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Existence of Distribution Null-Solutions for Every Fuchsian Partial Differential Operator

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Abstract. We construct a distribution null-solution for *every* Fuchsian partial differential operator in the sense of Baouendi-Goulaouic with real-analytic coefficients, if the initial surface is characteristic. This construction is valid also for Fuchsian hyperbolic operators with C^∞ coefficients considered by H. Tahara, and for some non-Fuchsian operators with real-analytic coefficients.

1. Introduction

We consider a Fuchsian partial differential operator with weight $m-k$ defined by M. S. Baouendi and C. Goulaouic[1].

$$(1.1)\quad P = t^k\partial_t^m + \sum_{j=1}^{k} a_j(x)t^{k-j}\partial_t^{m-j} + \sum_{j+|\alpha|\le m, j<m} b_{j,\alpha}(t,x)t^{d(j)}\partial_t^j\partial_x^\alpha,$$

where $(t,x) = (t,x_1,\dots,x_n)$ are variables in $\boldsymbol{R}\times\boldsymbol{R}^n$ $(n\ge 1)$ and $d(j) := \max\{0, j-m+k+1\}$. We assume, for the time being, that m, k are integers satisfying $0\le k\le m$ and that a_j (resp. $b_{j,\alpha}$) are real-analytic in a neighborhood of $0\in\boldsymbol{R}^n$ (resp. $(0,0)\in\boldsymbol{R}\times\boldsymbol{R}^n$). (When $m=k$, M. Kashiwara and T. Oshima([5], Definition 4.2) called such an operator "to have regular singularity in a weak sense along $\Sigma_0 := \{t=0\}$.")

Baouendi and Goulaouic[1] gave a theorem about the unique solvability of the Cauchy problem

$$(CP)\qquad \begin{cases} Pu &= f(t,x) \\ \partial_t^j u|_{t=0} &= g_j(x) \quad (0\le j\le m-k) \end{cases}$$

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in the category of real-analytic (or holomorphic) functions. They also showed a uniqueness theorem in a wider class of functions. From this result, it easily follows that if P is a Fuchsian partial differential operator with real-analytic coefficients, then P does not have any sufficiently smooth null-solutions. Here, a Schwartz distribution u in a neighborhood of $(0,0)$ is called a *null-solution* of P at $(0,0)$, if $Pu = 0$ in a neighborhood of $(0,0)$ and $(0,0) \in \operatorname{supp} u \subset \Sigma_+ := \{t \geq 0\}$, where $\operatorname{supp} u$ is the support of u.

If $k = 0$, then the initial surface $\Sigma_0 := \{t = 0\}$ is noncharacteristic for P, and hence there exist *no* distribution null-solutions by (a modern version of) the well-known Holmgren's uniqueness theorem ([3], Theorem 8.6.5).

When $k \geq 1$, K. Igari[4] constructed a distribution null-solution under an additional assumption (see Theorem 2.2). In this article, we show that this additional assumption can be removed; thus, the following is the main theorem.

THEOREM 1.1. *If P is a Fuchsian partial differential operator* (1.1) *with real-analytic coefficients and $k \geq 1$, then there exists a distribution u in a neighborhood of* $(0,0)$ *such that*

$$Pu = 0 \quad \textit{and} \quad (0,0) \in \operatorname{supp} u \subset \Sigma_+.$$

The proof is divided into 2 steps. In the first step, we construct a formal series $\sum_{h=0}^{\infty} v_h$ of distributions v_h such that the partial sum $V_N := \sum_{h=0}^{N} v_h$ satisfies $PV_N \in C_+^{r_0+N+1}(-T,T;\mathcal{O}(\Omega))$ for every $N \in \boldsymbol{N}$, where $T > 0$, Ω is a domain including 0 in $\boldsymbol{C}^n$, r_0 is a constant (maybe negative) independent of N, $\mathcal{O}(\Omega)$ denotes the space of holomorphic functions on Ω, and $C_+^N(-T,T;\mathcal{O}(\Omega)) := \{\, \phi \in C^N(-T,T;\mathcal{O}(\Omega)) \ ; \ \phi(t) = 0 \ \text{ in } \mathcal{O}(\Omega) \text{ for } \ t < 0 \,\}$. This construction is valid for a far wider class of operators than that of Fuchsian operators.

In the second step, we show the existence of $u \in \mathcal{D}'_+(-T',T';\mathcal{O}(\Omega'))$ such that $Pu = 0$ and $u - V_N \in C_+^{r_0+\omega+N+1}(-T',T';\mathcal{O}(\Omega'))$ for every $N \in \boldsymbol{N}$, where $0 < T' \leq T$, Ω' is a subdomain of Ω also including 0. This step is also valid for Fuchsian hyperbolic operators with C^∞ coefficients considered by H. Tahara([9], [10], [11], [12], and so on), and for some non-Fuchsian operators with real-analytic coefficients considered by the author([6]).

Notations.

(i) The set of all integers (resp. nonnegative integers) is denoted by $\boldsymbol{Z}$ (resp. $\boldsymbol{N}$).

(ii) The real part of a complex number z is denoted by $\operatorname{Re} z$.

(iii) Put $\vartheta := t\partial_t$ and $(\lambda)_l := \prod_{j=0}^{l-1}(\lambda - j)$ for $l \in \boldsymbol{N}$.

(iv) For a domain Ω in $\boldsymbol{C}^n$, we denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions on Ω. For a complete locally convex topological vector space X, we put $\mathcal{O}(\Omega; X) := \{ f \in C^0(\Omega; X) \ ; \ \langle \phi, f \rangle_X \in \mathcal{O}(\Omega) \ \text{for every} \ \phi \in X' \}$, where X' is the dual space of X and $\langle \cdot, \cdot \rangle_X$ denotes the duality between X' and X. Note that if D is a domain in $\boldsymbol{C}^l$ and Ω is a domain in $\boldsymbol{R}^n$, then $\mathcal{O}(D; C^\infty(\Omega)) = C^\infty(\Omega; \mathcal{O}(D))$.

