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Continuity of Topological Entropy of One Dimensional Maps with Degenerate Critical Points

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Abstract. One dimensional maps with degenerate critical points are studied and we give a larger class of $\mathcal{C}^r$ functions whose topological entropy is continuous.

§Introduction

There are many studies of topological entropy of one dimensional maps. M. Misiurewicz and W. Szlenk [7] showed that topological entropy is lower semi-continuous for piecewise monotone $\mathcal{C}^0$ maps with $\mathcal{C}^0$ topology. Moreover, if none of the critical points of a $\mathcal{C}^1$ map are degenerate, its entropy is continuous with $\mathcal{C}^1$ topology [4] [7]. But when critical points are degenerate, there is a counter example of $\mathcal{C}^r (r \geq 1)$ functions whose topological entropy is not continuous [7].

Our aim in this paper is to show the continuity of topological entropy for maps with finitely degenerate critical points. In this paper I means an interval $[0, 1]$.

Theorem. *Let $r \geq 2$ and $\mathcal{F}^r$ be a set of $\mathcal{C}^r$ functions which have non zero k-th derivative and k is smaller than or equal to r at every point in I, i.e.*

$$\mathcal{F}^r = \{f \in \mathcal{C}^r(I, I); \forall x \in I, 1 < \exists k \leq r \text{ such that } f^{(k)}(x) \neq 0\}.$$

Then the topological entropy is continuous in $\mathcal{F}^r$.

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Considering the counter example by Misiurewicz and Szlenk [7], $\mathcal{F}^r$ is the widest class whose topological entropy is continuous.

To study topological entropy of one dimensional maps, the kneading theory due to J. Milnor and W. Thurston [4] is useful. But when degeneration occurs, we can not apply this theory directly because the lap number changes. There are essentially two types of degeneration and every degeneration is written by a combination of these two types. One is a degeneration of critical points into one critical point, and the other is that into a saddle. We improve the kneading theory to be applied to the functions with degenerate critical points and show that the topological entropy is continuous in $\mathcal{F}^r$.

§0. Preliminaries

Here we give fundamental theorems and definitions about the kneading theory. See [3] or [4] for proofs and details.

In this paper, I denotes the closed interval $[0,1]$. We name the critical points of f $c_1, \cdots, c_{l-1}$ and $c_0 = 0$, $c_l = 1$.

Definition 0.1. Let $f : I \to I$ be continuous, piecewise monotone. A **lap** of f is a maximal interval on which f is monotonous, the **lap number** l of f is the number of laps of f, i.e.,

$$l(f) = \#\{J \subset I; J \text{ is a maximal interval on which } f \text{ is monotonous}\}$$

and the **growth rate** of f is defined by

$$s(f) = \lim_{N\to\infty} l^{\frac{1}{N}}(f^N)$$

where $f^N = f \circ f^{N-1}$ and $f^0 = \mathrm{id}$, the identity map.

When the lap number of f is l, we use $I_1, \cdots, I_l$ to denote laps of f.

Theorem 0.2 (Misiurewicz and Szlenk [7]). *The topological entropy* $h(f)$ *of* f *is equal to* $\log(s(f))$.

DEFINITION 0.3. Let $\epsilon_0(x) = 1$ and $\epsilon_n(x) = \epsilon(x) \times \cdots \times \epsilon(f^{n-1}(x))$ where $\epsilon(x)$ is the sign of x, i.e.,

$$\epsilon(x) = \begin{cases} 1 \text{ if } x \in I_i \text{ and } f \text{ is increasing on } I_i \\ -1 \text{ if } x \in I_i \text{ and } f \text{ is decreasing on } I_i \\ 0 \text{ if } x \in \{c_1, \cdots, c_{l-1}\}. \end{cases}$$

The **invariant coordinate of** $x \in I$ is a formal power series

$$\theta^i(x;t) = a_0 + a_1 t + a_2 t^2 + \cdots$$

and a_k is defined as the sign

$$a_k = \begin{cases} 1 & f^k(x) \in \mathrm{Int} I_i \text{ and } \epsilon_k(x) = 1 \\ -1 & f^k(x) \in \mathrm{Int} I_i \text{ and } \epsilon_k(x) = -1 \\ \frac{1}{2} & f^k(x) = c_{i-1} \text{ or } c_i \text{ and } \epsilon_k(x) = 1 \\ -\frac{1}{2} & f^k(x) = c_{i-1} \text{ or } c_i \text{ and } \epsilon_k(x) = -1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\theta(x;t) = (\theta^1(x;t), \cdots, \theta^l(x;t)).$$

Thus in particular, $\theta(c_i) = (0, \cdots, 0, \frac{1}{2}, \frac{1}{2}, 0, \cdots, 0)$.

DEFINITION 0.4. The **kneading matrix** $[N_i^j(t;f)]_{\substack{1 \le i \le l-1 \\ 1 \le j \le l}}$ of f is defined by

$$N_i^j(t;f) = \theta^j(c_i^+;t) - \theta^j(c_i^-;t)$$

where

$$\theta^j(c_i^+;t) = \lim_{x \to c_i + 0} \theta^j(x;t)$$
$$\theta^j(c_i^-;t) = \lim_{x \to c_i - 0} \theta^j(x;t) \text{ with formal power series topology.}$$

From the previous definition, every coefficient of $N_i^j(t)$'s is in $\{0, \pm 1, \pm 2\}$, and we define the **kneading increment** ν_i by

$$\nu_i = N_i^1 I_1 + \cdots + N_i^l I_l \qquad i = 1, 2, \cdots, l-1.$$

We also use l-dimensional vector N_i as

$$N_i = (N_i^1, \cdots, N_i^l).$$

Lemma 0.5.

$$\theta^j(c_i^+; t) = \theta^j(c_i; t) + \frac{1}{2} N_i^j(t)$$
$$\theta^j(c_i^-; t) = \theta^j(c_i; t) - \frac{1}{2} N_i^j(t).$$

Lemma 0.6 [3, Chp. II, Lemma 8.1]. *For every x in I we have*

$$\sum_{k=1}^{l+1} (1 - \epsilon(I_k)t)\theta^k(x; t) = 1$$

where $\epsilon(I_k) = 1$ when $f|I_k$ is increasing and where $\epsilon(I_k) = -1$ when $f|I_k$ is decreasing.

