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On a Zelevinsky Theorem and the Schur Indices of the Finite Unitary Groups

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Abstract. Let G be the finite unitary group $U_n(\boldsymbol{F}_q)$ over a finite field $\boldsymbol{F}_q$ of characteristic p. Let U be a Sylow p-subgroup of G. We prove that, for any irreducible character χ of G that is contained in a certain class, there is a linear character λ of U such that $(\lambda^G, \chi)_G = 1$. As an application, we shall determine the local Schur indices of an irreducible character of G which belongs to such class.

1. Introduction

Let $\boldsymbol{F}_q$ be a finite field with q elements of characteristic p. In [4] I. M. Gel'fand and M. I. Graev proved:

Theorem A (Gel'fand-Graev [4, Theorems 1, 2]). *Let H be the special linear group $SL_n(\boldsymbol{F}_q)$ over $\boldsymbol{F}_q$, and let U be the upper-triangular maximal unipotent subgroup of H. Then*

(i) *For any irreducible character χ of H, there is a linear character λ of U such that $(\lambda^H, \chi)_H \neq 0$.*

(ii) *If λ is a linear character of U in "general position", then λ^H is multiplicity-free.*

It is well known that the assertion (ii) of Theorem A holds for any finite group of Lie type (T. Yokonuma [22], R. Steinberg [21, Theorem 49]; cf. R. W. Carter [1, Theorem 8.1.3]). But the assertion (i) of Theorem A does not hold generally for a finite group of Lie type (e.g. for $U_n(\boldsymbol{F}_q)$, $Sp_{2n}(\boldsymbol{F}_q)$, etc.).

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In [23] A. V. Zelevinsky proved:

THEOREM B (Zelevinsky [23, 12.5]). *Let H be the general linear group $GL_n(\boldsymbol{F}_q)$ and let U be the upper-triangular maximal unipotent subgroup of H. Then, for any irreducible character χ of H, there is a linear character λ of U such that $(\lambda^H, \chi)_H = 1$.*

As an application, Zelevinsky proved:

THEOREM C (Zelevinsky [23, 12.6], A. A. Kljačko [10]; cf. [15] for $p \neq 2$). *The Schur index $m_{\boldsymbol{Q}}(\chi)$ of any irreducible character χ of $GL_n(\boldsymbol{F}_q)$ with respect to $\boldsymbol{Q}$ is equal to one.*

The purpose of this paper is to show that Zelevinsky's Theorem B holds for a certain class of irreducible characters of $U_n(\boldsymbol{F}_q)$ (Theorem 1), and, as an application, we show that, for any irreducible character χ of $U_n(\boldsymbol{F}_q)$ contained in such class, we can determine the local Schur indices of χ in principle (Theorem 4).

As to the Schur indices of the irreducible characters of $G = U_n(\boldsymbol{F}_q)$, it is known that $m_{\boldsymbol{Q}}(\chi) \leqq 2$ for any irreducible character χ of G (R. Gow [5]) and that, for any irreducible character χ of G, we have $m_{\boldsymbol{Q}_l}(\chi) = 1$ for any prime number $l \neq p$ ([16]; for $p = 2$, we use some properties of the generalized Gelfand-Graev characters of G [9]). For $n \leqq 5$, all the local Schur indices of every irreducible character of G are completely determined ([17, 6]). Our result here is a certain contribution to the complete determination of the local Schur indices of all the irreducible characters of G. (In another paper [19], we give some sufficient conditions subject for that $m_{\boldsymbol{Q}}(\chi) = 1$.)

As to the use of Kawanaka's generalized Gelfand-Graev characters of a finite group of Lie type for the study of the rationality-properties of the irreducible character of such a group, we refer [18].

2. The unipotent values

2.1. Partitions

Let m be a positive integer. Let $\mathcal{P}_m$ be the set of all partitions of m. If μ is a partition of m, then we write $|\mu| = m$. We denote by 0 the unique partition of the number 0. $\mathcal{P}_m$ has the lexicographical ordering.

If $\mu = (m_1, \dots, m_s)$ is a partition of $m > 0$ and $\mu' = (m'_1, \dots, m'_{s'})$ is a partition of $m' > 0$, then we denote by $\mu + \mu'$ the partition $(m_1, \dots, m_s, m'_1, \dots, m'_{s'})$ of $m + m'$. If $\mu = (m_1, m_2, \dots, m_s)$ is a partition of m such that $m_1 \geqq m_2 \geqq \cdots \geqq m_s \geqq 0$ and $\mu' = (m'_1, m'_2, \dots, m'_s)$ is a partition of m' such that $m'_1 \geqq m'_2 \geqq \cdots \geqq m'_s \geqq 0$, then we denote by $\mu \cdot \mu'$ the partition $(m_1 + m'_1, m_2 + m'_2, \dots, m_s + m'_s)$ of $m + m'$. If d, v are positive integers and if $\pi = (p_1, p_2, \dots, p_s)$ is a partition of v, then we denote by $d \cdot \pi$ the partition $(dp_1, dp_2, \dots, dp_s)$ of dv. If μ is a partition of m, then $\tilde{\mu}$ will denote the conjugate partition of μ.

Let S_m denote the symmetric group of order $m!$. Then, as is well known, the conjugacy classes of S_m and the irreducible characters of S_m can be naturally parametrized by the partitions of m. For $\lambda, \rho \in \mathcal{P}_m$, let χ^λ_ρ or $\chi^\lambda(\rho)$ denote the value of the irreducible character χ^λ of S_m corresponding to λ at the class of S_m corresponding to ρ. It is well known that $\chi^{(m)} = 1_{S_m}$, $\chi^{(1^m)} = \operatorname{sgn}$ and $\chi^{\tilde{\lambda}} = \operatorname{sgn} \cdot \chi^\lambda$ and it is easy to see by induction on v that $\operatorname{sgn}(d \cdot \pi) = (-1)^{(d-1)v} \operatorname{sgn}(\pi)$, $\pi \in \mathcal{P}_v$.

2.2. The irreducible characters of $U_n(\boldsymbol{F}_q)$

Let $G = U_n(\boldsymbol{F}_q)$. Then, as to the character theory, by thanks to the truth of Ennola conjecture ([3]; R. Hotta and T. A. Springer [8], G. Lusztig and B. Srinivasan [13], G. Lusztig, D. Kazhdan, N. Kawanaka [9]), we can use V. Ennola's formulation in [3].