(v) The space of test functions on an open interval I of $\boldsymbol{R}$ is denoted by $\mathcal{D}(I)$ and the space of distributions by $\mathcal{D}'(I)$. The space of rapidly decreasing C^∞ functions is denoted by $\mathcal{S}(\boldsymbol{R})$ and the space of tempered distributions by $\mathcal{S}'(\boldsymbol{R})$. The duality between each pair of these spaces is denoted by $\langle \cdot, \cdot \rangle$. More generally, for a complete locally convex topological vector space X, the space of all X-valued distributions is denoted by $\mathcal{D}'(I; X) := \mathcal{L}(\mathcal{D}(I), X)$, where $\mathcal{L}(X, Y)$ denotes the space of all continuous linear mappings from X to Y (See [8]). Note that $\mathcal{D}'(I; \mathcal{O}(\Omega)) = \mathcal{O}(\Omega; \mathcal{D}'(I))$. Put $\mathcal{D}'_+(I; X) := \{ f \in \mathcal{D}'(I; X) \ ; \ f(t) = 0 \ \text{in} \ X \ \text{for} \ t < 0 \}$. Also, for $N \in \boldsymbol{N}$ put

$$
\begin{aligned}
C_+^N(I; X) &:= \{ f \in C^N(I; X) \ ; \ f(t) = 0 \ \text{in} \ X \ \text{for} \ t < 0 \}, \\
C_+^{-N}(I; X) &:= \{ \partial_t^N(f) \in \mathcal{D}'_+(I; X) \ ; \ f \in C_+^0(I; X) \}.
\end{aligned}
$$

(vi) For $z \in \boldsymbol{C}$ with $\operatorname{Re} z > -1$, we put

$$
t_+^z := \begin{cases} t^z & (t > 0) \\ 0 & (t \le 0) \end{cases},
$$

which is a locally integrable function of t with holomorphic parameter z, and hence belongs to $\mathcal{D}'_+(\boldsymbol{R}; \mathcal{O}(\{ z \in \boldsymbol{C} \ ; \ \operatorname{Re} z > -1 \}))$. By $\partial_t(t_+^z) = z t_+^{z-1}$, this distribution t_+^z is extended to $z \in \boldsymbol{C} \setminus \{-1, -2, \dots\}$ meromorphically with simple poles at $z = -1, -2, \dots$ ([2]).

(vii) For a commutative ring R, the ring of polynomials of λ with the coefficients belonging to R is denoted by $R[\lambda]$. The degree of $F(\lambda) \in R[\lambda]$ is denoted by $\deg_\lambda F$. Also, the ring of formal power series of t with the coefficients belonging to R is denoted by $R[[t]]$.

2. Igari's Result and Main Difficulties

As we already stated in the introduction, K.Igari([4]) showed the existence of distribution null-solutions under a weak additional condition.

For a Fuchsian operator (1.1), we put

$$\mathcal{C}(x;\lambda) := (\lambda)_m + \sum_{j=1}^{k} a_j(x)(\lambda)_{m-j} = t^{-\lambda+\omega} P(t^\lambda)|_{t=0}, \tag{2.1}$$

where $\omega := m-k$. This is called the *indicial polynomial* of P. Note that $\mathcal{C}$ is decomposed as $\mathcal{C}(x;\lambda) = \widetilde{\mathcal{C}}(x;\lambda-\omega)(\lambda)_\omega$, where $\widetilde{\mathcal{C}}(x;\lambda) := (\lambda)_k + \sum_{j=1}^{k} a_j(x)(\lambda)_{k-j}$.

DEFINITION 2.1. A holomorphic function $\rho(x)$ in a neighborhood of $x=0$ is called a *normal root* of $\widetilde{\mathcal{C}}$, if it satisfies

$$\widetilde{\mathcal{C}}(x;\rho(x)) \equiv 0, \quad \widetilde{\mathcal{C}}(0;\rho(0)+l) \neq 0 \quad (l = 1, 2, \ldots).$$

THEOREM 2.2 ([4]). *If P is a Fuchsian operator with the real-analytic coefficients and there exists a normal root of $\widetilde{\mathcal{C}}$, then there exists a distribution null-solution of P at $(0,0)$.*

Our main theorem(Theorem 1.1) asserts that we can replace the condition "there exists a normal root of $\widetilde{\mathcal{C}}$" by a trivial condition "$k \geq 1$". The solution constructed by Igari is analytic in x. We also want to construct such a solution. We cannot make the proof easier even if we don't require the analyticity in x for solutions, because the proof is deeply connected with the analyticity in x.

The following very simple examples show the main difficulties and basic ideas to overcome them in the construction of a formal solution $\sum_{h=0}^{\infty} v_h$.

Example 2.3. (1) Let $P = \vartheta - x + 1$. We have $\mathcal{C}(x;\lambda) = \widetilde{\mathcal{C}}(x;\lambda) = \lambda - x + 1$.

By freezing x in a neighborhood of $x = 0$ and by considering P as an ordinary differential operator with respect to t, we can solve the equation $Pu = 0$ in $\mathcal{D}'_+(\boldsymbol{R})$. The solutions make a one-dimensional space and the seemingly easiest base is $\{ t_+^{-1+x} \}$ for $x \neq 0$, and $\{ \delta(t) \}$ for $x = 0$. This base, however, has a jump at $x = 0$, while we want a solution which is holomorphic in x. This difficulty is already overcome by K. Igari and we use a little modified idea, that is, we make a solution which is holomorphic in x by considering $u = t_+^{-1+x}/\Gamma(x)$, where Γ is the Gamma function. The pole $x = 0$ of both t_+^{-1+x} and $\Gamma(x)$ cancel out each other and u becomes $\delta(t)$ at $x = 0$.
(2) Let $P = \vartheta^2 - x$. We have $\mathcal{C}(x; \lambda) = \widetilde{\mathcal{C}}(x; \lambda) = \lambda^2 - x$.

Also by freezing $x \neq 0$, the solutions in $\mathcal{D}'_+(\boldsymbol{R})$ make a two-dimensional space with a base $\{ t_+^{\sqrt{x}}, t_+^{-\sqrt{x}} \}$. The distribution $t_+^{\pm\sqrt{x}}$ is not holomorphic at $x = 0$. This is why the definition of the normal root includes the holomorphy of $\rho(x)$. Though this example does not satisfy the Igari's condition, we can make a solution which is holomorphic in x in a neighborhood of $x = 0$ by considering $u = t_+^{\sqrt{x}} + t_+^{-\sqrt{x}}$.

These two very simple examples suggest that we need to consider distributions like

$$\frac{t_+^{-1+\sqrt{x}}}{\Gamma(\sqrt{x})} + \frac{t_+^{-1-\sqrt{x}}}{\Gamma(-\sqrt{x})} \in \mathcal{D}'_+(\boldsymbol{R}; \mathcal{O}(\boldsymbol{C})).$$

In Section 4., we shall introduce this kind of distributions of t with holomorphic parameter x.

3. Extension of Fuchsian Operators

In this section, we introduce a class of operators wider than that of the Fuchsian operators.

Consider an operator of m-th order

$$P = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(t, x) \partial_t^j \partial_x^\alpha. \tag{3.1}$$

As for the regularity of the coefficients, we consider two cases.

Case (I): $Y(\Omega_0) := \mathcal{O}(\Omega_0)$, where Ω_0 is a domain in $\boldsymbol{C}^n$ including 0.

Case (II): $Y(\Omega_0) := C^\infty(\Omega_0)$, where Ω_0 is a domain in $\boldsymbol{R}^n$ including 0.

