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On a Family of Subgroups of the Teichmüller Modular Group of Genus Two Obtained from the Jones Representation

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Abstract. We study the image of the reduction of the specialized Jones representation of the Teichmüller modular group Γ of genus 2. As a result, we give a family of normal subgroups of Γ with finite unitary groups as quotients. We also show that they do not contain the Torelli subgroup of Γ.

0. Introduction

The purpose of the present paper is to give a family of "non-congruence" subgroups of the Teichmüller modular group of genus 2 by confirming a conjecture, posed by Takayuki Oda, on the image of the Jones representation.

In [J], Jones attached to a Young diagram a Hecke algebra representation of the braid group B_n on n strings. As was shown in [ibid,10], the Jones representation of B_6 corresponding to the three by two rectangular Young diagram factors through the Teichmüller modular group Γ of genus 2, namely, the mapping class group of a closed orientable surface of genus 2, and we thus get the representation $\pi : \Gamma \longrightarrow GL_5(\mathbf{Z}[x, x^{-1}])$ which is explicitly given [ibid, p362]. Now, for a certain natural number n, specializing x to $exp(2\pi\sqrt{-1}/n)$, we get a representation $\pi_n : \Gamma \longrightarrow GL_5(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers in the n-th cyclotomic field K. Let F be the maximal real subfield of K and take a non-zero ideal I of $\mathcal{O}_F$, the ring of integers of F. The reduction of π_n modulo $I_K = I\mathcal{O}_K$ gives a representation $\pi_{n,I} : \Gamma \longrightarrow GL_5(\mathcal{O}_K/I_K)$. Then, Oda conjectured that the image of $\pi_{n,I}$ is a certain unitary group if I is prime to an ideal of $\mathcal{O}_F$ containing (n). (For the precise formulation, see Section 2).

The main result of this paper is to confirm Oda's conjecture when I is a product of prime ideals of $\mathcal{O}_F$ which are inert in K/F (Main Theorem

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4.3). The proof consists of two steps. We first show that $\pi_{n,\wp}$ is irreducible under certain conditions on n and a prime $\wp$, and then investigate the list of all irreducible subgroups of $PSL_5(\mathcal{O}_K/\wp_K)$ due to Martino and Wagoner [M-W]. For the case of a product of inert primes, we apply a criterion of Weisfeiler [W] on the approximation of a Zariski-dense subgroup in a semisimple group over a finite ring. This proof is similar to that of Oda and Terasoma ([O-T]) on the similar problem for the Burau representations, where they use the induction after working with 2×2 matrices (see also [Be]). Our case is more complicated in the respect that we work with 5×5 matrices and so the finite group theory is more involved.

We also check that the kernel of $\pi_{n,I}$ does not contain the Torelli group using its explicit generator given by Birmann [Bi1].

Since the Teichmüller modular group is the fundamental group of the moduli space $\mathcal{M}$ of compact Riemann surfaces of genus 2, our result gives a tower of 3-folds, namely, finite Galois coverings of $\mathcal{M}$ with the Galois groups of finite unitary groups.

Finally, we mention that Kasahara ([Ka]) studied the image of the Torelli group of genus 2 under the specialized Jones representation at the 4-th root of unity and its abelianization in connection with the Johnson homomorphism.

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Notation. For an associative ring R with identity , $M_n(R)$ denotes the total matrix algebra over R of degree n, and $GL_n(R)$ denotes the groups of invertible elements of $M_n(R)$. We write $R^\times$ for $GL_1(R)$. For $A \in M_n(R)$, ${}^tA$, $tr(A)$, and $det(A)$ stand for the transpose, trace, and determinant of A, respectively. We write 0_n and 1_n for the zero and identity matrix in $M_n(R)$, respectively, and e_{ij} for the matrix unit and $diag(\cdot)$ for the diagonal matrix.

In Section 3, we shall use the notations in the list of [M-W].

