\Delta_i \quad \text{if } |i-j| \geq 2, \\
&\Delta_i\Delta_{i+1}\Delta_i = \Delta_{i+1}\Delta_i\Delta_{i+1} \quad \text{for } i = 1, \dots g-2, \\
&\Delta_{g-1}\Delta_g\Delta_{g-1}\Delta_g = \Delta_g\Delta_{g-1}\Delta_g\Delta_{g-1}.
\end{aligned} \tag{2.20.1}$$

(The last relation is not used in the discussion below.)

Let $R = R_{k,s}(x_1, \dots, x_s) = (x_1 \cdots x_s)^{2k}$. The following formula, which describes the image of α_H in terms of the basis X_w, was found independently by Demazure [D] and Bernstein-Gelfand-Gelfand [BGG], (see also [H, IV]). (Actually, the geometric identification of the Schubert basis with the basis X_w in the formula is done by [BGG].)

$$\alpha_H(R) = \sum_{w \in W_H} \varepsilon(\Delta_w(R))X_w, \tag{2.20.2}$$

where ε denotes the evaluation at 0, $\varepsilon(f) = f(0)$ for $f \in \mathbb{C}[x_1, \dots, x_g]$.

Now by (2.20.2), in order to prove Lemma 2.19 it is enough to show that $\varepsilon(\Delta_w(R)) = 0$ for $w \neq w_\beta$ and that $\varepsilon(\Delta_{w_\beta}(R)) = 1$.

First we note that $\Delta_i(R) = 0$ unless $i = s$ since R is invariant under $W_H^{(s)}$ with $s \leq d_3 < g$. It is easy to see that

$$\Delta_s(R) = S_{2k-1}(x_s, x_{s+1})(x_1 \cdots x_{s-1})^{2k},$$

where $S_{2k-1}(x_s, x_{s+1})$ is the total symmetric function as before.

For the next step, there are only two possibilities for Δ_i such that $\Delta_i\Delta_s(R) \neq 0$, i.e., $i = s-1$ or $i = s+1$. In fact, this follows from the first and second relation in (2.20.1). We note that

(2.20.3) In the expression of $\Delta_w = \Delta_{i_1}\Delta_{i_2}\cdots\Delta_{i_k}$ such that $\Delta_w(R) \neq 0$, we may assume that the last two terms are $\Delta_{s+1}\Delta_s$.

In fact, suppose the next term for Δ_s is Δ_{s-1}. So we consider $\Delta_{s-1}\Delta_s(R)$. Then the next non-zero possibility is one of Δ_{s-2}, Δ_s or Δ_{s+1}. But if Δ_{s+1} appears, then

$$\Delta_{s+1}\Delta_{s-1}\Delta_s(R) = \Delta_{s-1}\Delta_{s+1}\Delta_s(R)$$

and this is reduced to the case given in (2.20.3). If Δ_s appears, then

$$\Delta_s\Delta_{s-1}\Delta_s(R) = \Delta_{s-1}\Delta_s\Delta_{s-1}(R) = 0$$

by third relation in (2.20.1). So, the only remaining case is Δ_{s-2}, and we have to consider $\Delta_{s-2}\Delta_{s-1}\Delta_s(R)$. The similar consideration works in general, and we see that if $\Delta_{i_1}\cdots\Delta_{i_k}$ is not equal to the expression in (2.20.3), the possible choice for Δ_i is given by

$$\Delta_{s-j}\cdots\Delta_{s-2}\Delta_{s-1}\Delta_s(R).$$

By repeating this procedure, we can finally reach

$$R' = \Delta_1\Delta_2\cdots\Delta_{s-1}\Delta_s(R).$$

But since the degree of R is equal to $2ks > s$, the degree of R' is positive if $R' \neq 0$. Hence we need to proceed to the next step. The next possibility is then unique and it is given by Δ_{s+1}. It follows that this case is also reduced to the case in (2.20.3). Thus (2.20.3) is verified.

Now the similar consideration as (2.20.3) holds in general, and we may assume that $\Delta_{i_1}\cdots\Delta_{i_r}$ has the form $\Delta_{s+j}\cdots\Delta_{s+2}\Delta_{s+1}\Delta_s$ for the last $j+1$ terms. Using the formula

$$\Delta_{s+j}(S_h(x_s, x_{s+1}, \dots, x_{s+j})) = S_{h-1}(x_s, x_{s+1}, \dots, x_{s+j+1}),$$

we see easily that the above procedure continues until $j = 2k-1$, and that

$$\Delta_{s+2k-1}\cdots\Delta_{s+1}\Delta_s(R) = (x_1\cdots x_{s-1})^{2k}.$$

This implies, by induction on s, that the only non-zero possibility for $\Delta_w = \Delta_{i_1}\cdots\Delta_{i_r}$ is given by

$$\Delta_w = (\Delta_{2k}\cdots\Delta_2\Delta_1)\cdots(\Delta_{s+2k-2}\cdots\Delta_s\Delta_{s-1})(\Delta_{s+2k-1}\cdots\Delta_{s+1}\Delta_s),$$

and in this case, $\Delta_w(R) = 1$. Then it is easy to check that

$$w = \begin{pmatrix} 1 & 2 & \cdots & s & \cdots \\ 2k+1 & 2k+2 & \cdots & 2k+s & \cdots \end{pmatrix}.$$

Since $w \in \mathcal{D}$, w is written as $w = w_\beta$ with $\beta = (2k+1, 2k+2, \dots, 2k+s)$. This proves Lemma 2.19.

2.21. We now verify the statement (2.7.1) under the condition (i). In this case, we have

$$A = \begin{cases} (2^{2d_2}, 1^{2d_3}) & \text{if } G = Sp_{2n}, \\ (2^{2d_2}, 1^{2d_3+1}) & \text{if } G = SO_{2n+1}, \end{cases}$$

and $T_{j,t,k}(x_1, \dots, x_t) = x_1^{2k}$ with $k = d_2 + d_3$. Hence we have only to show that

(2.21.1) $\phi \circ \alpha(x_1^{2k})$ vanishes on $H^*(\mathcal{B}_A)$.

Let $V_0 = \operatorname{Ker} A$. We consider the Grassmann varieties $\mathcal{G}_1(V)$ and $\mathcal{G}_1(V_0)$. Note that $\mathcal{G}_1(V)$ is isomorphic to the variety $\mathcal{P}$ of parabolic subgroups of G conjugate to $P^{(1)}$, where $P^{(1)}$ is the maxiaml parabolic subgroup of G as given in the proof of Lemma 2.18 with $s = 1$. Then $\mathcal{G}_1(V_0)$ is isomorphic to $\mathcal{P}_A$, the subvariety of $\mathcal{P}$ consisting of parabolic subgroups whose Lie algebra contains A. We have the following commutative diagram.

$$\begin{array}{ccc} H^*(\mathcal{B}) & \xleftarrow{\pi_1^*} & H^*(\mathcal{P}) \\ \downarrow & & \downarrow \psi \\ H^*(\mathcal{B}_A) & \longleftarrow & H^*(\mathcal{P}_A). \end{array}$$

As discussed in the proof of Lemma 2.17, π_1^* gives an isomorphism $H^*(\mathcal{P})$ and $H^*(\mathcal{B}^{W^{(1)}})$, where $W^{(1)}$ is the Weyl group of type C_{n-1} with $n-1$ letters $\{2, \dots, n\}$. Since $\alpha(x_1^{2k})$ is $W^{(1)}$-invariant, we may assume that $\alpha(x_1^{2k})$ lies in $H^*(\mathcal{P})$. Hence, in order to prove (2.21.1), it is enough to show that $\psi \circ \alpha(x_1^{2k})$ vanishes on $H^*(\mathcal{P}_A)$. Now, $\dim V_0 = 2k$ (resp. $2k+1$), and $\mathcal{P}_A$ is isomorphic to the projective space $\mathbb{P}(V_0)$ (resp. a quadric in $\mathbb{P}(V_0)$ if $G = Sp_{2n}$ (resp. $G = SO_{2n+1}$), respectively. Hence $\dim \mathcal{P}_A = 2k-1$ for both cases. But since $\alpha(x_1^{2k}) \in H^{4k}(\mathcal{P})$, we must have $\psi \circ \alpha(x_1^{2k}) = 0$ on $H^*(\mathcal{P}_A)$. This proves (2.21.1) and so (2.7.1) is verified. This completes the proof of Proposition 2.1.

3. The Injectivity of θ

3.1. Let $\theta : H^{2i}(\mathcal{B}_{f(A)}) \to H^{4i}(\mathcal{B}_A)_1^D$ be the map constructed in Proposition 2.1. The aim of this section is to show that the map θ is injective, and to complete the proof of Theorem 1.9. We shall prove the injectivity of θ by passing to the situation where the groups are defined over a finite field so that one can make use of the Frobenius action on the cohomologies. So, we consider a finite field $\mathbb{F}_q$ with $p = ch(\mathbb{F}_q)$ large enough, and let $G, \bar{G}, \mathfrak{g}, \bar{\mathfrak{g}}$ be the similar objects as in Section 2, which are defined over $\mathbb{F}_q$. Then the l-adic cohomology group $H^*(\mathcal{B}_A, \overline{\mathbb{Q}}_l) = H^*(\mathcal{B}_A)$ together with the natural map $\phi_A : H^*(\mathcal{B}) \to H^*(\mathcal{B}_A)$ has the identical W-module structure with the corresponding cohomology in the complex case. This is true also for $\bar{G}$, and so by Proposition 2.1, one can construct a map θ in this setup. Since such a θ is unique, if one could prove the injectivity of θ in the case of $\mathbb{F}_q$, it implies the injectivity in the case of $\mathbb{C}$. Hence the following proposition will imply the theorem.

PROPOSITION 3.2. *Let $G, \bar{G}$ be groups defined over $\mathbb{F}_q$. Assume that θ is the map satisfying the commutative diagram (1.9.1). Then θ is injective.*

3.3. The remaining part of this section is devoted to the proof of Proposition 3.2. We use the similar notation as in Section 2, but replacing $\mathbb{C}$ by $\overline{\mathbb{F}}_q$, the algebraic closure of $\mathbb{F}_q$. Let $P = P^{(1)}$ be the maximal parabolic subgroup of G, with a Levi subgroup L of G such that the corresponding Weyl group W_L is of type C_{n-1}, and $\mathcal{P}$ the variety of parabolic subgroups of G conjugate to P as in 2.21. For a nilpotent element $A \in \mathcal{N}_{\mathfrak{g}}$, we have a natural map $\pi : \mathcal{B}_A \to \mathcal{P}_A$, which is the restriction of $\pi^{(1)}$ to $\mathcal{B}_A$ under the identification $\mathcal{B} \simeq \mathcal{F}(V)$ and $\mathcal{P} \simeq \mathcal{G}_1(V)$. As given in §2 in [Sh2], the map π has a locally trivial filtration. Each fibre of π is isomorphic to the variety $\mathcal{B}^L_{A'}$, where $\mathcal{B}^L$ is the variety of Borel subgroups in L, and A' is an element in $\mathcal{N}_{\mathfrak{l}}$ ($\mathfrak{l}$ is the Lie algebra of L), whose Young diagram is obtained from that of A by removing two boxes according to the filtration as explained below. In the following we shall show Proposition 3.2, by induction on the rank of G, by making use of this filtration.

3.4. We describe the filtration more precisely following [loc. cit.]. Let $A \in \mathcal{N}_{\mathfrak{g}}$ be of the type $A = (1^{m_1}, 2^{m_2}, \dots, h^{m_h})$, and put $V^{(s)} = \operatorname{Ker} A \cap \operatorname{Im} A^{s-1}$ for $s = 1, \dots, h$. We have a filtration of $\operatorname{Ker} A$ by subspaces $V^{(s)}$ with $\dim V^{(s)}/V^{(s+1)} = m_s$. Let $Y^{(s)} = \mathbb{P}(V^{(s)})$ be the projective space of $V^{(s)}$. Then we have a filtration

$$\mathbb{P}(\operatorname{Ker} A) = Y^{(1)} \supset Y^{(2)} \supset \cdots \supset Y^{(h)} \supset Y^{(h+1)} = \emptyset. \tag{3.4.1}$$

Assume that $m = m_s \neq 0$. As explained in §2 in [loc. cit.], one can consider a non-degenerate bilinear form defined over $\mathbb{F}_q$ on $V^{(s)}/V^{(s+1)}$, which is symmetric in the case where $G = Sp_{2n}$ and s is even, or $G = SO_{2n+1}$ and s is odd, and symplectic in other cases. In the case where the form on $V^{(s)}/V^{(s+1)}$ is symmetric, we define a closed subvariety $Q^{(s)}$ of $Y^{(s)}$ as the one associated to the subset of $V^{(s)}$ which is the pullback of the quadric on $V^{(s)}/V^{(s+1)}$ defined by its quadratic form. We put $C^{(s)} = Y^{(s)} - Q^{(s)}$. It is known that $\mathcal{P}_A$ is isomorphic to $Y^{(1)}$ (resp. $Q^{(1)}$) in the case where $G = Sp_{2n}$ (resp. $G = SO_{2n+1}$). Moreover, in each step of the filtration, the structure of $\pi^{-1}(x)$ for $x \in \mathcal{P}_A$ is given according to the types of $V^{(s)}$ as follows.

