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# *A Note on Four-Manifolds with Free Fundamental Groups*

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**Abstract.** In this paper we study the homotopy decomposition problem for closed connected 4-manifolds with free fundamental groups. For this we apply obstruction theory and give a detailed description of Whitehead's exact sequence for the named class of manifolds.

## 1. Introduction and results

If $M'$ is a closed simply-connected 4-manifold, then there is the well-known exact sequence (see [12]):

$$0 \longrightarrow H_4(M';\mathbb{Z}) \xrightarrow{b'} \Gamma(\Pi_2(M')) \longrightarrow \Pi_3(M';\mathbb{Z}) \longrightarrow 0.$$

Here $\Gamma(\cdot)$ is Whitehead's quadratic functor on abelian groups. One might think of $\Gamma(\Pi_2(M'))$ as a subgroup of $\Pi_2(M') \otimes_{\mathbb{Z}} \Pi_2(M')$. Let $[M'] \in H_4(M';\mathbb{Z})$ denote a fundamental class of $M'$. The element

$$b'([M']) \in \Gamma(\Pi_2(M')) \subset \Pi_2(M') \otimes_{\mathbb{Z}} \Pi_2(M')$$

can be interpreted as the intersection form $\lambda_{M'}$ over $H_2(M';\mathbb{Z})$, via Poincaré duality.

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In this note we shall prove a similar result for closed connected topological 4-manifolds with free fundamental groups. We shall always assume that the considered manifolds are all orientable although our results also work in the general case, provided the first Stiefel-Whitney classes coincide.

Let $M^4$ be a closed connected orientable TOP 4-manifold with free fundamental group $\Pi_1(M) \cong *_p\mathbb{Z}$ (free product of $p$ factors $\mathbb{Z}$). We can assume that $M$ is provided with a CW-structure, up to homotopy. Let $\Lambda = \mathbb{Z}[\Pi_1(M)]$ be the integral group ring of $\Pi_1(M)$. For a right $\Lambda$-module $A$, let $\overline{A}$ be the associated left $\Lambda$-module induced by the canonical anti-automorphism $-\colon \Lambda \to \Lambda$ (see [1] and [11]).

The following is our main theorem.

THEOREM 1.1. *Let $M^4$ be a closed connected orientable topological 4-manifold with $\Pi_1(M) \cong *_p\mathbb{Z}$. Then the sequence*

$$0 \to H_4(M;\mathbb{Z}) \xrightarrow{b} \Gamma(\Pi_2(M)) \otimes_\Lambda \mathbb{Z} \to \Pi_3(M) \otimes_\Lambda \mathbb{Z} \to H_3(M;\Lambda) \otimes_\Lambda \mathbb{Z} \to 0$$

*is exact. Moreover, $\Gamma(\Pi_2(M)) \otimes_\Lambda \mathbb{Z}$ is a subgroup of $\Pi_2(M) \otimes_\Lambda \overline{\Pi_2(M)}$ and the element $b([M]) \in \Pi_2(M) \otimes_\Lambda \overline{\Pi_2(M)}$ can be identified with the intersection form $\lambda_M\colon H_2(M;\Lambda) \times H_2(M;\Lambda) \to \Lambda$ via Poincaré duality.*

It is shown in Section 2 that $\Pi_2(M)$ is a free $\Lambda$-module. It can also be seen that $\Pi_3(M \backslash \overset{\circ}{D}{}^4)$ is $\Lambda$-free (Lemma 3.3). As a consequence (applying the Whitehead theorem [13]) we obtain an alternative proof of a result due to Matumoto and Katanaga (see [10]).

COROLLARY 1.2. *Any closed connected 4-manifold $M$ with $\Pi_1(M) \cong *_p\mathbb{Z}$ can be obtained by attaching a 4-disc to a bouquet $\vee_p(\mathbb{S}^1 \vee \mathbb{S}^3) \vee (\vee_q \mathbb{S}^2)$.*

Recently, Hambleton and Teichner (see [6]) have constructed a nonsingular Hermitian form $\lambda$ of rank 4 over $\Lambda = \mathbb{Z}[\mathbb{Z}]$ which is not extended from the integers. This together with the realization theorem of Freedman and Quinn (see [5]) yields an example of closed 4-manifold $M$ with $\Pi_1(M) \cong \mathbb{Z}$ which is not homotopy equivalent to the connected sum of $\mathbb{S}^1 \times \mathbb{S}^3$ with a simply-connected manifold.

Our main result is related to Theorem 1 of [3]. From this follows immediately a criterion for the homotopy decomposition of 4-manifolds with free fundamental groups.

COROLLARY 1.3. *Let $M^4$ be a closed connected orientable TOP 4-manifold with $\Pi_1(M) \cong *_p\mathbb{Z}$. Then $M$ is simple homotopy equivalent to the connected sum $\#_p(\mathbb{S}^1 \times \mathbb{S}^3)\#M'$ for some simply-connected closed 4-manifold $M'$ if and only if the intersection form $\lambda_M$ over $\Lambda$ is extendable from the integers.*

In our case any homotopy equivalence is *simple* because the Whitehead group of $\Pi_1(M) \cong *_p\mathbb{Z}$ vanishes (see for example [5]).

In particular, $M'$ is determined by the isomorphism $\lambda_{M'} \cong \lambda_M \otimes_\Lambda \mathbb{Z}$ over $\mathbb{Z}$ as shown in [2]. Moreover, $M'$ is unique, up to TOP homeomorphism, if $\lambda_{M'}$ is even (see [5]).