1. The Jones representation of the Teichmüller modular group of genus 2 and its unitarity

In [J], Jones attached to each Young diagram with n tiles a Hecke algebra representation of the braid group B_n on n strings. As was shown in [ibid, Section 10], the representation of B_6 corresponding to the three by two rectangular Young diagram factors through the Teichmüller modular group Γ of genus 2, namely, the mapping class group of a closed orientable surface of genus 2. It is known that Γ admits the following presentation ([Bi2], Theorem 4.8, p 183-4).

generators: $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$.

defining relations:

$$\begin{cases} \theta_i\theta_{i+1}\theta_i = \theta_{i+1}\theta_i\theta_{i+1} \;\; (1 \leq i \leq 4), \\ \theta_i\theta_j = \theta_j\theta_i \;\; (|i-j| \geq 2, 1 \leq i, j \leq 5), \\ (\theta_1\theta_2\theta_3\theta_4\theta_5)^6 = 1, \\ (\theta_1\theta_2\theta_3\theta_4\theta_5^2\theta_4\theta_3\theta_2\theta_1)^2 = 1, \\ \theta_1\theta_2\theta_3\theta_4\theta_5^2\theta_4\theta_3\theta_2\theta_1 \;\text{ commutes with }\; \theta_i \;\; (1 \leq i \leq 5). \end{cases}$$

The Jones representation of Γ mentioned above is given explicitly on generators as follows ([J], p362).

$$\pi : \Gamma \longrightarrow GL_5(\mathbf{Z}[x, x^{-1}]), \;\; x = t^{1/5};$$

$$\pi(\theta_1) = x^{-2} \begin{pmatrix} -1 & 0 & 0 & 0 & t \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & t \end{pmatrix},$$

$$\pi(\theta_2) = x^{-2} \begin{pmatrix} t & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 \\ 0 & t & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\pi(\theta_3) = x^{-2}\begin{pmatrix} -1 & 0 & 0 & t & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix},$$

$$\pi(\theta_4) = x^{-2}\begin{pmatrix} t & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & t \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & t \end{pmatrix},$$

$$\pi(\theta_5) = x^{-2}\begin{pmatrix} -1 & t & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

We see that $det\pi(\theta_i) = -1$, $1 \leq i \leq 5$.

Let $A = A(x) \in M_n(\mathbf{Z}[x, x^{-1}]), x = t^{1/5}$. We write A^* for ${}^tA(x^{-1})$ and call A x-hermitian if $A = A^*$ and $A \neq 0_n$. For an x-hermitian matrix A, we define the unitary group with respect to A by

$$U_n(A) := \{g \in GL_n(\mathbf{Z}[x, x^{-1}]) | g^*Ag = A\}.$$

Lemma 1.1. *Let π be the representation given in Section* 1. *Then, there is an x-hermitian matrix $H \in M_5(\mathbf{Z}[x, x^{-1}])$ so that the image of π is contained in $U_5(H)$. Moreover, such H is determined up to $\mathbf{Q}(x)^\times$-multiple.*

Proof. Let $H = (h_{ij}) \in M_5(\mathbf{Z}[x, x^{-1}])$. We just write down the system of linear equations for h_{ij}'s:

$$\pi(\theta_i)^* H \pi(\theta_i) = H,\ 1 \leq i \leq 5.$$

We then find the following x-hermitian matrix satisfies these equations:

$$\begin{pmatrix} (1+t)(1+t^{-1}) & -(1+t) & 2 & -(1+t) & -(1+t) \\ -(1+t^{-1}) & 1+t+t^{-1} & -(1+t^{-1}) & 1 & 1 \\ 2 & -(1+t) & (1+t)(1+t^{-1}) & -(1+t) & -(1+t) \\ -(1+t^{-1}) & 1 & -(1+t^{-1}) & 1+t+t^{-1} & 1 \\ -(1+t^{-1}) & 1 & -(1+t^{-1}) & 1 & 1+t+t^{-1} \end{pmatrix}.$$

If H' is such a matrix, then $H^{-1}H'$ commutes with $\pi(\theta_i), 1 \leq i \leq 5$. By solving the system of linear equation for a_{ij}'s : $A\pi(\theta_i) = \pi(\theta_i)A, A = (a_{ij}) \in M_5(\mathbf{Z}[x, x^{-1}]), 1 \leq i \leq 5$, we find $A \in \mathbf{Q}(x)1_5$. □

We write $h = h_t$ for the matrix in the proof. We see that $det(h_t) = (t + t^{-1})^4(1 + t + t^{-1})$.