Type I (the symmetric case).

Note that s is even if $G = Sp_{2n}$, or s is odd if $G = SO_{2n+1}$. In this case, for $x \in Q^{(s)} - Y^{(s+1)}$, $\pi^{-1}(x) \simeq \mathcal{B}^L_{A'}$, where the Young diagram of A' is obtained from that of A by replacing two rows of length s by two rows of length $s-1$. (Note if $Q^{(s)} \neq Y^{(s+1)}$, then $m \geq 2$.) If $x \in C^{(s)}$, $\pi^{-1}(x) \simeq \mathcal{B}^L_{A''}$, where the Young diagram of A'' is obtained from that of A by replacing one row of length s by one row of length $s-2$.

Type II (the symplectic case).

In this case, for $x \in Y^{(s)} - Y^{(s+1)}$, $\pi^{-1}(x)$ is isomorphic to $\mathcal{B}^L_{A'}$, where the Young diagram of A' is obtained from that of A by replacing two rows of length s by two rows of length $s-1$.

Fixing an $\mathbb{F}_q$-basis, we express the vector $v \in V^{(s)}/V^{(s+1)}$ by $v = (x_1, \dots, x_m)$. Let us define a subspace V_j $(0 \leq j \leq m)$ of $V^{(s)}$ containing $V^{(s+1)}$ by

$$V_j/V^{(s+1)} = \{v = (x_l) \in V^{(s)}/V^{(s+1)} \mid x_1 = \dots = x_j = 0\}.$$

By putting $Y_j = \mathbb{P}(V_j)$, we have a filtration of $Y^{(s)}$ by projective spaces Y_j,

$$Y^{(s)} = Y_0 \supset Y_1 \supset \dots \supset Y_m = Y^{(s+1)}, \tag{3.4.2}$$

where $Y_j - Y_{j+1} \simeq \mathbb{A}^{b-j-1}$ with $b = b_s = \dim V^{(s)}$. In the case of type I, we may assume that the quadratic form Q on $V^{(s)}/V^{(s+1)}$ is given as $Q(v) = 2x_1x_{2r} + 2x_2x_{2r-1} + \dots + 2x_rx_{r+1}$ if $m = 2r$ and $Q(v) = 2x_1x_{2r+1} + \dots + 2x_rx_{r+2} + x_{r+1}^2$ if $m = 2r+1$. We put $Q_j = Q^{(s)} \cap Y_j$. Thus we have the following filtration.

(3.4.3) If $m = 2r$, then

$$Q^{(s)} = Q_0 \supset Q_1 \supset \cdots \supset Q_{r-1} \supset Q_{r+1} \supset \cdots \supset Q_{2r-1} \supset Q_{2r} = Y^{(s+1)}.$$

(3.4.4) If $m = 2r+1$, then

$$Q^{(s)} = Q_0 \supset Q_1 \supset \cdots \supset Q_r \supset Q_{r+2} \supset \cdots \supset Q_{2r} \supset Q_{2r+1} = Y^{(s+1)}.$$

Note that $Q_j = Y_j$ if $j \ge [(m-1)/2] + 2$ for both cases. It follows that

$$\text{(3.4.5)} \qquad Q_j - Q_{j+1} \simeq \begin{cases} \mathbb{A}^{b-j-2} & \text{if } 0 \le j \le [(m-1)/2] - 1, \\ \mathbb{A}^{b-j-1} & \text{if } [(m-1)/2] + 2 \le j \le m-1, \end{cases}$$

and that

$$\text{(3.4.6)} \qquad Q_j - Q_{j+2} \simeq \begin{cases} \mathbb{A}^{b-j-2} \coprod \mathbb{A}^{b-j-2} \\ \qquad \text{if } j = [(m-1)/2] \text{ and } m = 2r, \\ \mathbb{A}^{b-j-2} \\ \qquad \text{if } j = [(m-1)/2] \text{ and } m = 2r+1. \end{cases}$$

We also consider the filtration of $C^{(s)}$ as follows. Let $C_j = C^{(s)} \cap Y_j$ for $j = 0, \dots, r$. Then we have

$$C^{(s)} = C_0 \supset C_1 \supset \cdots \supset C_{r-1} \supset \cdots$$

where $C_j - C_{j+1} \simeq \mathbb{A}^{b-j-1} - \mathbb{A}^{b-j-2}$ for $j = 0, \dots, r-2$ if $m = 2r$ and for $j = 0, \dots, r-1$ if $m = 2r+1$. Moreover, if $m = 2r$, the last term C_{r-1} is isomorphic to $\mathbb{A}^{b-r} - \mathbb{A}^{b-r-1}$, while if $m = 2r+1$, the last term C_r is isomorphic to $\mathbb{A}^{b-r-1}$.

3.5. As described in [Sh2], the map π is locally trivial with respect to the filtration of $\mathcal{B}_A$ considered in 3.4. First assume that $V^{(s)}$ is of type II. Then for $Z = Y_j - Y_{j+1}$, we have $\pi^{-1}(Z) \simeq Z \times \mathcal{B}^L_{A'}$, and so $H^k_c(\pi^{-1}(Z)) \simeq H^{k-2(b-j-1)}(\mathcal{B}^L_{A'})$. Note that $H^i_c(\pi^{-1}(Z))$ has a natural structure of W_L-module, and this action is compatible with the action on $H^{i'}(\mathcal{B}^L_{A'})$. This fact, and the corresponding statement for $Q_j - Q_{j+1}$ or $C_j - C_{j+1}$ were already shown in [Sh1]. But since the proof there is based on Springer's construction of Springer representations, we give in Appendix (cf. Proposition A) a proof of this fact based on Lusztig's construction, (in a more general form, by making use of Borho-MacPherson's results [BM]). Now each Z as above admits a natural action of $A_G(A)$, and the relationship with the action of $A_L(A')$ on $\mathcal{B}^L_{A'}$ is described in [Sh2]. It follows from this that we have

$$H^k_c(\pi^{-1}(Z))^{D'}_1 \simeq H^{k-2(b-j-1)}(\mathcal{B}^L_{A'})^{D'}_1 \tag{3.5.1}$$

for $Z = Y_j - Y_{j+1}$.

Next consider the case where $V^{(s)}$ is of type I. Then again we have $\pi^{-1}(Z) \simeq Z \times \mathcal{B}^L_{A'}$, for $Z = Q_j - Q_{j+1}$ or $Z = Q_j - Q_{j+2}$ according to the cases (3.4.5), (3.4.6). Thus $H^i_c(\pi^{-1}(Z))$ is isomorphic to $H^{i'}(\mathcal{B}^L_{A'})$ or a direct sum of copies of $H^{i'}(\mathcal{B}^L_{A'})$. In this case also $H^i_c(\pi^{-1}(Z))$ admits an action of W_L and of $A_G(A)$ as before. We now consider the Frobenius map $F : G \to G$. If X is an F-stable locally closed subvariety of $\mathcal{B}$, the map F induces an action F^* on $H^*_c(X)$. We denote by $H^*_c(X)_{\mathrm{ev}}$ the sum of generalized eigenspaces of F^* corresponding to the eigenvalues q^{2j} for $j \ge 0$. Note that $H^i(\mathcal{B}^L_{A'})^{D'}_1 = 0$ except when $i \equiv 0 \pmod 4$ by Lemma 1.3. Then the following formula is easily deduced from (3.4.5).

(3.5.2) Let $Z = Q_j - Q_{j+1}$. Then we have

$$H^k_c(\pi^{-1}(Z))^{D'}_{1,\mathrm{ev}} = \begin{cases} H^{k-2b+2j+4}(\mathcal{B}^L_{A'})^{D'}_1 & \text{if } b-j : \text{even} \\ & \quad \text{and } 0 \le j \le [(m-1)/2]-1, \\ H^{k-2b+2j+2}(\mathcal{B}^L_{A'})^{D'}_1 & \text{if } b-j : \text{odd} \\ & \quad \text{and } [(m-1)/2]+2 \le j \le m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Also it follows from (3.4.6), we have

(3.5.3) Let $Z = Q_j - Q_{j+2}$ for $j = [(m-1)/2]$. Then

$$H^k_c(\pi^{-1}(Z))^{D'}_{1,\mathrm{ev}} = \begin{cases} H^{k-2b+2j+4}(\mathcal{B}^L_{A'})^{D'}_1 & \text{if } b-j : \text{even}, \\ 0 & \text{if } b-j : \text{odd}. \end{cases}$$

In particular, we have

(3.5.4) Let $Z = Q_j - Q_{j+1}$ or $Q_j - Q_{j+2}$ as above. Then $H^k_c(\pi^{-1}(Z))^{D'}_{1,\mathrm{ev}} = 0$ except when $k \equiv 0 \pmod 4$.

Next we consider the filtration of $C^{(s)}$. Let $Z = C_j - C_{j+1}$ $(0 \le j \le r-1)$. Then by [Sh2], there exists a double covering $\widehat{Z} \to Z$ such that

$$\pi^{-1}(Z) \times_Z \widehat{Z} \simeq \widehat{Z} \times \mathcal{B}^L_{A''}.$$

Moreover, if $m = 2r+1$ and $Z = C_r$, we have $\pi^{-1}(Z) \simeq Z \times \mathcal{B}^L_{A''}$. We now assume that $Z = C_j - C_{j+1}$. Then by [loc. cit. §2], we have

$$H^*_c(Z)_\sigma \otimes H^*(\mathcal{B}^L_{A''})^{D'}_1 \simeq H^*_c(\pi^{-1}(Z))^{D'}_1,$$

where σ is an automorphism on Z as defined in 2.2 in [loc. cit.], and $H^*_c(Z)_\sigma$ denotes the σ-fixed subspace of $H^*_c(Z)$ with respect to the induced action of σ. It is easy to see that σ acts trivially on $H^*_c(Z)$, and we have

$$H^k_c(\pi^{-1}(Z))^{D'}_1 \simeq H^{k-2d}(\mathcal{B}^L_{A''})^{D'}_1(-d) \oplus H^{k-2d+1}(\mathcal{B}^L_{A''})^{D'}_1(-d+1)$$

where $d = b-j-1$ and $(\cdot)$ is the Tate twist. It follows from this and from Lemma 1.3 that

$$\text{(3.5.5)}\quad H^k_c(\pi^{-1}(Z))^{D'}_{1,\text{ev}} \simeq \begin{cases} H^{k-2d}(\mathcal{B}^L_{A''})^{D'}_1 & \text{if } k \equiv 0\,(4) \text{ and } d \text{ is even}, \\ H^{k-2d+1}(\mathcal{B}^L_{A''})^{D'}_1 & \text{if } k \equiv 1\,(4) \text{ and } d \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

In particular, we have

$$\text{(3.5.6)}\qquad H^{4k-1}_c(\pi^{-1}(Z))^{D'}_{1,\text{ev}} = 0.$$

This implies the following lemma.