We also remark that under the hypothesis of Corollary 1.3 the manifolds $M$ and $\#_p(\mathbb{S}^1 \times \mathbb{S}^3)\#M'$ are $s$-cobordant. This can be obtained by using some results of [7], Section 2. A complete proof can be found in [4]. Note that this fact was first proved for the case when $\chi(M) = 2\chi(K(\Pi_1, 1))$ by Hillman (see [8]). Finally, we observe that in case $\Pi_1 \cong \mathbb{Z}$, the manifolds in Corollary 1.3 are also topologically homeomorphic (apply the results of Freedman-Quinn's book [5]). This corrects a previous statement of Kawauchi (see Theorem 1.1 of [9]).

To prove Theorem 1.1 we first construct a map $\phi\colon M \to \#_p(\mathbb{S}^1 \times \mathbb{S}^3)$ of degree 1 (Lemma 2.1). This map serves to define maps $\alpha\colon \#_p(\mathbb{S}^1 \times \mathbb{S}^3)\backslash \mathring{D}^4 \to M$ and $\beta\colon M'\backslash \mathring{D}^4 \to M$ (see Section 2). A homotopy equivalence between

$$(\#_p(\mathbb{S}^1 \times \mathbb{S}^3)\#M')\backslash \mathring{D}^4 \quad \text{and} \quad (\#_p(\mathbb{S}^1 \times \mathbb{S}^3)\backslash \mathring{D}^4) \vee (M'\backslash \mathring{D}^4)$$

yields then a map

$$\alpha\#\beta \simeq \alpha \vee \beta\colon (\#_p(\mathbb{S}^1 \times \mathbb{S}^3)\#M')\backslash \mathring{D}^4 \to M$$

which induces isomorphisms on $\Pi_1$ and on $\Pi_2$ (see Section 2). Finally, in Section 3 we shall complete the proof of Theorem 1.1.

## 2. The map $\alpha\#\beta$

Let $M^4$ be a given closed connected orientable 4-manifold with $\Pi_1(M) \cong *_p\mathbb{Z}$. Choosing an isomorphism of $\Pi_1(M)$ with $*_p\mathbb{Z}$ yields a basis $(e_1, e_2, \ldots, e_p)$ of $H_1(M;\mathbb{Z})$. Let $(u_1, u_2, \ldots, u_p)$ be the dual basis in $H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(H_1(M;\mathbb{Z}), \mathbb{Z})$ and let $v_1, v_2, \ldots, v_p \in H^3(M;\mathbb{Z})$ be the Poincarè duals of $e_1, e_2, \ldots, e_p$, respectively. Then we have

$$u_i \cup v_j = \delta_{ij}\ \omega_M$$

for each $i, j = 1, 2, \ldots, p$. Here $\omega_M \in H^4(M;\mathbb{Z})$ is determined by the orientation of $M$, i. e. $\omega_M$ is the dual of the fundamental class $[M] \in H_4(M;\mathbb{Z})$. The cartesian product of the elements $u_i$ and $v_i$ defines a map

$$\varphi = \prod_{i=1}^{p}(u_i \times v_i)\colon M \to C = \prod_{1}^{p}(\mathbb{S}^1 \times K(\mathbb{Z}, 3)).$$

Since $K(\mathbb{Z}, 3) = \mathbb{S}^3 \cup \{\text{cells of dimension} \geq 5\}$, we can assume that

$$\varphi\colon M \to \prod_{1}^{p}(\mathbb{S}^1 \times \mathbb{S}^3).$$

The obstruction for deforming $\varphi$ to a map

$$M \to \vee_p(\mathbb{S}^1 \times \mathbb{S}^3)$$

belongs to (see also the appendix in Section 4)

$$H^3(M; \Pi_2(\vee_p(\mathbb{S}^1 \times \mathbb{S}^3))) \cong 0$$

and

$$H^4(M; \Pi_3(\vee_p(\mathbb{S}^1 \times \mathbb{S}^3))) \cong \Pi_3(\vee_p(\mathbb{S}^1 \times \mathbb{S}^3)) \otimes_\Lambda \mathbb{Z} \cong \oplus_p\mathbb{Z}.$$

Therefore the $i$-th component of this obstruction in $\oplus_p\mathbb{Z}$ is just the obstruction for extending the map $u_i \times v_i\colon M^{(3)} \to \mathbb{S}^1 \times \mathbb{S}^3$ to $M$, hence it is zero.

Now we consider the wedge $\vee_p(\mathbb{S}^1 \times \mathbb{S}^3)$ as the connected sum $\#_p(\mathbb{S}^1 \times \mathbb{S}^3)$ with $p-1$ four-dimensional discs adjoined along the 3-spheres which serve to define the connected sums. In other words, $\#_p(\mathbb{S}^1 \times \mathbb{S}^3)$ embeds into $\vee_p(\mathbb{S}^1 \times \mathbb{S}^3)$, up to homotopy.

LEMMA 2.1. *The map* $\varphi\colon M \to \vee_p(\mathbb{S}^1 \times \mathbb{S}^3)$ *can be deformed into a map*

$$\phi\colon M \to \#_p(\mathbb{S}^1 \times \mathbb{S}^3).$$

*Moreover,* $\phi$ *is of degree 1 by choosing an appropriate orientation of* $\#_p(\mathbb{S}^1 \times \mathbb{S}^3)$.

