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A Remark on a Limiting Behaviour of the Occupation Times on Unbounded Domains of Brownian Motion

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Abstract. We remark on a limitting behaviour of occupation times on unbounded domains of Brownian motions. It is a partial extention of the works by Mountford and Meyre.

1. Introduction

Let D be an unbounded domain with smooth boundary in $\mathbf{R}^n$ and assume that D^c; complement of D be unbounded. We consider a limiting behaviour of the mean occupation time of D; $L_t = \frac{1}{t}\int_0^t 1_D(B_s)ds$. T. S. Mountford and T. Meyre considered the case that D is a cone. Meyre showed the following. ([3]).

Theorem (Meyre [3]). *Let C_F be a cone in $\mathbf{R}^n$ defined by*

$$C_F = \{rx : x \in F, F \textit{ is an open subset of } S^{n-1} \textit{ and } r \geq 0\}.$$

If $q \leq \xi$,

$$\liminf_{t\to\infty} (\log t)^q \frac{1}{t}\int_0^t 1_{C_F}(B_s)ds = 0,$$

and if $q > \xi$,

$$\liminf_{t\to\infty} (\log t)^q \frac{1}{t}\int_0^t 1_{C_F}(B_s)ds = \infty,$$

where $\xi = \frac{2}{-(n-2)^2+\sqrt{\lambda_{F^c}+\frac{(n-2)^2}{4}}}$ and λ_{F^c} is the first eigenvalue of Laplacian $-\Delta$ of Dirichlet problem on F^c.

Mountford in [4] gave the result in case of $\mathbf{R}^2$ with $C_\alpha = \{re^{i\theta} : r \geq 0, \theta \in (-\alpha/2, \alpha/2)\}$ and $\xi = \frac{2}{\pi}(2\pi - \alpha)$ before the statement of the above style was given by Meyre and also discussed the behaviour of L_t as t tends

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to zero. In this note we treat the case that D is a more general domain than a cone and give such asymptotics of L_t as $t \to \infty$ as Meyre. The power ξ of $\log t$ appeared in the above Theorem reflects a geometric character of D or D^c. Instead of ξ we employ $\beta_D(a,r)$ and $\eta_D(a,r)$ defined as the following manner.

$$\beta_D(a,r) = \int_{\kappa a}^{\nu r} \alpha_r dr \qquad (0<\nu<1, 1<\kappa \text{ and } \kappa a < \nu r)$$

with

$$\alpha_r = -\frac{n-2}{2r} + \sqrt{\lambda_r^D + \frac{(n-2)^2}{4r^2}},$$

where λ_r^D is the first eigenvalue of the Laplacian $-\Delta_{S(r)}$ with Dirichlet boundary condition on $D \cap S(r)$, $S(r)$: the boundary of $B(r)$. We note that if D is a cone, $\lim_{r\to\infty} \beta(a,r)/\log r = \xi^{-1}$.

$$\eta_D(a,r) = \inf_{\phi \in \Gamma(a,r)} c \int_0^r \frac{|\dot{\phi}(t)|}{\rho(t)} dt,$$

where $\rho(t)$ is the distance from $\phi(t)$ to ∂D, c is a constant satisfying that $c(\log c - 1) = 1$ if $n = 2$ and $(n-1)^{\frac{n-1}{n-2}}$ if $n \geq 3$, and

$$\Gamma(a,r) = \{\phi : [0,r] \to D_r, \text{ simple smooth curve}, \\ \phi(r) \in D \cap S(r) \text{ and } \phi(0) = a\}.$$

We choose some growth orders of $\beta_D(a,r)$ and $\eta_D(a,r)$ in r to make them play the ξ's role. Our concern in this choice means chiefly that the domains are shaped asymptotically like a cone. We can show the following theorem in the similar way to the above Theorem. We say that an increasing function $f(x)$ is moderately increasing if there exists a constant $M > 0$ such that $f(x+y) \leq M(f(x) + f(y))$ for any $x, y \in \mathbf{R}$.