REMARK 1.2. The representation π multiplied by x^2 comes from the representation of Hecke algebra over $\mathbf{Z}[t, t^{-1}]$ which is known to be absolutely irreducible over $\mathbf{Q}(t)$. So, the uniqueness of H up to $\mathbf{Q}(x)^\times$-multiple in Lemma 1.1 follows also from Schur's lemma.

2. The reduction of the specialized Jones representation at a root of unity and the conjecture of Oda

Let n be a natural number. We assume that n is bigger than 6 and prime to 30. Let $\eta = exp(2\pi\sqrt{-1}/n)$ and $\zeta = \eta^5$. Set $K = \mathbf{Q}(\zeta), \mathcal{O}_K = \mathbf{Z}[\zeta], F = \mathbf{Q}(\zeta + \zeta^{-1})$ and $\mathcal{O}_F = \mathbf{Z}[\zeta + \zeta^{-1}]$.

By specializing $t \to \zeta, x = t^{1/5} \to \eta$ in the representation π, we get a representation

$$\pi_n : \Gamma \longrightarrow GL_5(\mathcal{O}_K).$$

Take a non-zero ideal I of $\mathcal{O}_F$ which is prime to n, and set $I_K = I\mathcal{O}_K$. The reduction of π_ζ modulo I_K defines the representation

$$\pi_{n,I} : \Gamma \longrightarrow GL_5(\mathcal{O}_K/I_K).$$

Then, $\pi_{n,I}$ certainly inherits the unitarity from π.

LEMMA 2.1. *The image of* $\pi_{n,I}$ *is contained in*

$$U_5(\mathcal{O}_K/I_K; h_{n,I}) := \{g \in GL_5(\mathcal{O}_K/I_K) \mid g^* h_{n,I} g = h_{n,I}\},$$

where $h_{n,I} := h_\zeta \bmod I_K$ *and* $g^* =^t g^\tau$, τ *is the involution induced from the generator of* $\mathrm{Gal}(K/F)$.

PROOF. This is immediate from Lemma 1.1. □

To formulate the conjecture, we twist π_I a little bit. Let $\chi : \Gamma \to \mathcal{O}_K^\times$ be the character defined by $\chi(\theta_i) = -1$, and set $\chi_I := \chi \bmod I_K$. We then consider $\rho_{n,I} := \pi_{n,I} \otimes \chi_I$. Since $det(\pi_\zeta(\theta_i)) = -1$, by Lemma 2.1, we have the inclusion

$$\rho_{n,I}(\Gamma) \subset SU_5(\mathcal{O}_K/I_K; h_{n,I}) := \{g \in U_5(\mathcal{O}_K/I_K; h_{n,I}) \mid det(g) = 1\}.$$

Then, the conjecture posed by Oda is formulated as follows.

Conjecture 2.2. *There is a non-zero ideal $\mathcal{C}$ of $\mathcal{O}_F$ containing (n) so that the image of $\rho_{n,I}$ coincides with $SU_5(h_{n,I})$ if I is prime to $\mathcal{C}$.*

3. The inert prime case

In this section, we verify Conjecture 2.2, when I is a maximal ideal $\wp$ of $\mathcal{O}_F$ which is inert in K/F. Set $\mathbf{F}_\wp = \mathcal{O}_F/\wp, \mathbf{F} = \mathbf{F}_{\wp K} = \mathcal{O}_K/\wp_K$ for simplicity. We simply write $\pi_\wp$ and $\rho_\wp$ for $\pi_{n,\wp}$ and $\rho_{n,\wp}$, respectively, also $h_\wp$ for $h_{n,\wp}$. As in Section 2, we assume that n is bigger than 6 and prime to 30. We note that $\wp$ is prime to an ideal of the form $(1-\zeta^m)$, m is prime to n, because $1-\zeta^m$ is a unit when n is a composite number and because $\wp$ is assumed to be inert and $(1-\zeta^m)$ is the maximal ideal of K over a totally ramified prime p when n is a power of p. In particular, $h_{n,\wp}$ is non-degenerate.

