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Real Shintani Functions and Multiplicity Free Property for the Symmetric Pair $\big(SU(2,1), S(U(1,1) \times U(1))\big)$

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Abstract. We shall present an explicit formula of "generalized" spherical functions on $SU(2,1)$ with respect to its reductive spherical subgroup $S\big(U(1,1) \times U(1)\big)$, which can be considered to be a real analogue of the Whittaker-Shintani functions introduced by Shintani and investigated by Murase and Sugano. At the same time, we shall prove a multiplicity one theorem for the corresponding space of intertwining operators.

§0. Introduction

In this paper we shall investigate a kind of generalized spherical functions on the real semisimple Lie group $SU(2,1)$ associated to a subgroup $S\big(U(1) \times U(1,1)\big)$, which will be called *the Shintani functions* in what follows. They were first introduced by Shintani in his unpublished work intending to study certain automorphic L-functions on symplectic groups, and later investigated by Murase and Sugano in setting of classical groups and their spherical subgroups over local or global fields to get new integral representations of automorphic L-functions for many examples in terms of the Shintani functions, or the Whittaker-Shintani functions in their terminology. Now we consider a situation that a classical group G and its spherical subgroup H, both defined over a local field k, are given. Their k-valued points are naturally considered to be locally compact groups. For a

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given irreducible admissible $G\times H$-module $\Pi\boxtimes\eta$, Whittaker-Shintani function of type $\Pi\boxtimes\eta$ is defined to be a function which belongs to the image of a $G\times H$-intertwining operator $\Phi:\Pi\boxtimes\eta\to C^\infty(G)$ and is finite under the action of some maximal compact subgroup of $G\times H$, where we regard $C^\infty(G)$ as a $G\times H$-module by the left G-action and the right H-action.

When k is non archimedean, G and H are both unramified over k and $\Pi\boxtimes\eta$ is of class one with respect to a certain maximal compact subgroup of $G\times H$, Murase, Sugano and Kato proved a multiplicity one theorem of intertwining operators Φ and an explicit formula of class one Whittaker-Shintani functions for a number of examples.

Compared with the unramified situation over non archimedean local fields, few facts are known when $k=\mathbb{R}$, i.e. G and H are real reductive Lie groups. Murase and Sugano consider automorphic forms whose archimedean components are special type of holomorphic or antiholomorphic discrete series representation and calculate the zeta integral by means of the Bergman kernel functions. To make theories on integral representations of automorphic L-functions applicable to automorphic forms with more general type of archimedean component, we have to study basic properties of the Shintani functions over real groups more.

On the other hand we can consider a representation theoretical problem that for given irreducible admissible representations η and Π of H and G respectively, under what condition Π is realized as a submodule of the induced representation $C^\infty\mathrm{Ind}_H^G(\eta)$. (Here induction is considered in the category of smooth representations.) In this context to consider the space $\mathcal{I}_{\eta,\Pi}=\mathrm{Hom}_G(\Pi,C^\infty\mathrm{Ind}_H^G(\eta))$ (or possibly $\mathrm{Hom}_{(\mathfrak{g},K)}\big(\Pi,C^\infty\mathrm{Ind}_H^G(\eta)\big)$ if $k=\mathbb{R}$) and functions in $\mathcal{S}_{\eta,\Pi}=\sum\mathrm{Im}(\Phi)$ with $\Phi\in\mathcal{I}_{\eta,\Pi}$, which we also call the Shintani functions, seems to be rather natural. As is seen naively, it is closely related to consider the space $\mathrm{Hom}_{G\times H}(\Pi\boxtimes\eta^*,C^\infty(G))$ (η^* means the contragredient representation) and the Shintani functions in the former sense.

In this paper, we shall investigate the Shintani functions in $\mathrm{Hom}_K(\tau,\mathcal{S}_{\eta,\Pi})$ with K-types τ in the case of $G=SU(2,1)$, $H=S(U(1)\times U(1,1))\cong U(1,1)$ giving a description of the intertwining space $\mathcal{I}_{\eta,\Pi}$ for every irreducible unitary representation η of H and a standard representation Π of G. Our result yields that the dimension of $\mathcal{I}_{\eta,\Pi}$ is not exceed one and gives a necessary and sufficient condition for $\dim_{\mathbb{C}}\mathcal{I}_{\eta,\Pi}=1$, or equivalently $\mathcal{S}_{\eta,\Pi}(\tau_0)\neq\{0\}$ for a K-type τ_0 of Π. (For precise definitions of

$\mathcal{I}_{\eta,\Pi}$ and $\mathcal{S}_{\eta,\Pi}$, see **(4.1)**.) We show that radial part of a Shintani function is expressed in terms of Gauss's hypergeometric function (Theorem 8.1.1, Theorem 8.1.2 and Theorem 8.1.3 for the discrete series representations and Theorem 9.2.1 for the principal series representations.) The method employed in this paper is that of Yamashita [Y], which gives a characterization of the space $\mathcal{S}_{\eta,\Pi}(\tau_0)$ in terms of the *Schmid operator*, when Π is a discrete series representation of G, τ_0 is its minimal K-type and η is an arbitrary irreducible unitary representation of H.

We should note that a description of the discrete part of the H-spectrum of $\Pi|H$ for a discrete series Π of G is already known by a work of Xie, [X] and this seems to have an intimate relation with one of our problem to determine the dimension of $\mathcal{I}_{\eta,\Pi}$ because of the 'Frobenius reciprocity'. But we need more concrete and precise formulas of functions in the spaces $\mathrm{Hom}_K(\tau, \mathcal{S}_{\eta,\Pi})$ for K-types τ of Π. First reason why we need such information is that by using the explicit formula we want to study a zeta integral of Shintani functions in the theory of Murese and Sugano that will give archimedean local factors of certain automorphic L-functions for unitary groups.

Secondly, though there are many works on spherical functions or special functions on real Lie groups, majority of them are of class one, or of one dimensional K-types. We believe that to see what happens when one removes this one dimensional assumption on K-types is interesting itself apart from number theoretical applications.

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§1. Basic notations

We establish basic notations, and recall root space decompositions of $SU(2,1)$.

(1.1) Groups and Lie algebras

For a given Lie group L we use the corresponding German letter $\mathfrak{l}$ to indicate its Lie algebra and $\mathfrak{l}_{\mathbb{C}}$ the complexification of $\mathfrak{l}$.

Let

$$G = \{g \in SL_3(\mathbb{C}) |\ {}^t\overline{g} I_{2,1} g = I_{2,1}\ \},$$

$$K = \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \middle| \; k_1 \in U(2),\ k_2 \in U(1),\ k_2 \det(k_1) = 1 \right\},$$

where $I_{2,1} = \mathrm{diag}(1,1,-1)$; G is a connected semisimple Lie group which is usually denoted by $SU(2,1)$ and K is a maximal compact subgroup of G. The Lie algebras of G and K are realized as

$$\begin{aligned} \mathfrak{g} &= \{X \in M_3(\mathbb{C}) |\ {}^t\overline{X} I_{2,1} + I_{2,1} X = O,\ \mathrm{tr}(X) = 0\}, \\ \mathfrak{k} &= \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \middle| \begin{array}{lll} X_1 \in M_2(\mathbb{C}), & X_2 \in \mathbb{C}, & \mathrm{tr}(X_1) + X_2 = 0 \\ {}^t\overline{X_1} + X_1 = O, & \overline{X_2} = -X_2 & \end{array} \right\}. \end{aligned}$$

Let θ be the Cartan involution of $\mathfrak{g}$ corresponding to the choice of K and $\mathfrak{p}$ the -1 eigenspace of θ. Then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Z \\ {}^t\overline{Z} & 0 \end{pmatrix} \middle| \; Z \in M_{21}(\mathbb{C}) \right\}.$$

Let σ be the involutive automorphism of G defined by $\sigma(g) = I_{1,2}^{-1} g I_{1,2}$ $(g \in G)$ with $I_{1,2} = \mathrm{diag}(1,-1,-1)$. The set consisting of all fixed points of σ forms a closed subgroup of G which we denote by H. We have

$$H = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \middle| h_1 \in U(1),\ h_2 \in U(1,1),\ h_1 \det(h_2) = 1 \right\},$$

where

$$U(1,1) = \{g' \in GL_2(\mathbb{C}) |\ {}^t\overline{g}' I_{1,1} g' = I_{1,1}\}$$

with $I_{1,1} = \mathrm{diag}(1,-1)$. The Lie algebra of H is realized as

$$\mathfrak{h} = \left\{ \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \middle| \begin{array}{lll} Y_1 \in \mathbb{C}, & Y_2 \in M_2(\mathbb{C}), & {}^t\overline{Y_2} I_{1,1} + I_{1,1} Y_2 = O, \\ Y_1 + \mathrm{tr}(Y_2) = 0, & \overline{Y_1} + Y_1 = 0 & \end{array} \right\}.$$

(1.2) Iwasawa decomposition

Let

$$A=\left\{a_r=\begin{pmatrix}(r+r^{-1})/2 & 0 & (r-r^{-1})/2\\ 0 & 1 & 0\\ (r-r^{-1})/2 & 0 & (r+r^{-1})/2\end{pmatrix}\,\middle|\, r>0\right\}.$$

Then its Lie algebra $\mathfrak{a}=\mathbb{R}H_1,\ H_1:=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$, is a maximal abelian subspace of $\mathfrak{p}$. Let us denote by M the centralizer of $\mathfrak{a}$ in K. We have

$$M=\left\{m_\theta=\begin{pmatrix}e^{\sqrt{-1}\theta} & 0 & 0\\ 0 & e^{-2\sqrt{-1}\theta} & 0\\ 0 & 0 & e^{\sqrt{-1}\theta}\end{pmatrix}\,\middle|\,\theta\in\mathbb{R}\right\}.$$

For every integer n, set $\mathfrak{g}_n:=\{X\in\mathfrak{g}|\ [H_1,X]=nX\ \}$. Then

$$\begin{aligned}
&\mathfrak{g}_0=\mathfrak{a}+\mathfrak{m}\\
&\mathfrak{g}_1=\mathbb{R}E_2^+\oplus\mathbb{R}E_2^-,\ E_2^+=\begin{pmatrix}0&-1&0\\1&0&-1\\0&-1&0\end{pmatrix},\\
&\qquad\qquad\qquad\qquad E_2^-=\begin{pmatrix}0&-\sqrt{-1}&0\\-\sqrt{-1}&0&\sqrt{-1}\\0&-\sqrt{-1}&0\end{pmatrix},\\
&\mathfrak{g}_2=\mathbb{R}E_1,\ E_1=\begin{pmatrix}\sqrt{-1}&0&-\sqrt{-1}\\0&0&0\\\sqrt{-1}&0&-\sqrt{-1}\end{pmatrix},\\
&\mathfrak{g}_{-1}=\theta\mathfrak{g}_1,\ \mathfrak{g}_{-2}=\theta\mathfrak{g}_2
\end{aligned}$$

and $\mathfrak{g}_n=\{0\}$ $(|n|>2)$. Since $[\mathfrak{g}_i,\mathfrak{g}_j]\subset\mathfrak{g}_{i+j}$, $\mathfrak{n}=\mathfrak{g}_1+\mathfrak{g}_2$ becomes a nilpotent Lie subalgebra of $\mathfrak{g}$, giving the Iwasawa decomposition $\mathfrak{g}=\mathfrak{n}+\mathfrak{a}+\mathfrak{k}$.

(1.3) Root space decompositions

Let T be the subgroup of G consisting of all diagonal matrices. Then T is a Cartan subgroup of G contained in K. For every pair of integers $\lambda=(l_1,l_2)$, we define the unitary character χ_λ of T by setting

(1.3.1) $$\chi_\lambda(\mathrm{diag}(t_1,t_2,t_3))=t_1^{l_1}t_2^{l_2},\quad \mathrm{diag}(t_1,t_2,t_3)\in T.$$

The character group of T is identified with $\mathbb{Z}^{\oplus 2}$ through the assignment $\lambda \to \chi_\lambda$. By taking derivative at the identity, we can embed $\widehat{T}$ into $\sqrt{-1}\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, \sqrt{-1}\mathbb{R})$. Then its image L_T becomes a lattice of $\sqrt{-1}\mathfrak{t}^*$. In what follows, we shall fix the identifications $\mathbb{Z}^{\oplus 2} \cong \widehat{T} \cong L_T$ so obtained. Let

$$\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\epsilon_i - \epsilon_j |\ 1 \leqslant i, j \leqslant 3, i \neq j\ \}$$

be the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with $\epsilon_i \in L_T$ defined by $\epsilon_i(X) = x_i,\ X = \mathrm{diag}(x_1, x_2, x_3) \in \mathfrak{t}_{\mathbb{C}}$. Put $\Sigma^+ = \{\epsilon_i - \epsilon_j |\ 1 \leqslant i < j \leqslant 3\ \}$, a set of positive roots in $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Let us denote by Σ_c and Σ_n the set of compact roots and the set of noncompact roots respectively, or explicitly

$$\begin{aligned} \Sigma_c &= \{\pm(\epsilon_1 - \epsilon_2)\}, \\ \Sigma_n &= \{\pm(\epsilon_1 - \epsilon_3),\ \pm(\epsilon_2 - \epsilon_3)\}. \end{aligned}$$

Set

$$X_{ij} = \begin{cases} E_{ij} & \big((i,j) \neq (2,1)\big) \\ -E_{ij} & \big((i,j) = (2,1)\big) \end{cases} \text{ and } H'_{ij} = E_{ii} - E_{jj},$$

where E_{ij} denotes the usual matrix element. The complexification of $\mathfrak{g}$ is identified with $\mathfrak{sl}(3, \mathbb{C})$ via the natural inclusion $\mathfrak{g} \subset \mathfrak{sl}(3, \mathbb{C})$. Then $X_{ij} \in \mathfrak{sl}(3, \mathbb{C})$ is a root vector with weight $\epsilon_i - \epsilon_j$ and $[X_{12}, X_{21}] = H'_{21}$, $[X_{13}, X_{31}] = H'_{13}$, $[X_{23}, X_{32}] = H'_{23}$. We have the root space decompositions:

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} &= \mathfrak{t}_{\mathbb{C}} + \sum_{\substack{1 \leqslant i,j \leqslant 3 \\ i \neq j}} \mathbb{C}X_{ij}, \\ \mathfrak{k}_{\mathbb{C}} &= \mathfrak{t}_{\mathbb{C}} + \mathbb{C}X_{12} + \mathbb{C}X_{21}, \\ \mathfrak{h}_{\mathbb{C}} &= \mathfrak{t}_{\mathbb{C}} + \mathbb{C}X_{23} + \mathbb{C}X_{32} \end{aligned}$$

with $\mathfrak{t}_{\mathbb{C}} = \mathbb{C}H'_{12} + \mathbb{C}H'_{13}$.

(1.4) Inner products

Let $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ denote the Killing form of G, or explicitly

$$B_{\mathfrak{g}}(X, Y) = 6\mathrm{trace}(XY), \quad X,\ Y \in \mathfrak{g}$$

in our case $\mathfrak{g} = \mathfrak{su}(2,1)$. Here trace$(X)$ means the trace of 3×3-matrix $X \in M_3(\mathbb{C})$ in the usual sense. Set

$$\langle X, Y \rangle_{\mathfrak{g}} = -\frac{1}{6} B_{\mathfrak{g}}(X, \theta Y), \quad X,\ Y \in \mathfrak{g}$$

with θ the Cartan involution of $\mathfrak{g}$. Then $\langle\ ,\ \rangle_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a positive definite $\mathbb{R}$-bilinear form of $\mathfrak{g}$. For every $\mathbb{R}$-subspace $\mathfrak{q}$ of $\mathfrak{g}$, we regard it as a Euclidean space equipped with the inner product, say $\langle\ ,\ \rangle_{\mathfrak{q}}$, induced from $\langle\ ,\ \rangle_{\mathfrak{g}}$. Especially

$$\langle X, X' \rangle_{\mathfrak{k}} = -\mathrm{trace}(XX'), \quad X,\ X' \in \mathfrak{k}, \tag{1.4.1}$$

$$\langle Y, Y' \rangle_{\mathfrak{p}} = \mathrm{trace}(YY'), \quad Y,\ Y' \in \mathfrak{p}. \tag{1.4.2}$$

Let $\mathrm{Ad}_{\mathfrak{p}}$ denote the natural action of K on $\mathfrak{p}$ induced from the adjoint action of K on $\mathfrak{g}$. Then $\mathrm{Ad}_{\mathfrak{p}}$ preserves the inner product (1.4.2).

Next we introduce an inner product of $\sqrt{-1}\mathfrak{t}^*$. By means of the inner product $\langle\ ,\ \rangle_{\mathfrak{t}}$ above, we define an isomorphism $\sqrt{-1}\mathfrak{t}^* \cong \mathfrak{t}$ assigning $H'_\lambda \in \mathfrak{t}$ to $\lambda \in \sqrt{-1}\mathfrak{t}^*$ such that

$$\frac{\lambda(H')}{\sqrt{-1}} = \langle H', H'_\lambda \rangle_{\mathfrak{t}}, \quad H' \in \mathfrak{t}.$$

Then the inner product of $\sqrt{-1}\mathfrak{t}^*$ is given by

$$\langle \lambda, \lambda' \rangle = \langle H'_\lambda, H'_{\lambda'} \rangle_{\mathfrak{t}}, \quad \lambda,\ \lambda' \in \sqrt{-1}\mathfrak{t}^*.$$

§2. Irreducible K-modules

As is well known, we can parametrize $\widehat{K}$, the set of equivalence classes of irreducible finite dimensional representations of $K \cong U(2)$, by the highest weight theory. But for our explicit computation, we need to realize irreducible representations of K more concretely by using *standard basis* below. In **(2.1)**, we introduce such realizations and in **(2.2)** recall an explicit formula of Clebsch-Gordan coefficients with respect to the standard basis.

(2.1) Parametrization of $\widehat{K}$

We shall specify a choice of positive root systems of Σ_c by putting $\Sigma_c^+ = \{\epsilon_1 - \epsilon_2\}$ and fix this in what follows. Then the set of Σ_c^+-dominant weights becomes

$$L_T^+ = \{\lambda = (l_1, l_2) \in \mathbb{Z}^{\oplus 2} |\ l_1 \geqslant l_2\ \}.$$

For every $\lambda \in L_T^+$, put $W_\lambda = \bigoplus_{i=0}^{i=d_\lambda} \mathbb{C} w_i^\lambda$ with $d_\lambda := l_1 - l_2$ and define the actions of elements H'_{12}, H'_{13}, X_{12}, X_{21} of $\mathfrak{k}_{\mathbb{C}}$ on the $\mathbb{C}$-vector space W_λ as follows:

$$\begin{aligned} &\tau_\lambda(H'_{12}) w_i^\lambda = (2i - d_\lambda) w_i^\lambda, \\ &\tau_\lambda(H'_{13}) w_i^\lambda = (i + l_2) w_i^\lambda, \\ &\tau_\lambda(X_{12}) w_i^\lambda = (i+1) w_{i+1}^\lambda, \\ &\tau_\lambda(X_{21}) w_i^\lambda = (i - d_\lambda - 1) w_{i-1}^\lambda \quad (i = 0, 1, \ldots, d_\lambda), \end{aligned} \tag{2.1.1}$$

here we understand $w_i^\lambda = 0$ for $i = -1,\ d_\lambda + 1$. It is easily checked that (2.1.1) defines a $\mathfrak{k}_{\mathbb{C}}$-module structure on W_λ and the action τ_λ of $\mathfrak{k}$ can be globalized to that of K giving a $d_\lambda + 1$-dimensional representation $(\tau_\lambda, W_\lambda)$ of K; τ_λ is an irreducible representation with Σ_c^+-highest weight $\lambda \in L_T^+$. The basis $\{w_i^\lambda\}_{i=0}^{i=d_\lambda}$ is called the *standard basis* of W_λ. Let $\widehat{K}$ denote the set of all equivalence classes of irreducible finite dimensional representations of K. Assigning the class of τ_λ to $\lambda \in L_T^+$, we get a map from L_T^+ to $\widehat{K}$ and highest weight theory tells us that this is a bijection.

(2.2) Tensor products with $\mathfrak{p}_{\mathbb{C}}$

Here we recall Clebsch-Gordan's decomposition of $\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_{\mathbb{C}}}$ for a given irreducible K-module τ_λ.

The vector space $\mathfrak{p}_{\mathbb{C}}$ becomes a K-module via the adjoint representation of K, which splits into two irreducible sub K-modules $\mathfrak{p}^+$ and $\mathfrak{p}^-$with

$$\mathfrak{p}^+ = \mathbb{C} X_{13} \oplus \mathbb{C} X_{23}, \quad \mathfrak{p}^- = \mathbb{C} X_{31} \oplus \mathbb{C} X_{32}.$$

Set $\beta_i = \epsilon_i - \epsilon_3$ $(i = 1, 2)$. It is easy to see that $\mathrm{Ad}_{\mathfrak{p}^+} \cong \tau_{\beta_1}$ and $\mathrm{Ad}_{\mathfrak{p}^-} \cong \tau_{-\beta_2}$. For a given irreducible representation τ_λ of K, Clebsch-Gordan's rule tells that the K-module $\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}^\pm}$ decomposes as follows:

$$\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}^+} \cong \tau_{\lambda+\beta_1} \oplus \tau_{\lambda+\beta_2}, \tag{2.2.1}$$

$$\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}^-} \cong \tau_{\lambda-\beta_1} \oplus \tau_{\lambda-\beta_2},$$

here we understand that $\tau_\lambda = \{0\}$ for non dominant $\lambda \in L_T$. This decomposition determines projections

$$p_\lambda^\beta : W_\lambda \otimes \mathfrak{p}_\mathbb{C} \to W_{\lambda+\beta}, \quad \beta \in \Sigma_n.$$

Proposition 2.2.1. *For every $\lambda \in L_T^+$ and $\beta \in \Sigma_n^+$, we have*

$$p_\lambda^\beta | W_\lambda \otimes_\mathbb{C} \mathfrak{p}^- = 0, \quad p_\lambda^{-\beta} | W_\lambda \otimes_\mathbb{C} \mathfrak{p}^+ = 0.$$

Furthermore we can choose the isomorphism (2.2.1) *so that the following identities hold.*

$$\begin{cases} p_\lambda^{\beta_1}(w_i^\lambda \otimes X_{13}) = (i+1) w_{i+1}^{\lambda+\beta_1}, \\ p_\lambda^{\beta_1}(w_i^\lambda \otimes X_{23}) = (d_\lambda - i + 1) w_i^{\lambda+\beta_1}, \end{cases} \tag{2.2.2}$$

$$\begin{cases} p_\lambda^{\beta_2}(w_i^\lambda \otimes X_{13}) = -w_i^{\lambda+\beta_2}, \\ p_\lambda^{\beta_2}(w_i^\lambda \otimes X_{23}) = w_{i-1}^{\lambda+\beta_2}, \end{cases} \tag{2.2.3}$$

$$\begin{cases} p_\lambda^{-\beta_1}(w_i^\lambda \otimes X_{32}) = w_i^{\lambda-\beta_1}, \\ p_\lambda^{-\beta_1}(w_i^\lambda \otimes X_{31}) = w_{i-1}^{\lambda-\beta_1}, \end{cases} \tag{2.2.4}$$

$$\begin{cases} p_\lambda^{-\beta_2}(w_i^\lambda \otimes X_{32}) = -(i+1) w_{i+1}^{\lambda-\beta_2}, \\ p_\lambda^{-\beta_2}(w_i^\lambda \otimes X_{31}) = (d_\lambda - i + 1) w_i^{\lambda-\beta_2} \end{cases} \tag{2.2.5}$$

for $i = 0, \ldots, d_\lambda$, where one should note that $d_{\lambda\pm\beta_1} = d_{\lambda\mp\beta_2} = d_\lambda \pm 1$.

Proof. [K-O, Proposition **(2.3)**]. □

§3. Representation theory of H

We note that $H \cong U(1,1)$ by the assignment

$$H \ni \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix} \to \begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix} \in U(1,1)$$

with $x_{11} = \big(x_{22}x_{33} - x_{23}x_{32}\big)^{-1}$ and correspondingly $T = H \cap K$, a maximal compact subgroup of H, is mapped onto $U(1) \times U(1)$ diagonally embedded in $U(1,1)$. Thus representation theory of H including a description of its unitary dual is wellknown. We recall it here briefly.