Put $X(T_0, \Omega_0) := C^\infty(-T_0, T_0; Y(\Omega_0))$, where $T_0 > 0$, and assume that

$$a_{j,\alpha} \in X(T_0, \Omega_0) \quad (j + |\alpha| \le m). \tag{3.2}$$

Note that $X(T_0, \Omega_0) = C^\infty(-T_0, T_0; \mathcal{O}(\Omega_0)) = \mathcal{O}(\Omega_0; C^\infty(-T_0, T_0))$ in Case (I), and $X(T_0, \Omega_0) = C^\infty((-T_0, T_0) \times \Omega_0)$ in Case (II).

Let $r(j, \alpha)$ be the *vanishing order* of $a_{j,\alpha}$ on the initial surface Σ_0, that is

$$r(j, \alpha) := \sup\{\, r \in \boldsymbol{Z} \;;\; t^{-r} a_{j,\alpha} \in X(T_0, \Omega_0) \,\} \in \boldsymbol{Z} \cup \{\infty\}. \tag{3.3}$$

From now on, we assume the following condition.

(A-0) There exists (j, α) such that $r(j, \alpha) < \infty$.

If $r(j, \alpha) < \infty$, then put

$$\widetilde{a}_{j,\alpha}(t, x) := t^{-r(j,\alpha)} a_{j,\alpha}(t, x) \quad (\in X(T_0, \Omega_0)). \tag{3.4}$$

Note that $\widetilde{a}_{j,\alpha}(0, x) \not\equiv 0$.

We associate a *weight* $\omega(j, \alpha) := j - r(j, \alpha)$ to each differential monomial $a_{j,\alpha}(t, x)\partial_t^j \partial_x^\alpha$, and put $\omega = \omega(P) := \sup\{\, \omega(j, \alpha) \in \boldsymbol{Z} \cup \{-\infty\} \;;\; j + |\alpha| \le m \,\} \in \boldsymbol{Z}$.

We assume the following conditions.

(A-1) $\omega(P) \ge 0$, and if $\omega(j, \alpha) = \omega(P)$, then $\alpha = 0$.

Put $J := \{\, j \in \{0, 1, \dots, m\} \;;\; \omega(j, 0) = \omega(P) \,\} (\ne \emptyset)$ and $m' = m'(P) := \max J$.

(A-2) $\widetilde{a}_{m',0}(0, x) \ne 0$ on Ω_0.

A Fuchsian operator (1.1) satisfies these conditions with $\omega(P) = m - k$, $m'(P) = m$, and $\widetilde{a}_{m,0}(t, x) \equiv 1$.

REMARK 3.1. (1) If $\omega(P) < 0$, then $t^{j-\omega(P)} \partial_t^j \partial_x^\alpha = t^{|\omega(P)|} (\vartheta)_j \partial_x^\alpha = (\vartheta - |\omega(P)|)_j \partial_x^\alpha \circ t^{|\omega(P)|}$. Hence, we can write $P = \widetilde{P} \circ t^{|\omega(P)|}$, where $\widetilde{P}$ has also the coefficients belonging to $X(T_0, \Omega_0)$. Thus, $u = \delta(t)$ is a null-solution of P at $(0, 0)$.
(2) If P is a Fuchsian hyperbolic operator considered by H. Tahara ([9], [10], [11], and so on) with coefficients in $C^\infty((-T_0, T_0) \times \Omega_0)$, where Ω_0 is

a domain in $\boldsymbol{R}^n$ including 0, then P satisfies the conditions (A-0)–(A-2) for Case (II).
(3) The operators of the class considered in [6] also satisfies the conditions (A-0)–(A-2) for Case (I). This class includes some non-Fuchsian operators, and have the coefficients belonging to $C^\infty(-T_0, T_0; \mathcal{O}(\Omega_0))$, where Ω_0 is a domain in $\boldsymbol{C}^n$ including 0.

Put

$$\mathcal{C}(x;\lambda) = \mathcal{C}[P](x;\lambda) := \sum_{j\in J} \widetilde{a}_{j,0}(0,x)(\lambda)_j = t^{-\lambda+\omega} P(t^\lambda)|_{t=0} \in Y(\Omega_0)[\lambda].$$

This polynomial $\mathcal{C}[P]$ of λ is called the *indicial polynomial* of P. Note that $\deg_\lambda \mathcal{C}(x;\lambda) = m'(P)$ and the coefficient $\widetilde{a}_{m',0}(0,x)$ of the highest degree never vanishes on Ω_0 by the assumption (A-2).

Lemma 3.2. (1) *There holds $m'(P) \geq \omega(P)$.*
(2) *Putting $k = k(P) := m'(P) - \omega(P)$ (≥ 0), we can decompose $\mathcal{C}(x;\lambda)$ as*

$$\mathcal{C}(x;\lambda) = \widetilde{\mathcal{C}}(x;\lambda-\omega)\,(\lambda)_\omega, \tag{3.5}$$

where $\widetilde{\mathcal{C}}(x;\rho) := \sum_{l+\omega\in J} \widetilde{a}_{l+\omega,0}(0,x)(\rho)_l \in Y(\Omega_0)[\rho]$, which is of degree k.

Note that for a Fuchsian operator (1.1), there holds $m'(P) - \omega(P) = k$, and hence the use of the letter k makes no confusion.

Proof. Since there holds

$$j \in J \Longrightarrow \omega(j,0) = \omega \Longrightarrow j = r(j,0) + \omega \geq \omega,$$

we have (1). Further, in the definition $\mathcal{C}(x;\lambda) = \sum_{j\in J} \widetilde{a}_{j,0}(0,x)(\lambda)_j$, there is no term with $j < \omega$. If $j \geq \omega$, then $(\lambda)_j = (\lambda)_\omega\ (\lambda-\omega)_{j-\omega}$. Hence, we have (2). □

Finally, we assume the following condition.

(A-3) $\quad k(P) \geq 1$.

The operators satisfying these four conditions (A-0)–(A-3) are the operators for which we can construct a *formal* solution $\sum_{h=0}^\infty v_h$ in Section 5.

4. Preliminaries

In this section, we give some preliminary results needed for the later argument.

DEFINITION 4.1. We put $G(z) = G(z;t) := \dfrac{t_+^z}{\Gamma(z+1)}$. We have $G(z;t) \in \mathcal{D}'_+(\boldsymbol{R}; \mathcal{O}(\boldsymbol{C})) \bigcap C^\infty(0,\infty; \mathcal{O}(\boldsymbol{C}))$, since the simple poles $z = -1$, $-2, \ldots$ of both t_+^z and $\Gamma(z+1)$ cancel out. (Normalization of t_+^z. See [2], Chapt.I, §3.5.) Note that $G(-d;t) = \delta^{(d-1)}(t)$ $(d = 1, 2, 3, \ldots)$. Also note that $G(z;\cdot) \in \mathcal{S}'(\boldsymbol{R})$ for every fixed $z \in \boldsymbol{C}$.