For a proof we refer to [3], Lemma 13.

REMARK. By [11], the $\Lambda$-module

$$H_2(M;\Lambda) = \mathrm{Ker}(H_2(M;\Lambda) \xrightarrow{\phi_*^\Lambda} H_2(\#_p(\mathbb{S}^1 \times \mathbb{S}^3);\Lambda))$$

is stably $\Lambda$-free, hence $\Lambda$-free. In particular, we have

$$H_2(M;\mathbb{Z}) \cong H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z} \cong H_2(\widetilde{M};\mathbb{Z}) \otimes_\Lambda \mathbb{Z} \cong \Pi_2(M) \otimes_\Lambda \mathbb{Z},$$

where $\widetilde{M}$ is the universal covering space of $M$. Therefore any element $x \in H_2(M;\mathbb{Z})$ can be represented by a map $\mathbb{S}^2 \to M$.

By Freedman's result (see for example [5]) there is a simply-connected closed 4-manifold $M'$ with integral intersection form $\lambda_{M'} \cong \lambda_M \otimes_\Lambda \mathbb{Z}$ (also use [2]).

By the above remark we can represent a basis

$$x_1, x_2, \ldots, x_r \in H_2(M';\mathbb{Z}) \cong H_2(M;\mathbb{Z}) \cong \oplus_r \mathbb{Z}$$

by maps of 2-spheres into $M$, i. e. there exists a map

$$\beta\colon M' \backslash \overset{\circ}{D}{}^4 \simeq \vee_r \mathbb{S}^2 \to M.$$

Obviously, the induced homomorphism

$$\beta_*\colon H_2(M' \backslash \overset{\circ}{D}{}^4;\mathbb{Z}) \to H_2(M;\mathbb{Z})$$

is bijective.

LEMMA 2.2. *There exists a map*

$$\alpha\colon (\#_p(\mathbb{S}^1 \times \mathbb{S}^3))\backslash \overset{\circ}{D}{}^4 \to M$$

*such that the composition*

$$(\#_p(\mathbb{S}^1 \times \mathbb{S}^3))\backslash \overset{\circ}{D}{}^4 \xrightarrow{\ \alpha\ } M \xrightarrow{\ \phi\ } \#_p(\mathbb{S}^1 \times \mathbb{S}^3)$$

*is homotopic to the inclusion.*

PROOF. For simplicity, we set $Y = \#_p(\mathbb{S}^1 \times \mathbb{S}^3)$ and denote the $q$-skeleton of $Y$ by $Y^{(q)}$. Then there is a map

$$\vee_p \mathbb{S}^1 = Y^{(1)} \to M$$

such that its composition with $\phi$ is the canonical inclusion. There is an obstruction map

$$H_2(\widetilde{Y}^{(2)}, \widetilde{Y}^{(1)}) \to \Pi_1(M)$$

for extending over the 2-skeleton of $Y$. Here $\widetilde{Y}^{(q)}$ denotes the universal covering space of $Y^{(q)}$. However, composition of this map with $\phi_*\colon \Pi_1(M) \underset{\cong}{\to} \Pi_1(Y)$ shows that we can extend it over the 2-skeleton of $Y$. There is then an obstruction in the cohomology group $H^3(Y; \Pi_2(M))$ with local coefficients for extending over $Y^{(3)} = Y\backslash \overset{\circ}{D}{}^4$. Since $\Pi_2(M) \cong H_2(\widetilde{M}) \cong H_2(M; \Lambda)$ is stably $\Lambda$-free, we have

$$H^3(Y; \Pi_2(M)) \cong H_1(Y; \Pi_2(M)) \cong 0.$$

The first isomorphism follows from Poincaré duality with local coefficients (see for example [1] and [11]). Obviously, one has to consider $\Pi_2(M)$ as $\Lambda$-right and as $\Lambda$-left module by making use of the involution $^-\colon \Lambda \to \Lambda$ defined by

$$\overline{\sum n_g g} = \sum n_g g^{-1}$$

for any $g \in \Pi_1(M)$ and $n_g \in \mathbb{Z}$. Therefore the map $\alpha\colon Y\backslash \mathring{D}^4 \to M$ can be defined. Now the obstructions for homotopy are in $H^2(Y;\Pi_2(M)) \cong 0$ and in $H^3(Y;\Pi_3(M))$. Looking at the diagram

$$\begin{array}{ccc} \Pi_3(M) & \xrightarrow{\phi_*} & \Pi_3(Y) \\ \downarrow & & \downarrow\cong \\ H_3(M;\Lambda) & \xrightarrow[\phi_*^\Lambda]{} & H_3(Y;\Lambda) \end{array}$$

one sees that $\phi_*\colon \Pi_3(M) \to \Pi_3(Y)$ is surjective because the homomorphism

$$\Pi_3(M) \to H_3(M;\Lambda) \cong H_3(\widetilde{M};\mathbb{Z})$$

is onto by the Hurewicz theorem. Therefore it is possible to construct an extension $\alpha\colon Y^{(3)} \to M$ such that $\phi \circ \alpha$ is homotopic to the inclusion $Y^{(3)} \subset Y$. This can be seen as follows. We choose $\alpha$ and consider the difference cochain

$$d(i,\phi\circ\alpha)\colon H_3(\widetilde{Y}^{(3)},\widetilde{Y}^{(2)}) \to \Pi_3(Y)$$

between $\phi\circ\alpha$ and the inclusion $i\colon Y^{(3)} \subset Y$. Since $H_3(\widetilde{Y}^{(3)},\widetilde{Y}^{(2)})$ is $\Lambda$-free and $\phi_*\colon \Pi_3(M) \to \Pi_3(Y)$ is surjective, we can lift $d(i,\phi\circ\alpha)$ to

$$\widetilde{d}\colon H_3(\widetilde{Y}^{(3)},\widetilde{Y}^{(2)}) \to \Pi_3(M).$$