Theorem 1. *If an increasing function* $\Phi(r)$ *is satisfying that*

$$\limsup_{r/a\to\infty} \frac{1}{\log \Phi(\frac{r}{a})} \eta_{D^c}(a,r) = 1,$$

$$\liminf \frac{Vol(S_r \cap D^c)}{r^{n-1}} > 0,$$

and $\Psi(r)$ *which is the inverse function of* Φ *is moderately increasing and satisfying that*

$$\limsup_{t\to\infty} \frac{(\log t)^2}{\Psi(t\log t)} < \infty,$$

then we have

$$\liminf_{t\to\infty} \Psi(\log t)\frac{1}{t}\int_0^t 1_D(B_s)ds = 0.$$

THEOREM 2. *Let* $\tilde{D}$ *be a subdomain of* D *satisfying that* $\tilde{D} \subset D$. *If*

$$0 < \limsup_{r/a\to\infty} \frac{1}{\log r/a}\eta_D(a,r) < \infty,$$

and

$$q > 2(\liminf_{r/a\to\infty} \frac{1}{\log r/a}\beta_{\tilde{D}^c}(a,r))^{-1} + 2,$$

then

$$\liminf_{t\to\infty}(\log t)^q\frac{1}{t}\int_0^t 1_D(B_s)ds = \infty.$$

In Theorem 2 q is larger than one expected from the result of Mountford and Meyre. It is because in our case the domains are lacking of some uniformity property unlike cones. With the following condition on D we can shorten this diference. Define D^*_δ for $\delta > 0$ which is swelled from D by

$$D^*_\delta = \{x; d(x,D) < \delta|x|\} \cup D.$$

THEOREM 3. *Assume that there exist* $\tilde{D} \subset D$ *and* $\delta > 0$ *such that* $\tilde{D}^*_\delta \subset D$. *If*

$$q > 2(\liminf_{r/a\to\infty} \frac{1}{\log r/a}\beta_{\tilde{D}^c}(a,r))^{-1},$$

then

$$\liminf_{t\to\infty}(\log t)^q\frac{1}{t}\int_0^t 1_D(B_s)ds = \infty.$$

Our results are weaker than Mountford's and Meyre's. Their method depends on skew product representation of Euclidean Brownian moitions.

We point out that in our method we only use estimates on hitting probability and do not use the other properties of Brownian motion. Then it is not essential that the process is a Brownian motion. We can obtain some variants of the above results for some diffusions and more general state spaces than Euclidean space, for example, some Riemannian manifolds ([1]).

We here give some simple examples of domains to which the above results is applicable. It seems difficult to treat them using the skew product representation of Brownian motions.

We first introduce *asymptotic cones*. For simplicity we consider the case that $n = 2$. We define an *asymptotic cone* by

$$\begin{aligned} C \quad = \quad & \{x = (r, \theta) : 0 < r < \infty,\ -\theta_r/2 < \theta < \theta_r/2 \text{ with} \\ & \theta_r = \alpha + o(1) \text{ as } r \to \infty.\} \end{aligned}$$

Let $D^c = C$ with $0 < \alpha < \pi$. In this case we have $\eta_C(a, r) = c \int_a^r \frac{dt}{t \sin \theta_t/2}$ and $\beta_C(a, r) = \pi \int_{\kappa a}^{\nu r} \frac{dt}{t\theta_t}$ where c is a root of $c(\log c - 1) = 1$. It is easy to see that

$$\lim_{r/a \to \infty} \eta_C(a, r)/\log \frac{r}{a} = c \sin \alpha/2 \quad \text{and} \quad \lim_{r/a \to \infty} \beta_C(a, r)/\log \frac{r}{a} = \alpha\pi.$$

In this case the assumptions of Theorem 1 and Theorem 3 are all satisfied. We note that if we set $C^*_\delta = \{x = (r, \theta) : -(\theta_r + \delta)/2 < \theta < (\theta_r + \delta)/2\}$, it plays the same role as D^*_δ in Theorem 3. Then we have a similar result to the usual cone case.

Second example is a case to which Theorem 1 can easily be applied and which we call a *modified cone*. Let $\varphi(t)$ $(0 \le t < \infty)$ be a simple smooth divergent curve and $k(t)$ a continuous increasing function. Define a modified cone C by

$$C = \{x \in \mathbf{R}^d \ :\ d(x, \varphi(t)) < k(t)|\dot{\varphi}(t)|\}.$$

If

$$\liminf_{t \to \infty} \frac{k(t)|\dot{\varphi}(t)|}{|\varphi(t)|} > 0 \text{ and } k(t) = \alpha t + O(1) \text{ as } t \to \infty,$$

then the assumptions of Theorem 1 are all satisfied and we can take $\Phi(r) = r^\alpha$, that is to say, $\Psi(r) = r^{1/\alpha}$.