The distribution $G(z;t)$ has the following basic properties. We omit the proofs of these two lemmas, since they are very easy.

LEMMA 4.2. (1) *For $l \in \boldsymbol{N}$, there hold $\partial_t^l G(z;t) = G(z-l;t)$ and $t^l G(z;t) = (z+l)_l\, G(z+l;t)$.*
(2) *For $E(\lambda) \in \boldsymbol{C}[\lambda]$, there holds $E(\vartheta)G(z;t) = E(z)G(z;t)$.*
(3) *For $\epsilon > 0$, there holds $\langle G(z;t), e^{-t/\epsilon} \rangle = \epsilon^{z+1}$.*

LEMMA 4.3. *For $M \in \boldsymbol{R}$, put $D_M := \{\, z \in \boldsymbol{C}\,;\, \operatorname{Re} z > M \,\}$. Then, for every $l \in \boldsymbol{N}$ and every $N \in \boldsymbol{Z}$, we have $t^l G(z;t) \in C_+^N(\boldsymbol{R}; \mathcal{O}(D_{N-l}))$. (Note that $\operatorname{Re} z + l > N$ on D_{N-l}.)*

Now, consider an operator (3.1) satisfying the conditions (A-0), (A-1), (A-2), and (A-3). Take a root $\lambda_0 \in \boldsymbol{C}$ of $\widetilde{\mathcal{C}}(0;\lambda) = 0$ that satisfies

$$\widetilde{\mathcal{C}}(0;\lambda_0 + l) \neq 0 \quad \text{for} \quad l = 1, 2, 3 \ldots. \tag{4.1}$$

For example, every root having the largest real part satisfies this condition.

LEMMA 4.4. *There exist a domain Ω including 0, an open ball $D \subset \boldsymbol{C}$ with the center λ_0, $r \in \boldsymbol{N}$ with $r \geq 1$, and polynomials $E(x;\lambda)$, $R(x;\lambda) \in Y(\Omega)[\lambda]$ for which the following conditions are satisfied.*

(a) $\widetilde{\mathcal{C}}(x;\lambda) = E(x;\lambda)R(x;\lambda)$.

(b) $\widetilde{\mathcal{C}}(x;\lambda + l) \neq 0$ *for every $(x,\lambda) \in \Omega \times D$ and $l = 1, 2, 3 \ldots$.*

(c) *$E(x;\lambda)$ is monic of degree r, and $E(0;\lambda) = (\lambda - \lambda_0)^r$.*

(d) *If $E(x;\lambda) = 0$ and $x \in \Omega$, then $\lambda \in D$.*

Note that if there exists a normal root $\rho(x)$ of $\widetilde{\mathcal{C}}$, then we can take $r = 1$ and $E(x;\lambda) = \lambda - \rho(x)$.

PROOF. By a well-known argument based on Rouché's theorem, we can take polynomials E, $R \in Y(\Omega)[\lambda]$ satisfying (a) and (c), by taking a sufficiently small domain Ω including 0. Since there holds (4.1), the condition (b) holds by retaking a smaller Ω and taking a sufficiently small D. Finally by (c), the condition (d) is satisfied by retaking again a smaller Ω. □

For $x \in \Omega$, let $\lambda_i(x)$ $(1 \leq i \leq r)$ be the roots of $E(x;\lambda) = 0$. Though $\lambda_i(x) \in D$ for every $x \in \Omega$, they are not necessarily holomorphic nor C^∞ in x, like in Example 2.3-(2).

We fix λ_0, Ω, D, r, E, R, and $\lambda_i(x)$ from now on. We can assume that the radius of D is smaller than 1.

DEFINITION 4.5. For $j \in \boldsymbol{Z}$ and $\phi \in \mathcal{O}(D; Y(\Omega))$, put

$$\mathcal{S}G_j[\phi](t,x) := \sum_{l=1}^{r} \phi(x;\lambda_l(x)) G(\lambda_l(x) + j; t) \quad (x \in \Omega).$$

PROPOSITION 4.6. *For every $j \in \boldsymbol{Z}$ and every $\phi \in \mathcal{O}(D; Y(\Omega))$, there holds*

$$\mathcal{S}G_j[\phi] \in \mathcal{D}'_+(\boldsymbol{R}; Y(\Omega)) \bigcap C^\infty(0,\infty; Y(\Omega)).$$

Note that $\mathcal{S}G_j[\phi](\cdot, x) \in \mathcal{S}'(\boldsymbol{R})$ for every fixed $x \in \Omega$.

PROOF. Let Γ be a closed curve in D enclosing $\{\, \lambda_l(x) \in D \ ;\ l = 1,\dots,r \,\}$ with positive direction. Since

$$\frac{(\partial_\lambda E)(x;\lambda)}{E(x;\lambda)} = \sum_{l=1}^{r} \frac{1}{\lambda - \lambda_l(x)},$$

there holds

$$SG_j[\phi](t,x) = \frac{1}{2\pi i}\int_\Gamma \frac{(\partial_\lambda E)(x;\lambda)}{E(x;\lambda)}\phi(x;\lambda)G(\lambda+j;t)\,d\lambda \tag{4.2}$$

by the Cauchy's integral formula. It is easy to show the lemma from this expression. □

LEMMA 4.7. (1) *For $j \in \boldsymbol{Z}$, $h \in \boldsymbol{N}$, $\phi \in \mathcal{O}(D; Y(\Omega))$, and $\epsilon > 0$, there hold*

$$\partial_t^h(SG_j[\phi]) = SG_{j-h}[\phi], \qquad t^h SG_j[\phi] = SG_{j+h}[(\cdot + h + j)_h\, \phi],$$
$$\langle SG_j[\phi](t,0), e^{-t/\epsilon}\rangle = r\phi(0;\lambda_0)\epsilon^{\lambda_0+j+1}.$$

(2) *For $F \in Y(\Omega)[\lambda]$, there holds $F(x;\vartheta)SG_j[\phi] = SG_j[F(\cdot;\cdot+j)\,\phi]$. Especially, for E in Lemma 4.4, there holds $E(x;\vartheta)SG_0[\phi] = 0$.*

PROOF. It is easy from Lemma 4.2 and the definition of $SG_j[\phi]$. □

DEFINITION 4.8. For $j, A \in \boldsymbol{N}$, put

$$\mathcal{G}_j := \{\, SG_j[\phi](t,x) \in \mathcal{D}'_+(\boldsymbol{R}; Y(\Omega))\ ;\ \phi(x;\lambda) \in \mathcal{O}(D; Y(\Omega))\,\},$$

and

$$\mathcal{G}_j^{(A)} := \{\sum_{|\alpha|\le A} a_\alpha(x)\partial_x^\alpha v_\alpha(t,x)\ ;\ a_\alpha \in Y(\Omega), v_\alpha \in \mathcal{G}_j\,\}.$$

Example 4.9. If $r = 1$ and $\rho(x) := \lambda_1(x) \notin \{-1,-2,\dots,\}$ $(x \in \Omega)$, then

$$\mathcal{G}_j^{(A)} = \{\sum_{l=0}^{A} a_l(x) t_+^{\rho(x)+j}(\log t)^l\ ;\ a_l \in Y(\Omega) \quad (0 \le l \le A)\,\},$$

where $t_+^z(\log t)^l$ $(z \ne -1,-2,\dots)$ is a distribution of t defined similarly to t_+^z.