We can now use $\widetilde{d}$ to change $\alpha$ in order to obtain a map $\alpha'\colon Y\backslash \mathring{D}^4 = Y^{(3)} \to M$ such that the difference cochain of $\alpha$ and $\alpha'$ is $\widetilde{d}$. Then $\phi\circ\alpha'$ is homotopic to the inclusion. This completes the proof. □

Now we observe that there is a homotopy equivalence

$$(\#_p(\mathbb{S}^1 \times \mathbb{S}^3)\backslash \mathring{D}^4) \vee (M'\backslash \mathring{D}^4) \simeq (\#_p(\mathbb{S}^1 \times \mathbb{S}^3)\#M')\backslash \mathring{D}^4,$$

hence the maps $\alpha$ and $\beta$ define a map

$$\alpha\#\beta \simeq \alpha \vee \beta\colon (\#_p(\mathbb{S}^1 \times \mathbb{S}^3)\#M')\backslash \mathring{D}^4 \to M.$$

Corollary 2.3. *The map $\alpha\#\beta$ induces isomorphisms on $\Pi_1$ and on $\Pi_2$.*

## 3. The homotopy type

In Section 2 we have constructed a map

$$\alpha\#\beta\colon (\#_p(\mathbb{S}^1\times\mathbb{S}^3)\#M')\backslash\overset{\circ}{D}{}^4 \to M.$$

In this section we are studying the problem of its extension to

$$\#_p(\mathbb{S}^1\times\mathbb{S}^3)\#M' = M_1.$$

First we observe that any extension must be a homotopy equivalence.

Lemma 3.1. *If $\alpha\#\beta$ extends to a map $h\colon M_1\to M$, then $h$ must be a homotopy equivalence.*

Proof. By the construction of

$$\alpha\vee\beta\colon (\vee_p(\mathbb{S}^1\times\mathbb{S}^3)\backslash\overset{\circ}{D}{}^4)\vee(M'\backslash\overset{\circ}{D}{}^4)\to M,$$

the following diagram commutes, up to homotopy:

$$\begin{array}{ccc} M_1 & \xrightarrow{h} & M \\ c\downarrow & & \downarrow\phi \\ \#_p(\mathbb{S}^1\times\mathbb{S}^3) & = & \#_p(\mathbb{S}^1\times\mathbb{S}^3). \end{array}$$

Here $c$ denotes the collapsing map. Both maps $\phi$ and $c$ are of degree one, hence also $h$ must be of degree one. The kernel $\operatorname{Ker} h_*^\Lambda$ of $h_*^\Lambda\colon H_2(M_1;\Lambda)\to H_2(M;\Lambda)$ is stably $\Lambda$-free and finitely generated (see [11]), hence $\Lambda$-free. On the other hand,

$$\operatorname{Ker} h_*^{\mathbb{Z}} = \operatorname{Ker}(H_2(M_1;\mathbb{Z})\xrightarrow{h_*^{\mathbb{Z}}} H_2(M;\mathbb{Z}))$$

is isomorphic to $\operatorname{Ker} h_*^\Lambda\otimes_\Lambda\mathbb{Z}$. But $\operatorname{Ker} h_*^{\mathbb{Z}}=0$ by Corollary 2.3. Therefore, $h_*^\Lambda$ is an isomorphism. It follows from duality that $h$ is a homotopy equivalence. □

The obstruction for extending $\alpha \vee \beta$ belongs to

$$H^4(\#_p(\mathbb{S}^1 \times \mathbb{S}^3)\#M'; \Pi_3(M)) \cong H_0(\#_p(\mathbb{S}^1 \times \mathbb{S}^3)\#M'; \Pi_3(M)) \\ \cong \Pi_3(M) \otimes_\Lambda \mathbb{Z}.$$

More precisely, it is the image of a generator by the composite map

$$\Pi_4(M_1, M_1 \backslash \overset{\circ}{D}{}^4) \otimes_\Lambda \mathbb{Z} \xrightarrow{\partial_* \otimes_\Lambda 1} \Pi_3(M_1 \backslash \overset{\circ}{D}{}^4) \otimes_\Lambda \mathbb{Z} \xrightarrow{(\alpha \vee \beta)_* \otimes_\Lambda 1} \Pi_3(M) \otimes_\Lambda \mathbb{Z}.$$

We are going to get information on the obstruction by using Whitehead's exact sequence for a 4-complex $X$ (see [12]):

$$H_4(X; \Lambda) \longrightarrow \Gamma(\Pi_2(X)) \longrightarrow \Pi_3(X) \xrightarrow{H_*} H_3(X; \Lambda) \longrightarrow 0.$$

Here $H_*$ is the Hurewicz homomorphism and $\Gamma(\cdot)$ is the quadratic functor on abelian groups. This is then an exact sequence of right $\Lambda$-modules. In our case the group $\Pi_2(X)$ is $\mathbb{Z}$-free, hence there is a natural inclusion $\tau\colon \Gamma(\Pi_2(X)) \to \Pi_2(X) \otimes_{\mathbb{Z}} \Pi_2(X)$. The $\Lambda$-module structure on $\Gamma(\Pi_2(X))$ is then compatible with this inclusion. Recall also that there is a natural identification of $\Gamma(\Pi_2(X))$ with $H_4(K(\Pi_2(X), 2); \mathbb{Z})$.