Lemma 3.6. *For any $j \ge 0$, we have*

$$H^{4k-1}_c(\pi^{-1}(C_j))^{D'}_{1,\text{ev}} = 0.$$

Proof. In the case where $m = 2r$, $H^{4k-1}_c(\pi^{-1}(C_{r-1}))^{D'}_1 = 0$ by (3.5.6). If $m = 2r+1$, $H^i_c(\pi^{-1}(C_r)) \simeq H^{i-2(b-r-1)}(\mathcal{B}^L_{A''})$ and so $H^{4k-1}_c(\pi^{-1}(C_r))^{D'}_{1,\text{ev}} = 0$ also. Now taking the 1-part, ev-part and D'-invariant part are exact functors and preserve the long exact sequence of cohomologies. Then the long exact sequence of cohomologies with respect to $C_{j-1} \subset C_j$ combined with (3.5.6) implies the lemma. □

REMARK 3.7. The irreducible components of $\mathcal{B}_A$ are parameterized by tableaux with $2n$ boxes, and with entries $1, 1, 2, 2, \dots, n, n$. This parameterization is compatible with the locally trivial fibration in the following sense. Suppose Z appears in the top part of the filtration, such as $Z = Y_0 - Y_1, Q_0 - Q_1, Q_0 - Q_2$ or $C_0 - C_1$. Then Z is irreducible or a disjoint union of two copies of irreducible subsets. Hence, if $\dim \pi^{-1}(Z) = \dim \mathcal{B}_A$, it is possible to construct irreducible components of $\mathcal{B}_A$ by making use of the irreducible components of $\mathcal{B}^L_{A'}$ or $\mathcal{B}^L_{A''}$. For example, if $\pi^{-1}(Z) \simeq Z \times \mathcal{B}^L_{A'}$, and Z is irreducible, the closure of $Z \times I$ gives an irreducible component of $\mathcal{B}_A$ for each irreducible component I of $\mathcal{B}^L_{A'}$. All the irreducible components of $\mathcal{B}_A$ are obtained in this way, and they were described precisely by Spaltenstein in his unpublished paper [Sous groupes de Borel contenant un unipotent donne], and were summarized in [S3]. In the following, we just give a list of Z which produce irreducible components of $\mathcal{B}_A$, (see also [Sh1, Prop. 2.6]).

(i) Type II. $Z = Y_0 - Y_1$ is irreducible.
(ii) Type I, $m_s > 2$. $Z = Q_0 - Q_1$ is irreducible.
(iii) Type I, $m_s = 2$. $Z = Q_0 - Q_2$ is a disjoint union of two copies of irreducible components.
(iv) Type I, $m_s \geq 1, m_{s-1} = 0$. $Z = C_0 - C_1$ is irreducible.

In cases (i), (ii), suppose we know the tableau corresponding to an irreducible component of $\mathcal{B}_{A'}$. Then the corresponding irreducible component of $\mathcal{B}_A$ is obtained by adding a vertical strip of 2 boxes containing n. In the case of (iii), we add a vertical strip in the same way as above but we get two components of $\mathcal{B}_A$ parameterized by the same tableau. In the case (iv), we get a tableau corresponding to an irreducible component of $\mathcal{B}_A$ by adding a horizontal strip of two boxes containing n to a tableau which parameterizes an irreducible component of $\mathcal{B}_{A''}$.

3.8. Let $f(A) = A_0 \in \mathcal{N}_{\bar{\mathfrak{g}}}$. As in 2.17, we consider $\bar{G} = GL(\overline{V})$. We can write A_0 as $A_0 = (1^{n_1}, 2^{n_2}, \dots)$. Put $\overline{V}^{(s)} = \operatorname{Ker} A_0 \cap \operatorname{Im} A_0^{s-1}$. We have a filtration of $\operatorname{Ker} A_0$ by subspaces $\overline{V}^{(s)}$ with $\dim \overline{V}^{(s)}/\overline{V}^{(s+1)} = n_s$. Let $\overline{Y}^{(s)} = \mathbb{P}(\overline{V}^{(s)})$. Assume that $n_s \neq 0$. As in 3.4, we define subspaces $\overline{V}_j$ $(0 \leq j \leq n_s)$, and associated projective spaces $\overline{Y}_j = \mathbb{P}(\overline{V}_j)$. Hence we have a filtration

$$\overline{Y}^{(s)} = \overline{Y}_0 \supset \overline{Y}_1 \supset \cdots \supset \overline{Y}_{n_s} = \overline{Y}^{(s+1)}.$$

Let $\overline{P}$ be the maximal parabolic subgroup of $\bar{G}$ with a Levi subgroup $\overline{L}$ such that the corresponding Weyl group (which we denote by $\overline{W}_L$ instead of $\overline{W}_{\overline{L}}$) isomorphic to S_{n-1}. Let $\overline{\mathcal{P}}$ be the variety of parabolic subgroups of $\bar{G}$ conjugate to $\overline{P}$. Then as in 3.3, we have a natural map $\overline{\pi} : \overline{\mathcal{B}}_{A_0} \to \overline{\mathcal{P}}_{A_0}$. As in 3.3, each fibre of $\overline{\pi}$ is isomorphic to the variety $\overline{\mathcal{B}}^L_{A'_0}$, where $\overline{\mathcal{B}}^L$ is the variety of Borel subgroups in $\overline{L}$, and A'_0 is a nilpotent element in $\bar{\mathfrak{l}}$ obtained by removing one square from the Young diagram of A_0. Note that $\overline{\mathcal{P}}_{A_0} \simeq \mathbb{P}(\operatorname{Ker} A_0) = \overline{V}^{(1)}$, and the map $\overline{\pi}$ is locally trivial with respect to the above filtration, i.e., for each $\overline{Z} = \overline{Y}_j - \overline{Y}_{j+1}$, we have $\overline{\pi}^{-1}(\overline{Z}) \simeq \overline{Z} \times \overline{\mathcal{B}}^L_{A'_0}$. Here $\overline{Z} \simeq \mathbb{A}^{\bar{b}-j-1}$ with $\bar{b} = \bar{b}_s = \dim \overline{V}^{(s)}$.

In what follows, we are interested in comparing the filtration of $\mathcal{B}_A$ and that of $\overline{\mathcal{B}}_{A_0}$. For this, we will make the construction of A_0 more transparent. Remember that $b_s = \dim V^{(s)}$. We define a number δ_s for each s by

$$\delta_s = \begin{cases} 0 & \text{if } b_s : \text{even}, \\ 1 & \text{if } b_s : \text{odd}. \end{cases}$$

Then the following statement is easily deduced from 1.5.

(3.8.1) Let $A = (1^{m_1}, \dots, s^{m_s}, \dots)$ and $A_0 = (1^{n_1}, \dots, s^{n_s}, \dots)$. Then we have

$$A_0 = \begin{cases} (\dots, s_1^{\delta_s}, s^{[(m_s-\delta_s)/2]}, \dots) & \text{if } V^{(s)} : \text{type I}, \\ (\dots, s_1^{\delta_s}, s^{m_s/2-\delta_s}, \dots) & \text{if } V^{(s)} : \text{type II}, \end{cases}$$

for some integer $s_1 \le s$. (Note that in the second case m_s is even.) Assume that b_s is odd and let $s' < s$ be the largest number such that $m_{s'} \ne 0$. If $s' \le s-2$, then $s' < s_1 < s$. If $s' = s-1$, then $s_1 = s$ or $s_1 = s-1$, and $V^{(s_1)}$ is of type II in each case.

(3.8.1) means that the Young diagram of A_0 is obtained from that of A by replacing the rectangle consisting of rows of length s by a smaller rectangle with rows the same length, and adding one row of length s_1 below this rectangle when b_s is odd.

The following fact is also easily verified from the definition of f.

(3.8.2) Let A' and A'' be as in 3.4. We express $f(A) = A_0$ as in (3.8.1). Then the Young diagram of $f(A')$ is obtained from that of A_0 by replacing one row of length s by a row of length $s-1$. In the case where $V^{(s)}$ is of type I and b_s is odd, the Young diagram of $f(A'')$ is obtained from that of A_0 by replacing one row of length s_1 by a row of length $s_1 - 1$.

3.9. We shall prove Proposition 3.2 using induction on the rank of W. For $n = 1$, the proposition is true. So we assume that the proposition holds for a group whose rank is smaller than n. In particular we assume, in the remainder of this section, that the proposition holds for L. We now consider each step $Y^{(s)} \supset Y^{(s+1)}$ of the filtration of $\mathcal{P}_A$ in (3.4.1) separately. First assume that $V^{(s)}$ is of type I. In the following discussion, we fix such s, and put $m = m_s, b = b_s, \bar{b} = \bar{b}_s$ and $\delta = \delta_s$, respectively. Along with $Y^{(s)} \subset \mathcal{P}_A$, we consider $\overline{Y}^{(s)} \subset \overline{\mathcal{P}}_{A_0}$. We shall construct maps ξ_j, which connects the cohomologies of the refinements of these filtrations, as follows.

LEMMA 3.10. *Let $V^{(s)}$ be of type I. Then for each j such that $0 \le j < [(m-\delta)/2]$, there exist maps*

$$\xi_j : H^{2k}(\bar{\pi}^{-1}(\overline{Y}_j)) \to \begin{cases} H^{4k}(\pi^{-1}(Q_{2j+\delta}))^{D'}_{1,\mathrm{ev}} \\ \qquad \text{if } \ 0 \le 2j+\delta \le [(m-1)/2], \\ H^{4k}(\pi^{-1}(Q_{2j+1+\delta}))^{D'}_{1,\mathrm{ev}} \\ \qquad \text{if } \ [(m-1)/2]+2 \le 2j+1+\delta < m, \end{cases}$$

so that the following diagram commutes.

(3.10.1)

$$\begin{array}{ccccccc} H^{2k}(\overline{\mathcal{B}}_{A_0}) & \longrightarrow & H^{2k}(\bar{\pi}^{-1}(\overline{Y}_0)) & \longrightarrow & H^{2k}(\bar{\pi}^{-1}(\overline{Y}_1)) & \longrightarrow \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\xi_0} & & \downarrow{\scriptstyle\xi_1} & \\ H^{4k}(\mathcal{B}_A)^D_1 & \longrightarrow & H^{4k}(\pi^{-1}(Q_{0+\delta}))^{D'}_{1,\mathrm{ev}} & \longrightarrow & H^{4k}(\pi^{-1}(Q_{2+\delta}))^{D'}_{1,\mathrm{ev}} & \longrightarrow \end{array}$$

where $H^{4k}(\mathcal{B}_A)^D_1 = H^{4k}(\mathcal{B}_A)^{D'}_{1,\mathrm{ev}}$ by Lemma 1.3, and the horizontal maps are the natural maps induced from the closed immersions $\pi^{-1}(Q_{2+\delta}) \hookrightarrow \pi^{-1}(Q_{0+\delta}) \hookrightarrow \mathcal{B}_A$, etc.

PROOF. Let $\bar{x}_i \in H^*(\overline{\mathcal{B}}_{A_0})$ be the image of x_i under the map $\bar{\mathbf{Q}}_l[x_1, x_2, \dots, x_n] \to H^*(\overline{\mathcal{B}}_{A_0})$. Then according to [CP, Lemma 4.5], it is known that the natural map $H^*(\overline{\mathcal{B}}_{A_0}) \to H^*(\bar{\pi}^{-1}(\overline{Y}_j))$ is surjective, and that the kernel of it is the principal ideal generated by $\bar{x}_1^{\bar{b}-j}$. (Although Lemma 4.5 is stated for the groups over $\mathbb{C}$, it works also for the case of characteristic p with l-adic cohomology.)

We have a diagram, for $j = 0, 1, 2, \dots$

$$\begin{array}{ccccc} H^*(\overline{\mathcal{B}}) & \xrightarrow{\bar{\phi}} & H^*(\overline{\mathcal{B}}_{A_0}) & \longrightarrow & H^*(\bar{\pi}^{-1}(\overline{Y}_j)) \\ {\scriptstyle\theta_0}\downarrow & & \downarrow{\scriptstyle\theta} & & \\ H^*(\mathcal{B})^D & \xrightarrow{\phi_D} & H^*(\mathcal{B}_A)^D_1 & \longrightarrow & H^*(\pi^{-1}(Q_{2j+\delta}))^{D'}_{1,\mathrm{ev}} \end{array}$$

where $H^*(\pi^{-1}(Q_{2j+\delta}))^{D'}_{1,\mathrm{ev}}$ is a subring of the cohomology ring $H^*(\pi^{-1}(Q_{2j+\delta}))$.