Let s be a positive integer. Then a set $g = \{k, k(-q), k(-q)^2, \dots, k(-q)^{s-1}\}$ of integers will be called an s-simplex with the roots $k(-q)^i$, $0 \leqq i \leqq s-1$, if the $k(-q)^i$ are all distinct modulo $q - (-1)^s$; we write $d(g) = s$. Let $\mathcal{Y}$ be the set of all s-simplexes for $s \geqq 1$. Put $\mathcal{P} = \bigcup_{m \geqq 0} \mathcal{P}_m (\mathcal{P}_0 = \{0\})$. Let X be the set of functions $\nu \colon \mathcal{Y} \to \mathcal{P}$ such that

$$\sum_{g \in \mathcal{Y}} |\nu(g)| d(g) = n.$$

For $\nu \in X$, set (formally)

$$\chi_\nu = (\cdots g^{\nu(g)} \cdots) = (g_1^{\nu_1} \cdots g_N^{\nu_N}),$$

where $g_1, \cdots, g_N$ are all the $g \in \mathcal{Y}$ such that $\nu(g) \neq 0$ and, for $1 \leqq i \leqq N$, $\nu_i = \nu(g_i)$. Then the χ_ν, $\nu \in X$, parametrize the irreducible characters of G.

For $\nu \in X$, we identify χ_ν with the irreducible character of G corresponding to it.

Let $Q_\rho^\lambda(q)$ be the Green polynomial of $GL_n(\boldsymbol{F}_q)$ ([7]). For $\pi = (1^{r_1}2^{r_2}3^{r_3}\cdots) \in \mathcal{P}_v$, put $z_\pi = 1^{r_1}r_1!2^{r_2}r_2!3^{r_3}r_3!\cdots$. If $n_1, \dots, n_N$ are positive integers, then we put $\mathcal{P}_{(n_1,\dots,n_N)} = \mathcal{P}_{n_1} \times \cdots \times \mathcal{P}_{n_N}$.

PROPOSITION 1. *Let $\chi = (g_1^{\nu_1}\cdots g_N^{\nu_N})$ be any irreducible character of $G = U_n(\boldsymbol{F}_q)$. For $1 \leqq i \leqq N$, put $d_i = d(g_i)$ and $v_i = |\nu_i|$. Let λ be a partition of n, and let u_λ be any unipotent element of G of type λ. Then we have:*

$$\chi(u_\lambda) = \eta(\chi) \sum_{(\pi_1,\dots,\pi_N)\in\mathcal{P}_{(v_1,\dots,v_N)}} \frac{1}{z_{\pi_1}\cdots z_{\pi_N}} \chi_{\pi_1}^{\nu_1}\cdots\chi_{\pi_N}^{\nu_N} \times Q^\lambda_{d_1\cdot\pi_1+\cdots+d_N\cdot\pi_N}(-q),$$

where $\eta(\chi) = \pm 1$ such that $\chi(u_\lambda) > 0$ if $\lambda = (1^n)$.

REMARK. We remark here about the relation between Ennola's parametrization of the irreducible characters of $G = U_n(\boldsymbol{F}_q)$ and G. Lusztig's parametrization ([11, 12]; also see [1, pp. 391–2]). Let $\boldsymbol{G} = GL_n(\bar{\boldsymbol{F}}_q)$, where $\bar{\boldsymbol{F}}_q$ is an algebraic closure of $\boldsymbol{F}_q$, and let $F'\colon \boldsymbol{G} \to \boldsymbol{G}$ be the endomorphism of $\boldsymbol{G}$ given by $F'([g_{ij}]) = {}^t[g_{ij}^q]^{-1}$ for $[g_{ij}] \in \boldsymbol{G}$. Then F' is the Frobenius map relative to some $\boldsymbol{F}_q$-structure on $\boldsymbol{G}$, and the group $\boldsymbol{G}(\boldsymbol{F}_q) = \boldsymbol{G}^{F'}$ of F-fixed points of $\boldsymbol{G}$ is isomorphic to G. The dual group $\boldsymbol{G}^{\#}$ of $\boldsymbol{G}$ is isomorphic to $\boldsymbol{G}$. Let $\chi = (g_1^{\nu_1}\cdots g_N^{\nu_N})$ be an irreducible character of G. Then χ is a unipotent character of G if and only if $N = 1$, $d(g_1) = 1$ and 0 is the root of g_1. χ is a semisimple character of G (i.e. $p \nmid \chi(1)$) if and only if, for $1 \leqq i \leqq N$, $\nu_i = (v_i)$ ($v_i = |\nu_i|$). And χ is a regular character of G (i.e. an irreducible component of the Gelfand-Graev character Γ_G of G) if and only if, for $1 \leqq i \leqq N$, $\nu_i = (1^{v_i})$. Generally, the dual class $(g_1^{(1^{v_1})}\cdots g_N^{(1^{v_N})})$ determines the unique semisimple conjugacy class (s) of $G = \boldsymbol{G}^{\#}(\boldsymbol{F}_q)$ (see [3, pp. 6–7]). The partitions $\nu_1, \dots, \nu_N$ determine a unique unipotent character ρ of $H(s) = (Z_{\boldsymbol{G}^{\#}}(s))^{\#}(\boldsymbol{F}_q)$ ($Z_{\boldsymbol{G}^{\#}}(s)$ is the centralizer of s in $\boldsymbol{G}^{\#}$). We see easily that $\chi(1) = \chi_s(1)\rho(1)$, where χ_s is the semisimple character $(g_1^{(v_1)}\cdots g_N^{(v_N)})$. This may be regarded as the "Jordan decomposition" of χ. Thus we can regard the mapping $(s,\rho) \to \chi$ as Lusztig's parametrization mapping for the irreducible characters of G (cf. [11]).

3. Linear characters of U

3.1.

We say that a partition μ of n is involutive if the parts of μ are arranged so that $\mu = (n_1, n_2, \dots, n_s, n_{s+1}, n_s, \dots, n_2, n_1)$ (possibly $n_{s+1} = 0$). For example, if $n = 4$, then (4), (2^2), (21^2) and (1^4) are the involutive partitions of 4 and (31) is not involutive.

Let $\boldsymbol{G} = GL_n(\bar{\boldsymbol{F}}_q)$, and let $F\colon \boldsymbol{G} \to \boldsymbol{G}$ be the endomorphism of $\boldsymbol{G}$ given by $F([g_{ij}]) = w_0^t[g_{ij}^q]^{-1}w_0$, where $w_0 = \begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & 0 \end{bmatrix}$. Then $G = \boldsymbol{G}^F \simeq U_n(\boldsymbol{F}_q)$.