First, we show that the representation $\pi_\wp$ is irreducible. This can be obtained from the irreducibility of the corresponding specialized representation of the Hecke algebra over a finite field (This was communicated by the referee). Here, we give an elementary proof based on the observation that each $\pi_\wp(\theta_i)$ is a quasi-reflection given explicitly as follows.

Lemma 3.1. *Let $V = \mathbf{F}^{\oplus 5}$ be the representation space of $\pi_\wp$. For each $1 \le i \le 5$, there are subspaces X_i and Y_i of V such that*

$$V = X_i \oplus Y_i, \qquad dimX_i = 3,\ dimY_i = 2,$$
$$\pi_\wp(\theta_i)|_{X_i} = -\eta^{-2} id_{X_i}, \qquad \pi_\wp(\theta_i)|_{Y_i} = \eta^3 id_{Y_i},$$

where η denotes a primitive n-th root of 1 in $\mathbf{F}$ by abuse of notation. Note that $-\eta^{-2} \neq \eta^3$ for $(n, 10) = 1$

PROOF. Using the explicit matrix form of each $\pi(\theta_i)$ given in Section 1, we easily find the eigenspaces X_i and Y_i as follows:

$$\begin{array}{ll} X_1 = \{{}^t(x_1, x_2, 0, x_4, 0)\}, & Y_1 = \{{}^t(y_1, y_2, (1+\zeta)y_2, y_2, (1+\zeta^{-1})y_1)\} \\ X_2 = \{{}^t(0, 0, x_3, x_4, x_5)\}, & Y_2 = \{{}^t((1+\zeta)y_1, (1+\zeta^{-1})y_2, y_2, y_1, y_1)\} \\ X_3 = \{{}^t(x_1, x_2, 0, 0, x_5)\}, & Y_3 = \{{}^t(y_1, y_2, (1+\zeta)y_2, (1+\zeta^{-1})y_1, y_2)\} \\ X_4 = \{{}^t(0, x_2, x_3, x_4, 0)\}, & Y_4 = \{{}^t((1+\zeta)y_1, y_1, y_2, y_1, (1+\zeta^{-1})y_2)\} \\ X_5 = \{{}^t(x_1, 0, 0, x_4, x_5)\}, & Y_5 = \{{}^t(y_1, (1+\zeta^{-1})y_1, (1+\zeta)y_2, y_2, y_2)\}, \end{array}$$

