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A Base Point Free Theorem of Reid Type

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Abstract. Assume that X is a normal projective variety over $\mathbb{C}$, of dimension $\leqq 3$, and that (X, Δ) is a log variety that is weakly Kawamata log terminal. Let L be a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and log big on (X, Δ) for some $a \in \mathbb{N}$. Then $\mathrm{Bs}|mL| = \emptyset$ for every $m \gg 0$.

Introduction

We generally use the notation and terminology of [Utah].

Let X be a normal projective variety over $\mathbb{C}$. Let (X, Δ) be a log variety which is log canonical. We assume that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

Let $\Theta = \sum_{i=1}^{s} \Theta_i$ be a reduced divisor with only simple normal crossings on an n-dimensional non-singular complete variety over $\mathbb{C}$. We denote **Strata**$(\Theta) := \{\Gamma \mid 1 \leqq k \leqq n, 1 \leqq i_1 < i_2 < \cdots < i_k \leqq s$, Γ is an irreducible component of $\Theta_{i_1} \cap \Theta_{i_2} \cap \cdots \cap \Theta_{i_k} \neq \emptyset\}$.

First we recall the famous base point free theorem of Kawamata and Shokurov.

THEOREM 0. (Theorem 3-1-1 of [KMM]) *Assume that (X, Δ) is weakly Kawamata log terminal (wklt). Let L be a nef Cartier divisor such that $aL - (K_X + \Delta)$ is ample for some $a \in \mathbb{N}$. Then $\mathrm{Bs}|mL| = \emptyset$ for every $m \gg 0$.*

We note that, if $aL - (K_X + \Delta)$ is nef and big but not ample, there exists a counterexample (cf. Remark 3-1-2 of [KMM]).

In [Rd], Reid introduced the notion of "log big" and suggested that, if $aL - (K_X + \Delta)$ is nef and log big on (X, Δ), then the theorem holds.

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Definition. (due to Reid [Rd]) Let $f\colon Y \to X$ be a log resolution such that $K_Y = f^*(K_X+\Delta)+\sum a_j E_j$ (where $a_j \geqq -1$). Let L be a Cartier divisor on X. L is called *nef and log big* on (X,Δ) if L is nef and big and $(L\mid_{f(\Gamma)})^{\dim f(\Gamma)} > 0$ for any member Γ of $\mathbf{Strata}(\sum_{a_j=-1} E_j)$.

Remark. The definition of the notion of "nef and log big" does not depend on the choice of the log resolution f (Claim of [Fu]).

Now we state the main theorem of this paper.

Main Theorem. *Assume that* $\dim X \leqq 3$ *and that* (X,Δ) *is weakly Kawamata log terminal (wklt). Let L be a nef Cartier divisor such that* $aL-(K_X+\Delta)$ *is nef and log big on* (X,Δ) *for some* $a\in\mathbb{N}$. *Then* $\mathrm{Bs}|mL| = \emptyset$ *for every* $m \gg 0$.

Remark. In [Fu0], the author proved the theorem under the conditions that X is non-singular, Δ is a simple normal crossing divisor and $\dim X$ is arbitrary.

1. Preliminaries

We collect some results needed for the proof of Main Theorem.

Proposition 1. (Reid Type Vanishing, cf. [Fu]) *Assume that* (X,Δ) *is wklt. Let D be a $\mathbb{Q}$-Cartier integral Weil divisor. If* $D-(K_X+\Delta)$ *is nef and log big on* (X,Δ), *then* $H^i(X,\mathcal{O}_X(D))=0$ *for every* $i>0$.

Proof. The proof is analogous to that of the main theorem of [Fu]. But, when we read [Fu], we must take notice of the facts below.

Fact 1. *For real numbers a and b,* $\lceil a+b\rceil \geqq \lfloor a\rfloor + \lceil b\rceil$. *Hence* $\lceil f^*D+E\rceil \geqq \lfloor f^*D\rfloor + \lceil E\rceil$. *Thus* $f_*\mathcal{O}_Y(\lceil f^*D+E\rceil)=\mathcal{O}_X(D)$.