Let $\boldsymbol{U}$ be the upper triangular maximal unipotent subgroup of $\boldsymbol{G}$. Then $\boldsymbol{U}$ is F-stable and $U = \boldsymbol{U}^F$ is a Sylow p-subgroup of G. For $1 \leqq k \leqq n-1$, set $\boldsymbol{U}_k = \{u = [u_{ij}] \in \boldsymbol{U} \mid u_{i,i+1} = 0 \text{ for } i \neq k \text{ and } u_{ij} = 0 \text{ if } j - i \geqq 2\}$. Then, for $1 \leqq k \leqq n-1$, we have $F(\boldsymbol{U}_k) = \boldsymbol{U}_{n-k}$, so F acts on $\Delta = \{1, 2, \dots, n-1\}$ by $F(\boldsymbol{U}_k) = \boldsymbol{U}_{F(k)}$. Let I be the set of orbits of F on Δ. Let $\boldsymbol{U}.$ be the derived group of $\boldsymbol{U}$. Then $\boldsymbol{U}/\boldsymbol{U}. = \prod_{k\in\Delta} \boldsymbol{U}_k$. For $i \in I$, set $\boldsymbol{U}_i = \prod_{k\in i} \boldsymbol{U}_k (\subset \boldsymbol{U}/\boldsymbol{U}.)$. Then we have $\boldsymbol{U}^F/\boldsymbol{U}.^F = (\boldsymbol{U}/\boldsymbol{U}.)^F = \prod_{i\in I} \boldsymbol{U}_i^F$. For $i \in I$, we have $\boldsymbol{U}_i^F \simeq \boldsymbol{F}_{q^2}$ or $\boldsymbol{F}_q$.

Let $\mu = (n_1, \dots, n_s, n_{s+1}, n_s, \dots, n_1)$ be any involutive partition of n, and put

$$\boldsymbol{L}_\mu = \left\{ \begin{bmatrix} A_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & A_s & & & & \\ & & & A_{s+1} & & & \\ & & & & A'_s & & \\ & & & & & \ddots & \\ 0 & & & & & & A'_1 \end{bmatrix} \;\middle|\; A_i, A'_i \in GL_{n_i}(\bar{\boldsymbol{F}}_q), \; 1 \leqq i \leqq s, A_{s+1} \in GL_{n_{s+1}}(\bar{\boldsymbol{F}}_q) \right\}$$

(A_{s+1} does not occur in the above expression if $n_{s+1} = 0$). Put $\boldsymbol{P}_\mu = \boldsymbol{L}_\mu \boldsymbol{U}$. Then $\boldsymbol{P}_\mu$ is an F-stable parabolic subgroup of $\boldsymbol{G}$ and $\boldsymbol{L}_\mu$ is an F-stable Levi subgroup of $\boldsymbol{P}_\mu$. We put $P_\mu = \boldsymbol{P}_\mu^F$ and $L_\mu = \boldsymbol{L}_\mu^F$.

Let ϕ be a linear character of U. Then ϕ can be regarded as a character of $U/U.$ ($U. = \boldsymbol{U}.^F$). We say that ϕ is of type μ if, for $i \in I$, ϕ is non-trivial on $U_i = \boldsymbol{U}_i^F$ if $\boldsymbol{U}_i \subset \boldsymbol{L}_\mu$ and trivial on U_i if $\boldsymbol{U}_i \not\subset \boldsymbol{L}_\mu$. Conversely, it will be clear that if ϕ is any linear character of U, then there is uniquely determined involutive partition μ of n such that ϕ is of type μ.

Let ϕ be any linear character of U of type μ. Let Γ_μ be the Gelfand-Graev character of L_μ. Then we have

$$\phi^G = \mathrm{Ind}_{P_\mu}^G(\Gamma_\mu),$$

where we regard Γ_μ as a character of P_μ through the natural map $P_\mu \to P_\mu/V_\mu = L_\mu$ ($\boldsymbol{V}_\mu$ is the unipotent radical of $\boldsymbol{P}_\mu$ and $V_\mu = \boldsymbol{V}_\mu^F$).

4. Induced characters of $\boldsymbol{G}$

4.1.

Let $\boldsymbol{G} = GL_n(\bar{\boldsymbol{F}}_q)$ and let $F' \colon \boldsymbol{G} \to \boldsymbol{G}$ be the endomorphism of $\boldsymbol{G}$ given by $F'([g_{ij}]) = {}^t[g_{ij}^q]^{-1}$. Then F' is the Frobenius map of $\boldsymbol{G}$ corresponding to some $\boldsymbol{F}_q$-rational structure on $\boldsymbol{G}$. We have $\boldsymbol{G}^{F'} \simeq U_n(\boldsymbol{F}_q)$.

Let $\boldsymbol{T}_0$ be the diagonal maximal torus of $\boldsymbol{G}$. Then $\boldsymbol{T}_0$ is F'-stable. Let $W = W_{\boldsymbol{G}} = N_{\boldsymbol{G}}(\boldsymbol{T}_0)/\boldsymbol{T}_0$, where $N_{\boldsymbol{G}}(\boldsymbol{T}_0)$ is the normalizer of $\boldsymbol{T}_0$ in $\boldsymbol{G}$. Then F' acts on W trivially. W can be naturally identified with the symmetric group S_n. The $\boldsymbol{G}^{F'}$-conjugacy classes of F-stable maximal tori of $\boldsymbol{G}$ can be parametrized by the conjugacy classes of $W = S_n$, and the latter can be parametrized by the partitions of n. For $\rho \in \mathcal{P}_n$, let $\boldsymbol{T}_\rho$ denote one of the F'-stable maximal tori of $\boldsymbol{G}$ corresponding to ρ.

Let ρ be a partition of n, and suppose that $\boldsymbol{T}_\rho = y\boldsymbol{T}_0 y^{-1}$, $y \in \boldsymbol{G}$. Put $w = y^{-1}F'(y) \bmod \boldsymbol{T}_0 \in W$. Then ad y induces an identification: (F' on $\boldsymbol{T}_\rho$) = (ad $w \circ F'$ on $\boldsymbol{T}_0$), so we have:

$$|\boldsymbol{T}_\rho^{F'}| = |\boldsymbol{T}_\rho^{\mathrm{ad}\ w\circ F'}| = |c_\rho(-q)|,$$

where if $\rho = (1^{r_1}2^{r_2}3^{r_3}\cdots)$, then $c_\rho(q) = (q-1)^{r_1}(q^2-1)^{r_2}(q^3-1)^{r_3}\cdots$.

In the following, if $\boldsymbol{S}$ is a maximal torus of a connected reductive group $\boldsymbol{M}$, then we write $W_{\boldsymbol{M}}(\boldsymbol{S}) = N_{\boldsymbol{M}}(\boldsymbol{S})/\boldsymbol{S}$.