Lemma 3.2. *Let $X$ be either $M$ or $M_1$. Then tensoring the Whitehead sequence by $\otimes_\Lambda \mathbb{Z}$ yields the following exact sequence:*

$$H_4(X; \mathbb{Z}) \to \Gamma(\Pi_2(X)) \otimes_\Lambda \mathbb{Z} \to \Pi_3(X) \otimes_\Lambda \mathbb{Z} \to H_3(X; \Lambda) \otimes_\Lambda \mathbb{Z} \to 0.$$

Proof. Considering the universal coefficient spectral sequence

$$\mathrm{Tor}_p^\Lambda(H_q(X; \Lambda), \mathbb{Z}) \Rightarrow H_{p+q}(X; \mathbb{Z}),$$

we get $\mathrm{Tor}_1^\Lambda(H_3(X; \Lambda), \mathbb{Z}) \cong H_4(X; \mathbb{Z})$. Recall that $H_2(X; \Lambda)$ is $\Lambda$-free and that $H_4(\Pi_1(X); \mathbb{Z}) \cong 0$. Thus the result follows. $\square$

For $X = M$ or $M_1$, we consider the following diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & & & \Pi_4(X, X\backslash\overset{\circ}{D}{}^4)\otimes_\Lambda \mathbb{Z} & \xrightarrow{\cong} & H_4(X, X\backslash\overset{\circ}{D}{}^4;\Lambda)\otimes_\Lambda \mathbb{Z} & & \\
 & & & & \downarrow{\scriptstyle \partial_*\otimes 1} & & \downarrow{\scriptstyle \partial_*\otimes 1} & & \\
0 & \longrightarrow & \Gamma(\Pi_2(X\backslash\overset{\circ}{D}{}^4))\otimes_\Lambda \mathbb{Z} & \longrightarrow & \Pi_3(X\backslash\overset{\circ}{D}{}^4)\otimes_\Lambda \mathbb{Z} & \longrightarrow & H_3(X\backslash\overset{\circ}{D}{}^4;\Lambda)\otimes_\Lambda \mathbb{Z} & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
H_4(X;\mathbb{Z}) & \longrightarrow & \Gamma(\Pi_2(X))\otimes_\Lambda \mathbb{Z} & \longrightarrow & \Pi_3(X)\otimes_\Lambda \mathbb{Z} & \longrightarrow & H_3(X;\Lambda)\otimes_\Lambda \mathbb{Z} & \longrightarrow & 0.
\end{array}$$

The exactness of the middle row will be a consequence of the next result.

LEMMA 3.3. *$H_3(X\backslash\overset{\circ}{D}{}^4;\Lambda)$ is a free $\Lambda$-module*

PROOF. For $X = \#_p(\mathbb{S}^1\times\mathbb{S}^3)\#M'$ we have

$$X\backslash\overset{\circ}{D}{}^4 \simeq (\#_p(\mathbb{S}^1\times\mathbb{S}^3)\backslash\overset{\circ}{D}{}^4)\vee(M'\backslash\overset{\circ}{D}{}^4),$$

hence the result follows immediately.

For $X = M$, we consider the exact sequence of the pair $X\backslash\overset{\circ}{D}{}^4 = X^{(3)}\supset X^{(2)}$ :

$$0\to H_3(X\backslash\overset{\circ}{D}{}^4;\Lambda)\to H_3(X\backslash\overset{\circ}{D}{}^4, X^{(2)};\Lambda)\xrightarrow{\partial_*} H_2(X^{(2)};\Lambda)\xrightarrow{i_*} H_2(X^{(3)};\Lambda)\to 0.$$

Now $H_2(X^{(3)};\Lambda)\cong H_2(X;\Lambda)$ is $\Lambda$-free, hence $\operatorname{Ker} i_* = \operatorname{Im}\partial_*$ is a direct summand of $H_2(X^{(2)};\Lambda)$. But $X^{(2)}$ is a wedge of 1-spheres and 2-spheres so $H_2(X^{(2)};\Lambda)$ is $\Lambda$-free too. Therefore $\operatorname{Ker}\partial_*\cong H_3(X\backslash\overset{\circ}{D}{}^4;\Lambda)$ is a direct summand of the free $\Lambda$-module $H_3(X\backslash\overset{\circ}{D}{}^4, X^{(2)};\Lambda)$, hence it is free as $\Lambda$-module. □

LEMMA 3.4. *The homomorphism*

$$\partial_*\otimes_\Lambda 1\colon H_4(X, X\backslash\overset{\circ}{D}{}^4;\Lambda)\otimes_\Lambda\mathbb{Z}\to H_3(X\backslash\overset{\circ}{D}{}^4;\Lambda)\otimes_\Lambda\mathbb{Z}$$