2. Proof of Theorem 1

First we note key estimates of hitting probabilities. Let τ_D denote $\inf\{t > 0 : B_t \notin D\}$ throughout this paper.

LEMMA 2.1. *Let D be an unbounded domain in $\mathbf{R}^n$ with smooth boundary. If $x \in D_r = B(r) \cap D$, then we have the following etimates.*

i) $$P_x(B_{\tau_{D_r}} \in D \cap S(r)) \leq c_1 \exp(-\beta(|x|, r)).$$

ii) $$P_x(B_{\tau_{D_r}} \in D \cap S(r)) \geq c_2 \exp(-\eta(|x|, r)).$$

We can see the proofs of these estimates in case of $n = 2$ in Nevanlinna's and Tsuji's book ([5],[6]). We remark that they can be extended and we use them to see other properties of the first exit times of Brownian motion ([1]).

We define some notations. Let

$$r_n = e^{2n(\log n + \log \Psi(n))} \text{ and } \phi(r_n) = \frac{1}{\epsilon} r_n \Psi(\log r_n).$$

In this setting we note that for n large enough

$$r_n < \phi(r_n) < r_{n+1} \text{ and } \frac{\phi(r_n)}{r_{n+1}} \to 0 \text{ as } n \to \infty.$$

We show that

LEMMA 2.2.

$$\tau_{D^c} \circ \theta_{\tau_{r_n}} \geq \tau_{\phi(r_n)} \circ \theta_{\tau_{r_n}} \quad \textit{for infinitely many } n, \textit{ a.s.,}$$

where θ_t is a shift defined by $(\omega \circ \theta_t)(s) = \omega(t + s)$.

PROOF. Set $E_n = \{\tau_{D^c} \circ \theta_{\tau_{r_n}} \geq \tau_{\phi(r_n)} \circ \theta_{\tau_{r_n}}\}$.

We have only to check that Chung's Borell-Cantelli type lemma ([2]) is available for our case. Let $m > n$. By strong Markov property

$$\begin{aligned} P(E_n \cap E_m) &= E[1_{E_n} P(E_m | \mathcal{F}_{\tau_{\phi(r_n)}})] \\ &= E\left[1_{E_n} E_{B_{\tau_{\phi(r_n)}}}[P_{B_{\tau_{r_m}}}(\tau_D \geq \tau_{\phi(r_m)})]\right]. \end{aligned}$$

It is easy to see that if $f(x)$ is a bounded measurable function, then, by considering Poisson kernel,

$$E_{B_{\tau_r}}[f(B_{\tau_R})] = (1 + h(\frac{r}{R}))E[f(B_{\tau_R})] \text{ for } r < R,$$

where $h(t)$ satisfies that $h(t) \to 0$ as $t \to 0$.

Then we have

$$P(E_n \cap E_m) = (1 + h(\frac{\phi(r_n)}{r_m}))P(E_n)P(E_m),$$

that is to say,

$$\lim_{m,n\to\infty} \frac{P(E_n \cap E_m)}{P(E_n)P(E_m)} = 1.$$

By Lemma 2.1 and the assumption

$$\begin{aligned} P(E_n) &= E[P_{B_{\tau_{r_n}}}(\tau_{D^c} \geq \tau_{\phi(r_n)})] \\ &\geq const.\Phi(\frac{\phi(r_n)}{r_n})^{-1}P(B_{\tau_{r_n}} \in D^c) \\ &\geq const.(\log r_n)^{-1}. \end{aligned}$$

Then $\Sigma_n P(E_n) = \infty$ and we have known that Chung's Borel-Cantelli type lemma is usable in our case.