Let ρ be a partition of n, and let s_ρ be an element of S_n contained in the class of S_n corresponding to ρ. Then $W_{\boldsymbol{G}}(\boldsymbol{T}_\rho)^{F'}$ is isomorphic to $W_{\boldsymbol{G}}(\boldsymbol{T}_0)^{\mathrm{ad}\ w\circ F'} = Z_{S_n}(s_\rho)$, so we have

$$|W_{\boldsymbol{G}}(\boldsymbol{T}_\rho)^{F'}| = |Z_{S_n}(s_\rho)| = z_\rho.$$

Let $F\colon \boldsymbol{G} \to \boldsymbol{G}$ be as in §3. Then F acts on W by ad w_0, and the $\boldsymbol{G}^F$-conjugacy classes of F-stable maximal tori of $\boldsymbol{G}$ can be parametrized by the F-conjugacy classes in S_n ([2, p. 107]). For $w \in W$, let $\boldsymbol{T}(w)$ denote one of the F-stable maximal tori of $\boldsymbol{G}$ corresponding to w.

Let $w \in W$, and suppose that $\boldsymbol{T}(w) = z\boldsymbol{T}_0 z^{-1}$ with $z^{-1}F(z) \bmod \boldsymbol{T}_0 = w$. Then ad z induces an identification: (F on $\boldsymbol{T}(w)$) = (ad $w \circ F$ on $\boldsymbol{T}_0$), so we have

$$|\boldsymbol{T}(w)^F| = |\boldsymbol{T}_0^{\mathrm{ad}\ w\circ F}| = |\boldsymbol{T}_0^{\mathrm{ad}\ ww_0\circ F'}| = |\boldsymbol{T}_{\rho(ww_0)}^{F'}|,$$

where $\rho(ww_0)$ is the partition of n corresponding to the class of $W = S_n$ containing ww_0.

In the following, if $\boldsymbol{M}$ is an F-stable (resp. F'-stable) reductive subgroup of $\boldsymbol{G}$, then we denote by $\sigma(\boldsymbol{M})$ (resp. by $\sigma'(\boldsymbol{M})$) the $\boldsymbol{F}_q$-rank of $\boldsymbol{M}$ with respect to the $\boldsymbol{F}_q$-rational structure on $\boldsymbol{M}$ determined by F (resp. by F'). Then it is easy to see that, for $w \in W$, $\sigma(\boldsymbol{T}(w)) = \sigma'(\boldsymbol{T}_{\rho(ww_0)})$.

Let ρ be any partition of n. Then it is easy to see that $\sigma'(\boldsymbol{T}_\rho)$ is equal to the number of even parts of ρ. So we have

$$(-1)^{\sigma'(\boldsymbol{T}_\rho)} = \mathrm{sgn}(\rho).$$

We have $\sigma(\boldsymbol{G}) = \sigma(\boldsymbol{T}_0) = [n/2]$, the integral part of $n/2$, so we have

$$(-1)^{\sigma(\boldsymbol{G})-\sigma(\boldsymbol{T}(w))} = (-1)^{[n/2]}\,\mathrm{sgn}(\rho(ww_0)), \qquad w \in W.$$

4.2. Green function

Let $\boldsymbol{M}$ be a connected, reductive algebraic group, defined over $\boldsymbol{F}_q$, and let $F''\colon \boldsymbol{M} \to \boldsymbol{M}$ be the corresponding Frobenius endomorphism. If $\boldsymbol{S}$ is an F''-stable maximal torus of $\boldsymbol{M}$ and θ is a character of $S^{F''}$, then we denote by $R_{\boldsymbol{S}}^{\boldsymbol{M}}(\theta)$ the Deligne-Lusztig virtual character of $\boldsymbol{M}^{F''}$, and by $Q_{\boldsymbol{S},\boldsymbol{M}}$ the corresponding Green function. We shall often consider $Q_{\boldsymbol{S},\boldsymbol{M}}$ as a function on all $\boldsymbol{M}^{F''}$ by putting $Q_{\boldsymbol{S},\boldsymbol{M}}(x) = 0$ whenever x is not unipotent.

Now assume that $\boldsymbol{M} = \boldsymbol{G}$ with $F'' = F$ or F'. Let x be an element of $\boldsymbol{G}$ such that $x^{-1}F(x) = w_0$. Then ad x induces a bijection from $\boldsymbol{G}^{F'}$ onto $\boldsymbol{G}^{F}$, and we have $Q_{\boldsymbol{T}_{\rho(ww_0)},\boldsymbol{G}}(g) = Q_{\boldsymbol{T}(w),\boldsymbol{G}}(\text{ad } x(g))$, $g \in \boldsymbol{G}^{F'}$. Let λ be a partition of n, and let u_λ (resp. u'_λ) denote a unipotent element of $\boldsymbol{G}^{F}$ (resp. of $\boldsymbol{G}^{F'}$) of type λ. Then, by the result of Hotta-Springer-Kawanaka ([8], [9]), we have

$$Q_{\boldsymbol{T}(ww_0),\boldsymbol{G}}(u_\lambda) = Q^{\lambda}_{\rho(w)}(-q) = Q_{\boldsymbol{T}_{\rho(w)},\boldsymbol{G}}(u'_\lambda), \qquad w \in W.$$

4.3. The Gelfand-Graev character

Let $\Gamma_{\boldsymbol{G}}$ be the Gelfand-Graev character of $\boldsymbol{G}^{F'}$ (let ϕ' be the linear character of $U' = (\text{ad } x)^{-1}(U)$ corresponding (via the bijection ad $x\colon \boldsymbol{G}^{F'} \to \boldsymbol{G}^{F}$ in 4.2) to a linear character ϕ of U of type (n); then $\Gamma_{\boldsymbol{G}} = \text{Ind}_{U'}^{\boldsymbol{G}^{F'}}(\phi')$). Then, by Theorem 10.7 of [2], we have

$$\Gamma_{\boldsymbol{G}} = \sum_{(\boldsymbol{T},\theta) \bmod \boldsymbol{G}^{F'}} \frac{(-1)^{\sigma'(\boldsymbol{G})-\sigma'(\boldsymbol{T})}}{(R^{\boldsymbol{G}}_{\boldsymbol{T}}(\theta), R^{\boldsymbol{G}}_{\boldsymbol{T}}(\theta))_{\boldsymbol{G}^{F'}}} R^{\boldsymbol{G}}_{\boldsymbol{T}}(\theta),$$

where the sum is taken over all $\boldsymbol{G}^{F'}$-conjugacy classes of pairs $(\boldsymbol{T},\theta)$ of F'-stable maximal tori $\boldsymbol{T}$ of $\boldsymbol{G}$ and characters θ of $\boldsymbol{T}^{F'}$. Let ρ be any partition of n. Then, by using [2, Theorem 6.8], we see that

$$\sum_{\theta \bmod W_{\boldsymbol{G}}(\boldsymbol{T}_\rho)^{F'}} \frac{1}{(R^{\boldsymbol{G}}_{\boldsymbol{T}_\rho}(\theta), R^{\boldsymbol{G}}_{\boldsymbol{T}_\rho}(\theta))_{\boldsymbol{G}^{F'}}} = \frac{|\boldsymbol{T}_\rho^{F'}|}{z_\rho}.$$