Lemma 0.7. *Let $D_j(t)$ be the determinant of the matrix obtained from the kneading matrix deleting the j-th column, and $\epsilon_j = \epsilon(I_j)$. Then $D_f(t) = (-1)^{j+1} \dfrac{D_j(t)}{1 - \epsilon_j t}$ $j = 1, \cdots, l$ is independent of j. Furthermore, $D_f(t)$ satisfies the following conditions*:
1) $D_f(t)$ is holomorphic on the unit disk in $\mathbb{C}$,
2) $D_f(0) = 1$.

We call D_f in Lemma 0.7 the **kneading determinant of** f.

Lemma 0.8. *Let s be the growth rate of f. Then $D_f(t) \neq 0$ for every t satisfying $|t| < \dfrac{1}{s}$, and $t = \dfrac{1}{s}$ is a zero of $D_f(t)$.*

Now let us define the degeneration of critical points.

Definition 0.9. Let f be a $\mathcal{C}^r (r \geq 2)$ function on I. We say that a point x on I is **a k-degenerate point** if there is an integer k larger than 1 and smaller than r such that x satisfies

$$f'(x) = \cdots = f^{(k)}(x) = 0, f^{(k+1)}(x) \neq 0$$

and we call x **a non-degenerate critical point** if $k = 1$.

Our aim is to show the continuity of the topological entropy for maps with k-degenerate points. To avoid an infinite degeneration of the critical points, we restrict a class of $\mathcal{C}^r$ maps to those which have k-degenerate points and k is smaller than or equal to r. This class is endowed with $\mathcal{C}^r$ topology and denoted by $\mathcal{F}^r$ in this paper. (In particular, $\mathcal{F}^r$ consists of piecewise monotone maps.)

Note that all the critical points of f in $\mathcal{F}^r$ are isolated. Otherwise, there is a sequence $c_1, \cdots, c_n, \cdots$ converges to c such that $f'(c_i) = 0$ holds for every $i = 1, 2, \cdots, n, \cdots$ and $f'(c) = 0$. So by the mean value theorem, for every interval $]c_i, c_{i+1}[$, there is d_i satisfying $f''(d_i) = 0$. Repeating these operations for $f^{(k)}$, we conclude $f^{(r)}(x) = 0$, and this is a contradiction.

PROPOSITION 0.10. *Every critical point of f in $\mathcal{F}^r$ splits at most $r - 1$ critical points by small $\mathcal{C}^r$-perturbation.*

PROOF. First we note that if there is an open subinterval J of I satisfying that $f'(x) = 0$ for every x in J then $f^{(k)}(x) = 0$ also holds for every x in J. So we assume that every critical point of f is isolated. Thus it suffice to show that if c in I, a neighbourhood U of c in I, and some constant A satisfy $f'(c) = \cdots = f^{(k)}(c) = 0$, $|f^{(k+1)}(x)| > A \neq 0$ for every x in U and $f'(x) \neq 0$ in U then there is a neighbourhood N of f in $\mathcal{F}^r$ such that every g in N has at most k critical points on U. We show this by induction.

Let n be an positive integer smaller than k. Suppose $g^{(k-n)}(x)$ has n zeros in U. If $g^{(k-n-1)}(x)$ has $(n + 2)$ zeros in U, say $c_1, c_2, \cdots, c_{n+2}$ in this order, then by the mean value theorem we conclude that there are d_i in $]c_i, c_{i+1}[$, $i = 1, 2, \cdots, n + 1$, satisfying $g^{(k-n)}(d_i) = 0$. Since c_i's are in U, d_i's are also in U. This contradicts to the supposition.

Next we will show that $g^{(k-1)}(x)$ has at most one critical point in U. Suppose that $g^{(k-1)}(x)$ has two zeros, say c_1 and c_2 in this order, in U. Then by the mean value theorem there is d in $]c_1, c_2[$ such that $g^{(k)}(d) = 0$. But since g is ε close to f by $\mathcal{C}^r$ topology, $|g^{(k)}(x)| > A - \varepsilon$ holds and this is contradiction. $\square$

From now on, a function f is supposed to be in $\mathcal{F}^r$.

§1. Degenerate Systems with No Saddles

We consider the case three critical points are degenerate into one critical point.

Definition 1.1. Suppose f is a piecewise monotone map on I. We call a point x_0 a n-saddle of f if and only if x_0 is $2n$-degenerate point.

Definition 1.2. Let us choose a 3-degenerate critical point c_i, and make formal critical points d_1 and d_2 on c_i, and formal interval J_i and J_{i+1} such that $J_i = [d_1, c_i]$ and $J_{i+1} = [c_i, d_2]$. Then $\text{length}(J_i) = \text{length}(J_{i+1}) = 0$ holds. We define **the i-divided kneading matrix** $[\tilde{N}_i^j(t)] = [\tilde{\nu}_i]$ by

$$\begin{aligned} \tilde{\nu}_k(t) &= \quad \nu_k(t) && \text{if } k \neq i \\ \nu_{d_1}(t) &= -2\theta(d_1^-) + J_i + I_i \\ \nu_{\tilde{c}_i}(t) &= \quad 2\theta(\tilde{c}_i^+) - J_i - J_{i+1} \\ \nu_{d_2}(t) &= \quad 2\theta(d_2^+) - I_{i+1} - J_{i+1} \end{aligned}$$

and

$$\theta(d_1^-) + \theta(d_2^+) = I_{i+1} + I_i. \tag{1.2.1}$$

Moreover, since $d_1, d_2, \tilde{c}_i$ and c_i are the same point,

$$\begin{aligned} \theta(d_1^-) &= \quad \theta(c_i^-) = -\theta(c_i^+) + I_i + I_{i+1} \\ \theta(\tilde{c}_i^+) &= -\theta(c_i^+) + J_{i+1} + I_{i+1} \\ \theta(d_2^+) &= \quad \theta(c_i^+). \end{aligned}$$

Especially when c_i is periodic with period p and no other critical points are contained in this orbit, then one of $\theta(d_1^-)$ or $\theta(d_2^+)$ is periodic with period p and the other two be fallen in this periodic orbit.

Lemma 1.3. *Let $D_j(t)$ be the determinant of $(l+1) \times (l+1)$ matrix obtained from the renumbered i-divided kneading matrix by deleting the j-th column. Then*

$$\tilde{D}_f(t) = (-1)^{j+1} \frac{D_j(t)}{1 - \epsilon(I_j)t}$$

is independent of j*. Furthermore* $\tilde{D}_f(t)$ *satisfies the following conditions*:
1) $\tilde{D}_f$ *is holomorphic on the unit disk in* $\mathbb{C}$*,*
2) $\tilde{D}_f(0) = 1$*.*

Proof. We will go back to the definition of the kneading matrix. First we claim

Claim. Let every lap and critical point be renumbered. For i-divided kneading matrix,

$$\sum_{k=1}^{l+2}(1-\epsilon(I_k)t)N_j^k = 0$$

for every $j = 1, 2, \cdots, l+1$ holds.