(3.1) Non-unitary principal series representation

We first define subgroups A', M', N' and P' of H as follows:

$$A' = \left\{ a'_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (r+r^{-1})/2 & (r-r^{-1})/2 \\ 0 & (r-r^{-1})/2 & (r+r^{-1})/2 \end{pmatrix} \middle| \ r > 0 \right\},$$
$$M' = \left\{ m'_\theta = \begin{pmatrix} e^{-2\sqrt{-1}\theta} & 0 & 0 \\ 0 & e^{\sqrt{-1}\theta} & 0 \\ 0 & 0 & e^{\sqrt{-1}\theta} \end{pmatrix} \middle| \ \theta \in \mathbb{R} \right\},$$
$$N' = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\sqrt{-1}x & -\sqrt{-1}x \\ 0 & \sqrt{-1}x & 1-\sqrt{-1}x \end{pmatrix} \middle| \ x \in \mathbb{R} \right\},$$
$$P' = M'A'N'.$$

For $\epsilon \in \{0,\ 1\}$, set $\mathbb{Z}_\epsilon := \{ m \in \mathbb{Z} |\ m \equiv \epsilon \pmod 2 \}$. For every $\nu \in \mathbb{C}$ and $n \in \mathbb{Z}_\epsilon$, define the Hilbert space $\mathcal{V}_{n,\nu}$ as follows: it consists of all measurable functions $\varphi' : H \to \mathbb{C}$ satisfying

$$\varphi'(m'_\theta a'_r n' h) = e^{\sqrt{-1}n\theta} r^{\nu+1} \varphi'(h), \quad m'_\theta \in M',\ a'_r \in A',\ n' \in N',\ h \in H$$

and $\varphi' | K \cap H \in L^2(K \cap H)$, equipped with the inner product

$$\langle \varphi'_1,\ \varphi'_2 \rangle = \int_{(K\cap H)/M'} \varphi'_1(k') \overline{\varphi'}_2(k') dk'$$

with dk' the normalized Haar measure of $(K \cap H)/M'$.

The group H acts on $\mathcal{V}_{n,\nu}$ by the right translation and thus we get the non-unitary principal series representation $(\eta_{n,\nu}, \mathcal{V}_{n,\nu})$ of H. We shall describe the underlying $(\mathfrak{h}_\mathbb{C}, K \cap H)$-module structure of $\eta_{n,\nu}$ explicitly. For every $m \in \mathbb{Z}_\epsilon$, there exists the unique C^∞-function $v_m : H \to \mathbb{C}$ belonging to $\mathcal{V}_{n,\nu}$ such that

$$v_m(t) = t_2^{(n-m)/2} t_3^{(n+m)/2}, \quad t = \mathrm{diag}(t_1, t_2, t_3) \in T.$$

Note that $\eta_{n,\nu}(t) v_m = \chi_{(-(n+m)/2, -m)}(t) v_m$, $t \in T$ with $\chi_{(-(n+m)/2,-m)}$ the unitary character of T defined by (1.3.1). The family $\{ v_m |\ m \in \mathbb{Z}_\epsilon \}$ provides us with an orthonormal basis of the Hilbert space $\mathcal{V}_{n,\nu}$.

The following proposition describes explicitly the action of the Lie algebra $\mathfrak{h}_{\mathbb{C}}$ to $\{v_m\}$. It will be very important in computations carried out in section 6 and 7.

PROPOSITION 3.1.1. *Let $\nu \in \mathbb{C}$ and $n \in \mathbb{Z}_\epsilon$. The totality of all $H \cap K$-finite vectors of $\mathcal{V}_{n,\nu}$ coincides with $\mathcal{V}^0_{n,\nu} := \bigoplus_{m \in \mathbb{Z}_\epsilon} \mathbb{C} v_m$. Let $(\eta^0_{n,\nu}, \mathcal{V}^0_{n,\nu})$ be the underlying Harish-Chandra module of $(\eta_{n,\nu}, \mathcal{V}_{n,\nu})$. Then the actions of elements H'_{13}, H'_{12}, X_{23}, X_{32} on $\mathcal{V}^0_{n,\nu}$ are given as follows:*

$$\begin{aligned} \eta^0_{n,\nu}(H'_{13})v_m &= -\frac{n+m}{2} v_m, \\ \eta^0_{n,\nu}(H'_{12})v_m &= -\frac{n-m}{2} v_m, \\ \eta^0_{n,\nu}(X_{23})v_m &= \frac{\nu-m+1}{2} v_{m-2}, \\ \eta^0_{n,\nu}(X_{32})v_m &= \frac{\nu+m+1}{2} v_{m+2} \quad (m \in \mathbb{Z}_\epsilon). \end{aligned} \tag{3.1.1}$$

(3.2) Admissible representations of H

A bounded Hilbert representation $(\eta, \mathcal{F}_\eta)$ of H is called admissible if the $H \cap K$-module $\eta | H \cap K$ contains every irreducible representation with finite multiplicity. The center of H coincides with M' in our case. If there exists an integer $n \in \mathbb{Z}$ such that

$$\eta(m'_\theta) = e^{\sqrt{-1}n\theta} 1_{\mathcal{F}_\eta} \qquad (\theta \in \mathbb{R}),$$

then we say that η has central character n. For every admissible representation $(\eta, \mathcal{F}_\eta)$ of H, let us denote its underlying Harish-Chandra module by $(\eta^0, \mathcal{F}^0_\eta)$. Note that the nonunitary principal series representation $\eta_{n,\nu}$ with $(\nu, n) \in \mathbb{C} \times \mathbb{Z}$ is admissible and has central character n. Now we quote a result which is so called Casselman's subrepresentation theorem.

PROPOSITION 3.2.1. *Assume that an admissible representation η is irreducible and has central character $n \in \mathbb{Z}$. Then there exists an $(\mathfrak{h}_{\mathbb{C}}, H \cap K)$-inclusion $\eta^0 \to \eta^0_{n,\nu}$ for some $\nu \in \mathbb{C}$.*

Let $\widetilde{H}$ denote the set of infinitesimal equivalence classes of irreducible admissible representations of H. Let $\widehat{H}$ be the subset of $\widetilde{H}$ consisting of unitarizable classes. For a given integer n, $\widetilde{H}_n$ (resp. $\widehat{H}_n$) denotes the subset of $\widetilde{H}$ (resp. $\widehat{H}$) consisting of all classes with central character n. By Proposition 3.2.1, to parametrize the set $\widetilde{H}$, it suffices to describe how the non-unitary principal series $\eta_{n,\nu}$ decomposes. We recall it here.

Proposition 3.2.2. *Let $(n,\nu)\in\mathbb{C}\times\mathbb{Z}_\epsilon$ and $\{\epsilon,\epsilon'\}=\{0,1\}$.*
(1) *The $(\mathfrak{h}_{\mathbb{C}},H\cap K)$-module $\eta^0_{n,\nu}$ is irreducible if and only if $\nu\in\mathbb{C}-\mathbb{Z}_{\epsilon'}$.*
(2) *Let $k\in\mathbb{Z}_{\epsilon'}$ be a non negative integer.*
Let us define a subspace $\mathcal{D}^{+0}_{n,k}$ (resp. $\mathcal{D}^{-0}_{n,k}$) of $\mathcal{V}^0_{n,k}$ as the $\mathbb{C}$-linear span of $\{v_m|\ m\in\mathbb{Z}_\epsilon,\ m\geqslant k+1\}$ (resp. $\{v_m|\ m\in\mathbb{Z}_\epsilon,\ m\leqslant -k-1\}$). Then this is a $(\mathfrak{h}_{\mathbb{C}},H\cap K)$-invariant subspace of $\mathcal{V}^0_{n,k}$ and $(\mathfrak{h}_{\mathbb{C}},H\cap K)$-module $\delta^{\pm0}_{n,k}:=\eta^0_{n,k}|\mathcal{D}^{\pm0}_{n,k}$ is irreducible. The quotient $\eta^0_{n,\nu}/(\delta^{+0}_{n,k}\oplus\delta^{-0}_{n,k})$ is isomorphic to the k-dimensional irreducible representation with central character n. Let $\mathcal{D}^{\pm}_{n,k}$ be the closure of $\mathcal{D}^{\pm0}_{n,k}$ in $\mathcal{V}_{n,k}$ endowed with the naturally induced Hilbert space structure and $\delta^{\pm}_{n,\nu}$ denotes the corresponding representation of H.
(3) *Let $k\in\mathbb{Z}_{\epsilon'}$ be a negative integer. Let us define a subspace $\mathcal{E}^0_{n,-k}$ of $\mathcal{V}^0_{n,-k}$ as the $\mathbb{C}$-linear span of $\{v_m|\ m\in\mathbb{Z}_\epsilon,\ k+1\leqslant m\leqslant -k-1\}$. Then this is a $(\mathfrak{h}_{\mathbb{C}},H\cap K)$-invariant subspace of $\mathcal{V}^0_{n,-k}$ and the $(\mathfrak{h}_{\mathbb{C}},H\cap K)$-module $\sigma^0_{n,-k}:=\eta^0_{n,-k}|\mathcal{E}^0_{n,-k}$ is irreducible and isomorphic to the $-k$ dimensional representation with central character n. The quotient $\eta^0_{n,-k}/\sigma^0_{n,-k}$ is isomorphic to $\delta^{+0}_{n,k}\oplus\delta^{-0}_{n,k}$. $\mathcal{E}^0_{n,-k}$ is a closed H-invariant subspace of $\mathcal{V}_{n,-k}$. The irreducible $-k$-dimensional representation of H realized on $\mathcal{E}^0_{n,-k}$ is denoted by $\sigma_{n,-k}$.*
(4) *The set $\widetilde{H}_n$ is exhausted by the representations $\eta_{n,\nu}$ with $\nu\in\mathbb{C}-\mathbb{Z}_{\epsilon'}$, $\delta^{\pm}_{n,k}$ with $k\in\mathbb{Z}_{\epsilon'},k\geqslant 0$ and $\sigma_{n,-k}$ with $k\in\mathbb{Z}_{\epsilon'},k<0$.*

Proposition 3.2.3.
The set $\widehat{H}$ is exhausted by classes of the following representations.
(1) *(Unitary principal series) $\eta_{n,\nu}$ $(n\in\mathbb{Z}_\epsilon,\ \nu\in\sqrt{-1}\mathbb{R}-\mathbb{Z}_{\epsilon'}$ with $\{\epsilon,\epsilon'\}=\{0,1\})$.*
(2) *(Discrete series) $\delta^{\pm}_{n,k}$ $(n\in\mathbb{Z}_\epsilon,\ k\in\mathbb{Z}_{\epsilon'},\ k>0)$ with $\{\epsilon,\epsilon'\}=\{0,1\})$.*
(3) *(Limit of discrete series) $\delta^{\pm}_{n,0}$ $(n\in\mathbb{Z}_1)$.*
(4) *(Complimentary series) $\eta_{n,\nu}$ $(n\in\mathbb{Z}_0,\nu\in\mathbb{R},0<|\nu|<1)$.*
(5) *(One dimensional representations) $\sigma_{n,1}$ with $n\in\mathbb{Z}_0$.*

Remark. We should note that in case of **(2)**, **(3)** and **(4)**, we have to replace the Hermitian inner product by an appropriate one in order to make the action of H unitary. We will not present it here.

(3.3) Standard basis

Let $(\eta, \mathcal{F}_\eta)$ be an admissible Hilbert representation of H with central character $n \in \mathbb{Z}_\epsilon$. We assume that η is irreducible or isomorphic to a non-unitary principal series representation. From Proposition 3.1.1 and Proposition 3.2.2, we see that there exists a subset L_η of $\mathbb{Z}_\epsilon$ such that $K\cap H$-module $\eta|K\cap H$ decomposes into a multiplicity free direct sum of characters $\chi_{(-(n+m)/2,-m)}$ with $m \in L_\eta$. By Proposition 3.1.1 and Proposition 3.2.2, we have constructed a basis $\{v_m|\ m \in L_\eta\}$ of $\mathcal{F}_\eta^0$. Assume η^0 occurs as a submodule of $\eta_{n,\nu}^0$ with $(n,\nu) \in \mathbb{Z}_\epsilon \times \mathbb{C}$. Then this basis is characterized up to a multiplicative constant by requiring that v_m belongs to the character $\chi_{(-(n+m)/2,-m)}$ and

$$\eta^0(X_{23})v_m = \frac{\nu - m + 1}{2} v_{m-2}, \quad \eta^0(X_{32})v_m = \frac{\nu + m + 1}{2} v_{m+2}$$

for every $m \in L_\eta$.

We call any orthonormal basis $\{v_m|\ m \in L_\eta\}$ satisfying the above conditions *standard basis* of η.

§4. The Shintani functions on G

In the first subsection, we introduce a space of the Shintani functions $\mathcal{S}_{\eta,\Pi}(\tau)$ which is of our main interest in this paper. After recalling the Harish-Chandra parametrization of discrete series representations of G, we define Schmid operators (or shift operators) and quote a theorem of Yamashita about a characterization of the space $\mathcal{S}_{\eta,\Pi}(\tau_0)$ with Π a discrete series representation of G and τ_0 its minimal K-type in terms of the Schmid operators. In **(4.4)** we shall present commutation relations among the shift operators which will be used in §9.

(4.1) A space of Shintani functions

Let Π be an irreducible Harish-Chandra module of G and $(\eta, \mathcal{F}_\eta)$ be an admissible representation of H. Let $C_\eta^\infty(H\backslash G)$ denotes the $\mathbb{C}$-vector space consisting of all C^∞-functions $F : G \to \mathcal{F}_\eta$ with the following equivariant property:

$$F(hg) = \eta(h)F(g), \quad h \in H,\ g \in G.$$

Setting

$$R_X F(g) = \lim_{t\to 0} \frac{F(g\exp(tX)) - F(g)}{t}, \quad g \in G,$$

for every $X \in \mathfrak{g}$ and $F \in C_\eta^\infty(H\backslash G)$, we have an action of the Lie algebra $\mathfrak{g}$ on $C_\eta^\infty(H\backslash G)$. As is easily seen, this action of $\mathfrak{g}$ is compatible with the natural right action of K, thus $C_\eta^\infty(H\backslash G)$ becomes a $(\mathfrak{g},K)$-module.

Now we set

$$\mathcal{I}_{\eta,\Pi} = \mathrm{Hom}_{(\mathfrak{g},K)}\big(\Pi^*, C_\eta^\infty(H\backslash G)\big)$$

with Π^* the contragredient $(\mathfrak{g},K)$-module of Π. Let (τ, W) be an irreducible K-module. For every K-equivariant map $i : \tau^* \to \Pi^*|K$, we define a $\mathbb{C}$-linear map $[i]_{\eta,\Pi}$ by the composite of the following sequence of maps:

$$\mathcal{I}_{\eta,\Pi} \xrightarrow{i^*} \mathrm{Hom}_K\big(\tau^*, C_\eta^\infty(H\backslash G)\big) \cong C_{\eta,\tau}^\infty(H\backslash G/K),$$

where i^* denotes the pullback via i and $C_{\eta,\tau}^\infty(H\backslash G/K)$ is the space of smooth functions $F : G \to \mathcal{F}_\eta \otimes_{\mathbb{C}} W$ with the property

$$F(hgk) = \big(\eta(h)\otimes\tau(k)^{-1}\big)F(g), \quad h \in H,\ g \in G,\ k \in K. \tag{4.1.1}$$

In other words, $[i]_{\eta,\Pi}$ is characterized by the equations

$$e_{w^*}\big([i]_{\eta,\Pi}(\Phi)(g)\big) = \big(\Phi \circ i(w^*)\big)(g), \quad g \in G$$

for every $w^* \in W^*$ and $\Phi \in \mathcal{I}_{\eta,\Pi}$, where $e_{w^*} : \mathcal{F}_\eta \otimes_{\mathbb{C}} W \to \mathcal{F}_\eta$ is the contraction map via w^*. Now we set

$$\mathcal{S}_{\eta,\Pi} = \mathbb{C}\text{-span of } \bigcup \mathrm{Image}(\Phi),\ \Phi \text{ ranges over } \mathcal{I}_{\eta,\Pi},$$
$$\mathcal{S}_{\eta,\Pi}(\tau) = \mathbb{C}\text{-span of } \bigcup \mathrm{Image}([i]_{\eta,\Pi}),\ i \text{ ranges over } \mathrm{Hom}_K(\tau^*, \Pi^*|K).$$

Any function belonging to the space $\mathcal{S}_{\eta,\Pi}(\tau)$ is called *Shintani function with K-type τ*. Note that $\mathcal{S}_{\eta,\Pi}(\tau) \cong \mathrm{Hom}_K(\tau^*, \mathcal{S}_{\eta,\Pi})$ naturally and K-module $\mathcal{S}_{\eta,\Pi}$ decomposes to a direct sum of $\mathcal{S}_{\eta,\Pi}(\tau)\otimes\tau^*$ $(\tau \in \widehat{K})$.

(4.2) Discrete series representation of G

An irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module Π is said to be a *discrete series representation* of G if there exists an injective $(\mathfrak{g}_{\mathbb{C}}, K)$-module homomorphism $\Pi \to L^2(G)$. Here we recall the Harish-Chandra parametrization of the discrete series representations of G. Let Ξ be the set of all linear forms $\Lambda \in \sqrt{-1}\mathfrak{t}^*$ such that $\langle \Lambda, \epsilon_1 - \epsilon_2 \rangle > 0$ and $\langle \Lambda, \epsilon_i - \epsilon_3 \rangle \neq 0$ for $i = 1, 2$, or explicitly

$$\Xi = \{\Lambda = (\Lambda_1, \Lambda_2) \in \mathbb{Z}^{\oplus 2} |\ \Lambda_1 > \Lambda_2,\ \Lambda_1 \Lambda_2 \neq 0\ \}.$$

By Harish-Chandra's theorem, there exists the bijection $\Lambda \to \omega_\Lambda$ from Ξ to $\widehat{G}_d$, the set of all equivalence classes of discrete series representations of G. Any unitary representation belonging to the class ω_Λ is said to have *Harish-Chandra parameter* Λ. There exists the following three positive root systems of $\Sigma(\mathfrak{g}_{\mathbb{C}},\ \mathfrak{t}_{\mathbb{C}})$ which contains Σ_c^+:

$$\begin{aligned} \Sigma_I^+ &:= \{\epsilon_1 - \epsilon_2,\ \epsilon_2 - \epsilon_3,\ \epsilon_1 - \epsilon_3\ \}(= \Sigma^+), \\ \Sigma_{II}^+ &:= \{\epsilon_1 - \epsilon_2,\ \epsilon_3 - \epsilon_2,\ \epsilon_1 - \epsilon_3\ \}, \\ \Sigma_{III}^+ &:= \{\epsilon_1 - \epsilon_2,\ \epsilon_3 - \epsilon_2,\ \epsilon_3 - \epsilon_1\ \}. \end{aligned} \tag{4.2.1}$$

For every $J \in \{I, II, III\}$, set $\Xi_J := \{\Lambda \in \Xi |\ \langle \Lambda,\ \beta \rangle > 0,\ \beta \in \Sigma_J^+\}$. Then Ξ is a disjoint union of the following three subsets

$$\begin{aligned} \Xi_I &= \{(\Lambda_1,\ \Lambda_2) \in \mathbb{Z}^{\oplus 2} |\ \Lambda_1 > \Lambda_2 > 0\ \}, \\ \Xi_{II} &= \{(\Lambda_1,\ \Lambda_2) \in \mathbb{Z}^{\oplus 2} |\ \Lambda_1 > 0 > \Lambda_2\ \}, \\ \Xi_{III} &= \{(\Lambda_1,\ \Lambda_2) \in \mathbb{Z}^{\oplus 2} |\ 0 > \Lambda_1 > \Lambda_2\ \}. \end{aligned}$$

Discrete series representations with Harish-Chandra parameter belonging to Ξ_I (resp. Ξ_{III}) are called *holomorphic* (resp. *antiholomorphic*). The remaining discrete series representations of G, namely those whose Harish-Chandra parameter belong to Ξ_{II}, are said to be *large* in the sense of Vogan [V]. Let ρ_J (resp. ρ_c) be the half sum of roots in Σ_J^+ (resp. Σ_c^+). By (4.2.1), we easily have $\rho_I = \epsilon_1 - \epsilon_3$, $\rho_{II} = \rho_c = \epsilon_1 - \epsilon_2$ and $\rho_{III} = \epsilon_3 - \epsilon_2$. It is known that the Σ_c^+-highest weight of the minimal K- type of Π with Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2) \in \Xi_J$ is $\lambda = \Lambda + \rho_J - 2\rho_c$ called the *Blattner parameter* of Π. The explicit form of λ is as follows:

$$\lambda = (\Lambda_1 + 1, \Lambda_2 + 2) \quad (\text{ if } J = I\ ),$$

$$\lambda = (\Lambda_1, \Lambda_2) \quad (\text{ if } J = II\ ),$$
$$\lambda = (\Lambda_1 - 2, \Lambda_1 - 1) \quad (\text{ if } J = III\ ).$$

(4.3) Schmid operators

Let $(\eta, \mathcal{F}_\eta)$ be an admissible representation of H and (τ, W) a finite dimensional representation of K. For every $F \in C^\infty_{\eta,\tau}(H\backslash G/K)$, set

$$\nabla_{\eta,\tau} F(g) := \sum_{i=1}^{4} R_{X_i} F(g) \otimes X_i \quad \bigl(g \in G\bigr),$$

where $\{X_i\}_{i=1}^{i=4}$ is an orthonormal basis of $\mathfrak{p}$ with respect to the inner product (1.4.2). It is easily checked that the right hand side of the above identity does not depend on the particular choice of $\{X_i\}$ and the resulting function $\nabla_{\eta,\tau} F$ belongs to $C^\infty_{\eta,\tau\otimes \mathrm{Ad}_{\mathfrak{p}_\mathbb{C}}}(H\backslash G/K)$. Thus we have a first order gradient type differential operator

$$\nabla_{\eta,\tau} : C^\infty_{\eta,\tau}(H\backslash G/K) \longrightarrow C^\infty_{\eta,\tau\otimes \mathrm{Ad}_{\mathfrak{p}_\mathbb{C}}}(H\backslash G/K).$$

We can take

$$\left\{ \frac{X_{13} + X_{31}}{\sqrt{2}},\ \frac{\sqrt{-1}(X_{13} - X_{31})}{\sqrt{2}},\ \frac{X_{23} + X_{32}}{\sqrt{2}},\ \frac{\sqrt{-1}(X_{23} - X_{32})}{\sqrt{2}} \right\}$$

as $\{X_i\}_{i=1}^4$ and consequently have

$$\nabla_{\eta,\tau} F(g) = \nabla^+_{\eta,\tau} F(g) + \nabla^-_{\eta,\tau} F(g)$$

with

$$\nabla^+_{\eta,\tau} F(g) = R_{X_{31}} F(g) \otimes X_{13} + R_{X_{32}} F(g) \otimes X_{23}$$
$$\nabla^-_{\eta,\tau} F(g) = R_{X_{13}} F(g) \otimes X_{31} + R_{X_{23}} F(g) \otimes X_{32}$$

for every $F \in C^\infty_{\eta,\tau}(H\backslash G/K)$. Note that

$$\nabla^+_{\eta,\tau} F(g) = (1_{\mathcal{F}_\eta} \otimes 1_W \otimes \pi^+) \nabla_{\eta,\tau} F(g),$$
$$\nabla^-_{\eta,\tau} F(g) = (1_{\mathcal{F}_\eta} \otimes 1_W \otimes \pi^-) \nabla_{\eta,\tau} F(g)$$

with $\pi^+ : \mathfrak{p}_{\mathbb{C}} \to \mathfrak{p}^+$ and $\pi^- : \mathfrak{p}_{\mathbb{C}} \to \mathfrak{p}^-$ the natural projectors and the assignments $F \to \nabla^{\pm}_{\eta,\tau} F$ define differential operators

$$\nabla^{\pm}_{\eta,\tau} : C^{\infty}_{\eta,\tau}(H\backslash G/K) \longrightarrow C^{\infty}_{\eta,\tau\otimes \mathrm{Ad}_{\mathfrak{p}^{\pm}}}(H\backslash G/K).$$

The differential operators $\nabla_{\eta,\tau}$ or $\nabla^{\pm}_{\eta,\tau}$ thus defined are called *Schmid operators*, [S], [Y]. For every noncompact root $\beta \in \Sigma$ and every dominant weight $\lambda \in L^+_T$, we define the *β-shift operator*

$$\nabla^{\beta}_{\eta,\lambda} : C^{\infty}_{\eta,\tau_\lambda}(H\backslash G/K) \longrightarrow C^{\infty}_{\eta,\tau_{\lambda+\beta}}(H\backslash G/K)$$

by setting

$$\nabla^{\beta}_{\eta,\lambda} F(g) := (1_{\mathcal{F}_\eta} \otimes p^{\beta}_{\lambda})\big(\nabla_{\eta,\tau_\lambda} F(g)\big).$$

Let Π be a discrete series representation of G with Harish-Chandra parameter $\Lambda \in \Xi_J$ and Blattner parameter $\lambda = \Lambda + \rho_J - 2\rho_c$. We want to know the space $\mathcal{S}_{\eta,\Pi}(\lambda)$. The following result due to Yamashita is our basic tool to investigate this space. It tells us that the space $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ is characterized as the C^∞-solution space of certain system of differential equations.