Thus we get

$$\Gamma_{\boldsymbol{G}} = (-1)^{[n/2]} \sum_{\rho \in \mathcal{P}_n} \text{sgn}(\rho) \frac{|\boldsymbol{T}_\rho^{F'}|}{z_\rho} Q_{\boldsymbol{T}_\rho,\boldsymbol{G}}. \tag{1}$$

4.4. Degenerate linear characters

Let $\mu = (n_1, \dots, n_s, n_{s+1}, n_s, \dots, n_1)$ be any involutive partition of n, and let ϕ be any linear character of $U = \boldsymbol{U}^F$ of type μ. We assume that $\mu \neq (n)$. Let $W_\mu = W_{\boldsymbol{L}_\mu}(\boldsymbol{T}_0)$ (a subgroup of $W = W_{\boldsymbol{G}}(\boldsymbol{T}_0)$). Then we have

$$W_\mu = S_\mu = S_{n_1} \times \cdots \times S_{n_s} \times S_{n_{s+1}} \times S_{n_s} \times \cdots \times S_{n_1}.$$

By [2, Theorem 10.7, Proposition 8.2], we have

$$\phi^{\boldsymbol{G}^F} = \sum_{\substack{\boldsymbol{T} \bmod L_\mu \\ (\boldsymbol{T} \subset \boldsymbol{L}_\mu)}} (-1)^{\sigma(\boldsymbol{G}) - \sigma(\boldsymbol{T})} \frac{|\boldsymbol{T}^F|}{|W_{\boldsymbol{L}_\mu}(\boldsymbol{T})^F|} Q_{\boldsymbol{T},\boldsymbol{G}},$$

where the sum is taken over all L_μ-conjugacy classes of F-stable maximal tori $\boldsymbol{T}$ of $\boldsymbol{L}_\mu$.

F acts on $W_{\boldsymbol{L}_\mu}(\boldsymbol{T}_0) = S_\mu$ by ad w_0. The L_μ-conjugacy classes of F-stable maximal tori of $\boldsymbol{L}_\mu$ can be parametrized by the F-conjugacy classes of S_μ. S_μ acts on $S_\mu w_0$ by conjugations. We see that, for $w_1, w_2 \in S_\mu$, w_1 is F-conjugate to w_2 in S_μ if and only if $w_1 w_0$ is S_μ-conjugate to $w_2 w_0$ in $S_\mu w_0$.

Let w be an element of S_μ, and suppose that $\boldsymbol{T}(w) = y \boldsymbol{T}_0 y^{-1}$, $y \in \boldsymbol{L}_\mu$ ($y^{-1}F(y) \bmod \boldsymbol{T}_0 = w$). Then ad y induces an identification: (F on $W_{\boldsymbol{L}_\mu}(\boldsymbol{T}(w))$) = (ad $w \circ F$ on $W_{\boldsymbol{L}_\mu}(\boldsymbol{T}_0)$). Therefore we have:

$$\begin{aligned} |W_{\boldsymbol{L}_\mu}(\boldsymbol{T}(w))^F| &= |W_\mu^{\mathrm{ad}\ w \circ F}| \\ &= |W_\mu^{\mathrm{ad}\ ww_0 \circ F'}| \\ &= |Z_{W_\mu}(ww_0)| \qquad (F' = \text{id. on } W_\mu) \\ &= \frac{|W_\mu|}{|K_{W_\mu w_0}(ww_0)|}, \end{aligned}$$

where $K_{W_\mu w_0}(ww_0)$ is the W_μ-orbit of ww_0 in $W_\mu w_0$ under the conjugate action of W_μ.

Therefore we have

$$\phi^{\boldsymbol{G}^F} = \sum_{\substack{ww_0 \bmod W \\ (w \in W_\mu)}} (-1)^{\sigma(\boldsymbol{G}) - \sigma(\boldsymbol{T}(w))} \frac{|\boldsymbol{T}(w)^F|}{|W_{\boldsymbol{L}_\mu}(\boldsymbol{T}(w))^F|} Q_{\boldsymbol{T}(w),\boldsymbol{G}},$$

so, if ϕ' is the linear character of $U' = (\mathrm{ad}\ x)^{-1}(U)$ corresponding to the linear character ϕ of U, we have

$$\begin{aligned}
\phi'^{\boldsymbol{G}^{F'}} &= \sum_{\substack{ww_0 \bmod W_\mu \\ (w\in W_\mu)}} (-1)^{\sigma'(\boldsymbol{G})-\sigma'(\boldsymbol{T}_{\rho(ww_0)})} |\boldsymbol{T}^{F'}_{\rho(ww_0)}| \\
&\quad \times \frac{|K_{W_\mu w_0}(ww_0)|}{|W_\mu|} Q_{\boldsymbol{T}_{\rho(ww_0)},\boldsymbol{G}} \\
&= \sum_{\substack{w' \bmod W \\ (w'\in W)}} \Bigg\{ \sum_{\substack{ww_0 \bmod W_\mu \\ ww_0\sim w' \\ (w\in W_\mu)}} (-1)^{\sigma'(\boldsymbol{G})-\sigma'(\boldsymbol{T}_{\rho(w')})} \\
&\quad \times |\boldsymbol{T}^{F'}_{\rho(w')}| \frac{|K_{W_\mu w_0}(ww_0)|}{|W_\mu|} \Bigg\} Q_{\boldsymbol{T}_{\rho(w')},\boldsymbol{G}} \\
&\qquad (\text{“}\sim\text{” means conjugate in } W) \\
&= \sum_{\rho\in\mathcal{P}_n} (-1)^{[n/2]} \operatorname{sgn}(\rho) |\boldsymbol{T}^{F'}_\rho| \frac{|K_{S_n}(s_\rho)\cap S_\mu w_0|}{|S_\mu|} Q_{\boldsymbol{T}_\rho,\boldsymbol{G}},
\end{aligned}$$

where, for $\rho \in \mathcal{P}_n$, s_ρ is an element of S_n contained in the class of S_n corresponding to ρ and $K_{S_n}(s_\rho)$ denotes the class of s_ρ in S_n.