Proof of Claim. From Lemma 0.6, we have

$$\sum_{k=1}^{l+2}(1-\epsilon(I_k)t)\theta^k(x;t) = 1$$

for every x in I. So we get

$$\begin{aligned}
&\sum_{k=1}^{l+2}(1-\epsilon(I_k)t)\theta^k(c_p^+;t) - \sum_{k=1}^{l+2}(1-\epsilon(I_k)t)\theta^k(c_p^-;t)\\
=&\sum_{k=1}^{l+2}(1-\epsilon(I_k)t)(\theta^k(c_p^+;t) - \theta^k(c_p^-;t))\\
=&\sum_{k=1}^{l+2}(1-\epsilon(I_k)t)N_p^k\\
=&0. \ \square
\end{aligned}$$

Now from the Claim, we have

$$(1-\epsilon(I_r)t)N_j^r = -\sum_{k\neq r}(1-\epsilon(I_k)t)N_j^k$$

for every $r = 1, 2, \cdots, l+2$.

Let $[N^i]$ denote $[N^i_j]_{j=1,\cdots,l+3}$ then we have

$$\begin{aligned}
&(-1)^{l+3}\frac{D_{l+2}(t)}{1-\epsilon(I_{l+2})t}\\
&=\frac{(-1)^{l+3}}{1-\epsilon(I_{l+2})t}\det[N^1,\cdots,N^j,\cdots,N^{l+1},\hat{N}^{l+2}]\\
&=\frac{(-1)^{l+3}}{1-\epsilon(I_{l+2})t}\frac{1}{1-\epsilon(I_r)t}\det[N^1,\cdots,-\sum_{k\neq r}(1-\epsilon(I_k)t)N^k,\cdots,N^{l+1},\hat{N}^{l+2}]\\
&=\frac{(-1)^{r}}{1-\epsilon(I_{l+2})t}\frac{1}{1-\epsilon(I_r)t}\det[N^1,\cdots,\hat{N}^r,\cdots,N^{l+1},-\sum_{k\neq r}(1-\epsilon(I_k)t)N^k]\\
&=\frac{(-1)^{r+1}}{1-\epsilon(I_{l+2})t}\frac{1}{1-\epsilon(I_r)t}\det[N^1,\cdots,\hat{N}^r,\cdots,N^{l+1},\sum_{k\neq r}(1-\epsilon(I_k)t)N^k]\\
&=\frac{(-1)^{r+1}}{1-\epsilon(I_{l+2})t}\frac{1}{1-\epsilon(I_r)t}\det[N^1,\cdots,\hat{N}^r,\cdots,N^{l+1},(1-\epsilon(I_{l+2})t)N^{l+2}]\\
&=\frac{(-1)^{r+1}}{1-\epsilon(I_r)t}\det[N^1,N^2,\cdots,\hat{N}^r,\cdots,N^{l+2}]
\end{aligned}$$

and we are done.

Next, we show that this determinant satisfies the rest properties of the kneading matrix, that is:

1) $\tilde{D}_f$ is holomorphic on the unit disk in $\mathbb{C}$,

2) $\tilde{D}_f(0)=1$.

1) is clearly holds. 2) is also true because $[N^j(0)]$ is lower triangular with a 1 in each term in the diagonal. □

Lemma 1.4. *Dividing never changes the kneading determinant.*

Proof. From the definition of the kneading increments of d_1, $\tilde{c}_i$ and

d_2, we can write the i-divided kneading matrix explicitly as

$$[\tilde{N}_i^j] = \begin{bmatrix} N_1^1 & N_1^2 & \cdots & N_1^i & 0 & 0 & N_1^{i+1} & \cdots & N_1^l \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N_i^1 & N_i^2 & \cdots & N_i^i + 1 & -1 & 0 & N_i^{i+1} & \cdots & N_i^l \\ -N_i^1 & -N_i^2 & \cdots & -N_i^i & 1 & -1 & -N_i^{i+1} & \cdots & -N_i^l \\ N_i^1 & N_i^2 & \cdots & N_i^i & 0 & 1 & N_i^{i+1} - 1 & \cdots & N_i^l \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N_{l-1}^1 & N_{l-1}^2 & \cdots & N_{l-1}^i & 0 & 0 & N_{l-1}^{i+1} & \cdots & N_{l-1}^l \end{bmatrix}.$$

Let us calculate the kneading determinant. Deleting the $(l+2)$-th column and we have

$$\begin{aligned} &(1-\epsilon(I_l)t)(-1)^{l+3}\tilde{D}_f(t) \\ &= \begin{vmatrix} N_1^1 & N_1^2 & \cdots & N_1^i & 0 & 0 & N_1^{i+1} & \cdots & N_1^{l-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N_i^1 & N_i^2 & \cdots & N_i^i & 1 & 0 & N_i^{i+1} - 1 & \cdots & N_i^{l-1} \\ 0 & 0 & \cdots & 1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -1 & 1 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N_{l-1}^1 & N_{l-1}^2 & \cdots & N_{l-1}^i & 0 & 0 & N_{l-1}^{i+1} & \cdots & N_{l-1}^{l-1} \end{vmatrix} \\ &= \begin{vmatrix} N_1^1 & N_1^2 & \cdots & N_1^i & 0 & 0 & N_1^{i+1} & \cdots & N_1^{l-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N_i^1 & N_i^2 & \cdots & N_i^i & 1 & 0 & N_i^{i+1} - 1 & \cdots & N_i^{l-1} \\ 0 & 0 & \cdots & 1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N_{l-1}^1 & N_{l-1}^2 & \cdots & N_{l-1}^i & 0 & 0 & N_{l-1}^{i+1} & \cdots & N_{l-1}^{l-1} \end{vmatrix} \\ &= \begin{vmatrix} N_1^1 & N_1^2 & \cdots & N_1^i & 0 & 0 & N_1^{i+1} & \cdots & N_1^{l-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N_i^1 & N_i^2 & \cdots & N_i^i & 0 & 0 & N_i^{i+1} & \cdots & N_i^{l-1} \\ 0 & 0 & \cdots & 1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N_{l-1}^1 & N_{l-1}^2 & \cdots & N_{l-1}^i & 0 & 0 & N_{l-1}^{i+1} & \cdots & N_{l-1}^{l-1} \end{vmatrix} \end{aligned}$$

$$=(1-\epsilon(I_l)t)(-1)^{l+1}D_f(t)$$

and we are done. □

§2. Degenerate Systems with Saddles

In Definition 1.2, we define the i-divided kneading matrix which is useful for degenerate systems. Here we will expand the definition of the i-divided kneading matrix for the systems with saddles. For simplicity, we call a 1-saddle of f a “saddle”.