where x_i's and y_i's run over $\mathbf{F}$ and $\zeta = \eta^5$. □

LEMMA 3.2. *The representation $\pi_\wp$ is irreducible.*

PROOF. Suppose that V has $\pi_\wp(\Gamma)$-invariant subspace $W \neq 0, V$. First, assume $\dim(W) = 1$. Let w be a base of W and write $w = x + y, x \in X_1, y \in Y_1$. If $\pi_\wp(\theta_1)w = \alpha w, \alpha \in \mathbf{F}^\times$, by Lemma 4.1, we have $(\alpha + \eta^2)x + (\alpha - \eta^3)y = 0$, from which we see that $w \in X_1$ or $w \in Y_1$. Let $w = {}^t(x_1, x_2, 0, x_4, 0) \in X_1$. Then, $\pi_\wp(\theta_2)w = \eta^{-2}\ {}^t(\zeta x_1, \zeta x_2, \zeta x_2, x_1 - x_4, x_1)$ should be in X_1 and so we get $w = 0$. This is a contradiction. Similarly, w can not be in Y_1. Hence, $\dim(W) > 1$. We may assume $\dim(W) = 2$, since the orthogonal complement of W with respect to $h_{n,\wp}$ is $\pi_\wp(\Gamma)$-invariant. For this case, consider the exterior square representation $\bigwedge^2 \pi_\wp : \ \Gamma \longrightarrow GL(\bigwedge^2 V)$. Then, $\bigwedge^2 W$ is an invariant subspace of $\bigwedge^2 V$ and $\dim(\bigwedge^2 W\ ) = 1$, and the similar argument to the above can be applied. Fix a basis of $X_1; v_1 = {}^t(1, 0, 0, 0, 0), v_2 = {}^t(0, 1, 0, 0, 0), v_3 = {}^t(0, 0, 0, 1, 0)$ and a basis of $Y_1; v_4 = {}^t(1, 0, 0, 0, 1+\zeta^{-1}), v_5 = {}^t(0, 1, 1+\zeta, 1, 0)$ and set $V_1 = \mathbf{F}v_1 \wedge v_2 + \mathbf{F}v_2 \wedge v_3 + \mathbf{F}v_1 \wedge v_3$, $V_2 = \mathbf{F}v_4 \wedge v_5$, and $V_3 = \mathbf{F}v_1 \wedge v_4 + \mathbf{F}v_1 \wedge v_5 + \mathbf{F}v_2 \wedge v_4 + \mathbf{F}v_2 \wedge v_5 + \mathbf{F}v_3 \wedge v_4 + \mathbf{F}v_3 \wedge v_5$. Then, we get the decomposition $\bigwedge^2 V = V_1 \oplus V_2 \oplus V_3$, and by Lemma 3.1, $\pi_\wp(\theta_1)$ acts on V_1, V_2, V_3 by the scalar multiples $\eta^{-4}, \eta^6, -\eta$, respectively, from which we see that $\bigwedge^2 W$ sits in one of V_i's. Suppose $W = \mathbf{F}w \subset V_1$. Then, $\bigwedge^2 \pi_\wp(\theta_j)w$, $2 \leq j \leq 5$, should be in V_1. We write w as the linear combination of the above basis of V_1 and write down the condition of these coefficients so that $\bigwedge^2 \pi_\wp(\theta_j)w = 0, 2 \leq j \leq 5$. Then, we easily get $w = 0$. Similarly, W can not be in V_2, V_3, where we use the assumption on $\wp$. We conclude that $\pi_\wp$ is irreducible. □

Now, we shall determine the image of $\rho_\wp$ by investigating the list of irreducible subgroups of $PSL_5(\mathbf{F})$ due to Martino and Wagoner [M-W].

Further, we assume that $\wp$ is prime to 2. By abuse of notation we write $\rho_\wp$ for the associated projective representation and set $G = \rho_\wp(\Gamma)$, which is an irreducible subgroup of $PSL_5(\mathbf{F})$ by Lemma 3.2.

First, we have the following

LEMMA 3.3. *The group G can not be realized over $\mathbf{F}_{p^a}, a < 2f$, where p^{2f} is the cardinality of $\mathbf{F}$.*

PROOF. Suppose that G is a subgroup of $PSL_5(\mathbf{F}_{p^a}), a < 2f$. Then, the characteristic polynomial $(X - \eta^{-2})^3(X + \eta^3)^2$ of $\rho_\wp(\theta_1)$ is invariant under the action of the Galois group $\mathrm{Gal}(\mathbf{F}_{p^{2f}}/\mathbf{F}_{p^a}) = <\sigma>$, where $\sigma =$ Frobenius automorphism. Hence, $(\eta^{-2})^\sigma = \eta^{-2p^a} = \eta^{-2}$ and so $p^a \equiv 1 \bmod n$, for $(n, 30) = 1$. This contradicts to the minimality of $2f$ so that $p^{2f} \equiv 1 \bmod n$. □

By Lemma 3.2, the following groups in the list of Martino-Wagoner can not be G: (1.3)-(a), (1.5), (1.7), (1.10)-(a), (1.12), (1.13), (1.14)-(a), (1.15), (1.16), where the numbers are those in [M-W].

Next, since the image of $\rho_\wp$ is contained in $SU_5(\mathcal{O}_K/\wp_K; h_\wp) \simeq SU_5(\mathbf{F})$, G can not be $PSL_5(\mathbf{F}), PSO_5(\mathbf{F})$ and $P\Omega_5(\mathbf{F})$, by comparing the orders. So, the groups (1.4), (1.8), (1.9) and (1.10)-(b) in [M-W] are excluded.