Let us express the $|K_{S_n}(s_\rho)\cap S_\mu w_0|/|S_\mu|$ in terms of characters of S_n. Let $H = \langle S_\mu, w_0\rangle$. Then $(H : S_\mu) = 2$ (note that $w_0 \notin S_\mu$ and $w_0 S_\mu w_0 = S_\mu$). Let ξ be the linear character of H defined by

$$\xi(x) = \begin{cases} 1 & \text{if } x \in S_\mu, \\ -1 & \text{if } x \in H - S_\mu. \end{cases}$$

Let

$$\chi = 1_H - \xi.$$

Then we have

$$\chi^{S_n}(s_\rho) = \frac{|K_{S_n}(s_\rho)\cap S_\mu w_0|}{|S_\mu|} |Z_{S_n}(s_\rho)|.$$

It is well known that one has:

$$1_{S_\mu}^{S_n} = \chi^\mu + \sum_{\nu>\mu} k_\mu^\nu \chi^\nu,$$

where the k_μ^ν are certain non-negative integers. As $1_{S_\mu}^{S_n} = 1_H^{S_n} + \xi^{S_n}$, we see that the irreducible components of $1_H^{S_n}$ and ξ^{S_n} are contained in $1_{S_\mu}^{S_n}$; we have

$$\begin{aligned}\chi^{S_n} &= 1_H^{S_n} - \xi^{S_n}\\ &= \epsilon_\mu \chi^\mu + \sum_{\nu > \mu} c_\mu^\nu \chi^\nu,\end{aligned}$$

where $\epsilon_\mu = 1$ or -1 according as χ^μ is contained in $1_H^{S_n}$ or ξ^{S_n} respectively and the c_μ^ν are some integers. Thus we have:

$$(2) \qquad \phi'^{\boldsymbol{G}^{F'}} = \sum_{\rho \in \mathcal{P}_n} (-1)^{[n/2]} \operatorname{sgn}(\rho) \left(\epsilon_\mu \chi_\rho^\mu + \sum_{\nu > \mu} c_\mu^\nu \chi_\rho^\nu \right) \frac{|\boldsymbol{T}_\rho^{F'}|}{z_\rho} Q_{\boldsymbol{T}_\rho, \boldsymbol{G}} .$$

5. Inner products

5.1. Some preliminaries

Let m be a positive integer, and let $x_1, \ldots, x_m$ be m different variables over $\boldsymbol{Q}$. For a partition $\lambda = (l_1, \ldots, l_m)$ of m with $l_1 \geqq \cdots \geqq l_m \geqq 0$, set

$$s_\lambda(x_1, \ldots, x_m) = \frac{\det[x_i^{l_j + m - j}]_{1 \leqq i, j \leqq m}}{\det[x_i^{m-j}]_{1 \leqq i, j \leqq m}},$$

which we call the S-function in the variables $x_1, \ldots, x_m$ corresponding to λ (see Macdonald [14, p. 24]).

Let $m_1, \ldots, m_k$ be positive integers such that $m_1 + \cdots + m_k = m$, and, for $1 \leqq i \leqq k$, let λ_i be a partition of m_i. Let $x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}, \ldots, z_1, \ldots, z_{m_k}$ be independent variables. Suppose that

$$\begin{aligned}&s_{\lambda_1}(x_1, \ldots, x_{m_1}) s_{\lambda_2}(y_1, \ldots, y_{m_2}) \cdots s_{\lambda_k}(z_1, \ldots, z_{m_k})\\ &\qquad = \sum_{\lambda \in \mathcal{P}_m} c_{\lambda_1 \lambda_2 \cdots \lambda_k}^\lambda s_\lambda(x_1, \ldots, x_{m_1}; y_1, \ldots, y_{m_2}; \ldots; z_1, \ldots, z_{m_k}),\end{aligned}$$

where $c_{\lambda_1 \lambda_2 \cdots \lambda_k}^\lambda$'s are some non-negative integers. Then we have ([14, I, (7.3)]):

$$\operatorname{Ind}_{S_{m_1} \times S_{m_2} \times \cdots \times S_{m_k}}^{S_m} (\chi^{\lambda_1} \times \chi^{\lambda_2} \times \cdots \times \chi^{\lambda_k}) = \sum_{\lambda \in \mathcal{P}_m} c_{\lambda_1 \lambda_2 \cdots \lambda_k}^\lambda \chi^\lambda.$$

Lemma 1 (see, e.g., [15, (2.4)]). *If* $\lambda > \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_k$ *or* $\lambda < \lambda_1 + \lambda_2 + \cdots + \lambda_k$*, then we have* $c^{\lambda}_{\lambda_1\lambda_2\cdots\lambda_k} = 0$*, and, if* $\lambda = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_k$ *or* $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$*, then we have* $c^{\lambda}_{\lambda_1\lambda_2\cdots\lambda_k} = 1$.

By the Frobenius reciprocity law, we get:

$$\chi^{\lambda} \mid S_{m_1} \times \cdots \times S_{m_k} = \sum_{(\lambda_1,\dots,\lambda_k)\in\mathcal{P}_{m_1}\times\cdots\times\mathcal{P}_{m_k}} c^{\lambda}_{\lambda_1\cdots\lambda_k}\chi^{\lambda_1}\times\cdots\times\chi^{\lambda_k}.$$

5.2.

Let $G = \boldsymbol{G}^{F'} \simeq U_n(\boldsymbol{F}_q)$. Let $\chi = (g_1^{\nu_1}\cdots g_N^{\nu_N})$ be any irreducible character of G. For $1 \leqq i \leqq N$, put $d_i = d(g_i)$, $v_i = |\nu_i|$. For a partition ρ of n, we put $Q_{\rho,G} = Q_{\boldsymbol{T}_\rho,\boldsymbol{G}}$. Then, by Proposition 1, by the formula (1) and by the orthogonality relations for the Green functions of G, we have:

$$\begin{aligned}(\Gamma_{\boldsymbol{G}},\chi)_G &= (-1)^{[n/2]+\sum_{i=1}^N(d_i-1)v_i}\eta(\chi)\prod_{i=1}^N(\chi^{\tilde{\nu}_i}, 1_{S_{v_i}})_{S_{v_i}} \\ &= \begin{cases} 1 & \text{if } \nu_i = (1^{v_i}) \text{ for } 1 \leqq i \leqq N, \\ 0 & \text{if } \nu_i \neq (1^{v_i}) \text{ for some } i.\end{cases}\end{aligned}$$

This is a known result (see the remark in §2.2).