Definition 2.1. Let f be in $\mathcal{F}^r$ and with saddles. We choose one lap I_i with a saddle and divide I_i into two laps I_{iL} and I_{iR} at the saddle. Then make two formal critical points d_1 and d_2 on the saddle, and define null interval $J = [d_1, d_2]$, that is, length$(J) = 0$. We define **the lap-divided kneading matrix** $[\tilde{N}_i^j(t)] = [\tilde{\nu}_i]$ by

$$\begin{aligned}\tilde{\nu}_k(t) &= \quad \nu_k(t) \qquad \text{(if we identify } I_{iR} \text{ and } I_{iL} \text{ with } I_i)\\ \tilde{\nu}_{d_1}(t) &= -2\theta(d_1^-) + J + I_{iL}\\ \tilde{\nu}_{d_2}(t) &= \quad 2\theta(d_2^+) - J - I_{iR}\end{aligned}$$

and

$$\theta(d_2^+) = \theta(d_1^-) - I_{iL} + I_{iR}.$$

Note that when the orbit of some turning point c_j contains this saddle, since we take the limit of invariant coordinate, the new kneading increment of c_j is equal to the original when we identify I_{iL} and I_{iR} with I_i.

Lemma 2.2. *Let $D_j(t)$ be a determinant of the $l \times l$ matrix obtained from the renumbered lap-divided kneading matrix by deleting the j-th column. Then*

$$\tilde{D}_f(t) = (-1)^{j+1}\frac{D_j(t)}{1-\epsilon(I_j)t}$$

is independent of j. Further more, $\tilde{D}_f(t)$ satisfies the following conditions:
1) $\tilde{D}_f$ is holomorphic on the unit disk in $\mathbb{C}$,
2) $\tilde{D}_f(0) = 1$.

PROOF. The proof is just the same as Lemma 1.3, so we omit to show here. □

LEMMA 2.3. *Lap-division never changes zeros of the kneading determinant.*

PROOF. Let I_i be a lap with a saddle. For simplification, we suppose that f is monotone increasing on I_i. We divide I_i into two laps I_{iL} and I_{iR}, and define M^j_{iL} and M^j_{iR} as coefficients of I_{iL} and I_{iR} of $\tilde{\nu}_j$ respectively. Then $M^{iL}_k + M^{iR}_k = N^i_k$ for every $k = 1, 2, \cdots, l-1$. From Definition 2.1, we can write the kneading matrix of the lap-divided system as

$$[\tilde{N}^j_i] = \begin{bmatrix} N^1_1 & N^2_1 & \cdots & M^{iL}_1 & 0 & M^{iR}_1 & N^{i+1}_1 & \cdots & N^l_1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N^1_{i-1} & N^2_{i-1} & \cdots & M^{iL}_{i-1} & 0 & M^{iR}_{i-1} & N^{i+1}_{i-1} & \cdots & N^l_{i-1} \\ N^1_d & N^2_d & \cdots & N^{iL}_d + 1 & -1 & N^{iR}_d & N^{i+1}_d & \cdots & N^l_d \\ -N^1_d & -N^2_d & \cdots & -N^{iL}_d & 1 & -N^{iR}_d - 1 & -N^{i+1}_d & \cdots & -N^l_d \\ N^1_i & N^2_i & \cdots & M^{iL}_i & 0 & M^{iR}_i & N^{i+1}_i & \cdots & N^l_i \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N^1_{l-1} & N^2_{l-1} & \cdots & M^{iL}_{l-1} & 0 & M^{iR}_{l-1} & N^{i+1}_{l-1} & \cdots & N^l_{l-1} \end{bmatrix}.$$

If we delete the I_{l+2} column, we have the kneading determinant as

$$(-1)^{l+3}(1-\epsilon(I_l)t)\tilde{D}_f = \begin{vmatrix} N^1_1 & N^2_1 & \cdots & M^{iL}_1 & 0 & M^{iR}_1 & N^{i+1}_1 & \cdots & N^{l-1}_1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N^1_{i-1} & N^2_{i-1} & \cdots & M^{iL}_{i-1} & 0 & M^{iR}_{i-1} & N^{i+1}_{i-1} & \cdots & N^{l-1}_{i-1} \\ N^1_d & N^2_d & \cdots & N^{iL}_d & -1 & N^{iR}_d & N^{i+1}_d & \cdots & N^{l-1}_d \\ 0 & 0 & \cdots & 1 & 0 & -1 & 0 & \cdots & 0 \\ N^1_i & N^2_i & \cdots & M^{iL}_i & 0 & M^{iR}_i & N^{i+1}_i & \cdots & N^{l-1}_i \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ N^1_{l-1} & N^2_{l-1} & \cdots & M^{iL}_{l-1} & 0 & M^{iR}_{l-1} & N^{i+1}_{l-1} & \cdots & N^{l-1}_{l-1} \end{vmatrix}$$

$$
= \begin{vmatrix}
N_1^1 & N_1^2 & \cdots & M_1^{iL}+M_1^{iR} & 0 & M_1^{iR} & N_1^{i+1} & \cdots & N_1^{l-1} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
N_{i-1}^1 & N_{i-1}^2 & \cdots & M_{i-1}^{iL}+M_{i-1}^{iR} & 0 & M_{i-1}^{iR} & N_{i-1}^{i+1} & \cdots & N_{i-1}^{l-1} \\
N_d^1 & N_d^2 & \cdots & N_d^{iL}+N_d^{iR} & -1 & N_d^{iR} & N_d^{i+1} & \cdots & N_d^{l-1} \\
0 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\
N_i^1 & N_i^2 & \cdots & M_i^{iL}+M_i^{iR} & 0 & M_i^{iR} & N_i^{i+1} & \cdots & N_i^{l-1} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
N_{l-1}^1 & N_{l-1}^2 & \cdots & M_{l-1}^{iL}+M_{l-1}^{iR} & 0 & M_{l-1}^{iR} & N_{l-1}^{i+1} & \cdots & N_{l-1}^{l-1}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
N_1^1 & N_1^2 & \cdots & M_1^{iL}+M_1^{iR} & 0 & M_1^{iR} & N_1^{i+1} & \cdots & N_1^{l-1} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
N_{i-1}^1 & N_{i-1}^2 & \cdots & M_{i-1}^{iL}+M_{i-1}^{iR} & 0 & M_{i-1}^{iR} & N_{i-1}^{i+1} & \cdots & N_{i-1}^{l-1} \\
0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\
N_i^1 & N_i^2 & \cdots & M_i^{iL}+M_i^{iR} & 0 & M_i^{iR} & N_i^{i+1} & \cdots & N_i^{l-1} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
N_{l-1}^1 & N_{l-1}^2 & \cdots & M_{l-1}^{iL}+M_{l-1}^{iR} & 0 & M_{l-1}^{iR} & N_{l-1}^{i+1} & \cdots & N_{l-1}^{l-1}
\end{vmatrix}
$$