Proposition 4.3.1 ([Y, Theorem 2.4]). *Let Π be a discrete series representation of G with Harish-Chandra parameter $\Lambda \in \Xi_J$ and Blattner parameter $\lambda = \Lambda + \rho_J - 2\rho_c$. Let $(\eta, \mathcal{F}_\eta)$ be an irreducible unitary representation of H. If λ is far from the wall, then $\mathcal{S}_{\eta,\Pi}(\lambda)$ coincides with the totality of $F \in C^{\infty}_{\eta,\tau_\lambda}(H\backslash G/K)$ such that*

$$(\mathcal{D}_{\eta,\Lambda}) : \qquad \nabla^{-\beta}_{\eta,\lambda} F(g) = 0, \qquad \beta \in \Sigma^+_J \cap \Sigma_n.$$

In other words,

$$\mathcal{S}_{\eta,\Pi}(\tau_\lambda) = \bigcap_{\beta \in \Sigma^+_J \cap \Sigma_n} ker(\nabla^{-\beta}_{\eta,\lambda}).$$

(4.4) Casimir operator

Recall the inner product of $\mathfrak{k}$ and $\mathfrak{p}$ defined in **(1.4)**. Let $\{Y_i\}_{i=1}^{i=4}$ and $\{Z_i\}_{i=1}^{i=4}$ be arbitrary orthonormal $\mathbb{R}$-basis of $\mathfrak{p}$ and $\mathfrak{k}$ respectively. Then the Casimir element of G is the degree 2 element of $U(\mathfrak{g}_{\mathbb{C}})$, the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, defined by $\Omega_G = \sum_{i=1}^4 Y_i^2 - \sum_{i=1}^4 Z_i^2$. It is wellknown that Ω_G does not depend on the choice of $\{Y_i\}$ and $\{Z_i\}$, and that it belongs to the center of $U(\mathfrak{g}_{\mathbb{C}})$. Let $(\eta, \mathcal{F}_\eta)$ be an admissible representation of H and $\lambda \in L_T^+$ a dominant weight. Since Ω_G satisfies $\mathrm{Ad}(k)\Omega_G = \Omega_G$ for every $k \in K$, the operator R_{Ω_G} on $C_\eta^\infty(H\backslash G)$ induces that on $C_{\eta,\tau_\lambda}^\infty(H\backslash G/K)$, which will be denoted by $\Omega_{\eta,\lambda}$.

(4.5) Commutation relations of shift operators

Now we present commutation relations among β-shift operators with $\beta \in \Sigma_n$, whose proof is included in **Appendix 2**. We introduce a convention that for a non dominant weight $\lambda \in L_T$ the symbol τ_λ represent the zero K-module, and $\nabla_{\eta,\lambda}^\beta$ represent a zero operator if either λ or $\lambda + \beta$ is non dominant.

THEOREM 4.5.1. *Let $\lambda = (l_1, l_2) \in L_T^+$, a dominant weight and $F \in C_{\eta,\tau_\lambda}^\infty(H\backslash G/K)$.*
(1) *For every pair of non compact roots (β, β') such that $\beta \neq -\beta'$, we have*

$$\nabla_{\eta,\lambda+\beta'}^{\beta} \circ \nabla_{\eta,\lambda}^{\beta'} F(g) = \nabla_{\eta,\lambda+\beta}^{\beta'} \circ \nabla_{\eta,\lambda}^{\beta} F(g). \tag{4.5.1; β,β'}$$

(2) *For $\beta = \beta_1$ or β_2, we have*

$$\begin{aligned} &\nabla_{\eta,\lambda+\beta_i}^{-\beta_i} \circ \nabla_{\eta,\lambda}^{\beta_i} F(g) - \nabla_{\eta,\lambda-\beta_i}^{\beta_i} \circ \nabla_{\eta,\lambda}^{-\beta_i} F(g) \\ &= \frac{1}{2}\Omega_{\eta,\lambda} F(g) - \frac{1}{2}(1_{\mathcal{F}_\eta} \otimes \tau_\lambda(\Omega_K)) F(g) \\ &\quad + \big(\epsilon_i \frac{l_1 + l_2}{2} + d_\lambda + \epsilon_i d_\lambda l_i\big) F(g). \end{aligned} \tag{4.5.2; β}$$

Here ϵ_i is $+1$ or -1 according as $i = 1$ or $i = 2$.
(3) *For every $\beta \in \Sigma_n$, we have*

$$\Omega_{\eta,\lambda+\beta} \circ \nabla_{\eta,\lambda}^{\beta} F(g) = \nabla_{\eta,\lambda}^{\beta} \circ \Omega_{\eta,\lambda} F(g). \tag{4.5.3; β}$$

REMARK. All definitions in this section make sense and Theorem 4.5.1 remain true if we replace the pair (H, η) by a pair formed by an arbitrary closed subgroup of G and its continuous representation.

§5. Radial part of Schmid operators and Casimir operators

We first recall a decomposition $G = HAK$ with one dimensional split torus A of G, which is a kind of Cartan or Iwasawa decomposition. Using this or its infinitesimal version $\mathfrak{g}_{\mathbb{C}} = \mathrm{Ad}(a^{-1})\mathfrak{h}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}$ ($a \in A,\ a \neq 1$), we study the A-radial part of the differential operators $\nabla^{\pm}_{\eta,\tau}$ and $\Omega_{\eta,\lambda}$ defined in the previous section. The main result of this section is Proposition 5.2.1 and Proposition 5.3.1.

(5.1) Definition of radial part

Let M^* be the normalizer of A in K. Then M is a normal subgroup of M^* of index 2 and the quotient group $W_0 = M^*/M$ is isomorphic to the little Weyl group of the restricted root system of the pair $(\mathfrak{g}, \mathfrak{a})$. The coset $w_0 M$ with $w_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the non trivial element of W_0. First we recall the following result.

LEMMA 5.1.1. **(1)** *The multiplication map* $\Phi : H \times A \times K \to G$, $(h, a, k) \to hak$ *is a* C^∞*-surjection, and its tangent map at* (h, a, k) *is surjective if and only if* $a \neq 1$. *Moreover we have*

$$\mathfrak{g} = Ad(a^{-1})\mathfrak{h} + \mathfrak{a} + \mathfrak{k}$$

for $a \in A,\ a \neq 1$.
(2) *The fiber of* Φ *above* $g = hak$ *is given as follows:*

$$\Phi^{-1}(g) = \{(hl^{-1}, 1, lk) |\ l \in H \cap K\} \text{ if } a = 1,$$
$$\Phi^{-1}(g) = \{(hl^{-1}, Ad(l)a, lk) |\ l \in M^*\} \text{ if } a \neq 1.$$

PROOF. [R, Theorem 9], [R, Theorem 10]. □

For an admissible representation $(\eta, \mathcal{F}_\eta)$ of H and a finite dimensional K-module (τ, W), let us denote by $C^\infty_{W_0}(A\,; \mathcal{F}_\eta \otimes W)$ the totality of $\mathcal{F}_\eta \otimes W$-valued smooth functions on A satisfying

$$(5.1.1) \qquad \big(\eta(m) \otimes \tau(m)\big)\varphi(a) = \varphi(a) \qquad (m \in M,\ a \in A),$$

$$\big(\eta(w_0)\otimes\tau(w_0)\big)\varphi(a)=\varphi(a^{-1}) \qquad (a\in A), \tag{5.1.2}$$

$$(\eta(l)\otimes\tau(l))\varphi(1)=\varphi(1) \qquad (l\in H\cap K). \tag{5.1.3}$$

Lemma 5.1.2. *The restriction map*

$$res_A:\ C^\infty_{\eta,\tau}(H\backslash G/K)\longrightarrow C^\infty_{W_0}(A\,;\mathcal{F}_\eta\otimes W)$$

is an isomorphism of $\mathbb{C}$*-vector spaces.*

Proof. This follows from Lemma 5.1.1. □

Let (τ_i,W_i) $(i=1,2)$ be finite dimensional K-modules. The A-radial part of any $\mathbb{C}$-linear map $\mathcal{A}:C^\infty_{\eta,\tau_1}(H\backslash G/K)\longrightarrow C^\infty_{\eta,\tau_2}(H\backslash G/K)$ is defined by a unique $\mathbb{C}$-linear map $\rho(\mathcal{A}):C^\infty_{W_0}(A\,;\mathcal{F}_\eta\otimes W_{\tau_1})\to C^\infty_{W_0}(A\,;\mathcal{F}_\eta\otimes W_{\tau_2})$ satisfying $\mathrm{res}_A(\mathcal{A}F)=\rho(\mathcal{A})(\mathrm{res}_AF)$ for every $F\in C^\infty_{\eta,\tau_1}(H\backslash G/K)$.

(5.2) Radial part of shift operators

Let $(\eta,\mathcal{F}_\eta)$ be an admissible representation of H and (τ,W) a finite dimensional representation of K. We need a lemma.

Lemma 5.2.1. *Let* $F\in C^\infty_{\eta,\tau}(H\backslash G/K)$.
(1) $R_XF(g)=-(1_{\mathcal{F}_\eta}\otimes\tau(X))F(g)$ $(g\in G)$ *for every* $X\in\mathfrak{k}$.
(2) *The value* $F(g)$ *of* F *at an arbitrary* $g\in G$ *is a smooth vector of* $\mathcal{F}_\eta\otimes W$ *and we have*

$$R_{\mathrm{Ad}(g^{-1})Y}F(g)=(\eta(Y)\otimes 1_W)F(g)$$

for every $Y\in\mathfrak{h}$.

Proof. We omit the proof because it is rather easy. □

The following proposition gives the A-radial part of $\nabla^{\pm}_{\eta,\tau}$ explicitly.

Proposition 5.2.1. *For every* $\varphi\in C^\infty_{W_0}(A\,;\mathcal{F}_\eta\otimes W)$ *and* $r>0,\ r\neq 1$, *we have*

(5.2.2)

$$\rho(\nabla^+_{\eta,\tau})\varphi(a_r)$$
$$=\frac{1}{2}\bigg\{\partial_1-\frac{2r^2}{r^4-1}\eta(H'_{13})-\frac{r^4+1}{r^4-1}(\tau\otimes Ad_{\mathfrak{p}^+})(H'_{13})+2\frac{r^2-1}{r^2+1}+2\frac{r^4+1}{r^4-1}\bigg\}$$

$$\times(\varphi\otimes X_{13})+\left\{\frac{2r}{r^2+1}\eta(X_{32})-\frac{r^2-1}{r^2+1}(\tau\otimes Ad_{\mathfrak{p}^+})(X_{12})\right\}(\varphi\otimes X_{23}),$$

(5.2.3)

$$\begin{aligned}&\rho(\nabla^-_{\eta,\tau})\varphi(a_r)\\ &=\frac{1}{2}\left\{\partial_1+\frac{2r^2}{r^4-1}\eta(H'_{13})+\frac{r^4+1}{r^4-1}(\tau\otimes Ad_{\mathfrak{p}^-})(H'_{13})+2\frac{r^2-1}{r^2+1}+2\frac{r^4+1}{r^4-1}\right\}\\ &\quad\times(\varphi\otimes X_{31})+\left\{\frac{2r}{r^2+1}\eta(X_{23})-\frac{r^2-1}{r^2+1}(\tau\otimes Ad_{\mathfrak{p}^-})(X_{21})\right\}(\varphi\otimes X_{32}),\end{aligned}$$

where $\partial_1\varphi=R_{H_1}\varphi$ and we simply write $\eta(Y)$ and $\tau\otimes Ad_{\mathfrak{p}^\pm}(X)$ instead of $\eta(Y)\otimes 1_{W\otimes\mathfrak{p}^\pm}$ and $1_{\mathcal{F}_\eta}\otimes(\tau\otimes Ad_{\mathfrak{p}^\pm})(X)$ respectively for every $Y\in\mathfrak{h}_{\mathbb{C}}$ and $X\in\mathfrak{k}_{\mathbb{C}}$.

PROOF. We first note that

$$X_{13}=\frac{r^2}{r^4-1}\mathrm{Ad}(a_r^{-1})H'_{13}+\frac{1}{2}H_1-\frac{1}{2}\frac{r^4+1}{r^4-1}H'_{13}, \tag{5.2.4}$$

$$X_{31}=\frac{-r^2}{r^4-1}\mathrm{Ad}(a_r^{-1})H'_{13}+\frac{1}{2}H_1+\frac{1}{2}\frac{r^4+1}{r^4-1}H'_{13}, \tag{5.2.5}$$

$$X_{32}=\frac{2r}{r^2+1}\mathrm{Ad}(a_r^{-1})X_{32}+\frac{r^2-1}{r^2+1}X_{12}, \tag{5.2.6}$$

$$X_{23}=\frac{2r}{r^2+1}\mathrm{Ad}(a_r^{-1})X_{23}+\frac{r^2-1}{r^2+1}X_{21} \tag{5.2.7}$$

for every $r>0$, $r\neq 1$ corresponding to the decomposition $\mathfrak{g}_{\mathbb{C}}=\mathrm{Ad}(a_r^{-1})\mathfrak{h}_{\mathbb{C}}+\mathfrak{a}_{\mathbb{C}}+\mathfrak{k}_{\mathbb{C}}$. Take an $F\in C^\infty_{\eta,\tau}(H\backslash G/K)$ such that $\varphi=F|_A$. By using (5.2.5), (5.2.6) and Lemma 5.2.1, we have

$$\begin{aligned}&R_{X_{31}}F(a)\\ &=\frac{-r^2}{r^4-1}R_{\mathrm{Ad}(a^{-1})H'_{13}}F(a)+\frac{1}{2}R_{H_1}F(a)+\frac{1}{2}\frac{r^4+1}{r^4-1}R_{H'_{13}}F(a)\\ &=\frac{-r^2}{r^4-1}\big(\eta(H'_{13})\otimes 1_W\big)F(a)+\frac{1}{2}\partial_1F(a)-\frac{1}{2}\frac{r^4+1}{r^4-1}\big(1_{\mathcal{F}_\eta}\otimes\tau(H'_{13})\big)F(a)\end{aligned}$$

and

$$R_{X_{32}}F(a)$$

$$= \frac{2r}{r^2+1} R_{\mathrm{Ad}(a^{-1})X_{32}} F(a) + \frac{r^2-1}{r^2+1} R_{X_{12}} F(a)$$
$$= \frac{2r}{r^2+1} \big(\eta(X_{32}) \otimes 1_W\big) F(a) - \frac{r^2-1}{r^2+1} \big(1_{\mathcal{F}_\eta} \otimes \tau(X_{12})\big) F(a).$$

Hence we have

(5.2.8)
$$\begin{aligned}
&\rho(\nabla^+_{\eta,\tau})\varphi(a)\\
&= R_{X_{31}} F(a) \otimes X_{13} + R_{X_{32}} F(a) \otimes X_{23}\\
&= \left\{ \frac{-r^2}{r^4-1} \big(\eta(H'_{13}) \otimes 1_W\big) F(a) + \frac{1}{2} \partial_1 F(a) \right.\\
&\qquad \left. + \frac{-1}{2} \frac{r^4+1}{r^4-1} \big(1_{\mathcal{F}_\eta} \otimes \tau(H'_{13})\big) F(a) \right\} \otimes X_{13}\\
&\quad + \left\{ \frac{2r}{r^2+1} \big(\eta(X_{32}) \otimes 1_W\big) F(a) - \frac{r^2-1}{r^2+1} \big(1_{\mathcal{F}_\eta} \otimes \tau(X_{12})\big) F(a) \right\} \otimes X_{23}.
\end{aligned}$$

By using the relations $[H'_{13},\ X_{13}] = 2X_{13}$ and $[X_{12},\ X_{23}] = X_{13}$, we have

$$\begin{aligned}
\big(1_{\mathcal{F}_\eta} \otimes \tau(H'_{13})\big) F(a) \otimes X_{13} &= \big(1_{\mathcal{F}_\eta} \otimes \tau \otimes \mathrm{Ad}_{\mathfrak{p}^+}\big)(H'_{13})\big(F(a) \otimes X_{13}\big)\\
&\quad - 2\big(F(a) \otimes X_{13}\big)
\end{aligned}$$

and

$$\begin{aligned}
\big(1_{\mathcal{F}_\eta} \otimes \tau(X_{12})\big) F(a) \otimes X_{23} &= \big(1_{\mathcal{F}_\eta} \otimes \tau \otimes \mathrm{Ad}_{\mathfrak{p}^+}\big)(X_{12})\big(F(a) \otimes X_{23}\big)\\
&\quad - F(a) \otimes X_{13}.
\end{aligned}$$

By inserting these to (5.2.8) we finally obtain

$$\begin{aligned}
\rho(\nabla^+_{\eta,\tau})\varphi = &\left\{ \frac{1}{2}\partial_1 - \frac{r^2}{r^4-1} \big(\eta(H'_{13}) \otimes 1_{W\otimes\mathfrak{p}^+}\big) \right.\\
&\quad + \frac{-1}{2} \frac{r^4+1}{r^4-1} (1_{\mathcal{F}_\eta} \otimes \tau \otimes \mathrm{Ad}_{\mathfrak{p}^+})(H'_{13})\\
&\quad \left. + \frac{r^2-1}{r^2+1} + \frac{r^4+1}{r^4-1} \right\} \varphi \otimes X_{13}\\
&\quad + \left\{ \frac{2r}{r^2+1} \big(\eta(X_{32}) \otimes 1_{W\otimes\mathfrak{p}^+}\big) \right.\\
&\quad \left. - \frac{r^2-1}{r^2+1} \big(1_{\mathcal{F}_\eta} \otimes \tau \otimes \mathrm{Ad}_{\mathfrak{p}^+}(X_{12})\big) \right\} \varphi \otimes X_{23}.
\end{aligned}$$

The computation of $\rho(\nabla^-_{\eta,\tau})$ is similar. □

(5.3) Radial part of the Casimir operators

Let η be an irreducible admissible representation of H and $\tau = \tau_\lambda,\ \lambda \in L_T^+$ an irreducible finite dimensional representation of K. The A-radial part $\rho(\Omega_{\eta,\lambda})$ is given by the next proposition.

PROPOSITION 5.3.1. *For every $\varphi \in C^\infty_{W_0}(A\,;\mathcal{F}_\eta \otimes W_\lambda)$ and $r > 0,\ r \neq 1$, we have*

$$
\begin{aligned}
&2\rho(\Omega_{\tau,\lambda})\varphi(a_r)\\
&= \partial^2\varphi(a_r) + \big(3\frac{r^2-1}{r^2+1} + \frac{r^2+1}{r^2-1}\big)\partial\varphi(a_r)\\
&+ \Big[-\big(\frac{2r^2}{r^4-1}\big)^2\eta(H'_{13})^2 + \big(\frac{2r}{r^2-1}\big)^2\big\{-\frac{1}{2}\tau(H'_{13})\eta(H'_{13}) - \frac{1}{4}\tau(H'_{13})^2)\big\}\\
&+ \big(\frac{2r}{r^2+1}\big)^2\big\{-\frac{1}{2}\tau(H'_{13})\eta(H'_{13}) + \frac{1}{4}\tau(H'_{13})^2 - \eta(H'_{13}) + \tau(H'_{13})\\
&\qquad + 4\eta(X_{23})\eta(X_{32}) - 4\tau(X_{21})\tau(X_{12})\big\}\\
&+ 4\tau(X_{12})\tau(X_{21}) + 4\tau(X_{21})\tau(X_{12}) + \frac{1}{3}\tau(H'_{13} - 2H'_{23})^2\\
&- \frac{8r(r^2-1)}{(r^2+1)^2}\{\eta(X_{32})\tau(X_{21}) + \eta(X_{23})\tau(X_{12})\}\Big]\varphi(a_r),
\end{aligned}
$$

where $\partial_1\varphi = R_{H_1}\varphi$ and we simply write $\eta(Y)$ and $\tau \otimes Ad_{\mathfrak{p}^\pm}(X)$ instead of $\eta(Y) \otimes 1_{W\otimes\mathfrak{p}^\pm}$ and $1_{\mathcal{F}_\eta} \otimes (\tau \otimes Ad_{\mathfrak{p}^\pm})(X)$ respectively for every $Y \in \mathfrak{h}_\mathbb{C}$ and $X \in \mathfrak{k}_\mathbb{C}$.

PROOF. We can prove this in the same way as Proposition 5.2.1, using (5.2.i) ($i = 4, 5, 6, 7$). We omit it. □

§6. Differential equations for the A-radial parts

The computation carried out in this and the next sections is the technical main body of this paper, which is summarized as follows: since $F \in \mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ has the equivariance property (4.1.1), we may restrict our attention to the A-radial part $F|A$. We first show that $F|A$ is determined by a family of $d_\lambda + 1$ functions on $r > 0$, say $\{c_i(r)|\ 0 \leqslant i \leqslant d_\lambda\}$ (Lemma 6.1.1), and then rewrite the system of differential equations $(\mathcal{D}_{\eta,\Lambda})$ in Proposition 4.3.1 in terms of $\{c_i\}$ (Proposition 6.2.2, 6.2.3 and 6.2.4). In §7, we seek solutions of the system of difference-differential equations among c_i's thus obtained.

(6.1)

Let $(\eta, \mathcal{F}_\eta)$ be an admissible Hilbert representation of H with central character $n \in \mathbb{Z}$ that is irreducible or a non-unitary principal series representation. Let $\{v_m \mid m \in L_\eta\}$ be the standard basis of η.

For an integer n and a dominant weight $\lambda = (l_1, l_2) \in L_T^+$, set

$$\begin{aligned} \mu_{n,\lambda} &:= \frac{l_1 + l_2 + n}{3}, \\ \beta_{n,\lambda}(i) &:= -i - l_2 + \mu_{n,\lambda} \quad (i \in \mathbb{Z}), \\ m_{n,\lambda}(i) &:= -2i + \mu_{n,\lambda} + d_\lambda \quad (i \in \mathbb{Z}). \end{aligned} \tag{6.1.1}$$

It is easily checked that if $\mu_{n,\lambda} \in \mathbb{Z}$, then $m_{n,\lambda}(i) \equiv n \pmod 2$ for every $i \in \mathbb{Z}$.

Lemma 6.1.1.
(1) *If $\mu_{n,\lambda} \notin \mathbb{Z}$, then $C^\infty_{W_0}(A\,; \mathcal{F}_\eta \otimes W_\lambda) = \{0\}$.*
(2) *Assume that $\mu_{n,\lambda} \in \mathbb{Z}$. For a given C^∞-function $\varphi : A \to \mathcal{F}_\eta \otimes W_\lambda$, there exists a unique family of C^∞-functions $c_{mi}(r)$ $(m \in \mathbb{Z}_\epsilon,\ 0 \leqslant i \leqslant d_\lambda)$ on $r > 0$ satisfying $c_{mi}(r) = 0$ for $m \notin L_\eta$ and*

$$\varphi(a_r) = \sum_{i=0}^{d_\lambda} \sum_{m \in \mathbb{Z}_\epsilon} c_{mi}(r)(v_m \otimes w_i^\lambda) \tag{6.1.2}$$

for every $a_r \in A$. The right hand side converges in the Hilbert space $\mathcal{F}_\eta \otimes W_\lambda$. Furthermore φ belongs to the space $C^\infty_{W_0}(A\,; \mathcal{F}_\eta \otimes W_\lambda)$ if and only if the following conditions are satisfied:

(6.1.3) c_{mi} *is identically zero for* $m \notin \{m_{n,\lambda}(i) \mid 0 \leqslant i \leqslant d_\lambda\}$.

(6.1.4) $c_{m_{n,\lambda}(i),i}(r^{-1}) = (-1)^{\beta_{n,\lambda}(i)} c_{m_{n,\lambda}(i),i}(r)$ *for every* $r > 0$.

(6.1.5) $\beta_{n,\lambda}(i) c_{m_{n,\lambda}(i),i}(1) = 0$ *for* $i = 0, \dots, d_\lambda$.

Proof. Let $\{(w_i^\lambda)^*\}$ be the dual basis of $\{w_i^\lambda\}$ and $\gamma_i : \mathcal{F}_\eta \otimes_{\mathbb{C}} W_\lambda \to \mathcal{F}_\eta$ denotes the contraction by $(w_i^\lambda)^*$. Then $\varphi_i := \gamma_i \circ \varphi$ becomes a $\mathcal{F}_\eta$-valued C^∞-map on A and $\varphi(a) = \sum_{i=0}^{d_\lambda} \varphi_i(a) \otimes w_i^\lambda,\ a \in A$. Since $\{v_m | m \in L_\eta\}$ is an orthonormal basis of the Hilbert space $\mathcal{F}_\eta$, we can expand each $\varphi_i(a)$ as follows:

$$\varphi_i(a_r) = \sum_{m\in L_\eta} c_{mi}(r)v_m.$$

Hence $\varphi(a)$ can be expanded in $\mathcal{F}_\eta \otimes_{\mathbb{C}} W_\lambda$ in the form (6.1.2). Since $c_{mi} : A \to \mathbb{C}$ is obtained as the composite of C^∞-map φ_i and a bounded linear form $v \mapsto \langle v, v_m\rangle$ on $\mathcal{F}_\eta$, it is a C^∞-function also. Now we shall rewrite the condition (5.1.1) and (5.1.2) in terms of coefficients c_{mi}. Since M is the one parameter subgroup with an infinitesimal generator $\sqrt{-1}H' = \sqrt{-1}(2H'_{12} - H'_{13})$, and since we have

$$\eta_{n,\nu}(H')v_m = \frac{3m-n}{2}v_m \quad (m \in \mathbb{Z}_\epsilon)$$

and

$$\tau_\lambda(H')w_i^\lambda = (3i - 2l_1 + l_2)w_i^\lambda \quad (0 \leqslant i \leqslant d_\lambda),$$

the condition (5.1.1) means

$$\sum_{m\in\mathbb{Z}_\epsilon}\sum_{i=0}^{d_\lambda}\Big\{\exp\Big((\frac{3m-n}{2} + 3i - 2l_1 + l_2)\sqrt{-1}\theta\Big) - 1\Big\}c_{mi}(r)(v_m \otimes w_i^\lambda) = 0$$
$$(r > 0)$$

for all $\theta \in \mathbb{R}$. In other words, $c_{mi} = 0$ unless $\dfrac{3m-n}{2} + 3i - 2l_1 + l_2 = 0$.