We have the following basic properties of $\mathcal{G}_j^{(A)}$.

LEMMA 4.10. (1)
$$\partial_t^L(\mathcal{G}_j^{(A)}) \subset \mathcal{G}_{j-L}^{(A)}, \qquad t^L \times \mathcal{G}_j^{(A)} \subset \mathcal{G}_{j+L}^{(A)},$$
$$\vartheta^L(\mathcal{G}_j^{(A)}) \subset \mathcal{G}_j^{(A)}, \qquad Y(\Omega) \times \mathcal{G}_j^{(A)} \subset \mathcal{G}_j^{(A)},$$
$$\partial_x^\alpha(\mathcal{G}_j^{(A)}) \subset \mathcal{G}_j^{(A+|\alpha|)}.$$

(2) *If* $u \in \mathcal{G}_j^{(A)}$, *then* $\langle u(t,0), e^{-t/\epsilon}\rangle = o(\epsilon^{\Lambda+j+1})$ *for every* $\Lambda < \operatorname{Re}\lambda_0$.

Proof. (1) is trivial. By (4.2) and Lemma 4.2-(3), there holds

$$\langle SG_j[\phi](t,x), e^{-t/\epsilon}\rangle = \frac{1}{2\pi i}\int_\Gamma \frac{(\partial_\lambda E)(x;\lambda)}{E(x;\lambda)}\phi(x;\lambda)\epsilon^{\lambda+j+1}\,d\lambda,$$

and hence by Lemma 4.4-(c), there holds

$$\langle \partial_x^\alpha(SG_j[\phi])(t,0), e^{-t/\epsilon}\rangle = \frac{1}{2\pi i}\int_\Gamma w(\lambda)\epsilon^{\lambda+j+1}\,d\lambda,$$

with some w that is holomorphic in $D\setminus\{\lambda_0\}$. Since Γ can be taken arbitralily near to λ_0, we get the desired result. □

Lemma 4.11. *If* $r_0 \in \boldsymbol{Z}$ *and* $r_0 < \inf\{\operatorname{Re}\lambda_i(x)\ ;\ 1 \le i \le r, x \in \Omega\}$, *then*

$$\mathcal{G}_j^{(A)} \subset C_+^{r_0+j}(\boldsymbol{R}; Y(\Omega)) \bigcap C^\infty(0,\infty; Y(\Omega)) \quad (j \in \boldsymbol{N}).$$

Proof. If $A=0$, then this follows from (4.2) and Lemma 4.3. Since $SG_j[\phi]$ is holomorphic or C^∞ with respect to x, we have the result for general A. □

The following proposition is the most fundamental to the construction of v_h.

Proposition 4.12. *Let* $j, A \in \boldsymbol{N}$.
(1) *If* $F(x;\lambda) \in Y(\Omega)[\lambda]$ *and* $F(x;\lambda+j) \neq 0$ *for every* $(x,\lambda) \in \Omega \times D$, *then for every* $g \in \mathcal{G}_j^{(A)}$, *there exists a solution* $v \in \mathcal{G}_j^{(A)}$ *of* $F(x;\vartheta)v = g$.
(2) *Let* $L \in \boldsymbol{N}$. *For every* $g \in \mathcal{G}_j^{(A)}$, *there exists a solution* $w \in \mathcal{G}_{j+L}^{(A)}$ *of* $\partial_t^L w = g$.

Proof. We prove this proposition by an induction on A.

If $A=0$, then $g = SG_j[\phi]$ and hence $v = SG_j[\phi/F(\cdot;\cdot+j)]$ is a solution of $F(x;\vartheta)v = g$, and $w = SG_{j+L}[\phi]$ is a solution of $\partial_t^L w = g$, by Lemma 4.7.

We assume the result for A and let $g \in \mathcal{G}_j^{(A+1)}$. We may assume that $g = \partial_{x_l}\widetilde{g}$, $\widetilde{g} \in \mathcal{G}_j^{(A)}$, without loss of generality. By the induction hypothesis, there exists $\widetilde{v} \in \mathcal{G}_j^{(A)}$ such that $F(x;\vartheta)\widetilde{v} = \widetilde{g}$. Further, there exists $\widetilde{\widetilde{v}} \in \mathcal{G}_j^{(A)}$ such that $F(x;\vartheta)\widetilde{\widetilde{v}} = (\partial_{x_l} F)(x;\vartheta)\widetilde{v}$, since $(\partial_{x_l} F)(x;\vartheta)\widetilde{v} \in \mathcal{G}_j^{(A)}$. Thus, $v := \partial_{x_l}\widetilde{v} + \widetilde{\widetilde{v}} \in \mathcal{G}_j^{(A+1)}$ satisfies $F(x;\vartheta)v = \partial_{x_l}\widetilde{g} = g$. On the other hand, there exist $\widetilde{w} \in \mathcal{G}_{j+L}^{(A)}$ such that $\partial_T^L \widetilde{w} = \widetilde{g}$. Hence, $w := \partial_{x_l}\widetilde{w} \in \mathcal{G}_{j+L}^{(A+1)}$ satisfies $\partial_t^L w = \partial_{x_l}\widetilde{g} = g$. □

5. Construction of v_h

Is this section, we construct a formal solution $\sum_{h=0}^{\infty} v_h$ of $Pv = 0$.

Consider the operator (3.1), and assume the conditions (A-0), (A-1), (A-2) and (A-3). Then, we have a formal solution as follows.

THEOREM 5.1. *Put $v_0(t,x) := SG_\omega[1](t,x)$. Then, there exists $v_h \in \mathcal{G}_{\omega+h}^{(hm)}$ $(h \geq 1)$ such that*

$$P(\sum_{h=0}^{N} v_h) \in C_+^{r_0+N+1}(-T,T;Y(\Omega)) \qquad \textit{for every } N \in \boldsymbol{N},$$

where $r_0 \in \boldsymbol{Z}$ and $r_0 < \inf\{\operatorname{Re}\lambda_i(x)\ ;\ 1 \leq i \leq r, x \in \Omega\}$.