$$
=(-1)^{l+1}(1-\epsilon(I_l)t)D_f
$$

Thus we are done. □

§3. Continuity of Entropy

In this section, we use the same discussion as J. Milnor and W. Thurston [12][13] although their discussion is restricted to the $\mathcal{C}^1$ functions with fixed critical points. For every f in $\mathcal{F}^r$, we name the critical points and laps of f as $c_1, \cdots, c_{l-1}$ and $c_0 = 0$, $c_l = 1$, and $I_1, \cdots, I_l$, where l is the lap number of f. We use $\tilde{f}$ for divided system of f and $\tilde{c}_i$ or d_j for formal non-degenerate critical points of $\tilde{f}$. We also use c_i' and $\tilde{I}_i'$ for g in $\mathcal{F}^r$. For simplification, the critical point c_i of f is supposed to be a 3-degenerate critical point.

Lemma 3.1. *If $f^n(c_i) \notin \{c_1, \cdots, c_{l-1}\}$ for every $n \geq 1$ and every $i = 1, \cdots, l-1$ then the ij-element of the kneading matrix $N_i^j(t; \tilde{f})$ is continuous at $\tilde{f}$ for every $j = 1, 2, \cdots, l$. So the kneading determinant $D_{\tilde{f}}(t)$ of $\tilde{f}$ is also continuous at $\tilde{f}$.*

PROOF. First we note that $N_i^j(t;\tilde{f})$ is a holomorphic function on the unit disc $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$.

For any $\varepsilon > 0$ and every compact set K, we can choose ρ and N such that $|t| < \rho$ for every $t \in K$ and $4\dfrac{|\rho|^{N+1}}{1-|\rho|} < \varepsilon$ holds. Choose $\mathcal{N}$ as a $\mathcal{C}^r$ neighbourhood of $\tilde{f}$ satisfying that for every $g \in \mathcal{N}$, the lap number of g is equal to that of $\tilde{f}$ and itineraries of c_k and c_k' coincide up to the order N for every $k = 1, 2, \cdots, l-1$. Considering all the coefficients of the power series $N_i^j(t;\tilde{f})$ are in $\{0, \pm 1, \pm 2\}$, we have

$$|N_i^j(t;\tilde{f}) - N_i^j(t;g)| \leq \sum_{k=N+1}^{\infty} 4|t|^k = 4\frac{|t|^{N+1}}{1-|t|} < \varepsilon$$

for every $t \in K$. $\square$

LEMMA 3.2. *If c_i is not periodic or pre-periodic, the kneading determinant $D_{\tilde{f}}(t)$ is continuous at $\tilde{f}$.*

PROOF. For simplification, we suppose that $f^p(c_j) = c_i$ and there is no other turning point in the forward orbit of $c_k (k \neq j)$. Moreover, we suppose that $f^p(c_j^+) > c_i$. Other cases are just the same. When g has the same number of turning points as $\tilde{f}$, we claim

CLAIM. For every $\varepsilon > 0$, take a compact set K as in Lemma 3.1. Then there is $\delta > 0$ such that following conditions are satisfied for g satisfying $d(\tilde{f}, g) < \delta$ and every $t \in K$:
1) $|N_k^m(t;\tilde{f}) - N_k^m(t;g)| < \varepsilon$ for every $k \neq i$ and every $m = 1, 2, \cdots, l$,
2) For every $m = 1, \cdots, l$, one of the following holds.

$$|N_j^m(t;\tilde{f}) - N_j^m(t;g)| < \varepsilon$$

or $$|N_j^m(t;\tilde{f}) - 2t^p N_{d_2}^m(t;\tilde{f}) - N_j^m(t;g)| < \varepsilon$$

or $$|N_j^m(t;\tilde{f}) - 2t^p (N_{d_2}^m(t;\tilde{f}) + N_{\tilde{c}_i}^m(t;\tilde{f})) - N_j^m(t;g)| < \varepsilon$$

or $$|N_j^m(t;\tilde{f}) - 2t^p (N_{d_2}^m(t;\tilde{f}) + N_{\tilde{c}_i}^m(t;\tilde{f}) + N_{d_1}^m(t;\tilde{f})) - N_j^m(t;g)| < \varepsilon$$

PROOF. 1) is obvious from the last lemma. 2) is shown from a direct calculation. Because $N_j^m(t;\tilde{f})$ and $N_j^m(t;g)$ coincide at least up to order p, one of $g^p(c'_j) = c'_i$, $g^p(c'_j) \in I'_i$, $g^p(c'_j) \in J'_i$, $g^p(c'_j) \in J'_{i+1}$ or $g^p(c'_j) \in I'_{i+1}$ holds. In the first and the last cases, since $f(c_j^+) > c_i$,

$$N_j^m(t;\tilde{f}) \equiv N_j^m(t;g) \bmod t^N$$

holds. In the second case, since every term of $N_j^m(t;g)$ higher than p has the other sign of that of $N_j^m(t;\tilde{f})$, we add the term $2t^pN_{d_2}^m(t;\tilde{f})$ and the effect of 'anti-sign' can be canceled. The third case can be shown by using the method of the second case twice. The other cases can be got by the same reason. □

Now noting that elementary row operations of the matrix never change the value of its determinant, the continuity of the kneading determinant at f is easily shown from this Claim. □

Lemma 3.3. *If c_i is periodic with period p then for every g in the neighbourhood of $\tilde{f}$, $g^{np}(c'_i) - c'_i$ have the same sign for every $n \geq 1$, and also $g^{np}(d'_j) - d'_j$ have the same sign for every $n \geq 1$ and for every $j = 1, 2$.*