Note that, since $2\bar{b} = b-\delta$, we have $\theta(\bar{x}_1^{\bar{b}-j}) = \phi_D\circ\alpha(x_1^{b-\delta-2j})$. Therefore, in order to construct a map ξ_j, it is enough to show that the image of $\alpha(x_1)^{b-\delta-2j}$ under the map $H^*(\mathcal{B})^D \to H^*(\pi^{-1}(Q_{2j+\delta}))^{D'}_{1,\mathrm{ev}}$ vanishes. Here we have a commutative diagram

$$
\begin{array}{ccccc}
H^*(\mathcal{B})_{\mathrm{ev}} & \longrightarrow & H^*(\pi^{-1}(Y_{2j+\delta}))_{\mathrm{ev}} & \longrightarrow & H^*(\pi^{-1}(Q_{2j+\delta}))_{\mathrm{ev}} \\
\uparrow & & \uparrow & & \uparrow \\
H^*(\mathcal{P})_{\mathrm{ev}} & \longrightarrow & H^*(Y_{2j+\delta})_{\mathrm{ev}} & \longrightarrow & H^*(Q_{2j+\delta})_{\mathrm{ev}}.
\end{array}
\tag{3.10.2}
$$

Since $H^*(\mathcal{P})\xrightarrow{\sim}H^*(\mathcal{B})^{W_L}$, $\alpha(x_1)^2 \in H^*(\mathcal{B})^{D'}_{\mathrm{ev}}$ is in fact contained in $H^*(\mathcal{P})_{\mathrm{ev}}$. Furthermore, since $Y_{2j+\delta} = \mathbb{P}((V_{2j+\delta}))$, we see that $H^*(Y_{2j+\delta})_{\mathrm{ev}} \simeq \mathbb{Z}[x_1^2]/x_1^{b-\delta-2j}$. Hence the image of $\alpha(x_1)^{b-\delta-2j} \in H^*(\mathcal{P})_{\mathrm{ev}}$ vanishes on $H^*(Y_{2j+\delta})_{\mathrm{ev}}$, so it is zero on $H^*(\pi^{-1}(Q_{2j+\delta}))^{D'}_{\mathrm{ev}}$. This shows the vanishing of the image of $\alpha(x_1)^{b-\delta-2j}$. The above argument covers the cases $Q_{2j+\delta}$ for $0 \le 2j+\delta \le [(m-1)/2]$. Next consider $Q_{2j+1+\delta}$ for j such that $[(m-1)/2]+2 \le 2j+1+\delta < m$. In this case, again we have $H^*(Y_{2j+1+\delta})_{\mathrm{ev}} \simeq \mathbb{Z}[x_1^2]/x_1^{b-\delta-2j}$. So the image of $\alpha(x_1)^{b-\delta-2j} \in H^*(\mathcal{P})_{\mathrm{ev}}$ vanishes on $H^*(Y_{2j+1+\delta})_{\mathrm{ev}}$. Hence the similar diagram can be used to show the vanishing of $\alpha(x_1)^{b-\delta-2j}$ on $H^*(\pi^{-1}(Q_{2j+1+\delta}))^{D'}_{1,\mathrm{ev}}$. This proves the lemma. □

In view of (3.10.2), the above proof implies, in particular the following statement.

Corollary 3.11. *The map $\xi_j : H^*(\bar{\pi}^{-1}(\overline{Y}_j)) \to H^*(\pi^{-1}(Q_{2j+\delta}))^{D'}_{1,\mathrm{ev}}$ factors through the map $\xi'_j : H^*(\bar{\pi}^{-1}(\overline{Y}_j)) \to H^*(\pi^{-1}(Y_{2j+\delta}))^{D'}_{1,\mathrm{ev}}$ via the natural map $H^*(\pi^{-1}(Y_{2j+\delta}))^{D'}_{1,\mathrm{ev}} \to H^*(\pi^{-1}(Q_{2j+\delta}))^{D'}_{1,\mathrm{ev}}$.*

We note that using the similar argument as in the proof of Lemma 3.10, one can extend the definition of ξ_j for $j = [(m-\delta)/2]$ also, with a slight modification, i.e., we have a map

$$\xi_j : H^{2k}(\overline{\pi}^{-1}(\overline{Y}^{(s+1)})) \to H^{4k}(\pi^{-1}(Y^{(s+1)}))^{D'}_{1,\mathrm{ev}}$$

making the diagram (3.10.1) commutative, where $\overline{Y}^{(s+1)} = \overline{Y}_j$. We now prove the following proposition.

Proposition 3.12. *Assume that the map ξ_j is injective for $j = [(m-\delta)/2]$. Then ξ_j is injective for any j such that $0 \le j \le [(m-\delta)/2]$.*

Proof. We prove the lemma by backward induction on j. First we note that

$$H_c^{4k+1}(\pi^{-1}(Q_j - Q_i))_{1,\mathrm{ev}}^{D'} = 0 \quad \text{for any } j < i,$$

by (3.5.4). Also, we have

$$\text{(3.12.1)} \qquad H^{4k-1}(\pi^{-1}(Q_j))_{1,\mathrm{ev}}^{D'} = 0 \quad \text{for any } j \ge 0.$$

In fact, by using (3.5.4), this is reduced to showing that $H^{4k-1}(\pi^{-1}(Y^{(s+1)}))_{1,\mathrm{ev}}^{D'} = 0$. But in general, if $V^{(s)}$ is of type II, the vanishing of $H^{4k-1}(\pi^{-1}(Y^{(s)}))_{1,\mathrm{ev}}^{D'}$ is reduced to that of $\pi^{-1}(Y^{(s+1)})$ by 3.5. If $V^{(s)}$ is of type I, by making use of Lemma 3.6, it is reduced to showing that $H^{4k-1}(\pi^{-1}(Q^{(s)}))_{1,\mathrm{ev}}^{D'} = 0$ and so again reduced to the case $\pi^{-1}(Y^{(s+1)})$. Hence (3.12.1) follows by backward induction on s.

Then we get an exact sequence,

$$0 \to H_c^{4k}(\pi^{-1}(Z_j))_{1,\mathrm{ev}}^{D'} \to H^{4k}(\pi^{-1}(Q_{2j+\delta}))_{1,\mathrm{ev}}^{D'} \to H^{4k}(\pi^{-1}(Q_{2j+2+\delta}))_{1,\mathrm{ev}}^{D'} \to 0,$$

where $Z_j = Q_{2j+\delta} - Q_{2j+2+\delta}$, and a similar formula holds also for the closed immersion $Q_{2j+3+\delta} \hookrightarrow Q_{2j+1+\delta}$. We have another exact sequence, for $\overline{Z}_j = \overline{Y}_j - \overline{Y}_{j+1}$,

$$0 \to H_c^{2k}(\bar\pi^{-1}(\overline{Z}_j)) \to H^{2k}(\bar\pi^{-1}(\overline{Y}_j)) \to H^{2k}(\bar\pi^{-1}(\overline{Y}_{j+1})) \to 0.$$

Combining these two sequences together, we have

(3.12.2) There exists a unique map

$$\xi_j^* : H_c^{2k}(\bar\pi^{-1}(\overline{Z}_j)) \to H_c^{4k}(\pi^{-1}(Z_j))_{1,\mathrm{ev}}^{D'}$$

for any j such that $0 \le 2j+\delta \le [(m-1)/2] - 2$, which makes the following diagram commute.

$$\begin{array}{ccccccccc}
0 & \to & H_c^{2k}(\bar\pi^{-1}(\overline{Z}_j)) & \to & H^{2k}(\bar\pi^{-1}(\overline{Y}_j)) & \to & H^{2k}(\bar\pi^{-1}(\overline{Y}_{j+1})) & \to & 0 \\
 & & \xi_j^*\downarrow & & \xi_j\downarrow & & \xi_{j+1}\downarrow & & \\
0 & \to & H_c^{4k}(\pi^{-1}(Z_j))_{1,\mathrm{ev}}^{D'} & \to & H^{4k}(\pi^{-1}(Q_{2j+\delta}))_{1,\mathrm{ev}}^{D'} & \to & H^{4k}(\pi^{-1}(Q_{2j+\delta+2}))_{1,\mathrm{ev}}^{D'} & \to & 0.
\end{array}$$

The similar map ξ_j^* is defined also for j such that $[(m-1)/2]+2 \le 2j+1+\delta \le m-3$ by using $Q_{2j+3+\delta} \hookrightarrow Q_{2j+1+\delta}$, and for j such that $2j+\delta \le [(m-1)/2] < 2j+2+\delta$ by using $Q_{2j+3+\delta} \hookrightarrow Q_{2j+\delta}$ instead of $Q_{2j+\delta+2} \hookrightarrow Q_{2j+\delta}$.

Now by induction hypothesis, we may assume that ξ_{j+1} is injective. Since both of horizontal maps are exact, in order to show the injectivity of ξ_j it is enough to see that

(3.12.3) The map ξ_j^* is injective.

We show (3.12.3). We consider the following three cases according to the range of j;

(a) $0 \le 2j+\delta \le [(m-1)/2]-2$,
(b) $[(m-1)/2]+2 \le 2j+1+\delta \le m-3$,
(c) $2j+\delta \le [(m-1)/2] < 2j+2+\delta$.

First consider the case (a). We note that $H_c^{4k}(\pi^{-1}(Q_{2j+1+\delta} - Q_{2j+2+\delta}))_{1,\mathrm{ev}}^{D'} = 0$ by (3.5.2). It follows that, by (3.5.4), the map induced from the closed immersion $Q_{2j+\delta+1} - Q_{2j+\delta+2} \hookrightarrow Q_{2j+\delta} - Q_{2j+\delta+2}$ gives rise to an isomorphism

$$H_c^{4k}(\pi^{-1}(Q_{2j+\delta} - Q_{2j+\delta+1}))_{1,\mathrm{ev}}^{D'} \xrightarrow{\sim} H_c^{4k}(\pi^{-1}(Z_j))_{1,\mathrm{ev}}^{D'}$$

and so again by (3.5.2), we get

$$H_c^{4k}(\pi^{-1}(Z_j))_{1,\mathrm{ev}}^{D'} \xrightarrow{\sim} H^{4k-4a}(\mathcal{B}_{A'}^L)_1^{D'}, \tag{3.12.4}$$

where $a = (b-\delta)/2 - j - 1$. Similar formulae as (3.12.4) hold, using (3.5.2) and (3.5.3), by replacing $Q_{2j+\delta} - Q_{2j+\delta+2}$ by $Q_{2j+1+\delta} - Q_{2j+3+\delta}$ (in the case (b)) or $Q_{2j+\delta} - Q_{2j+\delta+3}$ (in the case (c)), respectively, where the right hand side remains unchanged.

Moreover, since $a = \bar{b} - j - 1$ we have a natural isomorphism

$$H_c^{2k}(\bar{\pi}^{-1}(\overline{Z}_j)) \xrightarrow{\sim} H^{2k-2a}(\overline{\mathcal{B}}_{f(A')}^L),$$

by (3.8.2). Therefore (3.12.3) is a consequence of the following statement.

(3.12.5) Under the above isomorphisms, the map ξ_j^* coincides with

$$\theta^L : H^{2k-2a}(\overline{\mathcal{B}}_{f(A')}^L) \to H^{4k-4a}(\mathcal{B}_{A'})_1^{D'}$$

up to a non-zero scalar, where θ^L denotes the map corresponding to θ in Theorem 1.9, defined by replacing G by L.

We shall prove (3.12.5). Let $\bar\pi_0 : \overline{\mathcal{B}} \to \overline{\mathcal{P}}$ be the natural map. Then $\bar\pi^{-1}(\overline{Y}_j)$ is a closed subset of $\bar\pi_0^{-1}(\overline{Y}_j)$. Moreover, we have

$$\bar\pi_0^{-1}(\overline{Z}_j) \simeq (\overline{Z}_j) \times \overline{\mathcal{B}}^L. \tag{3.12.6}$$

We know already $H_c^{2k}(\bar\pi^{-1}(\overline{Z}_j)) \simeq H^{2k-2a}(\overline{\mathcal{B}}^L_{f(A')})$. We also identify $H_c^{2k}(\bar\pi_0^{-1}(\overline{Z}_j))$ with $H^{2k-2a}(\overline{\mathcal{B}}^L)$ via (3.12.6). Since the inclusions $\bar\pi^{-1}(\overline{Z}_j) \hookrightarrow \bar\pi_0^{-1}(\overline{Z}_j)$ and $\overline{\mathcal{B}}^L_{f(A')} \hookrightarrow \overline{\mathcal{B}}^L$ are compatible with the isomorphism in (3.12.6), we see, under the above identification, that the map $H_c^{2k}(\bar\pi_0^{-1}(\overline{Z}_j)) \to H_c^{2k}(\bar\pi^{-1}(\overline{Z}_j))$ coincides with the map $H^{2k-2a}(\overline{\mathcal{B}}^L) \to H^{2k-2a}(\overline{\mathcal{B}}^L_{f(A')})$ induced from the closed immersion $\overline{\mathcal{B}}^L_{f(A')} \hookrightarrow \overline{\mathcal{B}}^L$.