PROOF. Since $\partial_t^\omega v_0 = SG_0[1]$, we have $\widetilde{\mathcal{C}}(x;\vartheta)\partial_t^\omega v_0 = 0$ by Lemma 4.7-(2).

Now, for every sufficiently large $N \in \boldsymbol{N}$, we consider the expansion of P by weight:

$$\begin{aligned} P \;=\; & \widetilde{\mathcal{C}}(x;\vartheta)\partial_t^\omega + \sum_{j=1}^{\omega} A_j(x,\partial_x;\vartheta)\partial_t^{\omega-j} + \sum_{l=1}^{N-\omega} t^l B_l(x,\partial_x;\vartheta) \\ & + t^{N-\omega+1} R_N(t,x,\partial_x;\vartheta), \end{aligned}$$

where $A_j(x,\xi;\lambda), B_l(x,\xi;\lambda) \in Y(\Omega)[\xi,\lambda]$, and $R_N(t,x,\xi;\lambda) \in X(T,\Omega)[\xi,\lambda]$. Note that for every h, there holds the following by Lemma 4.10-(1).

$$\begin{aligned} \widetilde{\mathcal{C}}(x;\vartheta)\partial_t^\omega :\ & \mathcal{G}_h^{(A)} \longrightarrow \mathcal{G}_{h-\omega}^{(A)}, \\ A_j(x,\partial_x;\vartheta)\partial_t^{\omega-j} :\ & \mathcal{G}_h^{(A)} \longrightarrow \mathcal{G}_{h-\omega+j}^{(A+m)} \quad (j = 1,2,\ldots,\omega), \\ t^l B_l(x,\partial_x;\vartheta) :\ & \mathcal{G}_h^{(A)} \longrightarrow \mathcal{G}_{h+l}^{(A+m)} \quad (l = 1,2,\ldots,N-\omega). \end{aligned}$$

Using Lemma 4.4-(b) and Proposition 4.12, we can take $v_h \in \mathcal{G}_{\omega+h}^{(hm)}$ $(h \geq 1)$ recursively as

$$\begin{aligned}\widetilde{\mathcal{C}}(x;\vartheta)\partial_t^{\omega} v_h &= -\sum_{j=1}^{\omega} A_j(x,\partial_x;\vartheta)\partial_t^{\omega-j} v_{h-j} \\ &\quad -\sum_{l=1}^{N-\omega} t^l B_l(x,\partial_x;\vartheta) v_{h-\omega-l} \quad (\in \mathcal{G}_h^{(hm)}).\end{aligned}$$

(Consider $v_h = 0$ if $h < 0$.)

Thus, we have

$$\begin{aligned}P\Big(\sum_{h=0}^{N} v_h\Big) &= \sum_{j=1}^{\omega} \sum_{p=N-j+1}^{N} A_j(x,\partial_x;\vartheta)\partial_t^{\omega-j} v_p \\ &\quad + \sum_{l=1}^{N-\omega} \sum_{q=N-\omega-l+1}^{N} t^l B_l(x,\partial_x;\vartheta) v_q \\ &\quad + R_N(t,x,\partial_x;\vartheta - N + \omega - 1)\Big(t^{N-\omega+1} \sum_{h=0}^{N} v_h\Big),\end{aligned}$$

and hence we have $P(\sum_{h=0}^{N} v_h) \in C_+^{r_0+N+1}(-T,T;Y(\Omega))$ by Lemma 4.10-(1) and 4.11. □

6. Realization of Solutions

Is this section, we show the existence of distribution null-solution u using v_h constructed in the previous section.

Consider an operator (3.1) and assume the conditions (A-0), (A-1), (A-2), and (A-3). In this section, we also assume the following condition.

(B) There exist $T' > 0$ and a domain Ω' including 0 for which the solvability of the flat Cauchy problem holds as follows.

For every $f \in C_+^{\infty}(-T,T;Y(\Omega))$, there exists $u \in C_+^{\infty}(-T',T';Y(\Omega'))$ such that $Pu = f$ in $(-T',T') \times \Omega'$.

REMARK 6.1. (1) Take $N \in \boldsymbol{N}$ such that $\mathcal{C}[P](x;l) \neq 0$ for every $l \in \boldsymbol{N}$ satisfying $l \geq \omega(P)+N$ and every $x \in \Omega$. If we put $\widetilde{P}_N := t^{-N} \circ P \circ t^{\omega(P)+N}$, then $\widetilde{P}_N$ is also an operator of the form (3.1) with $\omega(\widetilde{P}_N) = 0$. Further, $\widetilde{P}_N$ satisfies the conditions (A-0), (A-1), (A-2), and there holds $\mathcal{C}[\widetilde{P}_N](x;\lambda) = \mathcal{C}[P](x;\lambda+\omega(P)+N)$. Especially, $\mathcal{C}[\widetilde{P}_N](x;l) \neq 0$ for every $l \in \boldsymbol{N}$ and every $x \in \Omega$.

Now, assume that the solvability of the Cauchy problem for $\widetilde{P}_N$ holds in the following sense. (Note that we have no initial data since $\omega(\widetilde{P}_N) = 0$.)

> For every $f \in C^\infty([0,T);Y(\Omega))$, there exists $u \in C^\infty([0,T');Y(\Omega'))$ such that $\widetilde{P}_N u = f$ in $[0,T') \times \Omega'$.

Then, we can show that the condition (B) for P holds. In fact, for every $f \in C^\infty_+(-T,T;Y(\Omega))$, the equation $Pu = f$ is reduced to the equation $\widetilde{P}_N \widetilde{u} = \widetilde{f}$, where $\widetilde{u} := t^{-\omega-N}u$ and $\widetilde{f} := t^{-N}f$. Since $\widetilde{f} \in C^\infty([0,T);Y(\Omega))$, there exists $\widetilde{u} \in C^\infty([0,T');Y(\Omega'))$ such that $\widetilde{P}_N\widetilde{u} = \widetilde{f}$. By substituting the formal Taylor expansion $\widetilde{u} = \sum_{h=0}^\infty u_h(x)t^h$ into this equation, we can easily show that $u_h = 0$ for every $h \in \boldsymbol{N}$ by using $\mathcal{C}[\widetilde{P}_N](x;l) \neq 0$ for every $l \in \boldsymbol{N}$ and every $x \in \Omega$. By putting $\widetilde{u} = 0$ for $t < 0$, we can consider $\widetilde{u}$ as $\widetilde{u} \in C^\infty_+(-T',T';Y(\Omega'))$. Thus $u = t^N\widetilde{u} \in C^\infty_+(-T',T';Y(\Omega'))$.
(2) If P is a Fuchsian operator with coefficients in $C^\infty(-T,T;\mathcal{O}(\Omega))$, where Ω is a domain in $\boldsymbol{C}^n$ including 0, then P satisfies the condition (B) for Case (I) ($Y(\Omega) = \mathcal{O}(\Omega)$), by the result of Baouendi-Goulaoiuc([1]) and the remark (1) above.