The following useful lemma was suggested by Eiichi Bannai.

LEMMA 3.4. *The subgroup of G generated by $\rho_\wp(\theta_1)$ and $\rho_\wp(\theta_3)$ is isomorphic to $\mathbf{Z}/2n\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z}$.*

PROOF. Using Lemma 3.1, we easily see that the order of $\rho_\wp(\theta_i)$ is $2n$ and $<\rho_\wp(\theta_1)> \cap <\rho_\wp(\theta_3)> = id$. □

The group (1.2) in [M-W] is a subgroup of the group which is an extension of a cyclic subgroup by $\mathbf{Z}/5\mathbf{Z}$. So, by Lemma 3.4, G can not be this group. Next, (1.11) is $PSL_2(\mathbf{F})$ or $PGL_2(\mathbf{F})$. We have a list of subgroups of $PSL_2(\mathbf{F})$ due to Dickson, [H], p213, Satz 8.27. Looking at this, by Lemma 3.3, G can not be a subgroup of $PSL_2(\mathbf{F})$. Since $PGL_2(\mathbf{F})$ is an extension of $PSL_2(\mathbf{F})$ by a cyclic subgroup of order 2, G can not be in $PGL_2(\mathbf{F})$. The similar argument can be applied to the groups (1.3)-(b),(c).

Finally, the group (1.1) in [M-W] can be excluded as follows. The following argument was also communicated by E. Bannai. The group (1.1) is an irreducible subgroup of A, where A is a global stabilizer in $PSL_5(\mathbf{F})$ of a simplex. Note that A is a monomial group and has a normal abelian subgroup N (diagonal group) so that $A/N \simeq S_5$= the symmetric group on 5 letters. Assume that G is an irreducible subgroup of A. Then, $\bar{G} = G/(G \cap N)$ is a subgroup of S_5. Let ρ_i be the image of $\rho_{\wp_i}(\theta_i)$ in $\bar{G}$. Since the order of $\rho_{\wp_i}(\theta_i)$ is $2n$ and $(n, 15) = 1$, the order of ρ_i is 1 or 2. On the other hand, ρ_i's satisfies the relations $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$ $(1 \leq i \leq 4)$, $\rho_i\rho_j = \rho_j\rho_i$ $(|i-j| \geq 2, 1 \leq i, j \leq 5)$, in particular, ρ_i's are conjugate each other. Suppose all $\rho_i = 1$. Then, $G \subset N$. This is not the case. Suppose all ρ_i's are permutations. Then, it is clear that the second relation in the above is not satisfied. Thus, the group (1.1) is excluded.

Summing up the above, we have

THEOREM 3.5. *Assume that n is prime to* 30 *, bigger than* 6 *and that a prime ideal $\wp$ of $\mathcal{O}_F$ does not divide* 2 *and is inert in K/F. Then, the image of $\rho_\wp$ coincides with $SU_5(\mathcal{O}_K/\wp_K; h_\wp)$.*

4. The case of a product of inert primes

In this section, we extend Theorem 3.5 to the case where the ideal I is a product of non-split primes. For this, we apply a criterion of Weisfeiler [W] on the approximation of a Zariski-dense subgroup in a semisimple group scheme over a finite ring to our situation.

Let I be a product of different prime ideals $\wp_i$ of $\mathcal{O}_F$, $I = \prod_{i=1}^r \wp_i^{e_i}$, where each $\wp_i$ is inert in K/F and prime to 6. Set $A = \mathcal{O}_F/I$ and $B = \mathcal{O}_K/I_K, I_K = I\mathcal{O}_K$. We also set $A_i = \mathcal{O}_F/\wp_i^{e_i}, B_i = \mathcal{O}_K/\wp_{i_K}^{e_i}, \wp_{i_K} = \wp_i\mathcal{O}_K$, and $k_i = \mathcal{O}_F/\wp_i = \mathbf{F}_{q_i}, \mathcal{O}_K/\wp_{i_K} = \mathbf{F}_{q_i^2}, q_i = N\wp_i$. So, we have decompositions $A = \oplus_{i=1}^r A_i, B = \oplus_{i=1}^r B_i$. The radical of A is $R = \prod_{i=1}^r \wp_i$ modulo I. We write $B = A \oplus A\beta, \beta^2 \in A$. Since each $\wp_i$ is inert in K/F, we see that $\beta \in B^\times$.