Proof. Because the differential coefficient of $f^p(x)$ at c_i is 0, we can suppose

$$\left|\frac{d}{dx}g^p(x)\right| < 1$$

for every x in a neighbourhood of c'_i. So there exists an attractive fixed point x_0 in this neighbourhood such that $g^{np}(c'_i)$ converges to x_0 monotonically as n goes to infinity. So $g^{np}(c'_i) - c'_i$ has the same sign for every $n \geq 1$. The same discussions hold for d'_j. □

Lemma 3.4. *Let c_i be a periodic point of period p of f and suppose that the orbit of c_i contains no other turning points. Then:*
1. the i-th kneading vector $N_i(t;f)$ is of the form $\frac{1}{1-t^p}P(t)$ where $P : \mathbb{C} \to \mathbb{C}^l$ is a polynomial map of degree p;
2. if g is close enough to $\tilde{f}$ then the i-th kneading vector $N_i(t;g)$ of g is equal to

$$N_i(t;\tilde{f}) \text{ or to } \frac{1-t^p}{1+t^p}N_i(t;\tilde{f}).$$

PROOF. As we have seen before,

$$N_i^j(t) = \theta^j(c_i^+;t) - \theta^j(c_i^-;t) = 2\theta^j(c_i^+;t)$$

for $j \geq 1$. Moreover, since $f^p(c_i) = c_i$ we get from the previous lemma that the coefficients $\theta^j(c_i^{\pm})$, $j = 1, 2, \cdots$ have period p for x close to c_i. More precisely, we have that

$$\begin{aligned}
N_{d_1}(t;\tilde{f}) &= \left(0, \cdots, -1 - \frac{2t^p}{1-t^p}, \quad 1, \quad 0, 0, \cdots, 0\right) + \frac{t}{1-t^p}\tilde{Q}(t) \\
N_{c_i}(t;\tilde{f}) &= \left(0, \cdots, \qquad \frac{2t^p}{1-t^p}, -1, \quad 1, 0, \cdots, 0\right) - \frac{t}{1-t^p}\tilde{Q}(t) \\
N_{d_2}(t;\tilde{f}) &= \left(0, \cdots, \qquad -\frac{2t^p}{1-t^p}, \quad 0, -1, 1, \cdots, 0\right) + \frac{t}{1-t^p}\tilde{Q}(t)
\end{aligned}$$

when f^p has a local maximum at c_i and

$$\begin{aligned}
N_{d_1}(t;\tilde{f}) &= \left(0, \cdots, -1, \quad 1, \quad 0, \quad -\frac{2t^p}{1-t^p}, 0, \cdots, 0\right) + \frac{t}{1-t^p}\tilde{Q}(t) \\
N_{c_i}(t;\tilde{f}) &= \left(0, \cdots, \quad 0, -1, \quad 1, \quad \frac{2t^p}{1-t^p}, 0, \cdots, 0\right) - \frac{t}{1-t^p}\tilde{Q}(t) \\
N_{d_2}(t;\tilde{f}) &= \left(0, \cdots, \quad 0, \quad 0, -1, 1 + \frac{2t^p}{1-t^p}, 0, \cdots, 0\right) + \frac{t}{1-t^p}\tilde{Q}(t)
\end{aligned}$$

when f^p has a local minimum at c_i, where $\tilde{Q}(t)$ is a $\mathcal{C}^{l+1}$ valued polynomial of degree $p-2$. If f^p has a local maximum at c_i then for each g sufficiently close to $\tilde{f}$,

$$\begin{aligned}
N_{d_1}(t;g) &= N_{d_1}(t;\tilde{f}) \\
N_{c_i}(t;g) &= N_{c_i}(t;\tilde{f}) \\
N_{d_2}(t;g) &= N_{d_2}(t;\tilde{f})
\end{aligned}$$

when $g^p(d_1^+) \in \mathrm{Int} I_i$, or

$$\begin{aligned}
N_{d_1}(t;g) = &\left(0, \cdots, 0, \qquad -1, \quad 1 - \frac{2t^p}{1+t^p}, \quad 0, 0, \cdots, 0\right) \\
&+ \frac{t}{1+t^p}\tilde{Q}(t)
\end{aligned}$$

$$
\begin{aligned}
N_{c_i}(t;g) &= \left(0,\cdots,0,\frac{2t^p(1-t^{np})}{1-t},-1+\frac{2t^{(n+1)p}}{1+t^p},\quad 1,0,\cdots,0\right)\\
&\quad -\frac{t}{1+t^p}\tilde{Q}(t)\\
N_{d_2}(t;g) &= \left(0,\cdots,0,\qquad 0,\qquad -\frac{2t^p}{1+t^p},-1,1,\cdots,0\right)\\
&\quad +\frac{t}{1+t^p}\tilde{Q}(t)
\end{aligned}
$$

when $g^p(d_1^+)\in \mathrm{Int}J_i$, or

$$
\begin{aligned}
N_{d_1}(t;g) &= \left(0,\cdots,0,-1,\quad 1,\quad -\frac{2t^p}{1-t^p},0,\cdots,0\right)+\frac{t}{1-t^p}\tilde{Q}(t)\\
N_{c_i}(t;g) &= \left(0,\cdots,0,\quad 0,-1,\quad 1+\frac{2t^p}{1-t^p},0,\cdots,0\right)-\frac{t}{1-t^p}\tilde{Q}(t)\\
N_{d_2}(t;g) &= \left(0,\cdots,0,\quad 0,\quad 0,-1-\frac{2t^p}{1-t^p},1,\cdots,0\right)+\frac{t}{1-t^p}\tilde{Q}(t)
\end{aligned}
$$

when $g^p(d_1^+)\in \mathrm{Int}J_{i+1}$, or

$$
\begin{aligned}
N_{d_1}(t;g) &= \left(0,\cdots,0,-1,\quad 1,\qquad 0,\quad -\frac{2t^p}{1+t^p},0,\cdots,0\right)\\
&\quad +\frac{t}{1+t^p}\tilde{Q}(t)\\
N_{c_i}(t;g) &= \left(0,\cdots,0,\quad 0,-1,1+\frac{2t^p(1-t^{np})}{1-t},\quad \frac{2t^{(n+1)p}}{1+t^p},0,\cdots,0\right)\\
&\quad -\frac{t}{1+t^p}\tilde{Q}(t)\\
N_{d_2}(t;g) &= \left(0,\cdots,0,\quad 0,\quad 0,\qquad -1,1-\frac{2t^p}{1+t^p},0,\cdots,0\right)\\
&\quad +\frac{t}{1+t^p}\tilde{Q}(t)
\end{aligned}
$$

when $g^p(d_1^+)\in \mathrm{Int}I_{i+1}$. A similar statement holds when f^p has a local minimum at c_i and when we apply the elementary deformations of matrices to these formulae, we can calculate the kneading determinant of g. Statements 1) and 2) follow immediately from this. □

From this Lemma, we can show that

$$|D_g(t) - D_{\tilde{f}}(t)| < \varepsilon$$

or

$$\left| D_g(t) - \left(\prod_p \frac{1-t^p}{1+t^p} \right) D_{\tilde{f}}(t) \right| < \varepsilon.$$

Here p takes all the period of the periodic critical points. Thus from Cauchy integral formula, we can conclude that zeros of D_g and $D_{\tilde{f}}$ are close to each other in the unit disk $\mathbb{D}$.