Fact 2. $\lceil f^*D+E\rceil - (K_Y + f_*^{-1}\lfloor\Delta\rfloor + \{-f^*D-E\})$ *is nef and log big on* $(Y, f_*^{-1}\lfloor\Delta\rfloor+\{-f^*D-E\})$. $\square$

Proposition 2. (Kawamata, cf. the proofs of Lemma 3 of [Ka] and Theorem 3-1-1 of [KMM]) *Assume that* (X,Δ) *is wklt. Let L be a nef Cartier divisor such that* $aL-(K_X+\Delta)$ *is nef and big for some* $a\in\mathbb{N}$. *If* $\mathrm{Bs}|mL|\cap\lfloor\Delta\rfloor=\emptyset$ *for every* $m\gg 0$, *then* $\mathrm{Bs}|mL|=\emptyset$ *for every* $m\gg 0$.

2. Proof of Main Theorem

We prove the main theorem by induction on $\dim X$.

Step 0. Let $f : Y \to X$ be a log resolution of (X, Δ) such that the following conditions are satisfied:

(1) Y is projective (from the assumption that X is projective and the definition of the notion of wklt),

(2) $\mathrm{Exc}(f)$ consists of divisors,

(3) $K_Y + f_*^{-1}\Delta + F = f^*(K_X + \Delta) + E$,

(4) E and F are f-exceptional effective $\mathbb{Q}$-divisors such that $\mathrm{Supp}(E)$ and $\mathrm{Supp}(F)$ do not have common irreducible components,

(5) $\lfloor F \rfloor = 0$.

Note that, for any member $G \in \mathbf{Strata}(f_*^{-1}\lfloor\Delta\rfloor)$, $\mathrm{Exc}(f)$ does not include G (from an argument in p. 99 of [Sh], cf. the proof of Proposition of [Fu]).

Step 1. We shall show that we may assume that X is $\mathbb{Q}$-factorial.

Apply the relative log minimal model program to $f : (Y, f_*^{-1}\Delta + F) \to X$.

We end up with a $\mathbb{Q}$-factorial weakly Kawamata log terminal variety (over X) $\mu : (Z, \mu_*^{-1}\Delta + (F)_Z) \to X$ such that $K_Z + \mu_*^{-1}\Delta + (F)_Z$ is μ-nef.

Because $K_Z + \mu_*^{-1}\Delta + (F)_Z = \mu^*(K_X + \Delta) + (E)_Z$, $(E)_Z$ is μ-nef. Thus $(E)_Z$ is a μ-exceptional μ-nef effective divisor. Applying 1.1 of [Sh] to $-(E)_Z$, we obtain the fact that $(E)_Z = 0$. Hence $\mu^*(K_X + \Delta) = K_Z + \mu_*^{-1}\Delta + (F)_Z$.

We put $M := aL - (K_X + \Delta)$. Here we prove that μ^*M is nef and log big on $(Z, \mu_*^{-1}\Delta + (F)_Z)$. Let G_0 be a divisor of $\mathbb{C}(X)$ whose discrepancy with respect to $(Z, \mu_*^{-1}\Delta + (F)_Z)$ is -1. Let W be a non-singular projective variety and $f_1 : W \to Y$ and $\mu_1 : W \to Z$ birational morphisms, such that G_0 is a non-singular prime divisor on W and that $ff_1 = \mu\mu_1$. Then $f_1(G_0) \in \mathbf{Strata}(f_*^{-1}\lfloor\Delta\rfloor)$ (cf. the proof of Claim of [Fu]). So $\mathrm{Exc}(f)$ does not include $f_1(G_0)$.

Put $B := f(\mathrm{Exc}(f))$. By Zariski's Main Theorem, the set-theoretical inverse image $f^{-1}(B) = \mathrm{Exc}(f)$. We note that $Y \setminus \mathrm{Exc}(f) \cong Z \setminus \mu^{-1}(B)$.

Because $\mu_1(G_0) \cap (Z \setminus \mu^{-1}(B)) \cong f_1(G_0) \cap (Y \setminus \mathrm{Exc}(f)) \neq \emptyset$, $\mu_1(G_0)$ and $f_1(G_0)$ are birationally equivalent.

Thus $\mu_1(G_0)$ and $f(f_1(G_0))$ are birationally equivalent. Hence μ^*M is nef and big on $\mu_1(G_0)$. Therefore μ^*M is nef and log big on $(Z, \mu_*^{-1}\Delta + (F)_Z)$.

Thus we may assume that X is $\mathbb{Q}$-factorial.