Noting that $\dfrac{3m-n}{2} + 3i - 2l_1 + l_2 = \dfrac{3}{2}(m - m_{n,\lambda}(i))$, we finally know that the condition (5.1.1) is equivalent to $c_{mi} = 0$ for $m \notin \{m_{n,\lambda}(0), \ldots, m_{n,\lambda}(d_\lambda)\}$. Especially if $\mu_{n,\lambda} \notin \mathbb{Z}$, then $m_{n,\lambda}(i) \notin \mathbb{Z}$ for every $i = 0, \ldots, d_\lambda$ hence we have $c_{mi} = 0$ for all m and i. This proves **(1)**. Now we prove **(2)**. It suffices to check that for a given $\varphi \in C^\infty_{W_0}(A\,; \mathcal{F}_\eta \otimes W_\lambda)$, the conditions (5.1.1), (5.1.2) and (5.1.3) are equivalent to (6.1.3), (6.1.4) and (6.1.5). We have already seen the equivalence of (5.1.1) and (6.1.3). Now suppose that φ satisfies (5.1.1). As for the equivalence of (5.1.2) and (6.1.4), by using $w_0 = \exp\big(\pi\sqrt{-1}(2H'_{13} - H'_{12})\big)$, we have

$$\eta(w_0)v_m = \exp\big(\sqrt{-1}\pi(-\frac{3m+n}{2})\big)v_m$$

and

$$\tau_\lambda(w_0)w_i^\lambda = \exp\bigl(\sqrt{-1}\pi(l_1+l_2)\bigr)w_i^\lambda.$$

Hence the condition (5.1.2) can be written as follows:

$$c_{mi}(r) = (-1)^{\frac{3m+n}{2}+l_1+l_2}c_{mi}(r^{-1}) \quad (r>0).$$

After some computations, we can see that $\dfrac{3m_{n,\lambda}(i)+n}{2}+l_1+l_2 \equiv \beta_{n,\lambda}(i)$ (mod 2), thus we have done. Finally we show the equivalence of (5.1.3) and (6.1.5). Since we assume that φ satisfies (5.1.1) and the group $H\cap K$ is generated by M and $\exp(\sqrt{-1}\mathbb{R}(H'_{12}+H'_{13}))$, it suffices to require the condition (5.1.3)
for $l \in \exp(\sqrt{-1}\mathbb{R}(H'_{12}+H'_{13}))$. By using the relations

$$\eta(H'_{12}+H'_{13})v_m = -nv_m, \qquad \tau_\lambda(H'_{12}+H'_{13})w_i^\lambda = (3i-l_1+2l_2)w_i^\lambda,$$

we see that (5.1.3) for $l=\exp((H'_{12}+H'_{13})\sqrt{-1}\theta)$ ($\theta\in\mathbb{R}$) can be rewritten as

$$\sum_{m\in\mathbb{Z}_\epsilon}\sum_{i=0}^{d_\lambda}\Bigl\{\exp\bigl((-n+3i-l_1+2l_2)\sqrt{-1}\theta\bigr)-1\Bigr\}c_{mi}(1)v_m\otimes w_i^\lambda = 0,$$

or equivalently $\exp((-n+3i-l_1+2l_2)\sqrt{-1}\theta)c_{mi}(1) = c_{mi}(1)$ for all m and i. Noting $-3\beta_{n,\lambda}(i) = -n+3i-l_1+2l_2$ we have the conclusion. □

(6.2) Difference-differential equations

In Proposition 4.3.1 we present a system of differential equation $(\mathcal{D}_{\eta,\Lambda})$ with unknown function F which belongs to the space $C^\infty_{\eta,\tau_\lambda}(H\backslash G/K)$. On the other hand we have seen that any member F of this last space is completely determined by its A-radial part $\varphi = \mathrm{res}_A(F)$ or by the $d_\lambda+1$ functions $c_i(r) := c_{m_{n,\lambda}(i),i}(r)$ on $r>0$ defined in Lemma 6.1.1. We have to find C^∞-solutions of the A-radial part $\rho(\mathcal{D}_{\eta,\Lambda})$ of $(\mathcal{D}_{\eta,\Lambda})$ in terms of c_i's. So our first task to be carried out is to write down the A-radial part of shift operators $\rho(\nabla_{\eta,\lambda}^{\pm\beta_i})$ $(i=1,2)$ in the language of c_i's. In the rest of this subsection, we assume that $\eta=\eta_{n,\nu}$ with $(n,\nu)\in\mathbb{Z}_\epsilon\times\mathbb{C}$ for $\epsilon\in\{0,1\}$.

Proposition 6.2.1. *Let $\varphi \in C^\infty_{W_0}(A\,;\mathcal{F}_\eta \otimes W_\lambda)$ and let β be an arbitrary non compact root. In view of Lemma 6.1.1, we can express φ as*

$$\varphi(a_r) = \sum_{i=0}^{d_\lambda} c_i(r) v_{m_{n,\lambda}(i)} \otimes w_i^\lambda \tag{6.2.0}$$

with $d_\lambda + 1$ C^∞-functions $c_i(r)$ on $r > 0$. Then $\rho(\nabla^\beta_{\eta,\lambda})\varphi$ is given in terms of c_i's as follows:

$$\rho(\nabla^\beta_{\eta,\lambda})\varphi(a_r) = \sum_{i=0}^{d_{\lambda+\beta}} c[\beta]_i(r) v_{m_{n,\lambda+\beta}(i)} \otimes w_i^{\lambda+\beta}$$

with

$$\begin{aligned}
&c[+\beta_1]_{i+1} \\
&\quad = \frac{i+1}{2}\left\{\partial c_i + \left(\frac{4r^2}{r^4-1}\mu_{n,\lambda} - \frac{r^2+1}{r^2-1}(l_2+i) - 2\frac{r^2-1}{r^2+1}(d_\lambda - i)\right)c_i\right\} \\
&\quad + \frac{r(d_\lambda - i)}{r^2+1}(\nu + m_{n,\lambda}(i) - 1)c_{i+1} \quad (i = -1, \dots, d_\lambda), \\
&c[-\beta_1]_{i-1} \\
&\quad = \frac{1}{2}\left\{\partial c_i + \left(\frac{-4r^2}{r^4-1}\mu_{n,\lambda} + \frac{r^2+1}{r^2-1}(l_2+i) + 2\frac{r^2-1}{r^2+1}(d_\lambda - i + 1)\right)c_i\right\} \\
&\quad + \frac{r}{r^2+1}(\nu - m_{n,\lambda}(i) - 1)c_{i-1} \quad (i = 1, \dots, d_\lambda), \\
&c[+\beta_2]_i = \frac{-1}{2}\left\{\partial c_i + \left(\frac{4r^2}{r^4-1}\mu_{n,\lambda} - \frac{r^2+1}{r^2-1}(l_2+i) + 2\frac{r^2-1}{r^2+1}(i+1)\right)c_i\right\} \\
&\qquad + \frac{r}{r^2+1}(\nu + m_{n,\lambda}(i) - 1)c_{i+1} \quad (i = 0, \dots, d_\lambda - 1), \\
&c[-\beta_2]_i = \frac{d_\lambda - i + 1}{2}\left\{\partial c_i + \left(\frac{-4r^2}{r^4-1}\mu_{n,\lambda} + \frac{r^2+1}{r^2-1}(l_2+i) - 2\frac{r^2-1}{r^2+1}i\right)c_i\right\} \\
&\qquad - \frac{ri}{r^2+1}(\nu - m_{n,\lambda}(i) - 1)c_{i-1} \quad (i = 0, \dots, d_\lambda + 1).
\end{aligned}$$

PROOF. We prove the formulas for $\rho(\nabla_{\eta,\lambda}^{-\beta_1})\varphi$ and $\rho(\nabla_{\eta,\lambda}^{-\beta_2})\varphi$ only, because the other cases can be treated similarly. We write m_i instead of $m_{n,\lambda}(i)$ for simplicity. By using (5.2.3), we first have

(6.2.1)

$$\begin{aligned}
\rho(\nabla_{\eta,\lambda}^{-\beta_i})\varphi(a) &= (1\otimes p_\lambda^{-\beta_i})\rho(\nabla_{\eta,\lambda}^{-})\varphi \\
&= \frac{1}{2}\sum_{i=0}^{d_\lambda}\biggl\{\partial_1 c_i\bigl(v_{m_i}\otimes p_\lambda^{-\beta_i}(w_i^\lambda\otimes X_{31})\bigr) \\
&\qquad + \frac{2r^2}{r^4-1}c_i\bigl(\eta(H'_{13})v_{m_i}\otimes p_\lambda^{-\beta_i}(w_i^\lambda\otimes X_{31})\bigr) \\
&\qquad + \frac{r^4+1}{r^4-1}c_i\bigl(v_{m_i}\otimes \tau_{\lambda-\beta_i}(H'_{13})p_\lambda^{-\beta_i}(w_i^\lambda\otimes X_{31})\bigr) \\
&\qquad + 2\biggl(\frac{r^2-1}{r^2+1}+\frac{r^4+1}{r^4-1}\biggr)c_i\bigl(v_{m_i}\otimes p_\lambda^{-\beta_i}(w_i^\lambda\otimes X_{31})\bigr)\biggr\} \\
&\qquad + \sum_{i=0}^{d_\lambda}\biggl\{\frac{2r}{r^2+1}c_i\bigl(\eta(X_{23})v_{m_i}\otimes p_\lambda^{-\beta_i}(w_i^\lambda\otimes X_{32})\bigr) \\
&\qquad - \frac{r^2-1}{r^2+1}c_i\bigl(v_{m_i}\otimes \tau_{\lambda-\beta_i}(X_{21})p_\lambda^{-\beta_i}(w_i^\lambda\otimes X_{32})\bigr)\biggr\},
\end{aligned}$$

where $i=1,2$. We first consider the case $i=2$.

Tables (2.1.1) and (3.1.1) give us

$$\begin{aligned}
\tau_{\lambda-\beta_2}(H'_{13})w_i^{\lambda-\beta_2} &= (i+l_2-2)w_i^{\lambda-\beta_2}, \\
\tau_{\lambda-\beta_2}(X_{21})w_i^{\lambda-\beta_2} &= (i-d_\lambda-2)w_{i-1}^{\lambda-\beta_2}
\end{aligned}$$

and

$$\eta(H'_{13})v_m = -\frac{m+n}{2}v_m, \quad \eta(X_{23})v_m = \frac{\nu-m+1}{2}v_{m-2}.$$

Substituting these and the Clebsch-Gordan formulae (2.2.5) to (6.2.1), we see that $\rho(\nabla_{\eta,\lambda}^{-\beta_2})$ equals to the following expression:

$$\sum_{i=0}^{d_\lambda}\frac{1}{2}(d_\lambda-i+1)\biggl\{\partial_1 c_i(v_{m_i}\otimes w_i^{\lambda-\beta_2}) + \frac{2r^2}{r^4-1}c_i\biggl(-\frac{m+n}{2}v_{m_i}\biggr)\otimes w_i^{\lambda-\beta_2}$$

$$
\begin{aligned}
&+\frac{r^4+1}{r^4-1}c_i\big(v_{m_i}\otimes(i+l_2-2)w_i^{\lambda-\beta_2}\big)\\
&+2\left(\frac{r^2-1}{r^2+1}+\frac{r^4+1}{r^4-1}\right)c_i(v_{m_i}\otimes w_i^{\lambda-\beta_2})\Bigg\}\\
&+\sum_{i=0}^{d_\lambda}\Bigg\{\frac{2r^2}{r^2+1}c_i\left(\frac{\nu-m+1}{2}v_{m_i-2}\right)\otimes\big(-(i+1)w_{i+1}^{\lambda-\beta_2}\big)\\
&\qquad\qquad-\frac{r^2-1}{r^2+1}c_i\big(v_{m_i}\otimes\big(-(i+1)(i-d_\lambda-1)w_i^{\lambda-\beta_2}\big)\Bigg\}\\
&=\frac{1}{2}\sum_{i=0}^{d_\lambda}(d_\lambda-i+1)\Bigg\{\partial_1 c_i+\Bigg(-\frac{r^2}{r^4-1}(m+n)\\
&\qquad\qquad\qquad\qquad+\frac{r^4+1}{r^4-1}(l_2+i)-2\frac{r^2-1}{r^2+1}i\Bigg)\Bigg\}\\
&\quad\times(v_{m_i}\otimes w_i^{\lambda-\beta_2})-\sum_{i=0}^{d_\lambda+1}\frac{r}{r^2+1}(\nu-m_i-1)ic_{i-1}(v_{m_i}\otimes w_i^{\lambda-\beta_2}).
\end{aligned}
$$

Next we compute $\rho(\nabla_{\eta,\lambda}^{-\beta_1})\varphi(a)$.

Tables (2.1.1) and (3.1.1) give us

$$
\tau_{\lambda-\beta_1}(H'_{13})w_i^{\lambda-\beta_1}=(i+l_2-1)w_i^{\lambda-\beta_1},\quad \tau_{\lambda-\beta_1}(X_{21})w_i^{\lambda-\beta_1}=(i-d_\lambda)w_{i-1}^{\lambda-\beta_1}
$$

and

$$
\eta(H'_{13})v_m=-\frac{m+n}{2}v_m,\quad \eta(X_{23})v_m=\frac{\nu-m+1}{2}v_{m-2}.
$$

Inserting these and the Clebsch-Gordan formulae (2.2.4) to (6.2.1), we see that $\rho(\nabla_{\eta,\lambda}^{-\beta_1})$ equals to the following expression:

$$
\begin{aligned}
&\sum_{i=0}^{d_\lambda}\frac{1}{2}\Bigg\{\partial_1 c_i(v_{m_i}\otimes w_{i-1}^{\lambda-\beta_1})+\frac{2r^2}{r^4-1}c_i\left(-\frac{m+n}{2}v_{m_i}\right)\otimes w_{i-1}^{\lambda-\beta_1}\\
&+\frac{r^4+1}{r^4-1}c_i\big(v_{m_i}\otimes(i-1+l_2-1)w_{i-1}^{\lambda-\beta_1}\big)\\
&+2\left(\frac{r^2-1}{r^2+1}+\frac{r^4+1}{r^4-1}\right)c_i(v_{m_i}\otimes w_{i-1}^{\lambda-\beta_1})\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{d_\lambda}\left\{\frac{2r^2}{r^2+1}c_i\left(\frac{\nu-m+1}{2}v_{m_i-2}\right)\otimes w_i^{\lambda-\beta_1})\right.\\
&\qquad\left.-\frac{r^2-1}{r^2+1}c_i\big(v_{m_i}\otimes(i-d_\lambda)w_{i-1}^{\lambda-\beta_1}\big)\right\}\\
&=\frac{1}{2}\sum_{i=0}^{d_\lambda}\left\{\partial_1 c_i+c_i\left(-\frac{r^2}{r^4-1}(m_i+n)+\frac{r^4+1}{r^4-1}(l_2+i-2)\right.\right.\\
&\qquad\left.\left.-2\frac{r^2-1}{r^2+1}(i-d_\lambda)+2\frac{r^2+1}{r^2-1}+2\frac{r^4+1}{r^4-1}\right)\right\}(v_{m_i}\otimes w_{i-1}^{\lambda-\beta_1})\\
&\qquad+\sum_{i=1}^{d_\lambda}\frac{r}{r^2+1}(\nu-m_i-1)c_{i-1}(v_{m_i}\otimes w_{i-1}^{\lambda-\beta_1}).\ \square
\end{aligned}
$$

By using Proposition 6.2.1 we obtain the following.

Proposition 6.2.2. *Assume $\Lambda\in\Xi_I$, and let $\lambda=(l_1,l_2)$. Then $\rho(\mathcal{D}_{\eta,\lambda})$ is equivalent to the following system of differential equations of c_i's:*

$$
\begin{aligned}
(A_{\nu,\lambda}^-)_i:\quad &(d_\lambda-i+1)(\partial_1 c_i(r)+A_{\lambda i}^-(r)c_i(r))\\
&\qquad=\frac{2ir}{r^2+1}(\nu-m_{n,\lambda}(i)-1)c_{i-1}(r)\\
&(i=0,\dots,d_\lambda+1),\\
(B_{\nu,\lambda}^-)_i:\quad &\partial_1 c_i(r)+B_{\lambda i}^-(r)c_i(r)=\frac{-2r}{r^2+1}(\nu-m_{n,\lambda}(i)-1)c_{i-1}(r)\\
&(i=1,\dots,d_\lambda),
\end{aligned}
$$

where

$$
\begin{aligned}
A_{\lambda i}^-(r)&=\frac{-4r^2}{r^4-1}\mu_{n,\lambda}+\frac{r^2+1}{r^2-1}(l_2+i)-2\frac{r^2-1}{r^2+1}i,\\
B_{\lambda i}^-(r)&=\frac{-4r^2}{r^4-1}\mu_{n,\lambda}+\frac{r^2+1}{r^2-1}(l_2+i)+2\frac{r^2-1}{r^2+1}(d_\lambda-i+1).
\end{aligned}
$$

Proof. Since $\Lambda\in\Xi_I$, the system $\rho(\mathcal{D}_{\eta,\Lambda})$ consists of two differential equations $\rho(\nabla_{\eta,\lambda}^{-\beta_2})\varphi=0$ and $\rho(\nabla_{\eta,\lambda}^{-\beta_1})\varphi=0$. $\square$

PROPOSITION 6.2.3. *Assume $\Lambda \in \Xi_{II}$, and let $\lambda = (l_1, l_2)$. Then $\rho(\mathcal{D}_{\eta,\lambda})$ is equivalent to the following system of differential equations of c_i's:*

$$(A^+_{\nu,\lambda})_i : \quad \partial_1 c_i(r) + A^+_{\lambda i}(r)c_i(r) = \frac{2r}{r^2+1}(\nu + m_{n,\lambda}(i) - 1)c_{i+1}(r)$$
$$(i = 0, \ldots, d_\lambda - 1),$$
$$(B^-_{\nu,\lambda})_i : \quad \partial_1 c_i(r) + B^-_{\lambda i}(r)c_i(r) = \frac{-2r}{r^2+1}(\nu - m_{n,\lambda}(i) - 1)c_{i-1}(r)$$
$$(i = 1, \ldots, d_\lambda),$$

where

$$A^+_{\lambda i}(r) = \frac{4r^2}{r^4-1}\mu_{n,\lambda} - \frac{r^2+1}{r^2-1}(l_2+i) + 2\frac{r^2-1}{r^2+1}(i+1),$$
$$B^-_{\lambda i}(r) = \frac{-4r^2}{r^4-1}\mu_{n,\lambda} + \frac{r^2+1}{r^2-1}(l_2+i) + 2\frac{r^2-1}{r^2+1}(d_\lambda - i + 1).$$

PROOF. Since $\Lambda \in \Xi_{II}$, the system $\rho(\mathcal{D}_{\eta,\Lambda})$ consists of two differential equations $\rho(\nabla^{+\beta_2}_{\eta,\lambda})\varphi = 0$ and $\rho(\nabla^{-\beta_1}_{\eta,\lambda})\varphi = 0$. □

PROPOSITION 6.2.4. *Assume $\Lambda \in \Xi_{III}$, and let $\lambda = (l_1, l_2)$. Then $\rho(\mathcal{D}_{\eta,\lambda})$ is equivalent to the following system of differential equations of c_i's:*

$$(A^+_{\nu,\lambda})_i : \quad \partial_1 c_i(r) + A^+_{\lambda i}(r)c_i(r) = \frac{2r}{r^2+1}(\nu + m_{n,\lambda}(i) - 1)c_{i+1}(r)$$
$$(i = 0, \ldots, d_\lambda - 1),$$
$$(B^+_{\nu,\lambda})_i : \quad (i+1)(\partial_1 c_i(r) + B^-_{\lambda i}(r)c_i(r))$$
$$= \frac{-2(d_\lambda - i)r}{r^2+1}(\nu + m_{n,\lambda}(i) - 1)c_{i+1}(r)$$
$$(i = -1, \ldots, d_\lambda),$$

where

$$A^+_{\lambda i}(r) = \frac{4r^2}{r^4-1}\mu_{n,\lambda} - \frac{r^2+1}{r^2-1}(l_2+i) + 2\frac{r^2-1}{r^2+1}(i+1),$$
$$B^+_{\lambda i}(r) = \frac{4r^2}{r^4-1}\mu_{n,\lambda} - \frac{r^2+1}{r^2-1}(l_2+i) - 2\frac{r^2-1}{r^2+1}(d_\lambda - i).$$

PROOF. Since $\Lambda \in \Xi_{III}$, the system $\rho(\mathcal{D}_{\eta,\Lambda})$ consists of two differential equations $\rho(\nabla^{+\beta_2}_{\eta,\lambda})\varphi = 0$ and $\rho(\nabla^{+\beta_1}_{\eta,\lambda})\varphi = 0$. □

§7. An explicit formula of c_i

For given $n \in \mathbb{Z}$, $\lambda = (l_1, l_2) \in \mathbb{Z}^{\oplus 2}$ and $\nu \in \mathbb{C}$ such that $l_1 \geqslant l_2$ and $\mu_{n,\lambda} = 3^{-1}(l_1 + l_2 + n) \in \mathbb{Z}$, we have obtained the following three systems of differential equations with $d_\lambda + 1$ unknown functions $c_i(r)$ $(i = 0, \dots, d_\lambda)$ on $r > 0$.

$$
\begin{aligned}
(\mathcal{D})_I : &\qquad (A^-_{\nu,\lambda})_i,\ (B^-_{\nu,\lambda})_i \text{ for } i = 1, \dots, d_\lambda \text{ and } (A^-_{\nu,\lambda})_0,\ (A^-_{\nu,\lambda})_{d_\lambda+1},\\
(\mathcal{D})_{II} : &\qquad (A^+_{\nu,\lambda})_i,\ (B^-_{\nu,\lambda})_i \text{ for } i = 0, \dots, d_\lambda - 1 \text{ and } (A^+_{\nu,\lambda})_0,\ (B^-_{\nu,\lambda})_{d_\lambda},\\
(\mathcal{D})_{III} : &\qquad (A^+_{\nu,\lambda})_i,\ (B^+_{\nu,\lambda})_i \text{ for } i = 0, \dots, d_\lambda - 1 \text{ and } (B^+_{\nu,\lambda})_{-1},\ (B^+_{\nu,\lambda})_{d_\lambda}.
\end{aligned}
$$

Now we shall examine these systems of differential equations and find C^∞-solutions of them.

(7.1) Solution of $(\mathcal{D})_I$ or $(\mathcal{D})_{III}$

We consider the systems $(\mathcal{D})_I$ and $(\mathcal{D})_{III}$ first. Note that these were obtained from $(\mathcal{D}_{\eta,\Lambda})$ in Proposition 4.3.1 with Π being holomorphic or antiholomorphic discrete series representations of G respectively.