Hence we get a commutative diagram of exact sequences

(3.12.7)
$$\begin{CD}
0 @>>> H^{2k-2a}(\overline{\mathcal{B}}^L) @>>> H^{2k}(\bar\pi_0^{-1}(\overline{Y}_j)) @>>> H^{2k}(\bar\pi_0^{-1}(\overline{Y}_{j+1})) @>>> 0 \\
@. @VVV @VVV @VVV @. \\
0 @>>> H^{2k-2a}(\overline{\mathcal{B}}^L_{f(A')}) @>>> H^{2k}(\bar\pi^{-1}(\overline{Y}_j)) @>>> H^{2k}(\bar\pi^{-1}(\overline{Y}_{j+1})) @>>> 0
\end{CD}$$

where the vertical maps are those induced from the inclusions $\bar\pi^{-1}(\overline{Y}_j) \hookrightarrow \bar\pi_0^{-1}(\overline{Y}_j)$.

Next we consider the natural map $\pi_0 : \mathcal{B} \to \mathcal{P}$ and compare the sets such as $\pi^{-1}(Q_{2j+\delta})$ with $\pi_0^{-1}(Y_{2j+\delta})$ for various j. First we consider the case (a). We get the following commutative diagram.

(3.12.8)
$$\begin{CD}
0 @>>> H_c^{4k}(\pi_0^{-1}(Z_j'))^{D'}_{\mathrm{ev}} @>>> H^{4k}(\pi_0^{-1}(Y_{2j+\delta}))^{D'}_{\mathrm{ev}} @>>> H^{4k}(\pi_0^{-1}(Y_{2j+\delta+2}))^{D'}_{\mathrm{ev}} @>>> 0 \\
@. @VVV @VVV @VVV @. \\
0 @>>> H_c^{4k}(\pi^{-1}(Z_j))^{D'}_{1,\mathrm{ev}} @>>> H^{4k}(\pi^{-1}(Q_{2j+\delta}))^{D'}_{1,\mathrm{ev}} @>>> H^{4k}(\pi^{-1}(Q_{2j+\delta+2}))^{D'}_{1,\mathrm{ev}} @>>> 0
\end{CD}$$

where $Z_j = Q_{2j+\delta} - Q_{2j+\delta+2}$ as before and $Z_j' = Y_{2j+\delta} - Y_{2j+\delta+2}$. The horizontal maps are exact, and the vertical maps are those obtained from the inclusions such as $\pi^{-1}(Q_{2j+\delta}) \hookrightarrow \pi_0^{-1}(Y_{2j+\delta})$.

Let $\varphi : H_c^{4k}(\pi_0^{-1}(Z_j'))^{D'}_{\mathrm{ev}} \to H_c^{4k}(\pi^{-1}(Z_j))^{D'}_{1,\mathrm{ev}}$ be the map given in (3.12.8). According to the inclusions $\pi^{-1}(Z_j) \hookrightarrow \pi_0^{-1}(Z_j) \hookrightarrow \pi_0^{-1}(Z_j')$ the map φ factors through as

$$\varphi : H_c^{4k}(\pi_0^{-1}(Z_j'))^{D'}_{\mathrm{ev}} \xrightarrow{\varphi'} H_c^{4k}(\pi_0^{-1}(Z_j))^{D'}_{\mathrm{ev}} \xrightarrow{\varphi''} H_c^{4k}(\pi^{-1}(Z_j))^{D'}_{\mathrm{ev}}.$$

We know by (3.12.4) that $H_c^{4k}(\pi^{-1}(Z_j))_{1,\mathrm{ev}}^{D'} \xrightarrow{\sim} H^{4k-4a}(\mathcal{B}_{A'}^L)_1^{D'}$. A similar argument shows that $H_c^{4k}(\pi_0^{-1}(Z_j))_{\mathrm{ev}}^{D'} \xrightarrow{\sim} H^{4k-4a}(\mathcal{B}^L)_{\mathrm{ev}}^{D'}$. Furthermore, it is clear that under those isomorphisms, φ'' turns out to be a map

$$\phi^L : H^{4k-4a}(\mathcal{B}^L)^{D'} \to H^{4k-4a}(\mathcal{B}_{A'}^L)_1^{D'}$$

induced from the closed immersion $\mathcal{B}_{A'}^L \hookrightarrow \mathcal{B}^L$.

On the other hand, the closed immersion $\pi_0^{-1}(Y_{2j+1+\delta} - Y_{2j+2+\delta}) \hookrightarrow \pi_0^{-1}(Z_j')$ gives rise to an isomorphism

$$H_c^{4k}(\pi_0^{-1}(Z_j'))_{\mathrm{ev}}^{D'} \xrightarrow{\sim} H_c^{4k}(\pi_0^{-1}(Y_{2j+1+\delta} - Y_{2j+2+\delta}))_{\mathrm{ev}}^{D'},$$

and so it induces an isomorphism

$$H_c^{4k}(\pi_0^{-1}(Z_j'))_{\mathrm{ev}}^{D'} \xrightarrow{\sim} H^{4k-4a}(\mathcal{B}^L)^{D'}.$$

The similar argument holds also for $Z_j = Q_{2j+1+\delta} - Q_{2j+3+\delta}$ and $Z_j' = Y_{2j+1+\delta} - Y_{2j+3+\delta}$ in the case (b) and for $Z_j = Q_{2j+\delta} - Q_{2j+3+\delta}$ and $Z_j' = Y_{2j+\delta} - Y_{2j+3+\delta}$ in the case (c). Then we have the following lemma.

LEMMA 3.13. *Assume that j is in the case (a) or (c). Then under those isomorphisms given as above, the map $\varphi' : H_c^{4k}(\pi_0^{-1}(Z_j'))_{\mathrm{ev}}^{D'} \to H_c^{4k}(\pi_0^{-1}(Z_j))_{\mathrm{ev}}^{D'}$ turns out to be a non-zero scalar multiplication on $H^{4k-4a}(\mathcal{B}^L)^{D'}$. In particular, up to a non-zero scalar, the map φ coincides with the map ϕ^L.*

3.14. Assuming Lemma 3.13, we shall continue the proof of (3.12.5). First consider the case (a). The diagram (3.12.8) is now written as

(3.14.1)
$$\begin{array}{ccccccccc}
0 & \to & H^{4k-4a}(\mathcal{B}^L)^{D'} & \to & H^{4k}(\pi_0^{-1}(Y_{2j+\delta}))_{\mathrm{ev}}^{D'} & \to & H^{4k}(\pi_0^{-1}(Y_{2j+2+\delta}))_{\mathrm{ev}}^{D'} & \to & 0 \\
 & & {\scriptstyle\varphi}\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^{4k-4a}(\mathcal{B}_{A'}^L)_1^{D'} & \to & H^{4k}(\pi^{-1}(Q_{2j+\delta}))_{1,\mathrm{ev}}^{D'} & \to & H^{4k}(\pi^{-1}(Q_{2j+2+\delta}))_{1,\mathrm{ev}}^{D'} & \to & 0.
\end{array}$$

Then we have the following.

(3.14.2) There exist maps $\zeta_j : H^{2k}(\bar{\pi}_0^{-1}(\overline{Y}_j)) \to H^{4k}(\pi_0^{-1}(Y_{2j+\delta}))_{1,\mathrm{ev}}^{D'}$ such that ζ_j is compatible with the commutative diagrams (3.12.7), (3.14.1), and that the induced map $\zeta_j^* : H^{2k-2a}(\overline{\mathcal{B}}^L) \to H^{4k-4a}(\mathcal{B}^L)$ coincides with θ_0^L.

In fact, it is known by [CP, Lemma 4.5], applied to the case $0 \in \mathcal{N}_{\overline{\mathfrak{g}}}$, that the map $H^*(\overline{\mathcal{B}}) \to H^*(\overline{\pi}_0^{-1}(\overline{Y}_j))$ is surjective, and its kernel is the principal ideal generated by $\bar{x}_1^{\bar{b}-j}$. (In the case where $A_0 = 0$, we can choose, as a filtration given in 3.7, any filtration by successive projective subspaces. Hence $\overline{Y}_j$ can be regarded as the one appearing in this filtration with respect to $A_0 = 0$, and so Lemma 4.5 can be applied.) On the other hand, by using the similar argument as in [loc. cit.], one can show that the map $H^*(\mathcal{B}) \to H^*(\pi_0^{-1}(Y_{2j+\delta}))$ is surjective, and its kernel can be described. Hence we see that the map $H^*(\mathcal{B})_{\mathrm{ev}}^{D'} \to H^*(\pi_0^{-1}(Y_{2j+\delta}))_{\mathrm{ev}}^{D'}$ is surjective and the kernel is the principal ideal generated by $\alpha(x_1)^{b-\delta-2j}$. Since $b - \delta - 2j = 2(\bar{b} - j)$, this shows the existence of the map ζ_j compatible with the diagram (3.12.7) and (3.14.1). Moreover, under the above identification of $H^*(\pi_0^{-1}(Y_{2j+\delta}))_{\mathrm{ev}}^{D'}$ as the quotient of $H^*(\mathcal{B})_{\mathrm{ev}}^{D'}$, and similarly for $H^*(\bar{\pi}_0^{-1}(\overline{Y}_j))$, $H^*(\mathcal{B}^L)^{D'}$ (resp. $H^*(\overline{\mathcal{B}}^L)$) may be identified with $H^*(\mathcal{B})_{\mathrm{ev}}^{D'}/(\alpha(x_1)^2)$ (resp. $H^*(\overline{\mathcal{B}})/(\bar{x}_1)$). This implies that the induced map ζ_j^* coincides with θ_0^L. Hence (3.14.2) holds.

Now (3.14.2) implies (3.12.5). In fact, by (3.14.2), we obtain a commutative diagram

$$
\begin{CD}
H^{2k-2a}(\overline{\mathcal{B}}^L) @>{\theta_0^L}>> H^{4k-4a}(\mathcal{B}^L)^{D'} \\
@VVV @VV{\varphi}V \\
H^{2k-2a}(\overline{\mathcal{B}}^L_{f(A')}) @>{\xi_j^*}>> H^{4k-4a}(\mathcal{B}^L_{A'})_1^{D'}.
\end{CD}
$$

Then by Lemma 3.13 and by the uniqueness of θ^L, we see that $\xi_j^* = \theta^L$ up to a non-zero scalar.

The similar statement as in (3.14.2) holds also in the case (b) or (c). In fact, the kernel of the map $H^*(\mathcal{B})_{\mathrm{ev}}^{D'} \to H^*(\pi_0^{-1}(Y_{2j+1+\delta}))_{\mathrm{ev}}^{D'}$ is again the principal ideal generated by $\alpha(x_1)^{b-\delta-2j}$, and the above argument can be applied without change. Hence in the case (c), Lemma 3.13 is applied to get (3.12.5) in a similar way as above. While in the case (b), the statement of Lemma 3.13 is trivially true since we have $Q_{2j+1+\delta} = Y_{2j+1+\delta}$, and so again we obtain (3.12.5). This proves the proposition up to Lemma 3.13. □

3.15. We shall prove Lemma 3.13. First consider the case (a). We consider the following diagram.

(3.15.1)
$$
\begin{tikzcd}
 & H_c^{4k}(\pi_0^{-1}(Z_j'))^{D'}_{1,\mathrm{ev}} \arrow[r, "\sim"] \arrow[d, "\varphi'"'] & H_c^{4k}(\pi_0^{-1}(Y^0_{2j+1+\delta}))^{D'}_{1,\mathrm{ev}} \\
H_c^{4k}(\pi_0^{-1}(Q^0_{2j+\delta}))^{D'}_{1,\mathrm{ev}} \arrow[r, "\sim"] & H_c^{4k}(\pi_0^{-1}(Z_j))^{D'}_{1,\mathrm{ev}}, &
\end{tikzcd}
$$

where $Q^0_{2j+\delta} = Q_{2j+\delta} - Q_{2j+1+\delta}$ and $Y^0_{2j+1+\delta} = Y_{2j+1+\delta} - Y_{2j+2+\delta}$. Let U be one of the varieties such as $\pi_0^{-1}(Q_{2j+\delta} - Q_{2j+1+\delta})$ appearing in the diagram. Then we have a spectral sequence

$$H_c^i(\pi_0(U), R^l(\pi_U)_!\bar{\mathbf{Q}}_l) \Longrightarrow H_c^{i+l}(U, \bar{\mathbf{Q}}_l),$$

where $\pi_U = \pi_0|_U$. Taking their invariant parts, we also have

$$H_c^i(\pi_0(U), R^l(\pi_U)_!\bar{\mathbf{Q}}_l)^{D'}_{1,\mathrm{ev}} \Longrightarrow H_c^{i+l}(U, \bar{\mathbf{Q}}_l)^{D'}_{1,\mathrm{ev}}. \tag{3.15.2}$$