*is zero.*

Proof. Note that $H_4(X, X\backslash \mathring{D}^4; \Lambda) \otimes_\Lambda \mathbb{Z} \cong H_4(X, X\backslash \mathring{D}^4; \mathbb{Z})$. It will be sufficient to prove that $H_3(X\backslash \mathring{D}^4; \Lambda) \otimes_\Lambda \mathbb{Z} \cong H_3(X\backslash \mathring{D}^4; \mathbb{Z})$ because the boundary homomorphism $H_4(X, X\backslash \mathring{D}^4; \mathbb{Z}) \to H_3(X\backslash \mathring{D}^4; \mathbb{Z})$ is zero. But

$$H_3(X\backslash \mathring{D}^4; \Lambda) \otimes_\Lambda \mathbb{Z} \cong H_3(X\backslash \mathring{D}^4; \mathbb{Z})$$

follows again from the universal coefficient spectral sequence

$$\mathrm{Tor}_p^\Lambda(H_q(X\backslash \mathring{D}^4; \Lambda), \mathbb{Z}) \Rightarrow H_{p+q}(X\backslash \mathring{D}^4; \mathbb{Z}). \square$$

We can rewrite the above diagram as follows.

$$\begin{array}{ccccccccc}
 & & \Pi_4(X, X\backslash \mathring{D}^4) \otimes_\Lambda \mathbb{Z} & = & \Pi_4(X, X\backslash \mathring{D}^4) \otimes_\Lambda \mathbb{Z} & \xrightarrow{\cong} & H_4(X, X\backslash \mathring{D}^4; \Lambda) \otimes_\Lambda \mathbb{Z} & & \\
 & & \Delta_* \downarrow & & \partial_* \otimes 1 \downarrow & & \downarrow \partial_* \otimes 1 & & \\
0 & \longrightarrow & \Gamma(\Pi_2(X\backslash \mathring{D}^4)) \otimes_\Lambda \mathbb{Z} & \longrightarrow & \Pi_3(X\backslash \mathring{D}^4) \otimes_\Lambda \mathbb{Z} & \longrightarrow & H_3(X\backslash \mathring{D}^4; \Lambda) \otimes_\Lambda \mathbb{Z} & \longrightarrow & 0 \\
 & & \cong \downarrow & & \downarrow & & \downarrow & & \\
H_4(X; \mathbb{Z}) & \longrightarrow & \Gamma(\Pi_2(X)) \otimes_\Lambda \mathbb{Z} & \longrightarrow & \Pi_3(X) \otimes_\Lambda \mathbb{Z} & \longrightarrow & H_3(X; \Lambda) \otimes_\Lambda \mathbb{Z} & \longrightarrow & 0.
\end{array}$$

Corollary 3.5. *The image of a generator of* $\Pi_4(X, X\backslash \mathring{D}^4) \otimes_\Lambda \mathbb{Z} \cong \mathbb{Z}$ *under* $\Delta_*$ *coincides with the image of a generator of* $H_4(X; \mathbb{Z})$ *in*

$$\Gamma(\Pi_2(X)) \otimes_\Lambda \mathbb{Z} \cong \Gamma(\Pi_2(X\backslash \mathring{D}^4)) \otimes_\Lambda \mathbb{Z}.$$

Proof. Note first that $\Gamma(\Pi_2(X)) \cong H_4(K(\Pi_2(X), 2); \mathbb{Z})$. It was shown in [3], Proposition 8, that $\Gamma(\Pi_2(X))$ is $\Lambda$-free. The above diagram, before tensoring with $\otimes_\Lambda \mathbb{Z}$, is therefore a resolution of the bottom row. The result then follows from this and the identification of $\mathrm{Tor}_1^\Lambda(H_3(X; \Lambda), \mathbb{Z}) \cong H_4(X; \mathbb{Z})$. $\square$

Remark. The homomorphism $H_4(X; \mathbb{Z}) \to \Gamma(\Pi_2(X)) \otimes_\Lambda \mathbb{Z}$ must be injective. Otherwise, $\mathbb{S}^3 = \partial D^4 \xrightarrow{i} X\backslash \mathring{D}^4$ would be extendable over a disc

$D_1^4 \xrightarrow{i_1} X\backslash\overset{\circ}{D}{}^4$. Then $\mathbb{S}^4 = D^4 \cup D_1^4 \xrightarrow{i \cup i_1} X$ would be a degree one map, implying $X$ homotopy equivalent to $\mathbb{S}^4$.

It was shown in [3] (proof of Proposition 8) that $\Gamma(\Pi_2(X)) \subset \Pi_2(X) \otimes_{\mathbb{Z}} \Pi_2(X)$ induces an inclusion

$$\Gamma(\Pi_2(X)) \otimes_\Lambda \mathbb{Z} \subset (\Pi_2(X) \otimes_{\mathbb{Z}} \Pi_2(X)) \otimes_\Lambda \mathbb{Z} \cong \Pi_2(X) \otimes_\Lambda \overline{\Pi_2(X)}.$$

Here the bar denotes the left $\Lambda$-module structure provided by the canonical anti-automorphism on $\Lambda$. Via Poincaré duality the image of the generator of

$$\Pi_4(X, X\backslash\overset{\circ}{D}{}^4) \otimes_\Lambda \mathbb{Z}$$

under $\Delta_*$ is then the intersection form

$$\lambda_X \colon H_2(X;\Lambda) \otimes_\Lambda \overline{H_2(X;\Lambda)} \to \Lambda.$$

Summarizing we have obtained the following result.