Proposition 7.1.1. *The system of differential equations $(\mathcal{D})_I$ has a non trivial C^∞-solution if and only if the following conditions on λ, n and ν are satisfied:*
(1) There exists an integer q $(1 \leqslant q \leqslant d_\lambda + 1)$ such that $\nu = m_{n,\lambda}(q) + 1$.
(2) $\mu_{n,\lambda} \geqslant l_2 + q - 1$.
Under the above conditions, the system of differential equations $(\mathcal{D})_I$ has, up to a constant, a unique C^∞-solution $c_i(r)$ $(i = 0, \dots, d_\lambda)$ given by

$$
c_i(r) = L_i \left(\frac{r - r^{-1}}{2} \right)^{\beta_{n,\lambda}(i)} \left(\frac{r + r^{-1}}{2} \right)^{-\mu_{n,\lambda}} \qquad (r > 0),
$$

where

$$
\begin{aligned}
L_i &= 0 \ (i \geqslant q), \quad L_{q-1} = 1,\\
L_i &= \prod_{k=i+1}^{q-1} \frac{k - d_\lambda - 1}{k - q} \ (i < q - 1).
\end{aligned}
$$

PROOF. First assume that $\nu \neq m_{n,\lambda}(q)+1$ for all q $(1 \leqslant q \leqslant d_\lambda+1)$. From $(A^-_{\nu,\lambda})_{d_\lambda+1}$, we have

$$0 = \frac{2r}{r^2+1}(d_\lambda+1)(\nu - m_{n,\lambda}(d_\lambda+1)-1)c_{d_\lambda}(r).$$

Since $\nu \neq m_{n,\lambda}(d_\lambda+1)+1$, this implies that $c_{d_\lambda}(r)=0$. Now using $(B^-_{\nu,\lambda})_i$ successively and noting that $\nu \neq m_{n,\lambda}(i)+1$, we obtain $c_i(r)=0$ for $i=0,\dots,d_\lambda$. Next we consider the case of $\nu = m_{n,\lambda}(q)+1$ with some q $(q=1,\dots,d_\lambda+1)$. By adding $(A^-_{\nu,\lambda})_i$ and i-times $(B^-_{\nu,\lambda})_i$, we obtain

$$(d_\lambda+1)\partial_1 c_i(r) + \{(d_\lambda-i+1)A^-_{\lambda i}(r) + iB^-_{\lambda i}(r)\}c_i(r) = 0 \quad (i=1,\dots,d_\lambda),$$

hence

$$\text{(7.1.1)} \quad r\frac{d}{dr}c_i(r) + \left(\frac{-4r^2}{r^4-1}\mu_{n,\lambda} + \frac{r^2+1}{r^2-1}(l_2+i)\right)c_i(r) = 0 \quad (i=1,\dots,d_\lambda).$$

On the half interval $r>1$, this has a unique C^∞-solution $C_i(r)$ up to a multiplicative constant given by

$$C_i(r) = \left(\frac{r-r^{-1}}{2}\right)^{\beta_{n,\lambda}(i)}\left(\frac{r+r^{-1}}{2}\right)^{-\mu_{n,\lambda}} \quad (r>1).$$

If we define $C_0(r)$ by the right hand side of the above formula with $i=0$, then we can check by a direct computation that $C_i(r)$ $(i=0,\dots,d_\lambda)$ satisfies

$$\text{(7.1.2)} \quad \begin{aligned} \partial_1 C_i(r) + A^-_{\lambda i}(r)C_i(r) &= \frac{-4r}{r^2+1}iC_{i-1}(r), \\ \partial_1 C_i(r) + B^-_{\lambda i}(r)C_i(r) &= \frac{4r}{r^2+1}(d_\lambda-i+1)C_{i-1}(r). \end{aligned}$$

Since $c_i(r)$ $(i=0,\dots,d_\lambda)$ is a C^∞-solution of (7.1.1), $c_i(r)=K_iC_i(r)$ $(r>1)$ with a constant K_i. Inserting this to $(A^-_{\nu,\lambda})_i$, $(B^-_{\nu,\lambda})_i$ and using (7.1.2), we obtain the following recurrence relation:

$$-2(d_\lambda-i+1)K_i = (m_{n,\lambda}(q)-m_{n,\lambda}(i))K_{i-1} \quad (i=1,\dots,d_\lambda+1).$$

Using this, we can easily see that K_i $(i=0,\ldots,d_\lambda)$ is determined by K_{q-1} and $K_i = K_{q-1}L_i$ for $i=0,\ldots,d_\lambda$. Especially $K_i=0$ if $i \geqslant q$. Noting this fact, $K_iC_i(r)$ can be extended to $r>0$ smoothly if and only if $\beta_{n,\lambda}(i) \geqslant 0$ for $i=0,\ldots,q-1$, or equivalently $\mu_{n,\lambda} \geqslant l_2+q-1$. □

Proposition 7.1.2. *The system of differential equations $(\mathcal{D})_{III}$ has a nontrivial C^∞-solution if and only if the following conditions on λ, n and ν are satisfied:*
(1) There exists an integer q $(-1 \leqslant q \leqslant d_\lambda - 1)$ such that $\nu = -m_{n,\lambda}(q)+1$.
(2) $\mu_{n,\lambda} \leqslant l_2+q+1$.
When the above conditions are satisfied, the system of differential equations $(\mathcal{D})_{III}$ has a unique C^∞-solution $c_i(r)$ $(i=0,\ldots,d_\lambda)$ up to a multiplicative constant given by

$$c_i(r) = L_i\left(\frac{r-r^{-1}}{2}\right)^{-\beta_{n,\lambda}(i)}\left(\frac{r+r^{-1}}{2}\right)^{\mu_{n,\lambda}} \quad (r>0),$$

where

$$L_i = 0 \ (i \leqslant q), \quad L_{q+1}=1,$$
$$L_i = \prod_{k=q+1}^{i-1}\frac{k+1}{q-k} \ (i > q+1).$$

Proof. This can be proved similarly as Proposition 7.1.1. □

(7.2) Solutions of $(\mathcal{D})_{II}$

Next we treat the system $(\mathcal{D})_{II}$, which corresponds to the case of Π being a large discrete series representation of G in the situation of Proposition 4.3.1.

Proposition 7.2.1. *If a family of C^∞-functions $\{c_i|\ i=0,\ldots,d_\lambda\}$ is a solution of the system of differential equations $(\mathcal{D})_{II}$, then each $c_i(r)$ satisfies the following differential equation:*

$$(\Gamma_{\nu,\lambda})_i: \ \left(r\frac{d}{dr}\right)^2 w + \left(\frac{r^2+1}{r^2-1} + (2d_\lambda+3)\frac{r^2-1}{r^2+1}\right) r\frac{d}{dr}w + F_i(r)w = 0,$$

where

$$
\begin{aligned}
F_i(r) = &-\big(\frac{4r^2}{r^4-1}\mu_{n,\lambda}\big)^2 \\
&+\{\nu^2-(m_{n,\lambda}(i)-1)^2-4(i+1)(d_\lambda-i) \\
&\quad +2\mu_{n,\lambda}(d_\lambda-2i-1)\}\big(\frac{2r}{r^2+1}\big)^2 \\
&+\{2\mu_{n,\lambda}(l_2+i)-(l_2+i)^2\}\big(\frac{2r}{r^2-1}\big)^2-(i-l_2-2d_\lambda-2)(i-l_2+2).
\end{aligned}
$$

PROOF. For an integer i with $0 \leqslant i < d_\lambda$, we have two differential equations $(A^+_{\nu,\lambda})_i$ and $(B^-_{\nu,\lambda})_{i+1}$ with unknown functions c_i and c_{i+1}. We can obviously eliminate c_{i+1} from these to get a second order differential equation with only one unknown function c_i. After elementary computation we have $(\Gamma_{\nu,\lambda})_i$. We can also start with $(A^+_{\nu,\lambda})_{i-1}$ and $(B^-_{\nu,\lambda})_i$ $(0 < i \leqslant d_\lambda)$, eliminate c_{i-1} and get the same differential equation $(\Gamma_{\nu,\lambda})_i$ of c_i. □

Let $F(a,b\,;c\,;z)$ be the Gauss's hypergeometric function, which is given by the following power series expansion on the unit disc $|z|<1$:

$$
F(a,b\,;c\,;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}.
$$

Especially it is C^∞ on $0 \leqslant z < 1$ and $F(a,b\,;c\,;0)=1$.

PROPOSITION 7.2.2. **(1)** *If i is an integer such that $\beta_{n,\lambda}(i) \geqslant 0$, then $(\Gamma_{\nu,\lambda})_i$ has the unique C^∞-solution $C^+_i(\nu:r)$ up to a constant multiple on the closed half interval $r \geqslant 1$ given by*

$$
\begin{aligned}
C^+_i(\nu:r)=&\big(\frac{r+r^{-1}}{2}\big)^{-m_{n,\lambda}(i)-d_\lambda-2}\big(\frac{r-r^{-1}}{2}\big)^{\beta_{n,\lambda}(i)} \\
&\times F\bigg(\frac{m_{n,\lambda}(i)+1+\nu}{2},\frac{m_{n,\lambda}(i)+1-\nu}{2}\,; \\
&\qquad 1+\beta_{n,\lambda}(i)\,;\big(\frac{r-r^{-1}}{r+r^{-1}}\big)^2\bigg).
\end{aligned}
$$

(2) *If i is an integer such that $\beta_{n,\lambda}(i) \leqslant 0$, then $(\Gamma_{\nu,\lambda})_i$ has the unique C^∞-solution $C_i^-(\nu : r)$ up to a constant multiple on the closed half interval $r \geqslant 1$ given by*

$$\begin{aligned} C_i^-(\nu : r) = (\frac{r+r^{-1}}{2})^{m_{n,\lambda}(i)-d_\lambda-2}(\frac{r-r^{-1}}{2})^{-\beta_{n,\lambda}(i)} \\ \times F\Big(\frac{-m_{n,\lambda}(i)+1+\nu}{2}, \frac{-m_{n,\lambda}(i)+1-\nu}{2} ; \\ 1-\beta_{n,\lambda}(i) ; (\frac{r-r^{-1}}{r+r^{-1}})^2\Big). \end{aligned}$$

Proof. Let $w(r)$ be a C^∞-solution of $(\Gamma_{\nu,\lambda})_i$ on $r > 1$. We make change of variables from r to z by $z = (\frac{r-r^{-1}}{r+r^{-1}})^2$.

We simply write β_i instead of $\beta_{n,\lambda}(i)$. When $\beta_i \geqslant 0$ (resp. $\beta_i \leqslant 0$), set

$$u(z) = (\frac{r+r^{-1}}{2})^\alpha(\frac{r-r^{-1}}{2})^\beta w(r),$$

here $(\alpha, \beta) = (m_{n,\lambda}(i)+d_\lambda+2, -\beta_{n,\lambda}(i))$ (resp. $(-m_{n,\lambda}(i)+d_\lambda+2, \beta_{n,\lambda}(i))$). Then $u(z)$ is a C^∞ function on $0 < z < 1$ that satisfies the hypergeometric differential equation:

$$(\widetilde{\Gamma}_{\nu,\lambda})_i : \ z(1-z)\frac{d^2u}{dz^2} + \{(1+|\beta_i|) - (a_i+b_i+1)\}\frac{du}{dz} - a_ib_iu = 0,$$

with

$$\begin{aligned} a_i = \frac{m_{n,\lambda}(i)+1+\nu}{2}, \quad b_i = \frac{m_{n,\lambda}(i)+1-\nu}{2}. \\ (\textit{resp. } a_i = \frac{-m_{n,\lambda}(i)+1-\nu}{2}, \quad b_i = \frac{-m_{n,\lambda}(i)+1+\nu}{2}). \end{aligned}$$

By taking into account the condition that $w(r)$ is smooth at $r = 1$, we can conclude that $u(z)$ must be of the form

$$u(z) = F(a_i, b_i ; 1+|\beta_i| ; z), \qquad 0 < z < 1.$$

Thus we have done. □

Remark. Let i_0 be the integer such that $\beta_{n,\lambda}(i_0)=0$. Then we obtain $C^+_{i_0}(\nu:r)=C^-_{i_0}(\nu:r)$ by the following wellknown formula of Kummer:

$$F(a,b\,;c\,;z)=(1-z)^{c-a-b}F(c-a,c-b\,;c\,;z).$$

Lemma 7.2.3. *We have*

$$\begin{aligned}&\{(1-a-b)z+c-1\}F(a,b\,;c\,;z)+z(1-z)\frac{ab}{c}F(a+1\,;b+1\,;c+1\,;z)\\&=(c-1)F(a-1,b-1\,;c-1\,;z).\end{aligned}$$

Proof. This can be checked by comparing the Taylor series expansions at $z=0$ of both sides. □

Proposition 7.2.4. *Let $C^\pm_i(r)=C^\pm_i(\nu:r)$ $(i=0,\dots,d_\lambda)$ be the functions defined in Proposition 7.2.2.*

If $\beta_{n,\lambda}(i)\geqslant 0$, then

$$\partial C^+_i(r)+A^+_{\lambda i}(r)C^+_i(r)=2\beta_{n,\lambda}(i)\frac{2r}{r^2+1}C^+_{i+1}(r), \tag{7.2.1}$$

$$\begin{aligned}&\partial C^+_i(r)+B^-_{\lambda i}(r)C^+_i(r)\\&\qquad=\frac{(m_{n,\lambda}(i)+1-\nu)(m_{n,\lambda}(i)+1+\nu)}{2\beta_{n,\lambda}(i-1)}\frac{2r}{r^2+1}C^+_{i-1}(r).\end{aligned} \tag{7.2.2}$$

If $\beta_{n,\lambda}(i)\leqslant 0$, then

$$\begin{aligned}&\partial C^-_i(r)+A^+_{\lambda i}(r)C^-_i(r)\\&\qquad=\frac{(m_{n,\lambda}(i)-1-\nu)(m_{n,\lambda}(i)-1+\nu)}{-2\beta_{n,\lambda}(i+1)}\frac{2r}{r^2+1}C^-_{i+1}(r),\end{aligned} \tag{7.2.3}$$

$$\partial C^-_i(r)+B^-_{\lambda i}(r)C^-_i(r)=-2\beta_{n,\lambda}(i)\frac{2r}{r^2+1}C^-_{i-1}(r). \tag{7.2.4}$$

Proof. By making change of variables from r to $z = \left(\frac{r-r^{-1}}{r+r^{-1}}\right)^2$, we can easily deduce these identities from the previous lemma. □

We finally get the following, which gives all C^∞-solutions of the system $(\mathcal{D})_{II}$.

Proposition 7.2.5. *Put $q_{n,\lambda} := min(d_\lambda, max(0, i_0))$, where i_0 is the integer such that $\beta_{n,\lambda}(i_0) = 0$, or explicitly $i_0 = \mu_{n,\lambda} - l_2$.*

For $i = 0, \dots, d_\lambda$, set

$$\gamma_i(\nu : r) = L_i(\nu) C_i^+(\nu : r),\ L_i(\nu) = \prod_{k=i}^{q_{n,\lambda}-1} \frac{\nu + m_{n,\lambda}(k) - 1}{2\beta_{n,\lambda}(k)} \quad (i < q_{n,\lambda}),$$

$$\gamma_i(\nu : r) = L_i(\nu) C_i^-(\nu : r),\ L_i(\nu) = \prod_{k=q_{n,\lambda}+1}^{i} \frac{\nu - m_{n,\lambda}(k) - 1}{2\beta_{n,\lambda}(k)} \quad (i > q_{n,\lambda})$$

and $\gamma_{q_{n,\lambda}}(\nu : r) = C^+_{q_{n,\lambda}}(\nu : r)$ unless $q_{n,\lambda} = 0$ in which case $\gamma_{q_{n,\lambda}}(\nu : r) = C^-_{q_{n,\lambda}}(\nu : r)$. Then $\{\gamma_i(\nu : r) | \ i = 0, \dots, d_\lambda\}$ is,up to a constant, a unique C^∞-solution of the system of differential equations $(\mathcal{D})_{II}$.

Proof. Given a C^∞-solution $c_i(r)$ $(i = 0, \dots, d_\lambda)$ of $(A^+_{\nu,\lambda})_i$, $(B^-_{\nu,\lambda})_i$, then each $c_i(r)$ is a C^∞-solution of $(\Gamma_{\nu,\lambda})_i$. Hence, from Proposition 7.2.2, there exists a constant K_i satisfying $c_i(r) = K_i C_i^+(r)$ or $c_i(r) = K_i C_i^-(r)$ according as $\beta_{n,\lambda}(i) > 0$ or $\beta_{n,\lambda}(i) \leqslant 0$. If $0 \leqslant i < q_{n,\lambda}$, then $\beta_{n,\lambda}(i) > \beta_{n,\lambda}(i+1) \geqslant 0$. Inserting $c_i(r) = K_i C_i^+(r)$ and $c_{i+1}(r) = K_{i+1} C^+_{i+1}(r)$ to $(A^+_{\nu,\lambda})_i$ and $(B^-_{\nu,\lambda})_{i+1}$ and using Proposition 7.2.4, we obtain

$$2\beta_{n,\lambda}(i) K_i = (\nu + m_{n,\lambda}(i) - 1) K_{i+1}.$$

In the same manner, we have

$$2\beta_{n,\lambda}(i+1) K_{i+1} = (\nu - m_{n,\lambda}(i+1) - 1) K_i$$

if $q_{n,\lambda} \leqslant i \leqslant d_\lambda$. Using these we can see that K_i $(i = 0, \dots, d_\lambda)$ is determined uniquely by $K_{q_{n,\lambda}}$ and expressed as $K_i = K_{q_{n,\lambda}} L_i(\nu)$ $(i = 0, \dots, d_\lambda)$. □

§8. Main results

(8.1) Summing up the computations in the previous two sections, we can now tell our main results. Let Π be a discrete series representation of G with Harish-Chandra parameter $\Lambda \in \Xi$ and far from the wall Blattner parameter $\lambda = (l_1, l_2) \in L_T^+$. Recall the definitions of $\mu_{n,\lambda}$, $\beta_{n,\lambda}(i)$ and $m_{n,\lambda}(i)$. (See (6.1.1).)

THEOREM 8.1.1. *Assume that $\Lambda \in \Xi_I$, i.e. Π is a holomorphic discrete series representation of G. Then the intertwiner space $\mathcal{I}_{\eta,\Pi}$ for $\eta \in \widetilde{H}$ is non zero if and only if $\dfrac{l_1+l_2+n}{3} \in \mathbb{Z}$ and $\eta^0 \cong \delta_{n,k}^{+0}$ with n, k satisfying*

$$\begin{aligned} &k > 0, \quad \frac{3(k+1)+n}{2} \geqslant l_1 + l_2, \\ &2l_2 - l_1 \leqslant \frac{3(k+1)-n}{2} \leqslant 2l_1 - l_2. \end{aligned} \tag{8.1.1}$$

Under this condition, the space $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ is one dimensional and has the base F given by

$$F(a_r) = \sum_{i=0}^{d_\lambda} L_i \left(\frac{r-r^{-1}}{2}\right)^{\beta_{n,\lambda}(i)} \left(\frac{r+r^{-1}}{2}\right)^{-\mu_{n,\lambda}} (v_{m_{n,\lambda}(i)} \otimes w_i^\lambda) \quad (r > 0),$$

where p is the integer determined by $k-1 = m_{n,\lambda}(p)$ and

$$\begin{aligned} &L_i = 0 \ (i \geqslant p), \quad L_{p-1} = 1, \\ &L_i = \prod_{j=i+1}^{p-1} \frac{j-d_\lambda-1}{j-p} \ (i < p-1). \end{aligned}$$

PROOF. Let $(\eta, \mathcal{F}_\eta)$ be an irreducible admissible representation of H with central character $n \in \mathbb{Z}_\epsilon$. Let $\{\epsilon, \epsilon'\} = \{0, 1\}$. We have already seen in Lemma 6.1.1 that $\mu_{n,\lambda} \in \mathbb{Z}$ is a necessary condition for $\mathcal{S}_{\eta,\Pi}(\tau_\lambda) \neq \{0\}$. By Proposition 3.2.1, there exists an $(\mathfrak{h}_\mathbb{C}, K \cap H)$-module embedding $\eta^0 \hookrightarrow \eta_{n,\nu}^0$ to some non unitary principal series representation $\eta_{n,\nu}$ with $\nu \in \mathbb{C}$. Hence we have a natural injective linear map $\mathcal{S}_{\bar{\eta},\Pi}(\tau_\lambda) \hookrightarrow \mathcal{S}_{\eta_{n,\nu},\Pi}(\tau_\lambda)$ where $(\bar{\eta}, \overline{\mathcal{F}_\eta})$ means the closure of $\mathcal{F}_\eta^0$ in the Hilbert space $\mathcal{V}_{n,\nu}$.

One should note that the definition of two $\mathbb{C}$-vector spaces $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ and $\mathcal{S}_{\overline{\eta},\Pi}(\tau_\lambda)$ depends on topologies of $\mathcal{F}_\eta$ and $\overline{\mathcal{F}_\eta}$ respectively which may not be isomorphic. But as a result of Proposition 4.3.1, Lemma 5.1.2, Lemma 6.1.1 and Proposition 6.2.1, the $\mathbb{C}$-vector spaces $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ and $\mathcal{S}_{\overline{\eta},\Pi}(\tau_\lambda)$ are both isomorphic to the C^∞-solution space of the system of differential equations $(\mathcal{D})_I$ with unknown functions $c_i(r),\ i=0,\ldots,d_\lambda$, whose definition imvolves only numerical data determined by λ and the structure of $\eta^0 \cong (\overline{\eta})^0$. Thus $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ is isomorphic to $\mathcal{S}_{\overline{\eta},\Pi}(\tau_\lambda)$ as a $\mathbb{C}$-vector space. By this reason we may replace η by $\overline{\eta}$ and assume $\mathcal{F}_\eta$ is a closed sub H-module of $\mathcal{V}_{n,\nu}$ in what follows. We see that $\mathcal{S}_{\eta,\Pi}(\tau_\lambda) \neq \{0\}$ implies that $\nu = m_{n,\lambda}(q)+1$ with q satisfying $1 \leqslant q \leqslant d_\lambda + 1$ and $\mu_{n,\lambda} \geqslant q + l_2 - 1$ by using Proposition 7.1.1. Since $k := m_{n,\lambda}(q)+1 \in \mathbb{Z}_{\epsilon'}$, the possibility of η^0 is $\delta^{\pm 0}_{n,k}$ or $\sigma^0_{n,-k}$ according to $k \geqslant 0$ or $k<0$. We consider these cases separately. Let q' be the integer determined by $m_{n,\lambda}(q') = -m_{n,\lambda}(q)$.
(1) Assume that $\eta^0 \cong \sigma^0_{n,-k}$. Let $\varphi \in C^\infty_{W_0}(A\ ;\mathcal{F}_\eta \otimes W_\lambda)$ be a non zero function satisfying the differential equation $\rho(\mathcal{D}_{\eta,\lambda})$ and express it as (6.2.0) by means of the standard basis of $\sigma^0_{n,-k}$ and W_λ. Then $c_i \equiv 0$ for $i = 0,\ldots,d_\lambda$ such that $m_{n,\lambda}(i) \leqslant k-1$ or $m_{n,\lambda}(i) \geqslant -k+1$, or equivalently $i \geqslant q$ or $i \leqslant q'$. Since $\{c_i\}$ is a C^∞-solution of $(\mathcal{D})_I$, there exists a non zero constant K satisfying

$$c_i(r) = KL_i\left(\frac{r-r^{-1}}{2}\right)^{\beta_{n,\lambda}(i)}\left(\frac{r+r^{-1}}{2}\right)^{\mu_{n,\lambda}} \quad (r>0)$$

where L_i $(i=0,\ldots,d_\lambda)$ are the constants given in Proposition 7.1.1. From the definition, $L_i = 0$ for $i \geqslant q$. Since $L_i \neq 0$ for $i = 0,\ldots,q-1$, q' must be negative in order that $c_i \equiv 0$ for $i \leqslant q'$. On the other hand, since $q+q' = d_\lambda + \mu_{n,\lambda}$ and $\mu_{n,\lambda} \geqslant q + l_2 - 1$, we have $q' \geqslant d_\lambda + l_2 - 1 = l_1 - 1$. Furthermore noting that $\Lambda \in \Xi_I$, we have $l_1 - 1 > 0$ and consequently $q' > 0$. But this contradicts the negativity of q'. Thus the possibility of $\eta^0 \cong \sigma^0_{n,k}$ is excluded.
(2) Next we consider the case $\eta^0 \cong \delta^{+0}_{n,k}$ $(k \geqslant 0)$. Let φ and $\{c_i\}$ be as in (1). Then $c_i \equiv 0$ for $i=0,\ldots,d_\lambda$ such that $m_{n,\lambda}(i) \leqslant k-1$ or equivalently $i \geqslant q$. By the same argument as in the case (1), we have

$$c_i(r) = KL_i\left(\frac{r-r^{-1}}{2}\right)^{\beta_{n,\lambda}(i)}\left(\frac{r+r^{-1}}{2}\right)^{\mu_{n,\lambda}} \quad (r>0)$$

with a non zero constant K. Noting that $L_i = 0$ for $i \geqslant q$, this proves that the space $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ is one dimensional. It is easily checked that the condition $1 \leqslant q \leqslant d_\lambda + 1$ and $\mu_{n,\lambda} \geqslant q + l_2 - 1$ is equivalent to (8.1.1).
(3) Finally we treat the case $\eta^0 \cong \delta_{n,k}^{-0}$ $(k \geqslant 0)$. Let φ and $\{c_i\}$ be as above. Then $c_i \equiv 0$ for $i = 0, \ldots, d_\lambda$ such that $m_{n,\lambda}(i) \geqslant -k+1$, or equivalently $i \leqslant q'$. By using the same argument as (1), we again have a contradiction. □

THEOREM 8.1.2. *Assume that $\Lambda \in \Xi_{III}$, i.e. Π is an antiholomorphic discrete series representation of G. Then the intertwiner space $\mathcal{I}_{\eta,\Pi}$ for $\eta \in \widetilde{H}$ is non zero if and only if $\dfrac{l_1+l_2+n}{3} \in \mathbb{Z}$ and $\eta^0 \cong \delta_{n,k}^{-0}$ with n, k satisfying*

$$(8.1.2) \qquad \begin{aligned} k > 0, \quad & \frac{3(-k-1)+n}{2} \leqslant l_1 + l_2, \\ -l_2 + 2l_1 \geqslant & \frac{3(-k-1)-n}{2} \geqslant -l_1 + 2l_2. \end{aligned}$$

Under this condition, the space $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ is one dimensional and has the base F given by

$$F(a_r) = \sum_{i=0}^{d_\lambda} L_i \left(\frac{r-r^{-1}}{2}\right)^{-\beta_{n,\lambda}(i)} \left(\frac{r+r^{-1}}{2}\right)^{\mu_{n,\lambda}} (v_{m_{n,\lambda}(i)} \otimes w_i^\lambda) \quad (r > 0),$$

where p is the integer determined by $k+1 = m_{n,\lambda}(p)$ and

$$L_i = 0 \ (i \leqslant p), \quad L_{p+1} = 1,$$
$$L_i = \prod_{j=p+1}^{i-1} \frac{j+1}{p-j} \ (i > p+1).$$

THEOREM 8.1.3. *Assume $\Lambda \in \Xi_{II}$, i.e. Π is a large discrete series representation of G. If $\mu_{n,\lambda} \notin \mathbb{Z}$, then $\mathcal{I}_{\eta,\Pi} = 0$ for every $\eta \in \widetilde{H}$. Assume that $\mu_{n,\lambda} \in \mathbb{Z}$ and set $q_{n,\lambda} = min\ (d_\lambda, max(0, \mu_{n,\lambda} - l_2))$. Then the space $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ for $\eta \in \widehat{H}$ is described as follows:*

(1) *When η is the class of an irreducible unitary principal series representation $\eta_{n,\nu}$, then the space $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ is one dimensional and has the base $F(\nu)$ whose A-radial part is given by*

$$F(\nu : a_r) = \sum_{i=0}^{d_\lambda} \gamma_i(\nu : r)(v_{m_{n,\lambda}(i)} \otimes w_i^\lambda) \quad (r > 0),$$

$$\begin{aligned}\gamma_i(\nu : r) = L_i(\nu)\big(\frac{r+r^{-1}}{2}\big)^{-\epsilon_i m_{n,\lambda}(i) - d_\lambda - 2}\big(\frac{r-r^{-1}}{2}\big)^{|\beta_{n,\lambda}(i)|}\\ \times F\Bigg(\frac{\epsilon_i m_{n,\lambda}(i)+1+\nu}{2}, \frac{\epsilon_i m_{n,\lambda}(i)+1-\nu}{2} \ ;\\ 1+|\beta_{n,\lambda}(i)| \ ; \big(\frac{r-r^{-1}}{r+r^{-1}}\big)^2\Bigg)\end{aligned}$$

with

$$\begin{aligned}L_i(\nu) &= \prod_{k=i}^{q_{n,\lambda}-1} \frac{\nu + m_{n,\lambda}(k) - 1}{2\beta_{n,\lambda}(k)} \quad (i \leqslant q_{n,\lambda} - 1),\\ L_i(\nu) &= \prod_{k=q_{n,\lambda}+1}^{i} \frac{\nu - m_{n,\lambda}(k) - 1}{2\beta_{n,\lambda}(k)} \quad (i \geqslant q_{n,\lambda} + 1),\\ L_{q_{n,\lambda}}(\nu) &= 1,\end{aligned}$$

where ϵ_i expresses $+1$ or -1 according as $\beta_{n,\lambda}(i) \geqslant 0$ or $\beta_{n,\lambda}(i) < 0$ respectively.