We note that $R^l(\pi_U)_!\bar{\mathbf{Q}}_l$ is a constant sheaf $H^l(\mathcal{B}^L)$ on $\pi_0(U)$. In fact, by the base change theorem, we have

$$R^l(\pi_U)_!\bar{\mathbf{Q}}_l \simeq R^l(\pi_0)_!\bar{\mathbf{Q}}_l|_{\pi'(U)}.$$

But $\pi_0 : \mathcal{B} \to \mathcal{P}$ is a locally trivial fibration, and so $R^l(\pi_0)_!\bar{\mathbf{Q}}_l$ is a locally constant sheaf on $\mathcal{P}$. Since $\mathcal{P}$ is connected and simply connected, $R^l\pi'_!\bar{\mathbf{Q}}_l$ is a constant sheaf whose fibre is given by $(R^l(\pi_0)_!\bar{\mathbf{Q}}_l)_x \simeq H^l(\mathcal{B}^L)$ for $x \in \mathcal{P}$. It follows that

$$H_c^i(\pi_0(U), R^l(\pi_U)_!\bar{\mathbf{Q}}_l) \simeq H^l(\mathcal{B}^L) \otimes H_c^i(\pi_0(U)).$$

Hence, we have

$$H_c^i(\pi_0(U), R^l(\pi_U)_!\bar{\mathbf{Q}}_l)^{D'}_{1,\mathrm{ev}} \simeq H^l(\mathcal{B}^L)^{D'}_1 \otimes H_c^i(\pi_0(U))_{1,\mathrm{ev}}$$

since $H^l(\mathcal{B}^L)^{D'}_1 = H^l(\mathcal{B}^L)^{D'}_{1,\mathrm{ev}}$. But if $\pi_0(U) = Q^0_{2j+\delta}$ or $Y^0_{2j+1+\delta}$, then $\pi_0(U) \simeq \mathbb{A}^{2a}$ and so the left hand side of (3.14.2) vanishes except when $i = 4a$. This implies that

$$\begin{aligned} H_c^{4k}(U, \bar{\mathbf{Q}}_l)^{D'}_{1,\mathrm{ev}} &\simeq H^{4k-4a}(\mathcal{B}^L)^{D'}_1 \otimes H_c^{4a}(\pi_0(U)) \\ &\simeq H^{4k-4a}(\mathcal{B}^L)^{D'}_1. \end{aligned}$$

On the other hand, even if $\pi_0(U) = Q_{2j+\delta} - Q_{2j+2+\delta}$ ($\simeq \mathbb{A}^{2a} \cup \mathbb{A}^{2a-1}$), or $\pi_0(U) = Y_{2j+\delta} - Y_{2j+2+\delta}$ ($\simeq \mathbb{A}^{2a+1} \cup \mathbb{A}^{2a}$), then $H_c^j(\pi_0(U))_{\mathrm{ev}} = 0$ except when $j = 4a$. Thus, again we have

$$H_c^{4k}(U, \bar{\mathbf{Q}}_l)_{1,\mathrm{ev}}^{D'} \simeq H^{4k-4a}(\mathcal{B}^L)_1^{D'} \otimes H_c^{4a}(\pi_0(U))_{\mathrm{ev}}.$$

We now consider the following diagram,

$$\begin{array}{ccc} & & H_c^{4a}(Y_{2j+\delta} - Y_{2j+2+\delta})_{\mathrm{ev}} \xrightarrow{\sim} H_c^{4a}(Y_{2j+1+\delta} - Y_{2j+2+\delta})_{\mathrm{ev}} \\ & & \varphi_0 \downarrow \qquad\qquad \\ H_c^{4a}(Q_{2j+\delta} - Q_{2j+1+\delta})_{\mathrm{ev}} & \xrightarrow{\sim} & H_c^{4a}(Q_{2j+\delta} - Q_{2j+2+\delta})_{\mathrm{ev}}. \end{array}$$

In view of the previous discussion, in order to prove Lemma 3.13 we have only to show that φ_0 is an isomorphism, or since both have dimension one, enough to show that φ_0 is injective. Now the complement of $Q_{2j+\delta} - Q_{2j+2+\delta}$ in $Y_{2j+\delta} - Y_{2j+2+\delta}$ coincides with $C_{2j+\delta} - C_{2j+2+\delta}$. So it is enough to show that

$$H_c^{4a}(C_{2j+\delta} - C_{2j+2+\delta}) = 0.$$

Let

$$\widehat{C}_i = \{(y_{i+1}, \dots, y_b) \in \mathbb{A}^{b-i} \mid y_{i+1}y_{m-i} + y_{i+2}y_{m-i+1} \cdots = 1\}.$$

Then $\widehat{C}_i$ is a double covering of C_i. Moreover, $\widehat{C}_i \simeq \mathbb{A}^{i+b-m} \times \widehat{C}'_i$ with

$$\widehat{C}'_i = \{(y_{i+1}, \dots, y_{m-i}) \in \mathbb{A}^{m-2i} \mid y_{i+1}y_{m-i} + \cdots = 1\}.$$

Now using the result of Fary ([F, Th.3, page 35]) we have

$$H_c^l(\widehat{C}'_i) = \begin{cases} \bar{\mathbf{Q}}_l & \text{if } l = 2(m-2i-1), m-2i-1, \\ 0 & \text{otherwise.} \end{cases}$$

(Fary's result is concerned with the groups over $\mathbb{C}$. However, since $\widehat{C}'_i$ is smooth, it is also valid in the case where p is large enough.) It follows that $H_c^l(\widehat{C}_i) = 0$ except when $l = 2b - 2i - 2$ or $l = 2b - m - 1$. Since $4a = 2(b - \delta) - 4j - 4$, and $0 \le 2j + \delta \le [(m-1)/2] - 2$, we see that $H_c^{4a}(\widehat{C}_{2j+\delta}) = 0$. Also one can check that $H_c^{4a-1}(\widehat{C}_{2j+2+\delta}) = 0$. This implies that

$$H_c^{4a}(C_{2j+\delta}) = H_c^{4a-1}(C_{2j+2+\delta}) = 0.$$

By using the cohomology long exact sequence for $C_{2j+2+\delta} \hookrightarrow C_{2j+\delta}$, we obtain that $H^{4a}_c(C_{2j+\delta} - C_{2j+2+\delta}) = 0$. This proves the lemma in the case (a).

Next consider the case (c). In this case, we need to compare $Q_{2j+\delta} - Q_{2j+3+\delta}$ and $Y_{2j+\delta} - Y_{2j+3+\delta}$. But since $Q_{2j+3+\delta} = Y_{2j+3+\delta}$, we see that the complement of $Q_{2j+\delta} - Q_{2j+3+\delta}$ in $Y_{2j+\delta} - Y_{2j+3+\delta}$ coincides with $C_{2j+\delta}$. Then we can show that $H^{4a}_c(\widehat{C}_{2j+\delta}) = 0$ in a similar way as above. This proves the case (c), and so the lemma is proved. Now the proof of the Proposition 3.12 is complete.

3.16. We keep the assumption in Proposition 3.12. Then the map ξ_0 is injective. It follows from Corollary 3.11 that the map ξ'_0 is also injective. In particular, in the case where b is even, we have an injective map

$$\xi'_0 : H^{2k}(\overline{\pi}^{-1}(\overline{Y}_0)) \to H^{4k}(\pi^{-1}(Y_0))^{D'}_{1,\mathrm{ev}}. \tag{3.16.1}$$

We now consider the case where b is odd, i.e., $\delta = 1$. We have an injective map

$$\xi'_0 : H^{2k}(\overline{\pi}^{-1}(\overline{Y}_0)) \to H^{4k}(\pi^{-1}(Y_1))^{D'}_{1,\mathrm{ev}}.$$

Then A_0 is described as in (3.8.1). We have two cases where $s' \le s-2$ or $s' = s-1$. In each case we denote by $\overline{Y}_{-1}$ the variety preceding $\overline{Y}_0$ in the filtration of $\operatorname{Ker} A_0$ given in 3.8, as follows; if $s' \le s-2$, we put $\overline{Y}_{-1} = \overline{Y}^{(s_1)}_0$, and if $s' = s-1 = s_1$, we put $\overline{Y}_{-1} = \overline{Y}^{(s')}_j$ with $j = m_{s'}/2 - 1$. Then one can construct a map

$$\xi'_{-1} : H^{2k}(\overline{\pi}^{-1}(\overline{Y}_{-1})) \to H^{4k}(\pi^{-1}(Y_0))^{D'}_{1,\mathrm{ev}}$$

such that the following diagram commutes.

(3.16.2)

$$\begin{tikzcd}
H^{2k}(\overline{\mathcal{B}}_{A_0}) \arrow[r] \arrow[d, "\theta"'] & H^{2k}(\overline{\pi}^{-1}(\overline{Y}_{-1})) \arrow[r] \arrow[d, "\xi'_{-1}"] & H^{2k}(\overline{\pi}^{-1}(\overline{Y}_0)) \arrow[r] \arrow[d, "\xi'_0"] & {} \\
H^{4k}(\mathcal{B}_A)^D_1 \arrow[r] & H^{4k}(\pi^{-1}(Y_0))^{D'}_{1,\mathrm{ev}} \arrow[r] & H^{4k}(\pi^{-1}(Y_1))^{D'}_{1,\mathrm{ev}} \arrow[r] & {}.
\end{tikzcd}$$

In fact, the kernel of the map $H^*(\overline{\mathcal{B}}_{A_0}) \to H^*(\overline{\pi}^{-1}(\overline{Y}_1))$ is the principal ideal generated by $\bar{x}_1^{\bar{b}+1}$, with $2\bar{b} = b-1$, and $H^*(Y_0)_{\mathrm{ev}} \simeq \mathbb{Z}[x_1^2]/x_1^{b-1}$. Our assertion follows from this in a similar way as in the proof of Lemma 3.10. We shall show

Lemma 3.17. *Under the assumption in Proposition 3.12, the map ξ'_{-1} is injective.*

Proof. First we note that the closed immersion $Y_1 \hookrightarrow Y_0$ induces an exact sequence,
(3.17.1)

$$0 \to H_c^{4k}(\pi^{-1}(Y_0 - Y_1))_{1,\mathrm{ev}}^{D'} \to H^{4k}(\pi^{-1}(Y_0))_{1,\mathrm{ev}}^{D'} \to H^{4k}(\pi^{-1}(Y_1))_{1,\mathrm{ev}}^{D'} \to 0.$$

In fact, the complement of $Q_0 - Q_1$ in $Y_0 - Y_1$ coincides with $C_0 - C_1$. Since

$$H_c^{4k+1}(\pi^{-1}(Q_0 - Q_1))_{1,\mathrm{ev}}^{D'} = H_c^{4k+1}(\pi^{-1}(C_0 - C_1))_{1,\mathrm{ev}}^{D'} = 0$$

by (3.5.4) and (3.5.5), we see that $H_c^{4k+1}(\pi^{-1}(Y_0 - Y_1))_{1,\mathrm{ev}}^{D'} = 0$. On the other hand, by Lemma 3.6 we see that $H_c^{4k-1}(\pi^{-1}(C_1))_{1,\mathrm{ev}}^{D'} = 0$. Moreover $H_c^{4k-1}(\pi^{-1}(Q_1))_{1,\mathrm{ev}}^{D'} = 0$ by (3.5.4). This implies that $H_c^{4k-1}(\pi^{-1}(Y_1))_{1,\mathrm{ev}}^{D'} = 0$, and so (3.17.1) follows.

We put $Z_{-1} = C_0 - C_1$, $Z'_{-1} = Y_0 - Y_1$ and $\overline{Z}_{-1} = \overline{Y}_{-1} - \overline{Y}_0$. Then using (3.17.1), we can define a map

$$\xi^*_{-1} : H_c^{2k}(\overline{\pi}^{-1}(\overline{Z}_{-1})) \to H_c^{4k}(\pi^{-1}(Z'_{-1}))_{1,\mathrm{ev}}^{D'}$$

so that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & H_c^{2k}(\overline{\pi}^{-1}(\overline{Z}_{-1})) \arrow[r] \arrow[d, "\xi^*_{-1}"] & H^{2k}(\overline{\pi}^{-1}(\overline{Y}_{-1})) \arrow[r] \arrow[d, "\xi'_{-1}"] & H^{2k}(\overline{\pi}^{-1}(\overline{Y}_0)) \arrow[r] \arrow[d, "\xi'_0"] & 0 \\
0 \arrow[r] & H_c^{4k}(\pi^{-1}(Z'_{-1}))_{1,\mathrm{ev}}^{D'} \arrow[r] & H^{4k}(\pi^{-1}(Y_0))_{1,\mathrm{ev}}^{D'} \arrow[r] & H^{4k}(\pi^{-1}(Y_1))_{1,\mathrm{ev}}^{D'} \arrow[r] & 0.
\end{tikzcd}$$

As in the proof of Proposition 3.12, in order to prove ξ'_{-1} is injective it is enough to show that ξ^*_{-1} is injective.