THEOREM 3.6. *Let $X$ be an oriented closed connected TOP four-manifold with free fundamental group. Then there is the following exact sequence:*

$$0 \to H_4(X;\mathbb{Z}) \xrightarrow{b} \Gamma(\Pi_2(X)) \otimes_\Lambda \mathbb{Z} \to \Pi_3(X) \otimes_\Lambda \mathbb{Z} \xrightarrow{\bar{h}} H_3(X;\Lambda) \otimes_\Lambda \mathbb{Z} \to 0,$$

*where $\bar{h}$ is induced by the Hurewicz homomorphism and*

$$b([X]) \in \Gamma(\Pi_2(X)) \otimes_\Lambda \mathbb{Z} \subset \Pi_2(X) \otimes_\Lambda \overline{\Pi_2(X)}$$

*is determined by the intersection form*

$$\lambda_X \colon H_2(X;\Lambda) \otimes_\Lambda \overline{H_2(X;\Lambda)} \to \Lambda.$$

We can identify the obstruction for extending $\alpha \vee \beta$ to $M_1$ by using the above sequence. More precisely, we consider the following diagram:

$$\begin{array}{ccccccc}
 & & & & \Pi_4(M_1 \backslash \overset{\circ}{D^4}) \otimes_\Lambda \mathbb{Z} & = & \Pi_4(M_1 \backslash \overset{\circ}{D^4}) \otimes_\Lambda \mathbb{Z} \\
 & & & & \Delta_* \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & \Gamma(\Pi_2(M_1 \backslash \overset{\circ}{D^4})) \otimes_\Lambda \mathbb{Z} & \longrightarrow & \Pi_3(M_1 \backslash \overset{\circ}{D^4}) \otimes_\Lambda \mathbb{Z} \\
 & & & & (\alpha\vee\beta)_{**} \downarrow \cong & & \downarrow (\alpha\vee\beta)_* \\
0 & \longrightarrow & H_4(M;\mathbb{Z}) & \xrightarrow{b} & \Gamma(\Pi_2(M)) \otimes_\Lambda \mathbb{Z} & \longrightarrow & \Pi_3(M) \otimes_\Lambda \mathbb{Z}.
\end{array}$$

Let $\mathbb{S}^3_1 = \partial(M_1 \backslash \overset{\circ}{D^4})$, then

$$\theta = (\alpha \vee \beta)_{**} \circ \Delta_*([\mathbb{S}^3_1]) - b([M]) \in \Gamma(\Pi_2(M)) \otimes_\Lambda \mathbb{Z}$$

is the obstruction for extending the map $\alpha \vee \beta$.

If we consider the obstruction $\theta$ in $\Pi_2(M) \otimes_\Lambda \overline{\Pi_2(M)}$, then it can be interpreted via Poincaré duality as the difference of the intersection forms over $\Lambda$, i.e.

$$\theta = \lambda^{\mathbb{Z}}_{M'} \otimes_{\mathbb{Z}} \Lambda - \lambda^{\Lambda}_{M}$$

This links with the main theorem of [3].

## 4. Appendix

Now we consider the special case $H_2(M;\mathbb{Q}) \cong 0$ and explicitely realize a homotopy equivalence between $M$ and the connected sum $\#_p(\mathbb{S}^1 \times \mathbb{S}^3)$. The proof is much clearer and simpler than the one given in [3]. As shown in Section 2, $H_2(M;\Lambda)$ is $\Lambda$-free. Since $\Lambda$ is not Noetherian, we need to see why $H_2(M;\Lambda)$ is finitely generated. It follows from the spectral sequence of the universal covering $\widetilde{M} \to M$ and of $H_2(B\Pi_1;\mathbb{Z}) \cong 0$ that $H_2(M;\mathbb{Z}) \cong H_2(M;\Lambda) \otimes_\Lambda \mathbb{Z}$. Therefore, if $H_2(M;\mathbb{Z}) \cong \oplus_r \mathbb{Z}$, then $H_2(M;\Lambda) \cong \oplus_r \Lambda$. Now the assumption $H_2(M;\mathbb{Q}) \cong 0$ implies that $H_2(M;\Lambda) \cong H_2(\widetilde{M};\mathbb{Z}) \cong \Pi_2(M) \cong 0$, hence

$$H_3(M;\Lambda) \cong H_3(\widetilde{M};\mathbb{Z}) \cong \Pi_3(\widetilde{M}) \cong \Pi_3(M)$$

by the Hurewicz theorem.

Using the spectral sequence

$$\mathrm{Tor}_i^{\Lambda}(H_j(M;\Lambda),\mathbb{Z}) \Rightarrow H_{i+j}(M;\mathbb{Z}),$$

we easily obtain $H_3(M;\Lambda)\otimes_{\Lambda}\mathbb{Z} \cong H_3(M;\mathbb{Z}) \cong \oplus_p\mathbb{Z}$.

Let us choose generators

$$f = \vee_p f_i\colon\ \vee_p\,\mathbb{S}^3 \to M$$

and

$$e = \vee_p e_i\colon\ \vee_p\,\mathbb{S}^1 \to M$$

for $H_3(M;\mathbb{Z}) \cong \oplus_p\mathbb{Z}$ and $H_1(M;\mathbb{Z}) \cong \oplus_p\mathbb{Z}$, respectively. We can always assume that their intersection numbers satisfy $e_i \cdot f_j = \delta_{i\,j}$, for any $i,j = 1,2,\ldots,p$.