(2) *When η is the class of a (limit of) discrete series representation $\delta_{n,k}^{\pm 0}$ with (n,k) satisfying $n \equiv k+1 \pmod 2$ and $k \geqslant 0$. Then the space $\mathcal{I}_{\eta,\Pi}$ is non zero if and only if*

$$\eta^0 \cong \delta_{n,k}^{+0} \text{ with } k \text{ satisfying } 0 \leqslant k < -2q_{n,\lambda} + \mu_{n,\lambda} + d_\lambda + 1 \tag{8.1.3}$$

or

$$\eta^0 \cong \delta_{n,k}^{-0} \text{ with } k \text{ satisfying } -2q_{n,\lambda} + \mu_{n,\lambda} + d_\lambda - 1 < -k \leqslant 0. \tag{8.1.4}$$

Under this condition, the space $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ *is one dimensional and has the base given by* $F(k)$*, where* $F(k)$ *is given by the same formula in the case* **(1)**.
(3) *When* η *is the class of one dimensional representation* $\sigma_{n,1}$ $(n \in \mathbb{Z}_0)$*. Then the space* $\mathcal{I}_{\eta,\Pi}$ *is non zero if and only if*

$$-2q_{n,\lambda} + \mu_{n,\lambda} + d_\lambda = 0. \tag{8.1.5}$$

Under this condition, the space $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ *is one dimensional and has the base given by* $F(-1)$*,where* $F(-1)$ *is given by the same formula in the case* **(1)**.

Proof. Let$(\eta, \mathcal{F}_\eta)$ be an irreducible admissible representation with central character $n \in \mathbb{Z}$. It is already proved in Lemma 6.1.1 that $\mu_{n,\lambda} \in \mathbb{Z}$ is a necessary condition for $\mathcal{S}_{\eta,\Pi}(\tau_\lambda) \neq 0$. So we assume that $\mu_{n,\lambda} \in \mathbb{Z}$. We may also assume that η is a subrepresentation of $\eta_{n,\nu}$. Let $\varphi \in C^\infty_{W_0}(A\,; \mathcal{F}_\eta \otimes W_\lambda)$ be a non zero function satisfying the differential equation $\rho(\mathcal{D}_{\eta,\lambda})$ and express it as (6.2.0) by means of the standard basis of η and τ_λ. Since $\{c_i\}$ is a C^∞-solution of $(\mathcal{D})_{II}$, there exists a constant K satisfying

$$c_i(r) = K\gamma_i(\nu : r) \quad (i = 0, \ldots, d_\lambda)$$

in view of Proposition 7.2.5. The assertions of the case **(1)** are already proved in Proposition 7.2.6. We consider the case **(2)** for $\delta^{+0}_{n,k} (\subset \eta^0_{n,k})$. Let p be the integer such that $k - 1 = m_{n,\lambda}(p)$. We have $c_i \equiv 0$ for $i = 0, \ldots, d_\lambda$ such that $m_{n,\lambda}(i) \leqslant k - 1$, or equivalently $i \geqslant p$. If $q_{n,\lambda} \geqslant p$, then $c_{q_{n,\lambda}}(r) = KC^+_{q_{n,\lambda}}(k : r)$ must be identically zero. Hence $K = 0$, and $c_i \equiv 0$ for all i. If $q_{n,\lambda} < p$, then we have

$$c_i(r) = K \prod_{j=q_{n,\lambda}+1}^{i} \frac{m_{n,\lambda}(p) - m_{n,\lambda}(j)}{2\beta_{n,\lambda}(j)} C^-_i(\nu : r) = 0$$

for all $i \geqslant p$. It is easy to see that the condition $q_{n,\lambda} < p$ is equivalent to (8.1.3). This proves the assertions for $\eta^0 \cong \delta^{+0}_{n,k}$. The case $\eta^0 \cong \delta^{-0}_{n,k}$ is treated similarly.

Next we consider the case $\eta = \sigma_{n,k} (\subset \eta_{n,-k})$. Let p and p' be integers satisfying $-k - 1 = m_{n,\lambda}(p)$ and $k + 1 = m_{n,\lambda}(p')$. We have $c_i \equiv 0$ for

$m_{n,\lambda}(i) \leqslant -k-1$ or $m_{n,\lambda}(i) \geqslant k+1$, or equivalently $i \geqslant p$ or $i \leqslant p'$. If $q_{n,\lambda} \geqslant p$ or $q_{n,\lambda} \leqslant p'$, then we must have $c_{q_{n,\lambda}}(r) = K\gamma_{q_{n,\lambda}}(-k:r) \equiv 0$, and hence $K=0$. Thus in order that $\mathcal{I}_{\eta,\Pi} \neq 0$, it is necessary that $p' < q_{n,\lambda} < p$, or equivalently $-k-1 < m_{n,\lambda}(q) < k+1$ holds. On the other hand, we have $m_{n,\lambda}(q) \equiv k+1 \pmod 2$. Hence, $k=1$ means that $m_{n,\lambda}(q)=0$. □

§9. The case of principal series representations

In this section, we shall investigate the spaces of Shintani functions $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ when Π is a principal series representation of G.

(9.1) Principal series representations of G

We first parametrize the principal series representations of G as follows. Let $P = MAN$ be the minimal parabolic subgroup of G with the unipotent radical $N = \exp(\mathfrak{n})$ and the Levi part MA (**(1.2)**). For given $n \in \mathbb{Z}$ and $s \in \mathbb{C}$, let $\xi_{n,s} : P \to \mathbb{C}^*$ denote a quasi character of P defined by

$$\xi_{n,s}(ma_r n) = r^s \chi_n(m), \quad a_r \in A,\ m \in M,\ n \in N,$$

where $\chi_n : M \to \mathbb{C}^{(1)}$ stands for the unitary character of M defined by $\chi_n(m) = u^n$, $m = \mathrm{diag}(u, u^{-2}, u) \in M$. Then the non-unitary principal series representation of G with parameter $(n,s) \in \mathbb{Z} \times \mathbb{C}$ in C^∞ context is defined by the C^∞-induced module $\Pi^\infty_{n,s} = C^\infty\mathrm{Ind}_P^G(\xi_{n,s+2})$. Recall that the representation space of $\Pi^\infty_{n,s}$ is the Frechet space $\mathcal{F}^\infty_{n,s+2}$ consisting of all C^∞-functions $f : G \to \mathbb{C}$ with the equivariant property

$$f(pg) = \xi_{n,s+2}(p) f(g), \quad p \in P,\ g \in G$$

and the action $\Pi^\infty_{n,s}(g),\ g \in G$ on $\mathcal{F}^\infty_{n,s+2}$ is the right translation:

$$\big(\Pi^\infty_{n,s}(g) f\big)(x) = f(xg), \quad f \in \mathcal{F}^\infty_{n,s+2}.$$

We remark that in the definition of $\Pi^\infty_{n,s}$ the parameter s is shifted by 2 ('ρ-shift') so that the action $\Pi^\infty_{n,s}(g),\ g \in G$ is unitary for a purely imaginary s with respect to the following Hermitian inner product on $\mathcal{F}^\infty_{n,s+2}$:

$$\langle f_1, f_2 \rangle = \int_K f_1(k)\overline{f_2(k)}dk, \quad f_1,\ f_2 \in \mathcal{F}^\infty_{n,s+2},$$

with dk the normalized Haar measure of K. By passing to the completion with respect to the Hermitian inner product above, we get a bounded Hilbert representation $\Pi_{n,s}$ on a Hilbert space $\mathcal{F}_{n,s+2}$ which contains $\mathcal{F}^{\infty}_{n,s+2}$ as a dense subspace. It is known that $\mathcal{F}^{\infty}_{n,s+2}$ coincides with the space of C^{∞}-vectors of $\Pi_{n,s}$, [B-W, p.106 Theorm 7.5]. We note that the operators $\Pi_{n,s}(k)$ with $k \in K$ are unitary for all $n \in \mathbb{Z}$ and $s \in \mathbb{C}$. Moreover $\Pi^*_{n,s}$, the contragredient Hilbert representation of $\Pi_{n,s}$, is isomorphic to $\Pi_{-n,-s}$ through the pairing $\mathcal{F}_{n,s+2} \times \mathcal{F}_{-n,-s+2} \to \mathbb{C}$ given by

$$(f_1, f_2) \to \langle f_1, \overline{f_2} \rangle = \int_K f_1(k) f_2(k) dk, \qquad f_1 \in \mathcal{F}_{n,s+2}, \; f_2 \in \mathcal{F}_{-n,-s+2}. \tag{9.1.0}$$

The following Lemma is an easy consequence of the formula (2.1.1).

Lemma 9.1.1. *Let $\lambda = (l_1, l_2) \in L_T^+$, a dominant weight, and $\{w_i^{\lambda}\}_{i=0}^{d_\lambda}$ the standard basis of W_λ. Then*

$$\tau_\lambda(m) w_i^{\lambda} = \chi_{3i+l_2-2l_1}(m) w_i^{\lambda}, \quad m \in M \tag{9.1.1}$$

for $i = 0, \ldots, d_\lambda$. In other words the representation $\tau_\lambda | M$ is a direct sum of characters $\chi_{3i+l_2-2l_1}$ of M with $i = 0, \ldots, d_\lambda$.

Proof. The only point which should be pointed out is that $\mathfrak{m} = \sqrt{-1}(2H'_{12} - H'_{13})\mathbb{R}$. □

For a given $n \in \mathbb{Z}$, put

$$L(n) = \{(-n, -n) + n_1\beta_1 + n_2(-\beta_2) | \; n_1, \; n_2 \in \mathbb{Z}_{\geqslant 0}\}, \tag{9.1.2}$$

a subset of L_T^+. The proposition below describes how $\Pi_{n,s}$ decomposes to irreducible representations when restricted to K.

Proposition 9.1.2. *Let $\Pi_{n,s}$ be the principal series representation of G with $(n, s) \in \mathbb{Z} \times \mathbb{C}$. Then $\Pi_{n,s}|K$, a unitary representation of K, is a multiplicity free orthogonal direct sum of irreducible representations whose highest weights are in $L(n)$. In other words, $\dim_{\mathbb{C}} Hom_K(\tau_\lambda, \Pi_{n,s}|K)$ is 0 or 1 and it is indeed 1 if and only if $\lambda \in L(n)$.*

PROOF. We have a K-isomorphism $\Pi_{n,s}|K \cong \mathrm{Ind}_M^K(\chi_n)$ by restricting functions in $\mathcal{F}^\infty_{n,s+2}$ to K. Then we can prove the proposition by using the Frobenius reciprocity applied to representations of a compact group K and those of its subgroup M, combined with the branching law of $\tau_\lambda|M$ for $\lambda = (l_1, l_2) \in L_T^+$ given in Lemma 9.1.1. □

We shall specify a basis of the space $\mathcal{F}^0_{n,s+2}$. Take a $\lambda = (l_1, l_2) \in L(n)$ and for every $w \in W_\lambda$ define a function $\widetilde{j_\lambda}(w)$ on K by the equation

$$\widetilde{j_\lambda}(w)(k) = \left\langle (w^\lambda_{\gamma_\lambda})^*, \tau_\lambda(k)w \right\rangle, \quad k \in K, \tag{9.1.3}$$

where $\{(w_i^\lambda)^*\}$ is the dual basis of $\{w_i^\lambda\}$, $\langle\ ,\ \rangle : W_\lambda^* \otimes_{\mathbb{C}} W_\lambda \to \mathbb{C}$ stands for the canonical pairing and $\gamma_\lambda = 3^{-1}(n + 2l_1 - l_2)$. We note that the condition $\lambda \in L(n)$ implies that $\gamma_\lambda \in \{0, 1, \dots, d_\lambda\}$. Indeed, if we write $\lambda = (-n, -n) + n_1\beta_1 + n_2(-\beta_2)$ with non negative integers n_1 and n_2, then $d_\lambda = n_1 + n_2$ and $\gamma_\lambda = n_1$.

LEMMA 9.1.3. *The function $\widetilde{j_\lambda}(w) : K \to \mathbb{C}$ satisfies*

$$\widetilde{j_\lambda}(w)(mk) = \chi_n(m)\widetilde{j_\lambda}(w)(k), \quad m \in M,\ k \in K, \tag{9.1.4}$$

thus there exists a unique element $j_\lambda(w)$ of $\mathcal{F}^\infty_{n,s+2}$ whose restriction to K is $\widetilde{j_\lambda}(w)$. The application $w \to j_\lambda(w)$ gives an injective K-homomorphism j_λ from W_λ to $\mathcal{F}^0_{n,s+2}$. The family $\{f_i^\lambda |\ \lambda \in L(n),\ 0 \leqslant i \leqslant d_\lambda\}$ with $f_i^\lambda = j_\lambda(w_i^\lambda)$ provides us with a basis of the $\mathbb{C}$-vector space $\mathcal{F}^0_{n,s+2}$.

PROOF. For $m \in M$ and $k \in K$, we have

$$\begin{aligned}
\widetilde{j_\lambda}(w)(mk) &= \left\langle (w^\lambda_{\gamma_\lambda})^*, \tau_\lambda(m)\tau_\lambda(k)w \right\rangle \\
&= \left\langle \tau_\lambda^*(m^{-1})(w^\lambda_{\gamma_\lambda})^*, \tau_\lambda(k)w \right\rangle \\
&= \chi_{3\gamma_\lambda - 2l_1 + l_2}(m)\left\langle (w^\lambda_{\gamma_\lambda})^*, \tau_\lambda(k)w \right\rangle \\
&= \chi_n(m)\widetilde{j_\lambda}(w)(k).
\end{aligned}$$

This proves (9.1.4). We can easily confirm the equation $\widetilde{j_\lambda}(\tau_\lambda(k)w)(k') = \widetilde{j_\lambda}(k'k)$ for $k',\ k \in K$ by looking at the definition. Thus we get a K-homomorphism $\widetilde{j_\lambda} : \tau_\lambda \to C^\infty\mathrm{Ind}_M^K(\chi_n)$. Since $\widetilde{j_\lambda}(w^\lambda_{\gamma_\lambda})(1) = 1$, this map is not zero. Because τ_λ is irreducible, injectivity of $\widetilde{j_\lambda}$ follows. The remaining assertions are clear in view of Proposition 9.1.2. □

(9.2) Differential equations for principal series Shintani functions

Let $\Pi_{n,s}$ be the non-unitary principal series representation of G with $(n,s) \in \mathbb{Z} \times \mathbb{C}$. By Lemma 9.1.1 $\Pi_{n,s}$ has a unique one dimensional K-type, namely $\tau_{(-n,-n)}$, that is explicitly given by

$$\tau_{(-n,-n)}\big(\mathrm{diag}(k_1,k_2)\big) = \det(k_1)^{-n}, \quad k_1 \in U(2),\ k_2 = \det(k_1)^{-1} \in U(1).$$

By Lemma 9.1.2, the $\tau_{(-n,-n)}$-isotypic subspace of $\mathcal{F}_{n,s}$ is given by $\mathbb{C} f_0^{(-n,-n)}$. We note that the operator $\Pi_{n,s}^\infty(\Omega_G)$ on $\mathcal{F}_{n,s+2}^\infty$ is given by

$$(\frac{s^2}{2} + \frac{n^2}{6} - 2) 1_{\mathcal{F}_{n,s+2}^\infty}$$

as is calculated in [O-K, **(7.3)**]. The following is the main theorem of this subsection, which should be considered to be an analogue of Proposition 4.3.1.

Theorem 9.2.1. *Let $\Pi_{n,s}$ be a non-unitary principal series representation of G and η an irreducible admissible Hilbert representation of H. We assume that $\Pi_{n,s}^0$, the underlying $(\mathfrak{g},K)$-module of $\Pi_{n,s}$, is irreducible. Then the application $\Phi \to \Phi(f_0^{(-n,-n)})$ induces a bijective $\mathbb{C}$-linear map from $Hom_{(\mathfrak{g},K)}\big(\Pi_{n,s}^0, C_\eta^\infty(H\backslash G)\big)$ to the space of $F \in C_\eta^\infty(H\backslash G)$ satisfying the following equations:*

$$R_{\Omega_G} F(g) = \big(\frac{s^2}{2} + \frac{n^2}{6} - 2\big) F(g), \quad g \in G, \tag{9.2.1}$$

$$F(gk) = \tau_{(-n,-n)}(k) F(g), \quad k \in K,\ g \in G. \tag{9.2.2}$$

Proof. Given a function $F \in C_\eta^\infty(H\backslash G)$ of the form $\Phi(f_0^{(-n,-n)})$ with some $\Phi \in \mathrm{Hom}_{(\mathfrak{g},K)}\big(\Pi_{n,s}^0, C_\eta^\infty(H\backslash G)\big)$. Since $f_0^{(-n,-n)}$ is an eigen vector of $\Pi_{n,s}^0(\Omega_G)$ with eigenvalue $\frac{s^2}{2} + \frac{n^2}{6} - 2$ and belongs to the one dimensional K-type $\tau_{(-n,-n)}$, we get the equations (9.2.1) and (9.2.2) for F noting that Φ is a $(\mathfrak{g},K)$-homomorphism. The injectivity of $\Phi \to \Phi(f_0^{(-n,-n)})$ is a consequence of the irreducibility of $\Pi_{n,s}^0$. Indeed the equation $\Phi(f_0^{(-n,-n)}) =$

0 and a fact that $f_0^{(-n,-n)}$ is a cyclic vector of $\mathcal{F}^0_{n,s+2}$ imply $\Phi(\mathcal{F}^0_{n,s+2}) = \Phi(U(\mathfrak{g}_{\mathbb{C}})f_0^{(-n,-n)}) = R_{U(\mathfrak{g}_{\mathbb{C}})}\Phi(f_0^{(-n,-n)}) = \{0\}$.

Thus we only have to prove the surjectivity of the map $\Phi \to \Phi(f_0^{(-n,-n)})$. Take an arbitrary non zero function $F \in C^\infty_\eta(H\backslash G)$ which satisfies (9.2.1) and (9.2.2). Then it suffices to show that the $(\mathfrak{g}, K)$-submodule of $C^\infty_\eta(H\backslash G)$ generated by F, say $\mathcal{V}_F$, is irreducible and isomorphic to $\Pi^0_{n,s}$. Let $\mathcal{W}$ be a maximal proper $(\mathfrak{g}, K)$-submodule of $\mathcal{V}_F$ and $\overline{\mathcal{V}_F} = \mathcal{V}_F/\mathcal{W}$ the quotient $(\mathfrak{g}, K)$-module, that is irreducible. Since $\mathcal{V}_F$ is a cyclic $U(\mathfrak{g}_{\mathbb{C}})$-module generated by F and $\mathcal{W}$ is a proper $U(\mathfrak{g}_{\mathbb{C}})$-submodule, $F \notin \mathcal{W}$, or equivalently the image of F in $\overline{\mathcal{V}_F}$, say $\widetilde{F}$, is non zero. Thus $\overline{\mathcal{V}_F}$ contains a one dimensional K-type $\mathbb{C}\widetilde{F} \cong \tau_{(-n,-n)}$. By Casselman's subrepresentation theorem we can find a $(\mathfrak{g}, K)$-inclusion $\overline{\mathcal{V}_F} \to \Pi^0_{n',s'}$ with some $(n', s') \in \mathbb{Z} \times \mathbb{C}$, hence a K-inclusion $\tau_{(-n,-n)} \to \overline{\mathcal{V}_F} \to \Pi^0_{n',s'}|K$. Since the only one dimensional K-type of $\Pi^0_{n',s'}$ is $\tau_{(-n',-n')}$ (see Proposition 9.1.1), we conclude $n = n'$. Comparing the eigenvalues of the Casimir operators on $\overline{\mathcal{V}_F}$ and $\Pi^0_{n',s'}$, we have

$$\frac{s^2}{2} + \frac{n^2}{6} - 2 = \frac{s'^2}{2} + \frac{n'^2}{6} - 2.$$

Using this and the equation $n = n'$, we get $s = \pm s'$. Since $\Pi^0_{n,s}$ is assumed to be irreducible and $\overline{\mathcal{V}_F}$ is also irreducible by definition, we get $\overline{\mathcal{V}_F} \cong \Pi^0_{n,s} \cong \Pi^0_{n,-s}$. Now we use Theorem 9.2.2 below taking $(-n, -n)$ for λ_0 there. It claims that the K-spectrum of $\mathcal{V}_F$ is multiplicity free and contained in that of $\Pi^0_{n,s}$. Noting $\overline{\mathcal{V}_F} \cong \Pi^0_{n,s}$, we conclude that the K-spectrum of $\mathcal{V}_F$ and that of $\overline{\mathcal{V}_F}$ are same. Thus the natural surjection $\mathcal{V}_F \to \overline{\mathcal{V}_F}$ turns out to be bijective and we finally get an isomorphism $\mathcal{V}_F \cong \Pi^0_{n,s}$. $\square$

Theorem 9.2.2. *Let $\lambda_0 \in L_T^+$. Given a non zero function $F \in C^\infty_{\eta,\tau_{\lambda_0}}(H\backslash G/K)$ that satisfies the following equations with a constant $c_F \in \mathbb{C}$:*

(9.2.3) $$\nabla^{-\beta_1}_{\eta,\lambda_0} F(g) = 0,$$

(9.2.4) $$\nabla^{+\beta_2}_{\eta,\lambda_0} F(g) = 0,$$

(9.2.5) $$\Omega_{\eta,\lambda_0} F(g) = c_F F(g).$$

By means of the standard basis $\{w_i^{\lambda_0}\}$ *of* W_{λ_0}*, write* $F(g) = \sum_{i=0}^{d_{\lambda_0}} F_i(g) \otimes w_i^{\lambda_0}$*,* $g \in G$ *with* $F_i \in C_\eta^\infty(H\backslash G)$*,* $i = 0, \dots, d_{\lambda_0}$*. Let* $\mathcal{V}_F$ *denote the smallest* $(\mathfrak{g}, K)$*-submodule of* $C_\eta^\infty(H\backslash G)$ *which contains* $F_0, \dots, F_{d_{\lambda_0}}$*. Then the* K*-spectrum of* $\mathcal{V}_F$ *is multiplicity free and the highest weight of its arbitrary irreducible* K*-submodule is of the form* $\lambda_0 + n_1\beta_1 + n_2(-\beta_2)$ *with non negative integers* n_1 *and* n_2*.*

The rest of this subsection is devoted to give a proof of this theorem.