By the assumption on I, $h_I := h_\zeta$ modulo I_K defines a non-degenerate hermitian form on the free B-module $M = B^{\oplus 5}$. The following lemma which assures the existence of a unitary basis is well known for the case where I_K is a prime. For its proof, we refer to Propositions 2.3.1, 2.3.2 of [K-L]. For our general I, it follows easily by the argument using the decompositions $A = \oplus_{i=1}^r A_i, A^\times = \oplus_{i=1}^r A_i^\times$ etc.

Lemma 4.1. *Let J be the hermitian form on M given by the matrix*

$$\begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

Then, we have an isometry of hermitian modules

$$\phi: \ (M; h_I) \simeq (M; J).$$

Let SU_h and SU_J be the special unitary group schemes over A with respect to the hermitian forms h_I and J on M, respectively. Our task is to show $SU_h(A) = \rho_{n,I}(\Gamma)$. By Lemma 4.1, it is reduced to show $SU_J(A) = \Gamma'$, where $\Gamma' = \phi\rho_{n,I}(\Gamma)\phi^{-1}$.

Now, let us recall Weisfeiler's criterion for our pair $\Gamma' \subset SU_J(A)$.

The group A-scheme SU_J is a connected and simply-connected absolutely almost simple and quasi-split over A. Let T be a maximal A-torus of SU_J contained in a Borel A-subgroup. We say that $T(k_i)$ (= prime to p_i-part of $T(A_i)$, p_i = char. k_i) distinguishes the roots of T on SU_J if for two roots r_1, r_2 of T on SU_J, $r_1|T(k_i) = r_2|T(k_i)$ implies $r_1 = r_2$. For each i, $1 \leq i \leq r$, let N_i be the natural number such that if $q_i = N\wp_i > N_i$ then $T(k_i)$ distinguishes the roots of T.

Finally, let su_J be the Lie algebra of SU_J and $Ad : SU_J(A) \to GL(su_J(A))$ be the adjoint representation.

Lemma 4.2 ([W], Theorem (7.2)). *Notation being as above, assume that*

(1) $q_i \geq max(10, N_i)$, $1 \leq i \leq r$,

(2) The image of Γ' *in* $SU_J(k_i)$ *under the reduction modulo* $\wp_i/\wp_i^{e_i} \oplus \oplus_{j\neq i}\mathcal{O}_F/\wp_j^{e_j}$ *is the whole* $SU_J(k_i)$,

(3) $\mathbf{Z}[trAd(\Gamma')mod.R^2] = A/R^2$.

Then, we have the equality $\Gamma' = SU_J(A)$.

Remark 4.3. The assumption in (7.1) of [W] that the only proper ideal of the Lie algebra $su_J(k_i)$, $1 \leq i \leq r$, are central is satisfied for our su_J (*Remark* to (7.1) of [W]).

Let $R_{B/A}(\mathbf{G_{m}}_{,B})$ be the Weil restriction of scalers from B to A of the split B-torus $\mathbf{G_{m}}_{,B}$ of dimension 1 and $z \mapsto \bar{z}$ is the automorphism of $R_{B/A}(\mathbf{G_{m}}_{,B})$ induced from the non-trivial automorphism of B/A. For $z = x + y\beta \in B^{\times} = R_{B/A}(\mathbf{G_{m}}_{,B})(A),\ \bar{z} = x - y\beta$. Then, a maximal A-torus T of SU_J is given by

$$T := \{t = diag(t_1, t_2, t_3, \bar{t_2}^{-1}, \bar{t_1}^{-1}) |\ t_i \in R_{B/A}(\mathbf{G_{m}}_{,B}),\ t_3 = t_1^{-1}\bar{t_1}t_2^{-1}\bar{t_2}\},$$

which contains the maximal A-split torus

$$S := \{s = diag(s_1, s_2, 1, s_2^{-1}, s_1^{-1}) |\ s_i \in \mathbf{G_{m}}_{,A}\}.$$