Now in view of (3.5.5), the open immersion $\pi^{-1}(Z_{-1}) \hookrightarrow \pi^{-1}(Z'_{-1})$ induces an isomorphism

$$H_c^{4k}(\pi^{-1}(Z_{-1}))_{1,\mathrm{ev}}^{D'} \xrightarrow{\sim} H_c^{4k}(\pi^{-1}(Z'_{-1}))_{1,\mathrm{ev}}^{D'}.$$

It follows that

$$H_c^{4k}(\pi^{-1}(Z'_{-1}))_{1,\mathrm{ev}}^{D'} \xrightarrow{\sim} H_c^{4k-4a}(\mathcal{B}_{A''})_1^{D'},$$

with $a = (b-1)/2$. Also we have

$$H_c^{2k}(\overline{\pi}^{-1}(\overline{Z}_{-1})) \xrightarrow{\sim} H^{2k-2a}(\overline{\mathcal{B}}_{f(A'')}),$$

by (3.8.2). Hence, in order to show that ξ_{-1}^* is injective, it is enough to see that

(3.17.2) The map $H^{2k-2a}(\overline{\mathcal{B}}_{f(A'')}) \to H^{4k-4a}(\mathcal{B}_{A''})_1^{D'}$ induced from ξ_{-1}^* under the above isomorphism coincides with θ^L up to a non-zero scalar.

We show (3.17.2). As in the previous discussion, we consider a commutative diagram

$$\begin{CD}
0 @>>> H^{4k}(\pi_0^{-1}(Z'_{-1}))^{D'}_{1,\mathrm{ev}} @>>> H^{4k}(\pi_0^{-1}(Y_0))^{D'}_{1,\mathrm{ev}} @>>> H^{4k}(\pi_0^{-1}(Y_1))^{D'}_{1,\mathrm{ev}} @>>> 0 \\
@. @V{\varphi}VV @VVV @VVV @. \\
0 @>>> H^{4k}(\pi^{-1}(Z'_{-1}))^{D'}_{1,\mathrm{ev}} @>>> H^{4k}(\pi^{-1}(Y_0))^{D'}_{1,\mathrm{ev}} @>>> H^{4k}(\pi^{-1}(Y_1))^{D'}_{1,\mathrm{ev}} @>>> 0.
\end{CD}$$

So, (3.17.2) will follow if we can show that

(3.17.3) Under the above isomorphisms, the map φ coincides with ϕ^L : $H^{4k-4a}(\mathcal{B}^L)_1^{D'} \to H^{4k-4a}(\mathcal{B}^L_{A''})_1^{D'}$ up to a non-zero scalar.

Note that the identification of $H_c^{4k}(\pi^{-1}(Z'_{-1}))^{D'}_{1,\mathrm{ev}}$ with $H_c^{4k-4a}(\mathcal{B}^L_{A''})_1^{D'}$ is done via $H_c^{4k}(\pi^{-1}(Z_{-1}))^{D'}_{1,\mathrm{ev}}$. So, the map $\varphi' : H_c^{4k}(\pi_0^{-1}(Z_{-1}))^{D'}_{1,\mathrm{ev}} \to H_c^{4k}(\pi^{-1}(Z'_{-1}))^{D'}_{1,\mathrm{ev}}$ is nothing but the map ϕ^L under the above identification. On the other hand, by the locally trivial fibration, the map $\varphi'' : H_c^{4k}(\pi_0^{-1}(Z_{-1}))^{D'}_{1,\mathrm{ev}} \to H_c^{4k}(\pi_0^{-1}(Z'_{-1}))_1^{D'}$ induced from the open immersion $Z_{-1} \hookrightarrow Z'_{-1}$ clearly induces a non-zero scalar map on $H^{4k-4a}(\mathcal{B}^L)_1^{D'}$. Thus, $\varphi = \varphi' \circ \varphi''^{-1}$ coincides with θ^L up to a non-zero scalar. This proves (3.17.3) and so proves the lemma. □

3.18. We now assume that $V^{(s)}$ is of type II. Let $m = m_s$, $b = b_s$ and $\delta = \delta_s$ as before. We put $\bar{b} = (b-\delta)/2$. Note that, contrast to the previous cases, $\bar{b}$ does not necessarily coincide with $\bar{b}_s$. In order to get a uniform description in comparing the filtrations for the $\overline{Y}^{(s)}$ and $Y^{(s)}$, we shift the labeling of the filtration of $\overline{Y}^{(s)}$ as follows. If $s' \le s-2$ or b is even, we use the labeling given in the first part of 3.8, i.e.,

$$\overline{Y}^{(s)} = \overline{Y}_0 \supset \overline{Y}_1 \supset \cdots \supset \overline{Y}_{\bar{m}} = \overline{Y}^{(s+1)}.$$

While, if $s' = s-1$ and b is odd, we put

$$\overline{Y}^{(s)} = \overline{Y}_{-1} \supset \overline{Y}_0 \supset \cdots \supset \overline{Y}_{\bar{m}} = \overline{Y}^{(s+1)}.$$

Here $\bar{m}$ is given as follows; let $s'' > s$ be the smallest integer such that $m_{s''} \neq 0$. Then

$$\bar{m} = \begin{cases} m/2 & \text{if } b : \text{even}, \\ m/2 & \text{if } b : \text{odd and } s = s''-1, \\ m/2 - 1 & \text{if } b : \text{odd and } s \le s''-2. \end{cases}$$

In any case, we have $\overline{Y}_j - \overline{Y}_{j+1} \simeq \mathbb{A}^{\bar{b}-j-1}$. Then as in the proof of Lemma 3.10, one can construct, for each j such that $0 \le j \le m/2 - \delta$, a map

$$\xi'_j : H^{2k}(\bar{\pi}^{-1}(\overline{Y}_j)) \to H^{4k}(\pi^{-1}(Y_{2j+\delta}))^{D'}_{1,\mathrm{ev}}$$

so that the following diagram commutes.

$$\begin{tikzcd} H^{2k}(\overline{\mathcal{B}}_{A_0}) \arrow[r] \arrow[d, "\theta"'] & H^{2k}(\bar{\pi}^{-1}(\overline{Y}_0)) \arrow[r] \arrow[d, "\xi'_0"] & H^{2k}(\bar{\pi}^{-1}(\overline{Y}_1)) \arrow[r] \arrow[d, "\xi'_1"] & {} \\ H^{4k}(\mathcal{B}_A)^D_1 \arrow[r] & H^{4k}(\pi^{-1}(Y_{0+\delta}))^{D'}_{1,\mathrm{ev}} \arrow[r] & H^{4k}(\pi^{-1}(Y_{2+\delta}))^{D'}_{1,\mathrm{ev}} \arrow[r] & {}. \end{tikzcd}$$

Here we note that

(3.18.1) Assume that b is odd. Then $Y_{2j+\delta} = Y_{m-1}$ for $j = m/2 - \delta$. In this case the closed immersion $Y^{(s+1)} = Y_m \hookrightarrow Y_{m-1}$ induces an isomorphism

$$H^{4k}(\pi^{-1}(Y_{m-1}))^{D'}_{1,\mathrm{ev}} \xrightarrow{\sim} H^{4k}(\pi^{-1}(Y_m))^{D'}_{1,\mathrm{ev}}.$$

In fact, for $Z = Y_{m-1} - Y_m$, we have $Z \simeq \mathbb{A}^{b-m}$. Since $b-m$ is odd, we can verify, by using (3.5.1), that

$$H^{4k}(\pi^{-1}(Z))^{D'}_{1,\mathrm{ev}} = H^{4k+1}(\pi^{-1}(Z))^{D'}_{1,\mathrm{ev}} = 0.$$

(3.18.1) follows from this.

Then we have the following lemma.

Lemma 3.19. *Assume that the map ξ'_j is injective for $j = m/2 - \delta$. Then ξ'_j is injective for any j such that* $0 \le j \le m/2 - \delta$.

Proof. We prove the lemma in a similar way as in the proof of Proposition 3.12. Put $Z'_j = Y_{2j+\delta} - Y_{2j+2+\delta}$, and $\overline{Z}_j = \overline{Y}_j - \overline{Y}_{j+1}$ for $j = 0, 1, \dots, m/2 - 2$. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & H_c^{2k}(\bar\pi^{-1}(\overline{Z}_j)) & \to & H^{2k}(\bar\pi^{-1}(\overline{Y}_j)) & \to & H^{2k}(\bar\pi^{-1}(\overline{Y}_{j+1})) & \to & 0 \\
 & & \xi^*_j\downarrow & & \xi'_j\downarrow & & \xi'_{j+1}\downarrow & & \\
0 & \to & H_c^{4k}(\pi^{-1}(Z'_j))^{D'}_{1,\mathrm{ev}} & \to & H^{4k}(\pi^{-1}(Y_{2j+\delta}))^{D'}_{1,\mathrm{ev}} & \to & H^{4k}(\pi^{-1}(Y_{2j+2+\delta}))^{D'}_{1,\mathrm{ev}} & \to & 0.
\end{array}$$

It is enough to show that ξ^*_j is injective. As before we consider the following commutative diagram,

$$\begin{array}{ccccccccc}
0 & \to & H_c^{4k}(\pi_0^{-1}(Z'_j))^{D'}_{1,\mathrm{ev}} & \to & H^{4k}(\pi_0^{-1}(Y_{2j+\delta}))^{D'}_{1,\mathrm{ev}} & \to & H^{4k}(\pi_0^{-1}(Y_{2j+2+\delta}))^{D'}_{1,\mathrm{ev}} & \to & 0 \\
 & & \varphi\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_c^{4k}(\pi^{-1}(Z'_j))^{D'}_{1,\mathrm{ev}} & \to & H^{4k}(\pi^{-1}(Y_{2j+\delta}))^{D'}_{1,\mathrm{ev}} & \to & H^{4k}(\pi^{-1}(Y_{2j+2+\delta}))^{D'}_{1,\mathrm{ev}} & \to & 0.
\end{array}$$

Note that the closed immersion $\pi_0^{-1}(Y_{2j+1+\delta} - Y_{2j+2+\delta}) \hookrightarrow \pi_0^{-1}(Z'_j)$ induces an isomorphism

$$H_c^{4k}(\pi_0^{-1}(Z'_i))^{D'}_{1,\mathrm{ev}} \xrightarrow{\sim} H^{4k-4a}(\mathcal{B}^L)^{D'},$$

where $a = (b - \delta)/2 - j - 1$ as before. Also the closed immersion $\pi^{-1}(Y_{2j+1+\delta} - Y_{2j+2+\delta}) \hookrightarrow \pi^{-1}(Z'_j)$ induces an isomorphism

$$H_c^{4k}(\pi^{-1}(Z'_j))^{D'}_{1,\mathrm{ev}} \xrightarrow{\sim} H^{4k-4a}(\mathcal{B}^L_{A'})^{D'}_1.$$

It follows that, under the above isomorphism, the map φ coincides with the map

$$H^{4k-4a}(\mathcal{B}^L)^{D'} \to H^{4k-4a}(\mathcal{B}^L_{A'})^{D'}_1$$

induced from the closed immersion $\mathcal{B}^L_{A'} \hookrightarrow \mathcal{B}^L$. Then, a similar discussion as before implies that the map ξ^*_j coincides with θ^L under the above isomorphisms. Hence, ξ^*_j is injective. This proves the lemma. □

3.20. We keep the assumption in Lemma 3.19. Then the map ξ'_0 is injective. In particular, in the case where b is even, we have an injective map

$$\xi'_0 : H^{2k}(\overline{\pi}^{-1}(\overline{Y}_0)) \to H^{4k}(\pi^{-1}(Y_0))^{D'}_{1,\mathrm{ev}}.$$

We now assume that b is odd, i.e., $\delta = 1$. As discussed in 3.16, we define $\overline{Y}_{-1}$ as the variety preceding $\overline{Y}_0$ in the filtration of $\operatorname{Ker} A_0$ given in 3.18. Hence we have $\overline{Y}_{-1} = \overline{Y}_0^{(s_1)}$ if $s' \le s-2$, and $\overline{Y}_{-1} = \overline{Y}^{(s)}$ if $s' = s-1$. As in 3.15, we can construct a map

$$\xi'_{-1} : H^{2k}(\overline{\pi}^{-1}(\overline{Y}_{-1})) \to H^{4k}(\pi^{-1}(Y_0))^{D'}_{1,\mathrm{ev}}$$

making the similar diagram as (3.16.1) commutative. The following lemma is proved in a similar way as Lemma 3.17. (In some steps the proof becomes simpler since we don't need to use $C_0 - C_1$. Also we use the variety $\mathcal{B}_{A'}$ instead of $\mathcal{B}_{A''}$.)