Step 2. From 3.8 of [Sh], every irreducible component of $\lfloor\Delta\rfloor$ is normal.

Let S be an irreducible component of $\lfloor\Delta\rfloor$.

We put $S_0 := f_*^{-1}S$, $\mathrm{Diff}(0) := (f\mid_{S_0})_*(f^*(K_X+S)\mid_{S_0} - (K_Y+S_0)\mid_{S_0})$ and $\mathrm{Diff}(\Delta - S) := \mathrm{Diff}(0) + (\Delta - S)\mid_S$. We note that $\mathrm{Diff}(0) \geqq 0$ (the Subadjunction Lemma, 3.2.2 of [Sh], cf. 5-1-9 of [KMM]). Then $K_{S_0} + (f_*^{-1}\Delta - S_0)\mid_{S_0} + F\mid_{S_0} = f^*(K_S + \mathrm{Diff}(\Delta - S)) + E\mid_{S_0}$.

Because $(f\mid_{S_0})_*((f_*^{-1}\Delta - S_0)\mid_{S_0} + F\mid_{S_0} - E\mid_{S_0}) = \mathrm{Diff}(\Delta - S)$ and $\mathrm{Diff}(\Delta - S) = \mathrm{Diff}(0) + (\Delta - S)\mid_S \geqq 0$, $\lceil \mathrm{Diff}(\Delta - S)\rceil$ is reduced. Here $M\mid_S$ is nef and big. And, for any member Γ of $\mathbf{Strata}((f_*^{-1}\lfloor\Delta\rfloor - S_0)\mid_{S_0})$, $M\mid_{f(\Gamma)}$ is nef and big. Hence $(S, \mathrm{Diff}(\Delta - S))$ is log terminal and $M\mid_S$ is nef and log big on $(S, \mathrm{Diff}(\Delta - S))$.

We note that $\mathrm{Exc}(f\mid_{S_0})$ may not consist of divisors. But, because $\mathrm{Exc}(f) \cap S_0$ includes $\mathrm{Exc}(f\mid_{S_0})$, $\mathrm{Exc}(f\mid_{S_0})$ does not include any member of $\mathbf{Strata}((f_*^{-1}\lfloor\Delta\rfloor - S_0)\mid_{S_0})$. Hence, by a finite composition of blowing ups with centers included in $\mathrm{Exc}(f\mid_{S_0})$, we get a log resolution of $(S, \mathrm{Diff}(\Delta - S))$ such that the exceptional locus consists of divisors with discrepancies > -1 (because, for every divisor $\nu \in \mathbb{C}(S)$ whose discrepancy with respect to $(S, \mathrm{Diff}(\Delta - S))$ is -1, $\mathrm{center}_{S_0}(\nu) \in \mathbf{Strata}((f_*^{-1}\lfloor\Delta\rfloor - S_0)\mid_{S_0})$). Therefore $(S, \mathrm{Diff}(\Delta - S))$ is wklt (Lemma 4 of [Ka], [Sz]).

Thus $|mL\mid_S|$ is base point free for $m \gg 0$, by induction hypothesis.

Step 3. We consider the exact sequence:

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0.$$

Tensoring with $\mathcal{O}_X(mL)$ for $m \geqq a$, we have an exact sequence

$$0 \to \mathcal{O}_X(mL - S) \to \mathcal{O}_X(mL) \to \mathcal{O}_S(mL) \to 0$$

(Since the sheaf $\mathcal{O}_X(mL)$ is invertible, $\mathcal{O}_X(mL) \otimes \mathcal{O}_X(-S) = \mathcal{O}_X(mL - S)$).

Because $mL - S - (K_X + \Delta - S)$ is nef and log big on $(X, \Delta - S)$, $H^1(X, \mathcal{O}_X(mL - S)) = 0$ from Proposition 1. Hence the linear system

$|mL|$ on X cuts out a complete linear system $|mL\ |_S\ |$ on S. Therefore $\mathrm{Bs}|mL| \cap \lfloor\Delta\rfloor = \emptyset$ for $m \gg 0$.

Thus Proposition 2 implies the assertion. □

REMARK. The proof above implies that the main theorem holds also under the conditions that $\dim X = 4$ and that X is $\mathbb{Q}$-factorial.

More generally our proof shows the following implication:

If the log minimal model program works in all dimensions, then the main theorem holds without the condition that $\dim X \leqq 3$.

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