THEOREM 6.2. *If the operator* (3.1) *satisfies the conditions* (A-0), (A-1), (A-2), (A-3), *and* (B), *then there exists* $u \in \mathcal{D}'_+(-T',T';Y(\Omega')) \bigcap C^\infty(0,T';Y(\Omega'))$ *such that*

$$Pu = 0 \quad in \quad (-T',T') \times \Omega',$$

$$u - \sum_{h=0}^{N} v_h \in C_+^{r_0+\omega+N+1}(-T',T';Y(\Omega')) \qquad for\ every\ \ N \in \boldsymbol{N},$$

where $\{v_h\}_h$ *are the distributions constructed in Theorem* 5.1, $r_0 \in \boldsymbol{Z}$ *and* $r_0 < \inf\{\operatorname{Re}\lambda_i(x)\ ;\ 1 \leq i \leq r, x \in \Omega\}$. *Further, there holds* $(0,0) \in \operatorname{supp} u \subset \Sigma_+$, *that is,* u *is a null-solution of* P *at* $(0,0)$.

Theorem 1.1 follows from this theorem, by Remark 6.1-(2).

Remark 6.3. (1) If P is a Fuchsian hyperbolic operator considered by H. Tahara ([9], [10], [11], and so on) with coefficients in $C^\infty((-T_0,T_0)\times\Omega_0)$, then P satisfies the condition (B) for Case (II) ($Y(\Omega_0)=C^\infty(\Omega_0)$). For example, see Theorems 3.1, 4.1, 5.1, and 6.12 in [11]. Hence, if (A-3) is satisfied, then there exist distribution null-solutions.
(2) The operators of the class considered in [6] also satisfy the condition (B) for Case (I). See Theorem 1.5 in [6]. Hence, if (A-3) is satisfied, then there exist distribution null-solutions.

The key of the proof of this theorem is the following lemma.

Lemma 6.4. *For every $v_h \in \mathcal{G}^{(A_h)}_{\omega+h}$ ($A_h \in \boldsymbol{N}$; $h \in \boldsymbol{N}$), there exists*

$$v \in \mathcal{D}'_+(-T,T;Y(\Omega)) \bigcap C^\infty(0,T;Y(\Omega))$$

such that there holds

$$v - \sum_{h=0}^{N} v_h \in C_+^{r_0+\omega+N+1}(-T,T;Y(\Omega))$$

for every $N \in \boldsymbol{N}$.

Proof. Take $\psi(t) \in C^\infty(\boldsymbol{R})$ such that $\psi(t)=1$ for $(-\infty,1/2]$ and $\psi(t)=0$ for $[1,\infty)$. For a formal series $\sum_{h=0}^{\infty} v_h(t,x)$, we construct v in the form

$$v := \sum_{h=0}^{\infty} v_h(t,x)\psi(t/\epsilon_h) \tag{6.1}$$

for suitably chosen $\epsilon_h > 0$. We prove in Case (II). The proof in Case (I) is similar and easier, and hence omitted.

Take an increasing sequence $\{U_n\}_{n\in\boldsymbol{N}}$ of subdomains of Ω such that $K_n := \overline{U_n}$ are compact subsets of Ω and $\bigcup_{n\in\boldsymbol{N}} U_n = \Omega$. Put $||w||_h := \sum_{|\alpha|\le h} \sup_{x\in K_h} |\partial_x^\alpha w(x)|$.

We have

$$v_h \in \mathcal{G}^{(A_h)}_{\omega+h} \subset C_+^{r_0+\omega+h}(\boldsymbol{R};C^\infty(\Omega)) \bigcap C^\infty(0,\infty;C^\infty(\Omega)) \quad (h \in \boldsymbol{N})$$

by Lemma 4.11, and hence $\partial_t^l v_h \in C_+^{r_0+\omega+h-l}(\boldsymbol{R};C^\infty(\Omega))$ ($l \in \boldsymbol{N}$).

Thus, for every l and h satisfying $l \le r_0 + \omega + h$, the function $t^{l-h-\omega-r_0}||\partial_t^l v_h(t,\cdot)||_q$ of t is bounded on $(0,T]$ for every $q \in \boldsymbol{N}$. On the other hand, for every l, $q \in \boldsymbol{N}$, there holds $\sup_{0\le t\le T} |t^l \partial_t^q\{\psi(t/\epsilon)\}| \le C_{l,q}\epsilon^{l-q}$ with some constant $C_{l,q}$ independent of $\epsilon > 0$. Hence, we can easily show that for every h, $k \in \boldsymbol{N}$ satisfying $k \le r_0 + \omega + h$, there exists a constant $C_{h,k}$ independent of ϵ such that

$$\sup_{0\le t\le T} ||\partial_t^k(v_h(t,\cdot)\psi(t/\epsilon))||_h \le C_{h,k}\epsilon^{r_0+\omega+h-k} \quad \text{for every} \quad \epsilon > 0.$$

Thus, for every $h \in \boldsymbol{N}$ satisfying $h \ge 1 - r_0 - \omega$, we can take $\epsilon_h > 0$ such that

$$\sum_{k=0}^{r_0+\omega+h-1} \sup_{0\le t\le T} ||\partial_t^k(v_h(t,\cdot)\psi(t/\epsilon_h))||_h \le (\frac{1}{2})^h. \tag{6.2}$$

Put $\epsilon_h = 1$ for $h < 1 - r_0 - \omega$.

Finally, put

$$W_N := \sum_{h=0}^{N} v_h(t,x)\psi(t/\epsilon_h), \quad r_N := \sum_{h=N+1}^{\infty} v_h(t,x)\psi(t/\epsilon_h).$$

By the estimate (6.2), r_N converges in $C_+^{r_0+\omega+N}(-T,T;C^{N+1}(U_{N+1}))$ for every $N \in \boldsymbol{N}$ with $N \ge -\omega - r_0$. Hence, $v := \sum_{h=0}^{\infty} v_h(t,x)\psi(t/\epsilon_h) = W_N + r_N$ (independent of N) defines $v \in \mathcal{D}'_+(-T,T;C^\infty(\Omega)) \bigcap C^\infty(0,T;C^\infty(\Omega))$.