LEMMA 3.21. *Under the assumption in Lemma 3.19, the map ξ'_{-1} is injective.*

3.22. Proposition 3.12, (3.16.1), Lemma 3.17, Lemma 3.19 and Lemma 3.21 covers all the steps in the filtration of $\mathcal{B}_A$ and $\overline{\mathcal{B}}_{A_0}$ given in 3.4. Hence we see that the map ξ_0 or ξ'_0 for $V^{(0)}$ is injective. Since this map coincides with the map θ, Proposition 3.2 is now proved.

Appendix

In this Appendix, we use the same notation as before, but we consider reductive groups G in general. Let $P = LU_P$ be a parabolic subgroup of G containing B, where L is a Levi subgroup of P containing T and U_P is the unipotent radical of P. We denote by W_L the Weyl subgroup of W corresponding to L. Let $\mathcal{P} \simeq G/P$ be the variety of parabolic subgroups of G conjugate to P. For a nilpotent element $A \in \mathcal{N}_{\mathfrak{g}}$, we denote by $\mathcal{P}_A$ the subvariety of $\mathcal{P}$ consisting of parabolic subgroups whose Lie algebra contains A. The variety $\mathcal{B}_A$ is defined as before. Then we have a natural map $\pi : \mathcal{B}_A \to \mathcal{P}_A$. The following proposition is an easy consequence of the results of Borho and MacPherson [BM], and is applied in Section 3 for the special case where $P = P^{(1)}$ is the maximal parabolic subgroup of G with W_L of type C_{n-1}.

PROPOSITION A. *Let Y be a locally closed subvariety of $\mathcal{P}_A$. Then $H^i_c(\pi^{-1}(Y), \bar{\mathbf{Q}}_l) = H^i_c(\pi^{-1}(Y))$ admits a natural structure of W_L-modules satisfying the following.*

(i) *For a closed immersion $Y_1 \hookrightarrow Y_2$ in $\mathcal{P}_A$, the cohomology long exact sequence associated to $\pi^{-1}(Y_1) \hookrightarrow \pi^{-1}(Y_2)$ turns out to be a sequence of W_L-modules.*

(ii) *If* $Y = \mathcal{P}_A$, *then* $\pi^{-1}(Y) = \mathcal{B}_A$, *and the* W_L*-module structure of* $H^i(\pi^{-1}(Y))$ *coincides with the restriction to* W_L *of the Springer* (*W-*) *module* $H^i(\mathcal{B}_A)$.

(iii) *If* $Y = \{P'\}$ *with* $P' = gPg^{-1}$, *we have* $\pi^{-1}(Y) \simeq \mathcal{B}^L_{A'}$, *where* $A' \in \mathcal{N}_{\mathfrak{l}}$ *is the image of* $\mathrm{Ad}(g^{-1})A \in \mathrm{Lie}\,P$ *under the map* $\mathrm{Lie}\,P \to \mathrm{Lie}\,P/\,\mathrm{Lie}\,U_P \simeq \mathfrak{l}$. *Then the* W_L*-module structure of* $H^i(\pi^{-1}(Y))$ *coincides with the Springer* (*W_L-*) *module* $H^i(\mathcal{B}^L_{A'})$.

Proof. Let

$$\widetilde{\mathcal{N}}_{\mathfrak{g}} = \{(x, gB) \in \mathcal{N}_{\mathfrak{g}} \times G/B \mid \mathrm{Ad}(g^{-1})x \in \mathrm{Lie}\,B\},$$
$$\widetilde{\mathcal{N}}^P_{\mathfrak{g}} = \{(x, gP) \in \mathcal{N}_{\mathfrak{g}} \times G/P \mid \mathrm{Ad}(g^{-1})x \in \mathrm{Lie}\,P\}.$$

We consider a commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{N}}_{\mathfrak{g}} \arrow[rr, "\rho'"] \arrow[dr, "\eta"'] & & \mathcal{N}_{\mathfrak{g}} \\ & \widetilde{\mathcal{N}}^P_{\mathfrak{g}} \arrow[ur, "\xi"'] & \end{tikzcd}$$

where ρ', η and ξ are defined by

$$\rho'(x, gB) = x, \qquad \eta(x, gB) = (x, gP), \qquad \xi(x, gP) = x,$$

respectively. Let $\rho : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ be the map as given in the proof of Lemma 2.3. Then by the construction of Springer representations due to [L1] as explained in 2.3, the complex $\mathbb{R}\rho_*\bar{\mathbf{Q}}_l$ has a natural structure of W-complex. Since $\mathbb{R}\rho'_*\bar{\mathbf{Q}}_l \simeq \mathbb{R}\rho_*\bar{\mathbf{Q}}_l|_{\mathcal{N}_{\mathfrak{g}}}$, $\mathbb{R}\rho'_*\bar{\mathbf{Q}}_l$ also has a structure of W-complex. Now it is shown in [BM, Prop. 2.13] that $\mathbb{R}\eta_*\bar{\mathbf{Q}}_l$ has a natural structure of W_L-complex. Then for each locally closed subvariety Y' in $\widetilde{\mathcal{N}}^P_{\mathfrak{g}}$, $\mathbb{H}^i_c(Y', \mathbb{R}\eta_*\bar{\mathbf{Q}}_l) \simeq H^i_c(\eta^{-1}(Y'), \bar{\mathbf{Q}}_l)$ admits a structure of W_L-modules. If Y is a locally closed subvariety in $\mathcal{P}_A$, Y is isomorphic to $Y' = \{(A, gP) \in \widetilde{\mathcal{N}}^P_{\mathfrak{g}} \mid gPg^{-1} \in Y\}$, and $\pi^{-1}(Y) \simeq \eta^{-1}(Y')$. Hence $H^i_c(\pi^{-1}(Y))$ admits a structure of W_L-modules. Now (i) is clear from this construction.

We now assume that $Y = \mathcal{P}_A$. It is known by [loc. cit.] that the W_L-action on $\mathbb{R}\rho'_*\bar{\mathbf{Q}}_l \simeq \mathbb{R}\xi_*(\mathbb{R}\eta_*\bar{\mathbf{Q}}_l)$ induced from the W_L-action on $\mathbb{R}\eta_*\bar{\mathbf{Q}}_l$ coincides with the restriction to W_L of W-action (Springer action) on $\mathbb{R}\rho'_*\bar{\mathbf{Q}}_l$. If we put $Y' = \xi^{-1}(A)$, then $Y' \simeq \mathcal{P}_A$ and $\eta^{-1}(Y') \simeq \mathcal{B}_A$. Then we have

$$\mathbb{H}^i(\{A\}, \mathbb{R}\rho'_*\bar{\mathbf{Q}}_l) \simeq \mathbb{H}^i(Y', \mathbb{R}\eta_*\bar{\mathbf{Q}}_l) \simeq H^i(\eta^{-1}(Y'), \bar{\mathbf{Q}}_l).$$

Hence the W_L-module structure on $\mathbb{H}^i(\eta^{-1}(Y'))$ coincides with the W_L-module structure on $\mathbb{H}^i(\{A\}, \mathbb{R}\rho'_*\bar{\mathbf{Q}}_l) \simeq H^i(\mathcal{B}_A)$ which is nothing but the restriction to W_L of the Springer action of W. This shows (ii).

It remains to show (iii). We consider the following commutative diagram,

$$\begin{tikzcd}
\widetilde{\mathcal{N}}_{\mathfrak{l}} \arrow[r, "\tilde{i}"] \arrow[d, "\rho''"'] & \widetilde{\mathcal{V}}^P \arrow[d, "\zeta"] & \widetilde{\mathcal{N}}_{\mathfrak{g}} \arrow[l, "\tilde{q}"'] \arrow[d, "\eta"] \\
\mathcal{N}_{\mathfrak{l}} \arrow[r, "i"] & \mathcal{V}^P & \widetilde{\mathcal{N}}_{\mathfrak{g}}^P, \arrow[l, "q"']
\end{tikzcd}$$

where $\rho'' : \widetilde{\mathcal{N}}_{\mathfrak{l}} \to \mathcal{N}_{\mathfrak{l}}$ is the similar map as ρ' in the case for $\mathfrak{l} = \operatorname{Lie} L$, and

$$\begin{aligned}
\mathcal{V}^P &= \{(\bar{x}, gP) \mid \bar{x} \in \operatorname{Lie} gPg^{-1}/\operatorname{Lie} gU_Pg^{-1}\}, \\
\widetilde{\mathcal{V}}^P &= \{(\bar{x}, gB) \mid \bar{x} \in \operatorname{Lie} gBg^{-1}/\operatorname{Lie} gU_Pg^{-1}\},
\end{aligned}$$

and

$$\zeta(\bar{x}, gB) = (\bar{x}, gP), \qquad q(x, gP) = (\bar{x}, gP), \qquad \tilde{q}(x, gB) = (\bar{x}, gB),$$
$$i(\bar{x}) = (\bar{x}, P), \qquad \tilde{i}(\bar{x}, lB_L) = (\bar{x}, lB)$$

for $g \in G$, $l \in L$. Since the squares in the above diagram are cartesian, we have, by the proper base change theorem,

$$i^*(\mathbb{R}\zeta_*\bar{\mathbf{Q}}_l) \simeq \mathbb{R}\rho''_*\bar{\mathbf{Q}}_l, \qquad q^*(\mathbb{R}\zeta_*\bar{\mathbf{Q}}_l) \simeq \mathbb{R}\eta_*\bar{\mathbf{Q}}_l.$$

Then by [BM, Prop. 2.12] it is known that $\mathbb{R}\zeta_*\bar{\mathbf{Q}}_l$ admits a natural W_L-action, and the W_L-actions on $\mathbb{R}\rho''_*\bar{\mathbf{Q}}_l$ (the Springer action of W_L) and on $\mathbb{R}\eta_*\bar{\mathbf{Q}}_l$ (the one given in the proof of (i)) are inherited from the W_L-action on $\mathbb{R}\zeta_*\bar{\mathbf{Q}}_l$. We now consider a point $(A, gP) \in \widetilde{\mathcal{N}}_{\mathfrak{g}}^P$. Then $q(A, gP) = (\bar{A}, gP)$, where $\bar{A} \in \operatorname{Lie} gPg^{-1}/\operatorname{Lie} gU_Pg^{-1}$, and $(\bar{A}, gP)$ is G-conjugate to (A', P), where A' is the image of A in $\operatorname{Lie} P/\operatorname{Lie} U_P \simeq \mathfrak{l}$. Hence, by considering the stalks at these points, we have

$$(\mathbb{R}\rho''_*\bar{\mathbf{Q}}_l)_{A'} \simeq (\mathbb{R}\zeta_*\bar{\mathbf{Q}}_l)_{(A',P)}, \qquad (\mathbb{R}\eta_*\bar{\mathbf{Q}}_l)_{(A,gP)} \simeq (\mathbb{R}\zeta_*\bar{\mathbf{Q}}_l)_{(\bar{A},gP)}.$$

All the isomorphisms are W_L-equivariant. Since $(\mathbb{R}\zeta_*\bar{\mathbf{Q}}_l)_{(A',P)} \simeq (\mathbb{R}\zeta_*\bar{\mathbf{Q}}_l)_{(\bar{A},gP)}$ as W_L-complexes, we have $(\mathbb{R}\rho''_*\bar{\mathbf{Q}}_l)_{A'} \simeq (\mathbb{R}\eta_*\bar{\mathbf{Q}}_l)_{(A,gP)}$. On the other hand, we have $(R^i\rho''_*\bar{\mathbf{Q}}_l)_{A'} \simeq H^i(\mathcal{B}^L_{A'})$ and also,

$$(R^i\eta_*\bar{\mathbf{Q}}_l)_{(A,gP)} \simeq \mathbb{H}^i(Y', \mathbb{R}\eta_*\bar{\mathbf{Q}}_l) \simeq H^i(\eta^{-1}(Y'))$$

with $Y' = \{(A, gP)\}$, as W_L-modules. Hence for $Y = \{gPg^{-1}\} \subset \mathcal{P}_A$, the W_L-module $H^i(\pi^{-1}(Y))$ coincides with the Springer module $H^i(\mathcal{B}^L_{A'})$. This proves (iii), and so the proposition is proved. □

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