Then we have a map

$$\psi\colon\ \vee_p\,(\mathbb{S}^1\vee\mathbb{S}^3) \to M$$

which goes into the 3-skeleton of $M$.

LEMMA 4.1. *The restriction*

$$\psi\colon\ \vee_p\,(\mathbb{S}^1\vee\mathbb{S}^3) \to M^{(3)}$$

*is a homotopy equivalence.*

PROOF. Obviously $\psi$ induces isomorphisms on $\Pi_1$ and on $\Pi_2 \cong 0$.

Let $(g_1,g_2,\ldots,g_p)$ be a basis of $\Pi_1(M) \cong *_p\mathbb{Z}$.

The homology sequence of the pair $(M, M^{(3)})$

$$0 \to H_4(M,M^{(3)};\Lambda) \cong \Lambda \to H_3(M^{(3)};\Lambda) \to H_3(M;\Lambda) \cong (\oplus_p\Lambda)/\sigma\Lambda \to 0,$$

where $\sigma = (g_1-1, g_2-1, \ldots, g_p-1) \in \oplus_p\Lambda$, yields $H_3(M^{(3)};\Lambda) \cong \oplus_p\Lambda$.

On the other hand, $H_3(\vee_p(\mathbb{S}^1\vee\mathbb{S}^3);\Lambda) \cong \oplus_p\Lambda$ and $\psi$ induces an isomorphism on $H_3(\cdot : \Lambda)$ by construction. This completes the proof. □

Let $\gamma\colon M^{(3)} \to \vee_p(\mathbb{S}^1\vee\mathbb{S}^3)$ be a homotopy inverse of $\psi$.

Lemma 4.2. *The composition*

$$M^{(3)} \xrightarrow{\ \gamma\ } \vee_p(\mathbb{S}^1 \vee \mathbb{S}^3) \subset \vee_p(\mathbb{S}^1 \times \mathbb{S}^3)$$

*extends to a map* $\varphi\colon M \to \vee_p(\mathbb{S}^1 \times \mathbb{S}^3)$.

Proof. The obstruction for extending $\gamma$ belongs to

$$H^4(M; \Pi_3(\vee_p(\mathbb{S}^1 \times \mathbb{S}^3))) \cong \Pi_3(\vee_p(\mathbb{S}^1 \times \mathbb{S}^3)) \otimes_\Lambda \mathbb{Z} \cong H_3(\vee_p(\mathbb{S}^1 \times \mathbb{S}^3); \Lambda) \otimes_\Lambda \mathbb{Z}.$$

The spectral sequence

$$\mathrm{Tor}_i^\Lambda(H_j(\vee_p(\mathbb{S}^1 \times \mathbb{S}^3); \Lambda), \mathbb{Z}) \Rightarrow H_{i+j}(\vee_p(\mathbb{S}^1 \times \mathbb{S}^3); \mathbb{Z})$$

gives isomorphisms

$$H_3(\vee_p(\mathbb{S}^1 \times \mathbb{S}^3); \Lambda) \otimes_\Lambda \mathbb{Z} \cong H_3(\vee_p(\mathbb{S}^1 \times \mathbb{S}^3); \mathbb{Z}) \cong \oplus_p \mathbb{Z},$$

hence $\Pi_3(\vee_p(\mathbb{S}^1 \times \mathbb{S}^3)) \otimes_\Lambda \mathbb{Z} \cong \oplus_p \Pi_3(\mathbb{S}^1 \times \mathbb{S}^3)$.

Therefore the $i$-th component of the obstruction is just the obstruction for extending

$$M^{(3)} \xrightarrow{\ \gamma\ } \vee_p(\mathbb{S}^1 \vee \mathbb{S}^3) \xrightarrow{\ i\text{-th}\ } \mathbb{S}^1 \vee \mathbb{S}^3 \subset \mathbb{S}^1 \times \mathbb{S}^3$$

to a map $M \to \mathbb{S}^1 \times \mathbb{S}^3$. Since the above composition is

$$(u_i \times v_i)|_{M^{(3)}}\colon M^{(3)} \to \mathbb{S}^1 \times \mathbb{S}^3$$

(see Section 2), it extends to $M$, and hence the $i$-th component of the obstruction vanishes. Thus $\psi$ extends to a map $\varphi\colon M \to \vee_p(\mathbb{S}^1 \times \mathbb{S}^3)$ as required. □

Remark. The extension $\varphi\colon M \to \vee_p(\mathbb{S}^1 \times \mathbb{S}^3)$ when composed with the inclusion $\vee_p(\mathbb{S}^1 \times \mathbb{S}^3) \subset \prod_1^p(\mathbb{S}^1 \times \mathbb{S}^3)$ is homotopic to

$$\prod_{i=1}^{p}(u_i \times v_i)\colon M \to \prod_1^p(\mathbb{S}^1 \times \mathbb{S}^3)$$

by construction, i.e. $\varphi\colon M \to \vee_p(\mathbb{S}^1 \times \mathbb{S}^3)$ is a deformation as requested at the beginning of Section 2.

Now we can proceed as in Lemma 2.1 to obtain the following result.

Theorem 4.3. *Let $M$ be a closed connected orientable 4-manifold such that $\Pi_1(M) \cong *_p\mathbb{Z}$ and $H_2(M;\mathbb{Q}) \cong 0$. Then there exists a homotopy equivalence*

$$\phi\colon M \to \#_p(\mathbb{S}^1 \times \mathbb{S}^3).$$

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