Hence there exists a subsequence $\{\tilde{r}_n\}$ of $\{r_n\}$ such that the Brownian motion is not in D from $\tau_{\tilde{r}_n}$ to $\tau_{\phi(\tilde{r}_n)}$. Then making the same argument as [3] we have

$$L_{\tau_{\phi(\tilde{r}_n)}} \leq \frac{\tau_{\tilde{r}_n}}{\tau_{\phi(\tilde{r}_n)}}.$$

We note, as used later in §3,

$$P(\tau_r > t) \leq const.e^{-\frac{\pi^2}{r^2}t} \text{ and } P(\tau_r < t) \leq const.e^{-\frac{r^2}{4t}}.$$

Hence we have

$$\Sigma_n P(\tau_{r_n} > r_n^2 \log n) \leq \Sigma_n e^{-\pi^2 \log n} < \infty$$

and

$$\Sigma_n P(\tau_{\phi(r_n)} \leq 8\phi(r_n)^2/\log n) \leq \Sigma_n e^{-2\log n} < \infty.$$

In the same way $\log \tau_r / 2 \log r \to 1$ as $r \to \infty$.

By Borel-Cantelli lemma

$$\begin{aligned} \Psi(\log \tau_{\phi(\tilde{r}_n)}) L_{\tau_{\phi(\tilde{r}_n)}} &\leq \Psi(\log \tau_{\phi(\tilde{r}_n)}) \frac{\tau_{\tilde{r}_n}}{\tau_{\phi(\tilde{r}_n)}} \\ &\leq \frac{\{\Psi(\log \phi(\tilde{r}_n)^2) + \Psi(\log\log n)\}(\log n)^2}{\Psi(\log \tilde{r}_n)^2}. \end{aligned}$$

Assumption that Ψ is moderately increasing and the assumption on the growth of Ψ imply that

$$\begin{aligned} \text{the last term } &\leq \frac{const.\{\Psi(\log \tilde{r}_n^2) + \Psi(\log \Psi(\tilde{r}_n)^2) + \Psi(\log\log n)\}(\log n)^2}{\Psi(\log \tilde{r}_n)^2} \\ &\leq const.\epsilon \qquad a.s. \end{aligned}$$

This completes the proof of Theorem 1. □

3. Proof of Theorem 2 and 3

Let $\tilde{D} \subset\subset D$ and $o \in \tilde{D}$. Take $p < \liminf_{r\to\infty} \frac{1}{\log r/a} \beta_{\tilde{D}}(a,r)$ and $q/2 > p^{-1}$. T_n be the first hitting time to $(B(r_n) \setminus B(t^{1/2})) \cap \tilde{D}$, $t_n = 2^n$ and $r_n = t_n^{1/2}(\log t_n)^{q/2}$. Define $S_n = \tau_{D^c} \circ \theta_{T_n} + T_n$.

$\Gamma = S(t_n^{1/2}) \cap \tilde{D}^c$.

Lemma 3.1. *As $n \to \infty$*

$$T_n \leq t_n (\log t_n)^q (\log\log t_n) \quad a.s.$$

Proof. For a Brownian motion starting from a point on Γ $T_n \wedge \tau_{r_n}$ is the first exit time from the domain $B(r_n) \setminus ((B(r_n) \setminus B(t^{1/2})) \cap \tilde{D})$.

By Lemma 2.1 we have

$$\begin{aligned} P(T_n \geq \tau_{r_n}) &= P(B_{\tau_{t_n^{1/2}}} \in \Gamma, \tau_D > \tau_{r_n}) \\ &\leq E[P_{B_{\tau_{t_n^{1/2}}}}(\tau_{D^c} \geq \tau_{r_n})] \\ &\leq const. (\frac{r_n}{t_n^{1/2}})^{-p} \\ &\leq const. (\log t_n)^{-pq}. \end{aligned}$$

Then $\Sigma P(T_n \geq \tau_{r_n}) < \infty$. By Borel-Cantelli lemma there exists $n_0(\omega)$ such that for any $n > n_0(\omega), T_n \leq \tau_{r_n}$. It is known that

$$P(\tau_r > t) \sim e^{-\frac{\pi^2 t}{2r^2}}.$$

Hence

$$P(\tau_{r_n} > t_n(\log t_n)^q(\log\log t_n)) \sim (\log t_n)^{-\pi^2/2}.$$

Then we have

$$\Sigma_n P(\tau_{r_n} > t_n(\log t_n)^q(\log\log t_n)) < \infty.$$

There exists $n_1(\omega)$ such that for any $n > n_1(\omega)$,

$$\tau_{r_n} \leq t_n(\log t_n)^q(\log\log t_n).$$

We have

$$T_n \leq t_n(\log t_n)^q(\log\log t_n). \ \Box$$

LEMMA 3.2. *As $n \to \infty$ we have*

$$S_n - T_n \geq t_n(\log t_n)^{-2}(\log\log t_n)^{-3} \quad a.s.$$

PROOF. We may assume that c_2 appeared in Lemma 2.1 is less than 1. Choose c such that $c^{-\eta} = c_2$. Let $\alpha_n = t_n^{1/2}(\log t_n)^{-1}(\log\log t_n)^{-2}$.