We have $v - \sum_{h=0}^{N} v_h = (v - W_M) + (W_M - \sum_{h=0}^{N} v_h) = r_M + \sum_{h=0}^{N} v_h(t,x) \cdot \{\psi(t/\epsilon_h) - 1\} + \sum_{h=N+1}^{M} v_h(t,x)\psi(t/\epsilon_h) \in C_+^{r_0+\omega+N+1}(-T,T;C^{M+1}(U_{M+1}))$ for every $M \ge N+1$. Hence, $v - \sum_{h=0}^{N} v_h \in C_+^{r_0+\omega+N+1}(-T,T;C^\infty(\Omega))$ for every $N \in \boldsymbol{N}$ satisfying $N \ge -\omega - r_0 - 1$. If $r_0 + \omega + 1 < 0$ and $N < -\omega - r_0 - 1$, then $v - \sum_{h=0}^{N} v_h = v - \sum_{h=0}^{-\omega-r_0-1} v_h + \sum_{h=N+1}^{-\omega-r_0-1} v_h \in C_+^0(-T,T;C^\infty(\Omega)) + C_+^{r_0+\omega+N+1}(-T,T;C^\infty(\Omega)) \subset C_+^{r_0+\omega+N+1}(-T,T;C^\infty(\Omega))$. $\square$

We also have the following lemma, whose proof is easy and omitted.

LEMMA 6.5. *Let $N \in \boldsymbol{Z}$, $\delta > 0$ and Ω be a domain including* 0. *Let $\chi \in C^\infty(\boldsymbol{R})$ satisfy $\chi(t) = 1$ in a neighborhood of $t = 0$ and $\chi(t) = 0$ for $|t| \geq \delta/2$. If $u \in C_+^N(-\delta, \delta; Y(\Omega))$, then $\langle \chi u|_{x=0}, e^{-t/\epsilon}\rangle = O(\epsilon^{N+1})$ $(\epsilon \to +0)$.*

Now, we prove Theorem 6.2.

PROOF OF THEOREM 6.2. Since $R_N := \sum_{h=0}^N v_h(t,x)\{\psi(t/\epsilon_h) - 1\} \in C_+^\infty(\boldsymbol{R}; Y(\Omega))$ for every $N \in \boldsymbol{N}$, and since $1 \geq \omega - m$, we have

$$Pv = P(r_N) + P(\sum_{h=0}^{N} v_h) + P(R_N) \in C_+^{r_0+\omega+N-m}(-T,T; C^{N+1}(U_{N+1}))$$

in Case (II) for every $N \in \boldsymbol{N}$. That is, $g := Pv \in C_+^\infty(-T,T;Y(\Omega))$. This is valid also in Case (I). By the condition (B), there exists $w \in C_+^\infty(-T',T';Y(\Omega'))$ such that $Pw = g$. Thus, $u := v - w \in \mathcal{D}'_+(-T',T'; Y(\Omega'))$ satisfies $Pu = 0$. Further, $u - \sum_{h=0}^{N} v_h = v - \sum_{h=0}^{N} v_h - w \in C_+^{r_0+\omega+N+1}(-T',T';Y(\Omega'))$ for every $N \in \boldsymbol{N}$.

Finally, we show that u satisfies $(0,0) \in \operatorname{supp} u$.

By Lemma 4.7-(1), we have

$$\langle v_0|_{x=0}, e^{-t/\epsilon}\rangle = r\epsilon^{\lambda_0+\omega+1} \quad (\epsilon > 0).$$

Hence $\langle \chi v_0|_{x=0}, e^{-t/\epsilon}\rangle = r\epsilon^{\lambda_0+\omega+1} + \langle (\chi-1)v_0|_{x=0}, e^{-t/\epsilon}\rangle = r\epsilon^{\lambda_0+\omega+1} + o(\epsilon^N)$ for every $N \in \boldsymbol{N}$ by Lemma 6.5.

By Lemma 4.10-(2), we also have $\langle v_1|_{x=0}, e^{-t/\epsilon}\rangle = O(\epsilon^{\Lambda+\omega+2})$ $(\epsilon \to +0)$ for $\Lambda < \operatorname{Re}\lambda_0$, since $v_1 \in \mathcal{G}_{\omega+1}^{(m)}$. Hence $\langle \chi v_1|_{x=0}, e^{-t/\epsilon}\rangle = O(\epsilon^{\Lambda+\omega+2})$.

Since $w := u - v_0 - v_1 \in C_+^{r_0+\omega+2}$, we have

$$\langle \chi w|_{x=0}, e^{-t/\epsilon}\rangle = O(\epsilon^{r_0+\omega+3}) = o(\epsilon^{\operatorname{Re}\lambda_0+\omega+1}) \quad (\epsilon \to +0),$$

by Lemma 6.5, since we can take $r_0 > \operatorname{Re}\lambda_0 - 2$ in Lemma 4.11 by the assumption that the radius of D is smaller than 1.

Thus, $u = v_0 + v_1 + w$ satisfies

$$\langle \chi u|_{x=0}, e^{-t/\epsilon}\rangle = r\epsilon^{\lambda_0+\omega+1} + o(\epsilon^{\operatorname{Re}\lambda_0+\omega+1}) \quad (\epsilon \to +0).$$

Again by Lemma 6.5, this implies that $0 \in \operatorname{supp}(u|_{x=0})$ and hence that $(0,0) \in \operatorname{supp} u$. □

References

[1] Baouendi, M. S. and C. Goulaouic, Cauchy problems with characteristic initial hypersurface, Comm. Pure Appl. Math. **26** (1973), 455–475.

[2] Gel'fand, I. M. and G. E. Shilov, *Generalized Functions, Volume 1: Properties and Operations*, Academic Press, 1964, Transl. by Saletan, E.

[3] Hörmander, L., *The Analysis of Linear Partial Differential Operators I, Second Edition*, Grund. math. Wiss., vol. **256**, Springer-Verlag, 1983, 1990.

[4] Igari, K., Non-unicité dans le problème de Cauchy caractéristique—cas de type de Fuchs—, J. Math. Kyoto Univ. **25** (1985), 341–355.

[5] Kashiwara, M. and T. Oshima, Systems of differential equations with regular singularities and their boundary value problems, Ann. of Math. (2) **106** (1977), 145–200.

[6] Mandai, T., Characteristic Cauchy problems for some non-Fuchsian partial differential operators, J. Math. Soc. Japan **45** (1993), 511–545.

[7] Mandai, T., Distribution null-solutions for a class of non-Fuchsian partial differential operators, to appear.

[8] Schwartz, L., Théorie des distributions à valeurs vectorielles, Ann. Inst. Fourier (Grenoble) **7** (1957), 1–141.

[9] Tahara, H., Cauchy problems for Fuchsian hyperbolic partial differential equations, Proc. Japan Acad. Ser. A Math. Sci. **54** (1978), 92–96.

[10] Tahara, H., Fuchsian type equations and Fuchsian hyperbolic equations, Japan. J. Math. (N.S.) **5** (1979), 245–347.

[11] Tahara, H., Singular hyperbolic systems, III. on the Cauchy problem for Fuchsian hyperbolic partial differential equations, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 465–507.

[12] Tahara, H., Singular hyperbolic systems, V. asymptotic expansions for Fuchsian hyperbolic partial differential equations, J. Math. Soc. Japan **36** (1984), 449–473.

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