Lemma 3.5. *Let $c_{i(1)}$ be a periodic point of period p of f and suppose that the orbit of $c_{i(1)}$ contains the turning points $c_{i(1)}, c_{i(2)}, \cdots, c_{i(k)}$, $c_{i(k+1)} = c_{i(1)}$ (in this order). Then for g sufficiently close enough to f one has*

$$\begin{pmatrix} N_{i(0)}(t;g) \\ N_{i(1)}(t;g) \\ \vdots \\ N_{i(k)}(t;g) \end{pmatrix} = B(t) \begin{pmatrix} N_{i(0)}(t;\tilde{f}) \\ N_{i(1)}(t;\tilde{f}) \\ \vdots \\ N_{i(k)}(t;\tilde{f}) \end{pmatrix},$$

where $B(t)$ is $k \times k$ regular matrix with rational coefficients.

Proof. From the same reason as in Lemma 3.2, the j-th coefficient of the i-th kneading vector of g $(i \neq j)$ changes as the sign of $g^{a(j)}(c_{i(j)}) - c_{i(j+1)}$. So the (i,j)-th coefficient of $B(t)$ is equal to 0, $\dfrac{\pm 2t^{b(i,j)}}{1-t^p}$ or to $\pm 2t^{b(i,j)}$, where $b(i,j) > 0$ is the smallest integer so that $f^{b(i,j)}(c_i) = c_j$. Moreover, considering the periodic effect, with the same reason as in Lemma3.4, the (i,i)-th coefficient of $B(t)$ is equal to 1 or $\dfrac{1+t^p}{1-t^p}$. When we exchange the role between $\tilde{f}$ and g, we have that $B(t)$ is invertible and has non zero determinant. □

Thus we conclude that when $|f-g| < \delta$ and the orbit of a critical point c_i of f contains other critical points,

$$|\det B(t) D_{\tilde{f}}(t) - D_g(t)| < \varepsilon$$

and topological entropy of g is close to that of f.

Lemma 3.6. *When f has a saddle and this saddle is p-periodic, and suppose that the orbit of this saddle contains no other turning points. Let d_1 and d_2 be formal critical points of $\tilde{f}$. Then:*
1. the kneading vector $N_{d_i}(t;\tilde{f})$ is of the form $\frac{1}{1-t^p}P(t)$ where $P:\mathbb{C}\to\mathbb{C}^l$ is a polynomial map of degree p;
2. if g is close enough to $\tilde{f}$ then the kneading determinant of g is also close to that of $\tilde{f}$.

Proof. For simplification, we suppose that $f^p(d_2^+) > d_2$. The other case is shown by just the same method. From the definition of the kneading matrix, we have that

$$N_{d_1}(t;\tilde{f}) = \left(0,\cdots,-1-\frac{2t^p}{1-t^p},\quad 1,\qquad 0,0,\cdots,0\right)+\frac{t}{1-t^p}\tilde{Q}(t)$$
$$N_{d_2}(t;\tilde{f}) = \left(0,\cdots,\qquad 0,-1,1+\frac{2t^p}{1-t^p},0,\cdots,0\right)-\frac{t}{1-t^p}\tilde{Q}(t)$$

when f^p is increasing near the saddle and

$$N_{d_1}(t;\tilde{f}) = \left(0,\cdots,-1-\frac{2t^{2p}}{1-t^{2p}},\quad 1,\quad \frac{2t^p}{1-t^{2p}},\cdots,0\right)+\frac{t}{1-t^p}\tilde{Q}(t)$$
$$N_{d_2}(t;\tilde{f}) = \left(0,\cdots,\quad -\frac{2t^p}{1-t^{2p}},-1,1+\frac{2t^{2p}}{1-t^{2p}},\cdots,0\right)-\frac{t}{1-t^p}\tilde{Q}(t)$$

when f^p is decreasing near the saddle, where $\tilde{Q}(t)$ is a $\mathcal{C}^{l+1}$ valued polynomial of degree $p-2$. Then for each g sufficiently close to $\tilde{f}$, when $\tilde{f}$ is increasing near the saddle,

$$N_{d_1}(t;g) = \left(0,\cdots,0,-1-\frac{2t^p}{1-t^p},\quad 1,0,\cdots,0\right)+\frac{t}{1-t^p}\tilde{Q}(t)$$
$$N_{d_2}(t;g) = \left(0,\cdots,0,\qquad \frac{2t^p}{1-t^p},-1,1,\cdots,0\right)-\frac{t}{1-t^p}\tilde{Q}(t)$$

when $g^p(d_1^+) \in \mathrm{Int} I_{iL}$, or

$$N_{d_1}(t;g) = \left(0,\cdots,0,-1,\quad 1-\frac{2t^{(n+1)p}}{1+t^p},\frac{2t^p(1-t^{np})}{1-t^p},\cdots,0\right)+\frac{t}{1+t^p}\tilde{Q}(t)$$

$$N_{d_2}(t;g)=\left(0,\cdots,0,\quad 0,-1+\frac{2t^p}{1+t^p}\quad,\qquad\qquad 1,\cdots,0\right)-\frac{t}{1+t^p}\tilde{Q}(t)$$

when $g^p(d_1^+)\in \mathrm{Int}J$ and $g^{kp}(d_1^+)\in \mathrm{Int}I_{iR}$ for $k=1,\cdots,n$ and $g^{kp}(d_1^+)\in \mathrm{Int}J$ for $k>n$, or

$$N_{d_1}(t;g)=\left(0,\cdots,0,\qquad\quad -1,\qquad 1-\frac{2t^p}{1+t^p},0,\cdots,0\right)+\frac{t}{1+t^p}\tilde{Q}(t)$$
$$N_{d_2}(t;g)=\left(0,\cdots,0,\frac{2t^p(1-t^{np})}{1-t^p},-1+\frac{2t^{(n+1)p}}{1+t^p},1,\cdots,0\right)-\frac{t}{1+t^p}\tilde{Q}(t)$$

when $g^p(d_1^+)\in \mathrm{Int}J$ and $g^{kp}(d_1^+)\in \mathrm{Int}I_{iL}$ for $k=1,\cdots,n$ and $g^{kp}(d_1^+)\in \mathrm{Int}J$ for $k>n$, or

$$N_{d_1}(t;g)=\left(0,\cdots,0,-1,\quad 1,-\frac{2t^p}{1-t^p},0,\cdots,0\right)+\frac{t}{1-t^p}\tilde{Q}(t)$$
$$N_{d_2}(t;g)=\left(0,\cdots,0,0,-1,1+\frac{2t^p}{1-t^p},0,\cdots,0\right)-\frac{t}{1-t^p}\tilde{Q}(t)$$

when $g^p(d_1^+)\in \mathrm{Int}I_{iR}$, where $n\geq 0$ is the number of times the itinerary goes before it comes in the lap J.