For every $\lambda \in L_T^+$, let

$$\mathrm{ev}_\lambda : C^\infty_{\eta,\tau_\lambda}(H\backslash G/K) \otimes W_\lambda^* \to C_\eta^\infty(H\backslash G)$$

denote the natural contraction map. We first have the following

Lemma 9.2.1. *Let* $\lambda \in L_T^+$*, a dominant weight and* $F \in C^\infty_{\eta,\tau_\lambda}(H\backslash G/K)$ *be an arbitrary function. Take* $w^* \in W_\lambda^*$ *and* $X \in \mathfrak{p}$ *and let* w_β^**,* $\beta \in \Sigma_n$ *be the vectors in* $W^*_{\lambda+\beta}$ *determined from* w^* *and* X *by the equations*

$$\big\langle p_\lambda^\beta(w \otimes Y), w_\beta^* \big\rangle = \big\langle w^*, w \big\rangle \big\langle X, Y \big\rangle_{\mathfrak{p}}, \quad w \in W_\lambda,\ Y \in \mathfrak{p}.$$

Then

$$R_X\big(ev_\lambda(F \otimes w^*)\big)(g) = \sum_{\beta \in \Sigma_n} ev_{\lambda+\beta}\big(\nabla_\lambda^\beta F(g) \otimes w_\beta^*\big).$$

Proof. We can prove this by chasing definitions of various maps without any difficulty. □

For every non negative integer m, let $\mathcal{P}(m)$ denote the set of all sequence of non compact roots of length m, i.e.

$$\mathcal{P}(m) = \{I = (\alpha_1, \dots, \alpha_m) |\ \alpha_i \in \Sigma_n\},$$

and set $\mathcal{P} = \cup_{m\in\mathbb{N}}\mathcal{P}(m)$. For a given dominant weight $\lambda \in L_T^+$ and a sequence $I \in \mathcal{P}(m)$, define a sequence $\lambda^{(0)}, \dots, \lambda^{(m)}$ of elements in L_T by setting $\lambda^{(0)} = \lambda$, $\lambda^{(i+1)} = \lambda^{(i)} + \alpha_i$ for $i = 0, \dots, m-1$. Adapting a convention that the symbol $\tau_{\lambda'}$ represent the zero K-module for a non dominant $\lambda' \in L_T$ and $\nabla^{\beta}_{\lambda'}$ represent a zero map if either λ' or $\lambda'+\beta$ is non dominant, we define a linear operator

$$\nabla^I : C^{\infty}_{\eta,\tau_\lambda}(H\backslash G/K) \to C^{\infty}_{\eta,\tau_{\lambda\{I\}}}(H\backslash G/K)$$

as a composite $\nabla^{\alpha_m}_{\eta,\lambda^{(m-1)}} \circ \cdots \circ \nabla^{\alpha_1}_{\eta,\lambda^{(0)}}$, where $\lambda\{I\} = \lambda^{(m)}$.

If a given $I = (\alpha_1, \dots, \alpha_m) \in \mathcal{P}(m)$ is of the form $\alpha_1 = \cdots = \alpha_m = \beta$, then we write $\nabla^{m\beta}$ instead of ∇^I.

Lemma 9.2.2. *Let $\lambda \in L_T^+$ and $F \in C^{\infty}_{\eta,\tau_\lambda}(H\backslash G/K)$. Then we have*

$$\mathcal{V}_F \subset \sum_{I\in\mathcal{P}} ev_{\lambda\{I\}}\big(\nabla^I_\lambda F \otimes W^*_{\lambda\{I\}}\big). \tag{9.2.6}$$

Proof. Set $F_i = \mathrm{ev}_\lambda(F \otimes (w_i^\lambda)^*)$, $i = 0, \dots, d_\lambda$. Let $\mathcal{V}_F^{(m)}$ denote the $\mathbb{C}$-span of functions of the form $R_{X_1\cdots X_q}F_i$ with $i = 0, \dots, d_\lambda$ and $X_1, \dots, X_q \in \mathfrak{p}_{\mathbb{C}}$, $q \leqslant m$. Since the $\mathbb{C}$-span of elements F_i, $i = 0, \dots, d_\lambda$ is invariant under the action of $\mathfrak{k}_{\mathbb{C}}$, and $U(\mathfrak{g}_{\mathbb{C}})$ is a sum of subspaces of the form $X_1 \cdots X_q U(\mathfrak{k}_{\mathbb{C}})$ with $X_i \in \mathfrak{p}$, $\mathcal{V}_F$ coincides with the sum of $\mathcal{V}_F^{(m)}$, $m \in \mathbb{N}$. Using Lemma 9.2.1, we can show by an induction on m that $\mathcal{V}_F^{(m)}$ is contained in the right hand side of (9.2.6). $\square$

Lemma 9.2.3. *Let $\mathcal{Q}(m)$ denote the subset of $\mathcal{P}(m)$ consisting of all $J = (\alpha_i)_{1\leqslant i\leqslant m}$ with $\alpha_i = +\beta_1$ or $-\beta_2$. Take a $J = (\alpha_1, \dots, \alpha_m) \in \mathcal{Q}(m)$ and let n_1 (resp. n_2) be the number of i such that $\alpha_i = +\beta_1$ (resp. $\alpha_i = -\beta_2$). Let $F \in C^{\infty}_{\eta,\tau_\lambda}(H\backslash G/K)$ be an eigen function of $\Omega_{\eta,\lambda}$ with eigen value c_F.*
(1) *We have*

$$\nabla^J F = \nabla^{n_1\beta_1} \circ \nabla^{n_2(-\beta_2)} F = \nabla^{n_2(-\beta_2)} \circ \nabla^{n_1\beta_1} F$$

and $\nabla^J F \in C^{\infty}_{\eta,\tau_{\lambda\{J\}}}(H\backslash G/K)$ is an eigen function of $\Omega_{\eta,\lambda\{J\}}$ with the eigen value c_F.
(2) *There exist constants c'_F and c''_F such that*

$$\nabla^{-\beta_1} \circ \nabla^J F - \nabla^J \circ \nabla^{-\beta_1} F = c'_F \nabla^{n_2(-\beta_2)} \circ \nabla^{(n_1-1)\beta_1} F,$$
$$\nabla^{+\beta_2} \circ \nabla^J F - \nabla^J \circ \nabla^{+\beta_2} F = c''_F \nabla^{n_1\beta_1} \circ \nabla^{(n_2-1)(-\beta_2)} F.$$

Proof. This can be easily proved using Theorem 4.5.1 by an induction on the length of J. $\square$

Lemma 9.2.4. *Let F be as in Theorem 9.2.2. Take an arbitrary $I = (\alpha_1, \dots, \alpha_m) \in \mathcal{P}(m)$. Then $\nabla^I F$ can be expressed as a constant multiple of a function of the form $\nabla^{n_2(-\beta_2)} \circ \nabla^{n_1\beta_1} F$ with non negative integers n_1 and n_2.*

Proof. If $m \leqslant 1$, then the statement is true because of the differential equation (9.2.3) and (9.2.4) for F. Assume that $m > 1$ and the statement is true for any sequence I with length smaller than m. Let $I = (\alpha_1, \dots, \alpha_m) \in \mathcal{P}(m)$ and i denotes the smallest integer i such that $\alpha_i = -\beta_1$ or $+\beta_2$ when such i exists and $i = +\infty$ otherwise. If $i = +\infty$, i.e. there appears neither $-\beta_1$ nor $+\beta_2$ in I, then $I \in \mathcal{Q}(m)$ and by the Lemma 9.2.3 **(1)** the statement is true. Now suppose $i \leqslant m$. We can write $I = (J, \beta, I')$ with $J = (\alpha_1, \dots, \alpha_{i-1})$, $\beta = \alpha_i$ and $I' = (\alpha_{i+1} \dots, \alpha_m)$ and correspondingly

$$\nabla^I F = \nabla^{I'} \circ \nabla^\beta \circ \nabla^J F.$$

By the definition of i, we see that $J \in \mathcal{Q}(i-1)$. Let n_1 (resp. n_2) denote the number of appearances of $+\beta_1$ (resp. $-\beta_2$) in J. Now applying Lemma 9.2.3 **(2)**, there exists a constant c such that

$$\nabla^\beta \circ \nabla^J F = \nabla^J \circ \nabla^\beta F + c\nabla^{n_1'\beta_1} \circ \nabla^{n_2'(-\beta_2)} F$$

with $(n_1', n_2') = (n_1 - 1, n_2)$ or $(n_1, n_2 - 1)$ depending on $\beta = -\beta_1$ or β_2. Hence we get

$$\nabla^I F = \nabla^{I'} \circ \nabla^J \circ \nabla^\beta F + c\nabla^{I'} \circ \nabla^{n_1'\beta_1} \circ \nabla^{n_2'(-\beta_2)} F.$$

As a result of (9.2.3) and (9.2.4), the first term in the right hand side of the above identity is zero because $\beta = -\beta_1$ or $+\beta_2$. Thus we have $\nabla^I F = c\nabla^{(J',I')} F$ for $J' = (-\beta_2, \dots, -\beta_2, +\beta_1, \dots, +\beta_1)$ with β_1 appearing n_1' times and $-\beta_2$ appearing n_2' times. Since the length of (I', J') is smaller than that of I, the statement is now established for I because of the induction assumption. $\square$

By Lemma 9.2.3 and Lemma 9.2.4 we have

$$\mathcal{V}_F \subset \sum_{n_1,n_2\in\mathbb{Z}_+} \mathrm{ev}_{\lambda_0+n_1\beta_1+n_2(-\beta_2)}(\nabla^{n_1\beta_1}\circ\nabla^{n_2(-\beta_2)}F\otimes W^*_{\lambda_0+n_1\beta_1+n_2(-\beta_2)}).$$

This completes the proof of Theorem 9.2.2.

(9.3) Radial part of principal series Shintani functions

Let $\Pi_{n,s}$ be the principal series representation of G with $(n,s)\in\mathbb{Z}\times\mathbb{C}$ and $(\eta,\mathcal{F}_\eta)$ an irreducible admissible representation of H. Now we state the main result of this section.

Theorem 9.3.1. *Let η be an irreducible admissible representation of H realized as a submodule of $\eta_{\gamma,\nu}$ ($\gamma\in\mathbb{Z}$, $\nu\in\mathbb{C}$) and let $\{v_m|\ m\in L_\eta\}$ be its standard basis. Let $(\Pi_{n,s},\mathcal{F}_{n,s+2})$ be a non unitary principal series representation of G with $(n,s)\in\mathbb{Z}\times\mathbb{C}$. We identify the contragredient of $\Pi_{n,s}$ with $\Pi_{-n,-s}$ through the pairing (9.1.0). Let $\{f_i^{\lambda^*}|\ \lambda\in L(n),\ 0\leqslant i\leqslant d_\lambda\}$ be the basis of $\mathcal{F}^0_{-n,-s+2}$ constructed in Lemma 9.1.2. Here we set $\lambda^*=(-l_2,-l_1)$ for a given $\lambda=(l_1,l_2)\in L_T$.*

(1) *Then $\dfrac{\gamma-2n}{3}\in L_\eta$ is a necessary and sufficient condition for $\mathcal{I}_{\eta,\Pi_{n,s}}\neq\{0\}$. Under this condition, there exists a unique $\Phi_0\in\mathcal{I}_{\eta,\Pi_{n,s}}$ such that*

$$\begin{aligned}(9.3.1)\qquad &\Phi_0(f_0^{(n,n)})(a_r)\\ =&\Big(\frac{r-r^{-1}}{2}\Big)^{|\beta_{n,\gamma}|}\Big(\frac{r+r^{-1}}{2}\Big)^{s-2-|\beta_{n,\gamma}|}\\ &\times F\left(\frac{-s-\nu+|\beta_{n,\gamma}|+1}{2},\frac{-s+\nu+|\beta_{n,\gamma}|+1}{2}\ ;\right.\\ &\qquad\left. 1+|\beta_{n,\gamma}|,\Big(\frac{r-r^{-1}}{r+r^{-1}}\Big)^2\right)v_{\beta_{n,\gamma}-n}\end{aligned}$$

with $\beta_{n,\gamma}=3^{-1}(n+\gamma)$.

Proof. Since $3^{-1}(\gamma-2n)\in L_\eta$ is a necessary and sufficient condition for $C^\infty_{W_0}(A\ ;\mathcal{F}_\eta\otimes W_{(-n,-n)})\neq\{0\}$ (Lemma 6.1.1), we assume this in what follows and show that $\dim_{\mathbb{C}}\mathcal{I}_{\eta,\Pi}=1$. Every $F\in\mathcal{S}_{\eta,\Pi_{n,s}}(\tau_{(-n,-n)})$ satisfies

$$(9.3.2)\qquad \Omega_{\eta,(n,n)}F(g)=(\frac{1}{2}s^2+\frac{1}{6}n^2-2)F(g).$$

From this we get a differential equation of A-radial part $F|A$ by using Proposition 5.3.1. The function given by (9.3.1) is a C^∞-solution of this equation that is unique up to a constant multiple. By Theorem 9.2.1, the space $\mathcal{I}_{\eta,\Pi_{n,s}}$ is isomorphic to the C^∞-solution space of the differential equation (9.3.2) with unknown function $F \in C^\infty_{\eta,\tau_{(-n,-n)}}(H\backslash G/K)$. Thus $\dim_{\mathbb{C}}\mathcal{I}_{\eta,\Pi} = 1$. $\square$

Remark. Theorem 9.3.1 gives an explicit formula of principal series Shintani function with one dimensional K-type. As for Shintani functions with more general K-type, we can get their explicit form by applying the formulas (9.3.4) and (9.3.5) below sucsessively. For every $\lambda \in L(n)$, set

$$(9.3.3) \qquad F^\lambda(g) = \sum_{i=0}^{d_\lambda}(-1)^i \frac{i!(d_\lambda - i)!}{d_\lambda!}\Phi_0(f^{\lambda^*}_{d_\lambda - i})(g) \otimes w_i^\lambda, \quad g \in G.$$

Then $F^\lambda : G \to \mathcal{F}_\eta \otimes W_\lambda$ gives a basis of $\mathcal{S}_{\eta,\Pi}(\lambda)$ and the family $\{F^\lambda |\ \lambda \in L(n)\}$ satisfies

$$(9.3.4) \qquad a_i^+(\lambda)F^{\lambda+\beta_i}(g) = \nabla^{\beta_i}_{\eta,\lambda}F^\lambda(g),$$

$$(9.3.5) \qquad a_i^-(\lambda)F^{\lambda-\beta_i}(g) = \nabla^{-\beta_i}_{\eta,\lambda}F^\lambda(g)$$

for every $\lambda \in L(n)$ and $i = 1, 2$. Here $a_i^+(\lambda), a_i^-(\lambda),\ i = 1, 2$ are given by

$$a_1^+(\lambda) = \frac{s+n+2l_2+2}{2(d_\lambda+1)}, \quad a_2^+(\lambda) = \frac{(\gamma_\lambda - d_\lambda)(s+n+2l_1+4)}{2(d_\lambda+1)},$$
$$a_1^-(\lambda) = \frac{\gamma_\lambda(s-n-2l_2+4)}{2(d_\lambda+1)}, \quad a_2^-(\lambda) = \frac{s-n-2l_1+2}{2(d_\lambda+1)}.$$

Appendix 1

In this appendix, we shall give an explicit formula of matrix coefficients of discrete series representations of G. Let (Π, H) be a discrete series representation of G with Harish-Chandra parameter $\Lambda \in \Xi$. Let $(\tau_\lambda, W_\lambda)$ $(\lambda = (l_1, l_2) \in L_T^+)$ be the minimal K-type of Π. Then the K-module $\Pi|K$ contains τ_λ exactly once. We fix a K-embedding $\tau_\lambda \to \Pi|K$

such that the standard basis $\{w_i^\lambda\}$ becomes an orthonormal system in H. Set

$$c_{ij}(g) = \langle w_i^\lambda, \Pi(g) w_j^\lambda \rangle \ (g \in G, 0 \leqslant i, j \leqslant d_\lambda),$$

and

$$\Phi_\lambda(g) = \sum_{i,j=0}^{d_\lambda} c_{ij}(g)(w_i^\lambda)^* \otimes w_j^\lambda \ (g \in G),$$

where $\{(w_i^\lambda)^*\}$ is the dual basis of $\{w_i^\lambda\}$ and $\langle \ , \ \rangle$ is the Hermitian inner product of the Hilbert space H. The functions c_{ij} $(0 \leqslant i, j \leqslant d_\lambda)$ or Φ_λ are called matrix coefficients of Π with minimal K-type. It is easily checked that

$$\Phi_\lambda(k'gk) = (\tau_\lambda^*(k') \otimes \tau_\lambda(k^{-1}))\Phi_\lambda(g)$$

for every $k', k \in K$ and $g \in G$. Since we have the Cartan decomposition $G = KAK$, A-radial part $\Phi_\lambda | A$ completely determines Φ_λ.

Theorem A.1.1. *The radial part $\Phi_\lambda | A$ is given as follows:*
$c_{ij}|A = 0$ unless $i = j$.
If $\Lambda \in \Xi_I$, then

$$c_{ii}(a_r) = (\frac{r + r^{-1}}{2})^{-l_2 - i}.$$

If $\Lambda \in \Xi_{II}$, then

$$c_{ii}(a_r) = (\frac{r + r^{-1}}{2})^{l_2 - i - 2} F\Big(i+1, -l_2+1 \ ; d_\lambda + 2 \ ; (\frac{r - r^{-1}}{r + r^{-1}})^2\Big).$$

If $\Lambda \in \Xi_{III}$, then

$$c_{ii}(a_r) = (\frac{r + r^{-1}}{2})^{l_2 + i}.$$

Proof. Since the computation can be carried out in exactly the same manner with the case of Shintani functions, we shall give an outline briefly. The $W_\lambda^* \otimes W_\lambda$-valued function Φ_λ satisfies the equation $\mathcal{D}_{\tau_\lambda^*, \tau_\lambda} \Phi_\lambda = 0$, where $\mathcal{D}_{\tau_\lambda^*, \tau_\lambda}$ is the Schmid operator defined in §5. We first calculate the A-radial part of $\mathcal{D}_{\tau_\lambda^*, \tau_\lambda}$ using the Cartan decomposition $G = KAK$ in this case. By solving the radial part of the differential equations thus obtained, we have the result. $\square$

Appendix 2

In this appendix H stands for an arbitrary closed subgroup of G and $(\eta, \mathcal{F}_\eta)$ an arbitrary bounded Hilbert representation of H.

The aim of this appendix is to give a proof of Theorem 4.5.1. For that purpose, we introduce operators $\mathcal{T}_\lambda$, $\mathcal{S}_\lambda$ and $\mathcal{A}_\lambda$ associated to $T^2(\mathfrak{p}) = \mathfrak{p} \otimes_{\mathbb{R}} \mathfrak{p}$, $\mathrm{Sym}^2(\mathfrak{p})$ and $\mathrm{Alt}^2(\mathfrak{p})$ respectively, where $\mathrm{Sym}^2(\mathfrak{p})$ denotes the space of symmetric tensors of degree 2 over $\mathfrak{p}$ and $\mathrm{Alt}^2(\mathfrak{p})$ the space of alternating tensors of degree 2 over $\mathfrak{p}$. For $X,\ Y \in \mathfrak{p}$, the image of $X \otimes Y \in \mathrm{T}^2(\mathfrak{p})$ under the natural surjection $\mathrm{T}^2(\mathfrak{p}) \to \mathrm{Sym}^2(\mathfrak{p})$ (resp. $\mathrm{T}^2(\mathfrak{p}) \to \mathrm{Alt}^2(\mathfrak{p})$) is denoted by $X \cdot Y$ (resp. $X \wedge Y$). Let $j^s_{\mathfrak{p}} : \mathrm{Sym}^2(\mathfrak{p}) \to \mathrm{T}^2(\mathfrak{p})$ and $j^a_{\mathfrak{p}} : \mathrm{Alt}^2(\mathfrak{p}) \to \mathrm{T}^2(\mathfrak{p})$ stand for the natural maps such that

$$j^s_{\mathfrak{p}}(X \cdot Y) = \frac{1}{2}(X \otimes Y + Y \otimes X), \quad X \in \mathfrak{p},\ Y \in \mathfrak{p},$$
$$j^a_{\mathfrak{p}}(X \wedge Y) = \frac{1}{2}(X \otimes Y - Y \otimes X), \quad X \in \mathfrak{p},\ Y \in \mathfrak{p}.$$

Obviously $j^s_{\mathfrak{p}}$ (resp. $j^a_{\mathfrak{p}}$) provides a section of the natural surjection $\mathrm{T}^2(\mathfrak{p}) \to \mathrm{Sym}^2(\mathfrak{p})$ (resp. $\mathrm{T}^2(\mathfrak{p}) \to \mathrm{Alt}^2(\mathfrak{p})$); $j^s_{\mathfrak{p}}$ and $j^a_{\mathfrak{p}}$ are injective and $j^s_{\mathfrak{p}} \oplus j^s_{\mathfrak{p}}$ gives a K-isomorphism from $\mathrm{Sym}^2(\mathfrak{p}) \oplus \mathrm{Alt}^2(\mathfrak{p})$ onto $\mathrm{T}^2(\mathfrak{p})$. Let $\langle\ ,\ \rangle_{\mathrm{T}^2(\mathfrak{p})}$ denote the inner product of $\mathrm{T}^2(\mathfrak{p})$ induced by $\langle\ ,\ \rangle_{\mathfrak{p}}$, i.e.

$$\langle X \otimes Y, X' \otimes Y' \rangle_{\mathrm{T}^2(\mathfrak{p})} = \langle X, X' \rangle_{\mathfrak{p}} \langle Y, Y' \rangle_{\mathfrak{p}}, \quad X,\ Y,\ X',\ Y' \in \mathfrak{p}.$$

If $\{X_i\}_{i=1}^4$ is an orthonormal basis of $\mathfrak{p}$, then

$$X_i \otimes X_j \quad (1 \leqslant i,\ j \leqslant 4) \tag{A.2.0}$$

provides an orthonormal basis of $\mathrm{T}^2(\mathfrak{p})$.

By means of the injections $j^s_{\mathfrak{p}}$ and $j^a_{\mathfrak{p}}$, we can induce inner products $\langle\ ,\ \rangle_{\mathrm{Sym}^2(\mathfrak{p})}$ of $\mathrm{Sym}^2(\mathfrak{p})$ and $\langle\ ,\ \rangle_{\mathrm{Alt}^2(\mathfrak{p})}$ of $\mathrm{Alt}^2(\mathfrak{p})$ from $\langle\ ,\ \rangle_{\mathrm{T}^2(\mathfrak{p})}$. It is easy to see that

$$\langle X \cdot Y, X' \cdot Y' \rangle_{\mathrm{Sym}^2(\mathfrak{p})} = \frac{1}{2}\big(\langle X, X' \rangle_{\mathfrak{p}} \langle Y, Y' \rangle_{\mathfrak{p}} + \langle X, Y' \rangle_{\mathfrak{p}} \langle X', Y \rangle_{\mathfrak{p}}\big),$$
$$\langle X \wedge Y, X' \wedge Y' \rangle_{\mathrm{Alt}^2(\mathfrak{p})} = \frac{1}{2}\big(\langle X, X' \rangle_{\mathfrak{p}} \langle Y, Y' \rangle_{\mathfrak{p}} - \langle X, Y' \rangle_{\mathfrak{p}} \langle X', Y \rangle_{\mathfrak{p}}\big),$$

$$X,\ Y,\ X',\ Y' \in \mathfrak{p}.$$

Note that if $\{X_i\}_{i=1}^4$ is an orthonormal basis of $\mathfrak{p}$, then

$$X_i \cdot X_i\ (1 \leqslant i \leqslant 4), \quad \sqrt{2} X_i \cdot X_j\ (1 \leqslant i < j \leqslant 4) \tag{A.2.1}$$

provide orthonormal basis of $\mathrm{Sym}^2(\mathfrak{p})$ and

$$\sqrt{2} X_i \wedge X_j\ (1 \leqslant i < j \leqslant 4) \tag{A.2.2}$$

provides that of $\mathrm{Alt}^2(\mathfrak{p})$.

Let $\psi : \mathrm{T}^2(\mathfrak{p}) \to U(\mathfrak{g}_{\mathbb{C}})$ denote the map which sends each element $X \otimes Y$ of $\mathrm{T}^2(\mathfrak{p})$ to a degree 2 element YX of $U(\mathfrak{g}_{\mathbb{C}})$. It is easy to see that ψ is a K-homomorphism, where we consider $U(\mathfrak{g}_{\mathbb{C}})$ as a K-module by the action induced from the adjoint action of K on $\mathfrak{g}_{\mathbb{C}}$. Now we set

$$\mathcal{T}_\lambda F(g) = \sum_k R_{\psi(x_k)} F(g) \otimes x_k, \quad F \in C^\infty_{\eta,\tau_\lambda}(H\backslash G/K) \tag{A.2.3}$$

with $\{x_k\}$ an orthonormal basis of $\mathrm{T}^2(\mathfrak{p})$. Then the right hand side of the above identity does not depend on the choice of $\{x_k\}$ and $\mathcal{F}_\eta \otimes_{\mathbb{C}} W_\lambda \otimes_{\mathbb{R}} T^2(\mathfrak{p})$-valued C^∞-function $g \to \mathcal{T}_\lambda F(g)$ belongs to $C^\infty_{\eta,\tau_\lambda \otimes \mathrm{T}^2(\mathfrak{p}_{\mathbb{C}})}(H\backslash G/K)$. Thus we get an operator

$$\mathcal{T}_\lambda : C^\infty_{\eta,\tau_\lambda}(H\backslash G/K) \to C^\infty_{\eta,\tau_\lambda \otimes \mathrm{T}^2(\mathfrak{p}_{\mathbb{C}})}(H\backslash G/K).$$

In the same way, by setting

$$\mathcal{S}_\lambda F(g) = \sum_i R_{\psi \circ j^s_{\mathfrak{p}}(z_i)} F(g) \otimes z_i, \quad F \in C^\infty_{\eta,\tau_\lambda}(H\backslash G/K), \tag{A.2.4}$$

$$\mathcal{A}_\lambda F(g) = \sum_j R_{\psi \circ j^a_{\mathfrak{p}}(y_j)} F(g) \otimes y_j, \quad F \in C^\infty_{\eta,\tau_\lambda}(H\backslash G/K) \tag{A.2.5}$$

with $\{z_i\}$ and $\{y_j\}$ orthonormal basis of $\mathrm{Sym}^2(\mathfrak{p})$ and $\mathrm{Alt}^2(\mathfrak{p})$ respectively, we get operators

$$\mathcal{S}_\lambda : C^\infty_{\eta,\tau_\lambda}(H\backslash G/K) \to C^\infty_{\eta,\tau_\lambda \otimes \mathrm{Sym}^2(\mathfrak{p}_{\mathbb{C}})}(H\backslash G/K),$$

$$\mathcal{A}_\lambda : C^\infty_{\eta,\tau_\lambda}(H\backslash G/K) \to C^\infty_{\eta,\tau_\lambda\otimes \mathrm{Alt}^2(\mathfrak{p}_\mathbb{C})}(H\backslash G/K)$$

respectively.