Next, suppose that ϕ is any linear character of $U = \boldsymbol{U}^F$, of type μ, and suppose that $\mu \neq (n)$. Let ad $x\colon G \to \boldsymbol{G}^F$ be an isomorphism as before ($x^{-1}F(x) = w_0$), and let ϕ' be the linear character of $U' = (\mathrm{ad}\ x)^{-1}(U)$ corresponding to ϕ via ad x. Then, by Proposition 1 and the formula (2), we get:

$$\begin{aligned}(\phi'^G,\chi)_G = (-1)^{[n/2]}\sum_{\rho\in\mathcal{P}_n}\mathrm{sgn}(\rho)\left(\epsilon_\mu\chi^\mu_\rho + \sum_{\nu>\mu}c^\nu_\mu\chi^\nu_\rho\right)\frac{|c_\rho(-q)|}{z_\rho}\times\eta(\chi) \\ \times \sum_{\substack{(\pi_1,\dots,\pi_N)\in\mathcal{P}_{v_1}\times\cdots\times\mathcal{P}_{v_N} \\ \sigma = d_1\cdot\pi_1+\cdots+d_N\cdot\pi_N}} \frac{1}{z_{\pi_1}\cdots z_{\pi_N}}\chi^{\nu_1}_{\pi_1}\cdots\chi^{\nu_N}_{\pi_N}(Q_{\rho,G},Q_{\sigma,G})_G.\end{aligned}$$

By the orthogonality relations for the Green functions, we see that the latter expression of the above equality is equal to

$$
\begin{aligned}
&(-1)^{[n/2]}\eta(\chi) \sum_{\substack{\pi_1,\dots,\pi_N \\ \rho = d_1\cdot\pi_1+\cdots+d_N\cdot\pi_N}} \operatorname{sgn}(\rho)\left(\epsilon_\mu \chi^\mu_\rho + \sum_{\nu>\mu} c^\nu_\mu \chi^\nu_\rho\right) \\
&\qquad \times \frac{1}{z_{\pi_1}\cdots z_{\pi_N}} \chi^{\nu_1}_{\pi_1}\cdots\chi^{\nu_N}_{\pi_N} \\
&\quad = (-1)^{[n/2]+\sum_{i=1}^N (d_i-1)v_i}\eta(\chi) \sum_{\substack{\pi_1,\dots,\pi_N \\ \rho = d_1\cdot\pi_1+\cdots+d_N\cdot\pi_N}} \left(\epsilon_\mu \chi^\mu_\rho + \sum_{\nu>\mu} c^\nu_\mu \chi^\nu_\rho\right) \\
&\quad \times \frac{1}{z_{\pi_1}\cdots z_{\pi_N}} \chi^{\tilde\nu_1}_{\pi_1}\cdots\chi^{\tilde\nu_N}_{\pi_N}.
\end{aligned}
$$

Put $\eta(\chi)' = (-1)^{[n/2]+\sum_{i=1}^N (d_i-1)v_i}\eta(\chi)$. For $1 \leqq i \leqq N$, put $n_i = d_i v_i$. Then, by a remark in 5.1, we see that the last expression in the above equality is equal to:

$$
\begin{aligned}
&\eta(\chi)' \sum_{\pi_1,\dots,\pi_N} \sum_{(\xi_1,\dots,\xi_N)\in\mathcal{P}_{n_1}\times\cdots\times\mathcal{P}_{n_N}} \left(\epsilon_\mu c^\mu_{\xi_1\cdots\xi_N} + \sum_{\nu>\mu} c^\nu_\mu c^\nu_{\xi_1\cdots\xi_N}\right) \\
&\qquad \times \chi^{\xi_1}_{d_1\cdot\pi_1}\cdots\chi^{\xi_N}_{d_N\cdot\pi_N} \frac{1}{z_{\pi_1}\cdots z_{\pi_N}} \chi^{\tilde\nu_1}_{\pi_1}\cdots\chi^{\tilde\nu_N}_{\pi_N} \\
&\quad = \eta(\chi)' \Bigg\{ \epsilon_\mu \sum_{\xi_1,\dots,\xi_N} c^\mu_{\xi_1\cdots\xi_N} \prod_{i=1}^N \left(\sum_{\pi_i\in\mathcal{P}_{v_i}} \frac{1}{z_{\pi_i}} \chi^{\tilde\nu_i}_{\pi_i}\chi^{\xi_i}_{d_i\cdot\pi_i} \right) \\
&\qquad\qquad + \sum_{\nu>\mu} c^\nu_\mu \sum_{\xi_1,\dots,\xi_N} c^\nu_{\xi_1\cdots\xi_N} \prod_{i=1}^N \left(\sum_{\pi_i\in\mathcal{P}_{v_i}} \frac{1}{z_{\pi_i}} \chi^{\tilde\nu_i}_{\pi_i}\chi^{\xi_i}_{d_i\cdot\pi_i} \right) \Bigg\}.
\end{aligned}
$$

LEMMA 2 ([15, (2.8)]). *Let d, v be positive integers. Then one has*

$$
\sum_{\pi\in\mathcal{P}_v} \frac{1}{z_\pi} \chi^\nu_\pi \chi^\xi_{d\cdot\pi} = \begin{cases} 1 & \textit{if } \xi = d\cdot\nu, \\ 0 & \textit{id } \xi > d\cdot\nu. \end{cases}
$$

Assume that $\mu = (d_1 \cdot \tilde{\nu}_1) \cdot \cdots \cdot (d_N \cdot \tilde{\nu}_N)$. Then, by Lemmas 1, 2, we see easily that the last expression in the above equality is equal to $\eta(\chi)'\epsilon_\mu$. But, as $(\phi'^G, \chi)_G \geqq 0$, we must have $(\phi'^G, \chi)_G = 1$.

Thus we get

THEOREM 1. *Let $\chi = (g_1^{\nu_1} \cdots g_N^{\nu_N})$ be an irreducible character of $G = U_n(\boldsymbol{F}_q)$. Suppose that $\mu = (d(g_1) \cdot \tilde{\nu}_1) \cdot \cdots \cdot (d(g_N) \cdot \tilde{\nu}_N)$ is an involutive partition of n. Let ϕ be a linear character of a Sylow p-subgroup U' of G of "type μ". Then we have $(\phi^G, \chi)_G = 1$.*

6. The Schur index

6.1.