On the other side hand

$$\begin{aligned} P(S_n - T_n \leq \tau_{c(|B_{T_n}|+\alpha_n)} - T_n) &= E[P_{B_{T_n}}(\tau_D \leq \tau_{c(|B_0|+\alpha_n)})] \\ &= 1 - E[P_{B_{T_n}}(\tau_D \geq \tau_{c(|B_0|+\alpha_n)})] \\ &\leq 1 - E[(1 + \frac{\alpha_n}{|B_{T_n}|})^{-\eta}] \\ &= O(\frac{\alpha_n}{r_n^{1/2}}) = O(\frac{1}{n(\log n)^2}). \end{aligned}$$

Using the same estimate for one dimensional Brownian motion as a few lines above; $P(\sigma_r < t) \leq const.e^{-r^2/2t}$, where $\sigma_r = \inf\{t > 0; |B_t| = r\}$ with B_t

is a one dimensional Brownian motion starting from zero,

$$\begin{aligned} & P(\tau_{r_n+\alpha_n} - T_n \leq \frac{1}{8}\alpha_n^2(\log\log t_n)^{-1}) \\ & \leq E[P_{B_{T_n}}(\tau_{r_n+\alpha_n} \leq \frac{1}{8}\alpha_n^2(\log\log t_n)^{-1})] \\ & \leq P_0(\tau_{\alpha_n} \leq \frac{1}{8}\alpha_n^2(\log\log t_n)^{-1}) \\ & \leq P_0(\sigma_{\alpha_n/\sqrt{2}} \leq \frac{1}{8}\alpha_n^2(\log\log t_n)^{-1}) \\ & \leq const.(\log t_n)^{-2} = O(\frac{1}{n^2}). \end{aligned}$$

Since $\tau_{r_n+\alpha_n} \leq \tau_{c(|B_{T_n}|+\alpha_n)}$, by Borel-Cantelli lemma again

there exists $n_2(\omega)$ such that for any $n > n_2(\omega)$,
$S_n - T_n \geq \alpha_n^2(\log\log t_n)^{-1})$. □

Now we can make the same argument as [3] to end the proof of Theorem 2. Choose a sequence of integers $\{n(m)\}$ satisfying that

$$t_{m(n)} \leq \frac{t_n}{2(\log t_n)^q(\log\log t_n)} \leq 2t_{m(n)}.$$

Then from Lemma 3.1 it is easy to check that $T_{m(n)} \leq \frac{t_n}{2}$.

We first note that

$$t_n L_{t_n} \geq S_{m(n)} \wedge t_n - T_{m(n)}.$$

If $S_{m(n)} \geq t_n$, then $L_{t_n} \geq \frac{1}{2}$.

If $S_{m(n)} < t_n$, from Lemma 3.2 we have

$$\begin{aligned} t_n L_{t_n} & \geq S_{m(n)} - T_{m(n)} \\ & \geq t_{m(n)}(\log t_{m(n)})^{-2}(\log\log t_{m(n)})^{-3} \\ & \geq \frac{t_n}{4(\log t_n)^q(\log\log t_n)}(\log \frac{t_n}{2(\log t_n)^q(\log\log t_n)})^{-2} \\ & \quad \times(\log\log \frac{t_n}{2(\log t_n)^q(\log\log t_n)})^{-3}. \end{aligned}$$

Hence if $q' > q - 2$, we have

$$\begin{aligned}&\text{there exists } n_3(\omega) \text{ such that for any } n > n_3(\omega),\\&(\log t_n)^{q'} L_{t_n} \geq const. > 0. \ \square\end{aligned}$$

As for Theorem 3 we have only to note the following lemma playing the same role to Lemma 3.2.

Lemma 3.3. *Under the assumption of Theorem 3 we have*

$$S_n - T_n \geq t_n (\log\log t_n)^{-2} \quad \text{as } n \to \infty \quad a.s.$$

The proof is same as Lemma 3.4 in [3].

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