LEMMA A.2.1. *Let $\{X_i\}_{i=1}^4$ be an orthonormal basis of $\mathfrak{p}$. Then for every $F \in C^\infty_{\eta,\tau_\lambda}(H\backslash G/K)$, we have*

$$\text{(A.2.6)} \quad \mathcal{T}_\lambda F(g) = \sum_{i,j=1}^{4} R_{X_i}R_{X_j}F(g)\otimes (X_j\otimes X_i),$$

$$\text{(A.2.7)} \quad \mathcal{S}_\lambda F(g) = \sum_{i=1}^{4} R_{X_i}R_{X_i}F(g)\otimes (X_i\cdot X_i) + \sum_{1\leqslant i<j\leqslant 4}(R_{X_i}R_{X_j}F(g) + R_{X_j}R_{X_i}F(g))\otimes (X_i\cdot X_j),$$

$$\text{(A.2.8)} \quad \mathcal{A}_\lambda F(g) = \sum_{1\leqslant i<j\leqslant 4} R_{[X_i,X_j]}F(g)\otimes (X_j\wedge X_i)$$

and

$$\text{(A.2.9)} \quad \mathcal{T}_\lambda F(g) = (1_{\mathcal{F}_\eta\otimes W_\lambda}\otimes j^s_\mathfrak{p})\mathcal{S}_\lambda F(g) + (1_{\mathcal{F}_\eta\otimes W_\lambda}\otimes j^a_\mathfrak{p})\mathcal{A}_\lambda F(g).$$

PROOF. The formula (A.2.9) follows from the definitions (A.2.3), (A.2.4) and(A.2.5), because $j^s_\mathfrak{p}\oplus j^a_\mathfrak{p}$ gives an isometrical isomorphism from $\mathrm{Sym}^2(\mathfrak{p})\oplus \mathrm{Alt}^2(\mathfrak{p})$ onto $\mathrm{T}^2(\mathfrak{p})$. We can easily prove the formulas (A.2.6), (A.2.7) and (A.2.8) by direct computations taking orthonormal basis (A.2.0), (A.2.1) and (A.2.2) for $\{x_k\}$, $\{z_i\}$ and $\{y_j\}$ respectly. □

For a given $\lambda\in L^+_T$, let

$$j^s_\lambda : W_\lambda\otimes_\mathbb{R}\mathrm{Sym}^2(\mathfrak{p}) \to W_\lambda\otimes_\mathbb{R}\mathrm{T}^2(\mathfrak{p}),$$
$$j^a_\lambda : W_\lambda\otimes_\mathbb{R}\mathrm{Alt}^2(\mathfrak{p}) \to W_\lambda\otimes_\mathbb{R}\mathrm{T}^2(\mathfrak{p}),$$
$$\kappa_\lambda : W_\lambda\otimes_\mathbb{R}\mathrm{T}^2(\mathfrak{p}) \to W_\lambda$$

denote the natural maps defined by

$$\text{(A.2.10)} \qquad j^s_\lambda(w\otimes X\cdot Y) = w\otimes j^s_\mathfrak{p}(X\cdot Y),$$

$$j_\lambda^a(w\otimes X\wedge Y)=w\otimes j_{\mathfrak p}^a(X\wedge Y), \tag{A.2.11}$$

$$\kappa_\lambda(w\otimes X\otimes Y)=\frac{1}{2}\langle X,Y\rangle_{\mathfrak p}w \tag{A.2.12}$$

for $w\in W_\lambda,\ X,\ Y\in\mathfrak p$. These are obviously K-homomorphisms.

For a given pair of non compact roots (β,β') and a given dominant weight $\lambda\in L_T^+$, define a K-homomorphism $\pi_\lambda^{\beta,\beta'}: W_\lambda\otimes_{\mathbb R}\mathrm T^2(\mathfrak p)\to W_{\lambda+\beta+\beta'}$ by

$$\pi_\lambda^{\beta,\beta'}=p_{\lambda+\beta'}^{\beta}\circ(p_\lambda^{\beta'}\otimes 1_{\mathfrak p}), \tag{A.2.13}$$

where for a non compact root β and a dominant weight λ, $p_\lambda^\beta: W_\lambda\otimes_{\mathbb R}\mathfrak p\to W_{\lambda+\beta}$ denotes the K-projector specified in Proposition 2.2.1.

Lemma A.2.2. *Let $\lambda\in L_T^+$, a dominant weight and (β,β') a pair of non compact root.*
(1) *If $\beta\neq-\beta'$, then in the space $Hom_K(\tau_\lambda\otimes_{\mathbb R} Sym^2(\mathfrak p),\tau_{\lambda+\beta+\beta'})$, the following identity holds:*

$$\pi_\lambda^{\beta,\beta'}\circ j_\lambda^s=\pi_\lambda^{\beta',\beta}\circ j_\lambda^s. \tag{A.2.14; β,β'}$$

(2) *For $\beta=\beta_1$ or β_2, the following identity holds in the space $Hom_K(\tau_\lambda\otimes_{\mathbb R} Sym^2(\mathfrak p),\tau_\lambda)$:*

$$\pi_\lambda^{-\beta,\beta}\circ j_\lambda^s-\pi_\lambda^{\beta,-\beta}\circ j_\lambda^s=\kappa_\lambda\circ j_\lambda^s. \tag{A.2.15; β}$$

Proof. We only give a proof of (A.2.14; β_1,β_2) since the remaining identities can be proved quite similarly. Let $\{w_i^\lambda\}$ be the standard basis of W_λ. Then

$$\begin{gathered}
w_i^\lambda\otimes(X_{13}\cdot X_{13}),\ w_i^\lambda\otimes(X_{13}\cdot X_{23}),\ w_i^\lambda\otimes(X_{23}\cdot X_{23}),\\
w_\lambda\otimes(X_{13}\cdot X_{31}),\ w_i^\lambda\otimes(X_{13}\cdot X_{32}),\ w_i^\lambda\otimes(X_{23}\cdot X_{31}),\ w_i^\lambda\otimes(X_{23}\cdot X_{32}),\\
w_i^\lambda\otimes(X_{31}\cdot X_{31}),\ w_i^\lambda\otimes(X_{31}\cdot X_{32}),\ w_i^\lambda\otimes(X_{32}\cdot X_{32})
\end{gathered}$$

with $i=0,\dots,d_\lambda$ provides a basis of the $\mathbb C$-vector space $W_\lambda\otimes_{\mathbb R}\mathrm{Sym}^2(\mathfrak p)=W_\lambda\otimes_{\mathbb C}\mathrm{Sym}^2(\mathfrak p)_{\mathbb C}$. It suffices to check that $\pi_\lambda^{\beta_1,\beta_2}\circ j_\lambda^s$ and $\pi_\lambda^{\beta_2,\beta_1}\circ j_\lambda^s$ take the same value on the basis above. By using Proposition 2.2.1 and the definition (A.2.10) of the map j^s_λ, we easily prove

$$\begin{aligned}
\pi^{\beta_1,\beta_2}_\lambda \circ j^s_\lambda(w^\lambda_i \otimes (X_{13}\cdot X_{13})) &= \pi^{\beta_2,\beta_1}_\lambda \circ j^s_\lambda(w^\lambda_i \otimes (X_{13}\cdot X_{13})) \\
&= -(i+1)w^{\lambda'}_{i+1}, \\
\pi^{\beta_1,\beta_2}_\lambda \circ j^s_\lambda(w^\lambda_i \otimes (X_{13}\cdot X_{23})) &= \pi^{\beta_2,\beta_1}_\lambda \circ j^s_\lambda(w^\lambda_i \otimes (X_{13}\cdot X_{23})) \\
&= \frac{1}{2}(2i-d_\lambda)w^{\lambda'}_i, \\
\pi^{\beta_1,\beta_2}_\lambda \circ j^s_\lambda(w^\lambda_i \otimes (X_{23}\cdot X_{23})) &= \pi^{\beta_2,\beta_1}_\lambda \circ j^s_\lambda(w^\lambda_i \otimes (X_{23}\cdot X_{23})) \\
&= (d_\lambda - i + 1)w^{\lambda'}_{i-1}
\end{aligned}$$

with $\lambda' = \lambda + \beta_1 + \beta_2$ and the remaining values are all zero. Thus we have done. □

LEMMA A.2.3. *Let $F \in C^\infty_{\eta,\tau_\lambda}(H\backslash G/K)$. Then for every pair of non compact rooots (β,β'), we have*

$$\nabla^{\beta}_{\eta,\lambda+\beta'} \circ \nabla^{\beta'}_{\eta,\lambda} F(g) = \big(1_{\mathcal{F}_\eta} \otimes \pi^{\beta,\beta'}_\lambda\big)\mathcal{T}_\lambda F(g). \tag{A.2.16}$$

PROOF. Take an orthonormal basis $\{X_i\}_{i=1}^4$ of $\mathfrak{p}$. By definition we have

$$\begin{aligned}
\nabla^{\beta'}_{\eta,\lambda}F(g) &= (1_{\mathcal{F}_\eta} \otimes p^{\beta'}_\lambda)\nabla_{\eta,\lambda}F(g), \\
\nabla_{\eta,\lambda}F(g) &= \sum_{j=1}^4 R_{X_j}F(g) \otimes X_j
\end{aligned}$$

hence

$$\begin{aligned}
\nabla_{\eta,\lambda+\beta'} \circ \nabla^{\beta'}_{\eta,\lambda}F(g) &= \sum_{i=1}^4 R_{X_i}[(1_{\mathcal{F}_\eta} \otimes p^{\beta'}_\lambda)\nabla_{\eta,\lambda}F](g) \otimes X_i \\
&= (1_{\mathcal{F}_\eta} \otimes p^{\beta'}_\lambda \otimes 1_{\mathfrak{p}}) \sum_{i=1}^4 R_{X_i}\nabla_{\eta,\lambda}F(g) \otimes X_i \\
&= (1_{\mathcal{F}_\eta} \otimes p^{\beta'}_\lambda \otimes 1_{\mathfrak{p}}) \sum_{i=1}^4 R_{X_i}(\sum_{j=1}^4 R_{X_j}F(g) \otimes X_j) \otimes X_i
\end{aligned}$$

$$= (1_{\mathcal{F}_\eta} \otimes p_\lambda^{\beta'} \otimes 1_{\mathfrak{p}})(\mathcal{T}_\lambda F(g)),$$

where we have used (A.2.6) in the last step. Noting that

$$\nabla_{\eta,\lambda+\beta'}^{\beta} \circ \nabla_{\eta,\lambda}^{\beta'} F(g) = (1_{\mathcal{F}_\eta} \otimes p_{\lambda+\beta'}^{\beta}) \nabla_{\eta,\lambda+\beta'} \circ \nabla_{\eta,\lambda}^{\beta'} F(g)$$

we have done. □

We define a K-homomorphism $\gamma : \mathrm{Alt}^2(\mathfrak{p}) \to \mathfrak{k}$ by

$$\gamma : \mathrm{Alt}^2(\mathfrak{p}) \to \mathfrak{k}, \quad X \wedge Y \to \frac{1}{2}[Y, X].$$

For every dominant weight $\lambda \in L_T^+$, set

$$\iota_\lambda(w) = \sum_j \tau_\lambda(\gamma(y_j)) w \otimes y_j, \quad w \in W_\lambda$$

with $\{y_j\}$ an orthonormal basis of $\mathrm{Alt}^2(\mathfrak{p})$. Then we can prove that the right hand side of the above identity is independent of the choice of $\{y_j\}$ and get a K-equivariant linear map $\iota_\lambda : W_\lambda \to W_\lambda \otimes_{\mathbb{R}} \mathrm{Alt}^2(\mathfrak{p})$.

Lemma A.2.4. *Let $\lambda = (l_1, l_2) \in L_T^+$, a dominant weight and $F \in C_{\eta,\tau_\lambda}^\infty(H\backslash G/K)$.*
(1) *We have*

$$\mathcal{A}_\lambda F(g) = (1_{\mathcal{F}_\eta} \otimes \iota_\lambda) F(g), \quad g \in G. \tag{A.2.17}$$

(2) *For every pair of noncompact roots (β, β') such that $\beta \neq -\beta'$, we have*

$$[1_{\mathcal{F}_\eta} \otimes (\pi_\lambda^{\beta,\beta'} \circ j_\lambda^a)] \mathcal{A}_\lambda F(g) = 0, \tag{A.2.18; β,β'}$$

(3) *For $\beta = \beta_1$ or $-\beta_2$, we have*

$$[1_{\mathcal{F}_\eta} \otimes (\pi_\lambda^{\beta,-\beta} \circ j_\lambda^a)] \mathcal{A}_\lambda F(g) = -\frac{1}{2}(d_\lambda + d_\lambda l_1 + c) F(g), \tag{A.2.19; β}$$

$$[1_{\mathcal{F}_\eta} \otimes (\pi_\lambda^{-\beta,\beta} \circ j_\lambda^a)] \mathcal{A}_\lambda F(g) = \frac{1}{2}(l_1 + l_2 + d_\lambda + d_\lambda l_1 + c) F(g). \tag{A.2.20; β}$$

Here $c = -\delta_{\beta,-\beta_2} d_\lambda(d_\lambda + 2)$.

PROOF. **(1)** Let $\{y_\nu\}$ be an orthonormal basis of $\mathrm{Alt}^2(\mathfrak{p})$. By the definition (A.2.5), we have

$$\mathcal{A}_\lambda F(g) = \sum_\nu R_{\psi\circ j^a_{\mathfrak{p}}(y_\nu)} F(g) \otimes y_\nu.$$

Since $\psi \circ j^a_{\mathfrak{p}}(y_\nu) = \gamma(y_\nu) \in \mathfrak{k}(\subset U(\mathfrak{g}_{\mathbb{C}}))$, the K-equivariant property of F means $R_{\psi\circ j^a_{\mathfrak{p}}(y_\nu)} F(g) = \tau_\lambda(\gamma(y_\nu))F(g)$, thus the above identity can be rewritten as

$$\mathcal{A}_\lambda F(g) = \sum_\nu \tau_\lambda(\gamma(y_\nu))F(g) \otimes y_\nu. \tag{A.2.21}$$

Now by means of the standard basis $\{w^\lambda_i\}$ of W_λ, we can write $F(g)$ as

$$F(g) = \sum_{i=0}^{d_\lambda} F_i(g) \otimes w^\lambda_i, \qquad g \in G$$

with $F_i \in C_\eta(H\backslash G)$ for $i = 0, \ldots, d_\lambda$. Substituting this for $F(g)$ in (A.2.21), we have

$$\begin{aligned}\mathcal{A}_\lambda F(g) &= \sum_i F_i(g) \otimes \sum_\nu \tau_\lambda(\gamma(y_\nu))w^\lambda_i \otimes y_\nu \\ &= \sum_i F_i(g) \otimes \iota_\lambda(w^\lambda_i) \\ &= (1_{\mathcal{F}_\eta} \otimes \iota_\lambda)F(g).\end{aligned}$$

(2) Since the map $\pi^{\beta,\beta'}_\lambda \circ j^a_\lambda \circ \iota_\lambda$ provides a K-homomorphism from τ_λ to $\tau_{\lambda+\beta+\beta'}$, it must be zero if $\beta + \beta' \neq 0$. Thus using (A.2.17), we have

$$[1_{\mathcal{F}_\eta} \otimes \pi^{\beta,\beta'}_\lambda \circ j^a_\lambda]\mathcal{A}_\lambda F(g) = [1_{\mathcal{F}_\eta} \otimes \pi^{\beta,\beta'}_\lambda \circ j^a_\lambda \circ \iota_\lambda]F(g) = 0.$$

(3) Let $\mathfrak{q}$ denote the orthogonal complement of $\ker(\gamma)$ with respect to $\langle\ ,\ \rangle_{\mathrm{Alt}^2(\mathfrak{p})}$. Then $\mathfrak{q}$ has an orthonormal basis given by

$$\sqrt{-2}(X_{31} \wedge X_{13}), \qquad \sqrt{-2}(X_{32} \wedge X_{23}),$$

$$X_{13} \wedge X_{32} - X_{23} \wedge X_{31}, \qquad \sqrt{-1}(X_{13} \wedge X_{32} + X_{23} \wedge X_{31}).$$

In the definition of the map ι_λ we can take an orthonormal basis $\{y_j\}$ of $\mathrm{Alt}^2(\mathfrak{p})$ which contains the above basis of $\mathfrak{q}$ as a subset. As a result, we have

$$\begin{aligned}\iota_\lambda(w) = &-\tau_\lambda(H'_{13})w \otimes (X_{31} \wedge X_{13}) - \tau_\lambda(H'_{23})w \otimes (X_{32} \wedge X_{23})\\ &- \tau_\lambda(X_{21})w \otimes (X_{13} \wedge X_{32}) + \tau_\lambda(X_{12})w \otimes (X_{23} \wedge X_{31})\end{aligned}$$

after some computations. Taking w_i^λ for w and using (2.1.1), we have

$$\begin{aligned}\iota_\lambda(w_i^\lambda) = -\,&\{(i+l_2)w_i^\lambda \otimes (X_{31} \wedge X_{13})\\ &- (i-l_1)w_i^\lambda \otimes (X_{32} \wedge X_{23})\\ &+ (i-d_\lambda-1)w_{i-1}^\lambda \otimes (X_{13} \wedge X_{32})\\ &- (i+1)w_{i+1}^\lambda \otimes (X_{23} \wedge X_{31})\}.\end{aligned} \tag{A.2.22}$$

Now we prove (A.2.18; β_1). By using the formulas in Proposition 2.2.1, we can easily obtain the following:

$$\begin{aligned}\pi_\lambda^{\beta_1,-\beta_1} \circ j_\lambda^a(w_i^\lambda \otimes (X_{31} \wedge X_{13})) &= \frac{1}{2} i w_i^\lambda,\\ \pi_\lambda^{\beta_1,-\beta_1} \circ j_\lambda^a(w_i^\lambda \otimes (X_{32} \wedge X_{23})) &= \frac{1}{2}(d_\lambda - i) w_i^\lambda,\\ \pi_\lambda^{\beta_1,-\beta_1} \circ j_\lambda^a(w_{i-1}^\lambda \otimes (X_{13} \wedge X_{32})) &= -\frac{1}{2} i w_i^\lambda,\\ \pi_\lambda^{\beta_1,-\beta_1} \circ j_\lambda^a(w_{i+1}^\lambda \otimes (X_{23} \wedge X_{31})) &= -\frac{1}{2}(d_\lambda - i) w_i^\lambda.\end{aligned} \tag{A.2.23}$$

Using (A.2.17), (A.2.22) and (A.2.23), we proceed as follows to get the desired identity:

$$\begin{aligned}&-2[1_{\mathcal{F}_\eta} \otimes \pi_\lambda^{\beta_1,-\beta_1} \circ j_\lambda^a] \circ \mathcal{A}_\lambda F(g)\\ &= \sum_{i=0}^{d_\lambda} F_i(g) \otimes \{(i+l_2)i - (i-l_1)(d_\lambda - i)\\ &\qquad - i(i-d_\lambda-1) + (d_\lambda - i)(i+1)\} w_i^\lambda\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{d_\lambda} F_i(g) \otimes d_\lambda(l_1+1) w_i^\lambda \\
&= d_\lambda(l_1+1)F(g).
\end{aligned}
$$

The remaining identities in **(3)** can be proved in the same way. □

Now we give a proof of (4.5.1; β, β'). By Lemma A.2.3, (A.2.9), (A.2.14; β, β') and (A.2.18; β, β'), we proceed as follows to get the conclusion:

$$
\begin{aligned}
&\nabla^{\beta}_{\eta,\lambda+\beta'} \circ \nabla^{\beta'}_{\eta,\lambda} F(g) - \nabla^{\beta'}_{\eta,\lambda+\beta} \circ \nabla^{\beta}_{\eta,\lambda} F(g) \\
&= [1_{\mathcal{F}_\eta} \otimes \pi_\lambda^{\beta,\beta'} - 1_{\mathcal{F}_\eta} \otimes \pi_\lambda^{\beta',\beta}] \circ \mathcal{T}_\lambda F(g) \\
&= [1_{\mathcal{F}_\eta} \otimes \pi_\lambda^{\beta,\beta'} - 1_{\mathcal{F}_\eta} \otimes \pi_\lambda^{\beta',\beta}] \circ (1_{\mathcal{F}_\eta} \otimes j_\lambda^s \circ \mathcal{S}_\lambda F(g) + 1_{\mathcal{F}_\eta} \otimes j_\lambda^a \circ \mathcal{A}_\lambda F(g)) \\
&= 0.
\end{aligned}
$$

The formula (4.5.2; β_1) can be verified as below: First using Lemma A.2.3, (A.2.9), (A.2.15; β_1) and (A.2.19; β_1), we have

$$
\begin{aligned}
\text{(A.2.24)}\quad &\nabla^{-\beta}_{\eta,\lambda+\beta} \circ \nabla^{\beta}_{\eta,\lambda} F(g) - \nabla^{\beta}_{\eta,\lambda-\beta} \circ \nabla^{-\beta}_{\eta,\lambda} F(g) \\
&= [1_{\mathcal{F}_\eta} \otimes \pi_\lambda^{-\beta,\beta} - 1_{\mathcal{F}_\eta} \otimes \pi_\lambda^{\beta,-\beta}] \\
&\quad \circ \{(1_{\mathcal{F}_\eta} \otimes j_\lambda^s)\mathcal{S}_\lambda F(g) + (1_{\mathcal{F}_\eta} \otimes j_\lambda^a)\mathcal{A}_\lambda F(g)\} \\
&= [1_{\mathcal{F}_\eta} \otimes \kappa_\lambda \circ j_\lambda^s]\mathcal{S}_\lambda F(g) + \frac{1}{2}(l_1 + l_2 + 2d_\lambda + 2l_1 d_\lambda)F(g)
\end{aligned}
$$

By using (A.2.7), we have

$$
\begin{aligned}
\text{(A.2.25)}\quad &1_{\mathcal{F}_\eta} \otimes \kappa_\lambda \circ j_\lambda^s \mathcal{S}_\lambda F(g) \\
&= [1_{\mathcal{F}_\eta} \otimes \kappa_\lambda]\{\sum_{i=1}^{4} R_{X_i} R_{X_i} F(g) \otimes j_{\mathfrak{p}}^s (X_i \cdot X_i) \\
&\quad + \sum_{1 \leqslant i < j \leqslant 4} (R_{X_i} R_{X_j} F(g) + R_{X_j} R_{X_i} F(g)) \otimes j_{\mathfrak{p}}^s (X_i \cdot X_j)\}.
\end{aligned}
$$

Since $\kappa_\lambda(w \otimes j^s_{\mathfrak{p}}(X_i \cdot X_j)) = \frac{1}{2}\delta_{ij} w$ for every $w \in W_\lambda$ (see (A.2.12)), it turns out that the second term of the right hand side of (A.2.25) is zero and (A.2.25) becomes

$$[1_{\mathcal{F}_\eta} \otimes \kappa_\lambda \circ j^s_\lambda]\mathcal{S}_\lambda F(g) = \frac{1}{2}\sum_{i=1}^{4} R^2_{X_i} F(g).$$

Since $\{X_i\}$ is an orthonormal basis of $\mathfrak{p}$, this equals to

$$\frac{1}{2}(\Omega_{\eta,\lambda}F(g) - 1_{\mathcal{F}_\eta} \otimes \tau_\lambda(\Omega_K)F(g)).$$

Summing up the computations, we finally obtain

$$\begin{aligned}
&\nabla^{-\beta_1}_{\eta,\lambda+\beta_1} \circ \nabla^{\beta_1}_{\eta,\lambda} F(g) - \nabla^{\beta_1}_{\eta,\lambda-\beta_1} \circ \nabla^{-\beta_1}_{\eta,\lambda} F(g) \\
&= \frac{1}{2}(\Omega_{\eta,\lambda}F(g) - 1_{\mathcal{F}_\eta} \otimes \tau_\lambda(\Omega_K))F(g) + \big(\frac{l_1 + l_2}{2} + d_\lambda + l_1 d_\lambda\big)F(g).
\end{aligned}$$

Thus we have done. $\square$

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