Let $G = U_n(\boldsymbol{F}_q)$. Then the following two results are known:

THEOREM 2 (R. Gow [5]). *The Schur index $m_{\boldsymbol{Q}}(\chi)$ of any irreducible character χ of G with respect to $\boldsymbol{Q}$ is at most two.*

THEOREM 3 (cf. [16] for $p \neq 2$). *Let χ be any irreducible character of G. Then, for any prime number $l \neq p$, we have $m_{\boldsymbol{Q}_l}(\chi) = 1$.*

In [16] Theorem 2 is proved for $p \neq 2$. We give here a proof of this theorem which is valid for all p. Let χ be any irreducible character of G. Then, by a result of Kawanaka [9], there is a generalized Gelfand-Graev character γ_μ of G such that $(\gamma_\mu, \chi)_G = 1$ ([9, (3.2.18), (3.3.24)(i)]). γ_μ is of $\boldsymbol{Q}$-valued ([9, (3.2.14)]) and is supported by a set of unipotent elements of G (this is clear from the construction of γ_μ). Then, by [20, Theorem 34 in p. 145, Proposition 33 in p. 106], we see that, for any prime number $l \neq p$, γ_μ is realizable in $\boldsymbol{Q}_l$. Thus we have $m_{\boldsymbol{Q}_l}(\chi) = 1$.

6.2.

Let us review some results in [17, §3]. Let G be as above, and let U be a Sylow p-subgroup of G. Let $U.$ be the derived group of U. If $p = 2$, then $U/U.$ is an elementary abelian 2-group, so that any linear character of U is realizable in $\boldsymbol{Q}$.

Assume that $p \neq 2$. Let ζ_p be a fixed primitive p-th root of unity, and let α be a certain generator of $\mathrm{Gal}(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q})$.

First, assume that n is odd. Then there is an element t in $N_G(U)$, of order $p-1$, such that $\phi^t = \phi^\alpha$ for any linear character ϕ of U, where ϕ^t is the linear character of U defined by $\phi^t(u) = \phi(tut^{-1})$, $u \in U$. Put $M = U\langle t \rangle$. Then we see that, if ϕ is any non-principal linear character of U, ϕ^M is an irreducible character of M which is realizable in $\boldsymbol{Q}$. Therefore, for any linear character ϕ of U, ϕ^G is realizable in $\boldsymbol{Q}$.

Next, assume that n is even ($p \neq 2$). We use the notation in 3.1. Let $\mu = (n_1, \ldots, n_s, n_{s+1}, n_s, \ldots, n_1)$ be any involutive partition of n, and let ϕ be any linear character of U of type μ. Then we have $\phi^G = \mathrm{Ind}_{P_\mu}^G(\Gamma_\mu)$. We have

$$L_\mu \simeq \prod_{i=1}^{s} GL_{n_i}(\boldsymbol{F}_{q^2}) \times U_{n_{s+1}}(\boldsymbol{F}_q).$$

Therefore, if $n_{s+1} = 0$, then, by Gow's theorem ([5]), Γ_μ is realizable in $\boldsymbol{Q}$, so ϕ^G is realizable in $\boldsymbol{Q}$.

Assume that $n_{s+1} \neq 0$. Then, as n is even, n_{s+1} is even. There is an element t' in $N_G(U)$, of order $(p-1)(q+1)$, such that $\phi^{t'} = \phi^\alpha$ and $c = t'^{p-1}$ is a generator of the centre Z of G. Put $M' = U\langle t' \rangle$. For $0 \leqq j \leqq q$, let ϕ_j be the extension of ϕ to $U\langle c \rangle$ given by $\phi_j(c) = \zeta_{q+1}^j$, where ζ_{q+1} is a previously fixed primitive $(q+1)$-th root of unity. For $0 \leqq j \leqq q$, let $\nu_j = \phi_j^{M'}$. Then we see that the ν_j are irreducible characters of M' and $\phi^{M'} = \nu_0 + \cdots + \nu_q$. For $0 \leqq j \leqq q$, let $k_j = \boldsymbol{Q}(\nu_j)$, the field generated over $\boldsymbol{Q}$ by the values of ν_j. Then we have $k_j = \boldsymbol{Q}(\zeta_{q+1}^j)$, $0 \leqq j \leqq q$. For $0 \leqq j \leqq q$, let A_j be the simple component of the group algebra $k_j[M']$ of M' over k_j associated with ν_j. Then, for $0 \leqq j \leqq q$, if $j \neq (q+1)/2$, A_j splits in k_j, and if $j = (q+1)/2$, k_j has non-zero Hasse invariants ($\equiv \frac{1}{2} \bmod 1$) only at the places ∞, p of $k_j = \boldsymbol{Q}$.

We have:

Theorem 4. *Let $\chi = (g_1^{\nu_1} \cdots g_N^{\nu_N})$ be an irreducible character of $G = U_n(\boldsymbol{F}_q)$. Let $\mu = (d(g_1) \cdot \tilde{\nu}_1) \cdot \cdots \cdot (d(g_N) \cdot \tilde{\nu}_N)$. Assume that μ is an involutive partition of n, and suppose that $\mu = (n_1, \ldots, n_s, n_{s+1}, n_s, \ldots, n_1)$. Then:*

(i) *If $p = 2$, or n is odd, or $n_{s+1} = 0$, then we have $m_{\boldsymbol{Q}}(\chi) = 1$.*

(ii) *Assume that $p \neq 2$, n is even, and $n_{s+1} \neq 0$.*

Recall that c is a generator of the centre of G. Then, if $\chi(c) \neq -\chi(1)$, we have $m_{\boldsymbol{Q}}(\chi) = 1$. Assume that $\chi(c) = -\chi(1)$. Then we have $m_{\boldsymbol{R}}(\chi) = 2$ or 1 according as χ is real or not respectively, and we have $m_{\boldsymbol{Q}_p}(\chi) = 2$ or

1 according as $[\boldsymbol{Q}_p(\chi) : \boldsymbol{Q}_p]$ is odd or even respectively ($\boldsymbol{Q}_p(\chi)$ is the field generated over $\boldsymbol{Q}_p$ by the values of χ).

PROOF. We use the notation in 3.1. Let ϕ be any linear character of U of type μ. Then, by Theorem 1, we have $(\phi^G, \chi)_G = 1$. Then, as we have observed above, if $p = 2$, or n is odd, or $n_{s+1} = 0$, ϕ^G is realizable in $\boldsymbol{Q}$, so we have $m_{\boldsymbol{Q}}(\chi) = 1$. Assume therefore that $p \neq 2$, n is even, and $n_{s+1} \neq 0$. We have $\chi(c) = \zeta_{q+1}^j \chi(1)$ for some j, $0 \leqq j \leqq q$. Then, by Schur's lemma, we must have $(\chi, \nu_j)_{M'} = 1$. If $j \neq (q+1)/2$, then ν_j is realizable in k_j and $\boldsymbol{Q}(\chi) \supset k_j$, so we have $m_{\boldsymbol{Q}}(\chi) = m_{k_j}(\chi) = 1$. Suppose that $j = (q+1)/2$. Then we have $m_{\boldsymbol{R}}(\nu_j) = m_{\boldsymbol{Q}_p}(\nu_j) = 2$. Therefore the last assertion follows from properties of Hasse invariants. □

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