When $\tilde{f}$ is decreasing near the saddle,

$$N_{d_1}(t;g)=\left(0,\cdots,0,-1+\frac{2t^p}{1+t^p},\quad 1,0,\cdots,0\right)+\frac{t}{1+t^p}\tilde{Q}(t)$$
$$N_{d_2}(t;g)=\left(0,\cdots,0,\qquad -\frac{2t^p}{1+t^p},-1,1,\cdots,0\right)-\frac{t}{1+t^p}\tilde{Q}(t)$$

when $g^p(d_1^+)\in \mathrm{Int}I_{iL}$, or

$$N_{d_1}(t;g)=\left(0,\cdots,0,-1,\quad 1+\frac{2t^p}{1-t^p},0,\cdots,0\right)+\frac{t}{1-t^p}\tilde{Q}(t)$$
$$N_{d_2}(t;g)=\left(0,\cdots,0,\quad 0,-1-\frac{2t^p}{1-t^p},1,\cdots,0\right)-\frac{t}{1-t^p}\tilde{Q}(t)$$

when $g^p(d_1^+) \in \mathrm{Int}J$, or

$$\begin{aligned} N_{d_1}(t;g) &= \left(0,\cdots,0,-1,1+\frac{2t^p(1-t^{np})}{1-t^p},\frac{2t^{(n+1)p}}{1+t^p},0,\cdots,0\right) \\ &\quad + \frac{t}{1+t^p}\tilde{Q}(t) \\ N_{d_2}(t;g) &= \left(0,\cdots,0,\quad 0, \qquad\qquad -1,1-\frac{2t^p}{1+t^p},\cdots,0\right) \\ &\quad - \frac{t}{1+t^p}\tilde{Q}(t) \end{aligned}$$

when $g^p(d_1^+) \in \mathrm{Int}I_{iR}$. For the every case, each of $N_{d_1}(t;g)$ and $N_{d_2}(t;g)$ is expressed by the linear combination of $N_{d_1}(t;\tilde{f})$ and $N_{d_2}(t;\tilde{f})$ and from this expression, statements 1) and 2) follows. □

LEMMA 3.7. *Let $c_{i(1)}$ be a periodic point of period p of f and suppose that the orbit of $c_{i(1)}$ contains a saddle and all the turning points of $\tilde{f}$ in this orbit are $c_{i(1)}, c_{i(2)}, \cdots, c_{i(k)}, c_{i(k+1)} = c_{i(1)}$ (in this order). Then for g sufficiently close enough to $\tilde{f}$ one has*

$$\begin{pmatrix} N_{i(0)}(t;g) \\ N_{i(1)}(t;g) \\ \vdots \\ N_{i(k)}(t;g) \end{pmatrix} = B(t) \begin{pmatrix} N_{i(0)}(t;\tilde{f}) \\ N_{i(1)}(t;\tilde{f}) \\ \vdots \\ N_{i(k)}(t;\tilde{f}) \end{pmatrix},$$

where $B(t)$ is $k \times k$ regular matrix with rational coefficients.

PROOF. With the same discussion as Lemma 3.5 for lemmas 3.2 and 3.6, we have the same conclusion. □

Now we can prove the following proposition from previous lemmas and Lemma 0.8.

THEOREM 3.8. *The topological entropy is continuous in $\mathcal{F}^r$.*

PROOF. Note that every degenerate critical point is written by a finite combination of 3-degenerate critical points and 1-saddles. Suppose g is the

$\mathcal{C}^r$-perturbation of f and the number of critical points of g is more than f by $2n$. Let $||\cdot||_r$ be a $\mathcal{C}^r$ norm and suppose $||g-f||_r < \delta$. Let us take a sequence of functions f_i $(i = 0, 1, \cdots, m)$ which satisfy the following conditions:

$$\begin{cases} f_0 = f \\ 0 \leq \#(\text{critical points of } f_{i+1}) - \#(\text{critical points of } f_i) \leq 2 \\ ||f_{i+1} - f_i||_r < \delta \\ f_m = g \\ m < C \cdot n \text{ for some constant } C > 0. \end{cases}$$

From the previous lemmas, we can suppose that the difference of the topological entropy of f_i and f_{i+1} is less than ε for each i. Then we have

$$|h(g) - h(f)| \leq \sum_{i=0}^{m} |h(f_{i+1}) - h(f_i)| < m\varepsilon$$

and we have that the topologcal entropy is continuous between f and g. □

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References

[1] Block, L. S. and W. A. Coppel, Dynamics in One Dimension, Springer Lecture Notes in Mathematics **1513**, Springer: Berlin, 1992.

[2] Collet, P. and J. P. Eckmann, Iterated Maps on the Interval as Dynamical Systems, Birkhäuser: Basel, 1980.

[3] de Melo, W. and S. J. van Strien, One-Dimensional Dynamics, Springer: Berlin, 1993.

[4] Milnor, J. and W. Thurston, On Iterated Maps of the Interval, in *Dynamical Systems*, Springer Lecture Note in Mathematics **1342**, Springer: Berlin, 1988, 465–563.

[5] Misiurewicz, M., Jumps of entropy in one dimension, Fund. Math. **132** (1989), 215–226.

[6] Misiurewicz, M. and S. V. Shlyachkov, Entropy of piecewise continuous interval maps, European Conference of Iteration Theory, '89, World Scientific: Singapore, 1991, pp. 239–245.

[7] Misiurewicz, M. and W. Szlenk, Entropy of piecewise monotone mappings, Studia Math. **67** (1980), 45–63.

[8] Zdunik, A., Entropy of transformations of the unit interval, Fund. Math. **124** (1984), 235–241.

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