Next, a A-basis of the Lie algebra $su_J(A)$ is given as follows: $e_{11} - e_{55}, e_{22} - e_{44}, \beta(e_{11} + e_{55} - 2e_{33}), \beta(e_{22} + e_{44} - 2e_{33}), e_{12} - e_{45}, \beta(e_{12} + e_{45}), e_{14} - e_{25}, \beta(e_{14} + e_{25}), e_{13} - e_{35}, \beta(e_{13} + e_{35}), e_{23} - e_{34}, \beta(e_{23} + e_{34}), e_{21} - e_{54}, \beta(e_{21} + e_{54}), e_{41} - e_{52}, \beta(e_{41} + e_{52}), e_{31} - e_{53}, \beta(e_{31} + e_{53}), e_{32} - e_{43}, \beta(e_{32} + e_{43}), \beta e_{15}, \beta e_{24}, \beta e_{51}, \beta e_{42}$.

Now, by the computation using this basis, we can describe the roots of T on SU_J. Define the characters $\epsilon_i, 1 \leq i \leq 5$, of T by the following: for $t = diag(t_1, t_2, t_3, \bar{t_2}^{-1}, \bar{t_1}^{-1}) \in T(A),\ \epsilon_i(t) = t_i, 1 \leq i \leq 3, \epsilon_4(t) = \bar{t_2}^{-1}, \epsilon_5(t) = \bar{t_1}^{-1}$. Then, $\pm(\epsilon_i - \epsilon_j), 1 \leq i < j \leq 5$, are all roots of T and positive roots $\epsilon_i - \epsilon_j, 1 \leq i \leq 5$, correspond to a Borel A-subgroup consisting of upper triangular matrices and containing T. The relative roots of S are $\pm(\epsilon_1 - \epsilon_2), \pm\epsilon_i, \pm 2\epsilon_i, 1 \leq i \leq 2$. Then, we first see that each $q_i > 5$ by our assumptions and so k_i is large enough so that $S(k_i)$ distinguishes the roots of S for any i. Next, we also see easily that $T(k_i)$ distinguishes 2 roots of T restricted to the same root of S for each i. Hence, the assumption (1) of Lemma 4.2 is just $q_i \geq 10,\ 1 \leq i \leq r$. The assumption (2) is a consequence of Theorem 3.5.

Finally, using the above basis of $su_J(A)$, a straightforward calculation shows that $tr(Ad(g)) = N_{B/A}(tr(g)) - 1$ for $g \in SU_J(A)$, where $N_{B/A}$ is the norm map attached to B/A and $N_{B/A}(tr(\rho_{n,I}(\theta_1))) = 13 - 6(\zeta + \zeta^{-1})$ modulo I. Therefore, the assumption (3) is certified.

Hence, by Lemma 4.2, we have

Main Theorem 4.4. *Assume that n is prime to* 30, *bigger than* 6. *Let I be a product of prime ideals $\wp_i$ of $\mathcal{O}_F$. Assume that each $\wp_i$ is inert in K/F and prime to* 6 *and $N\wp_i \geq 10$. Then, the image of $\rho_{n,I}$ coincides with $SU_5(\mathcal{O}_K/I_K, h_I)$.*

5. Comparison with the Torelli group and coverings of the moduli space of compact Riemann surfaces of genus 2

Let $Sp_2(\mathbf{Z})$ be the Siegel modular group of degree 4, namely, the group consisting of all $X \in GL_4(\mathbf{Z})$ satisfying

$$X \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} {}^tX = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}.$$

Let $\theta : \Gamma \to Sp_2(\mathbf{Z})$ be the canonical homomorphism induced by the abelianization map of Γ and the Nielsen isomorphism. We call the kernel of θ the Torelli group of genus 2 and write $\Gamma(N)$ for $\theta^{-1}(Sp_2(\mathbf{Z}; N))$, where $Sp_2(\mathbf{Z}; N)$ is the principal congruence subgroup of $Sp_2(\mathbf{Z})$ modulo a natural number N. The following result of Birmann allows us to compare our groups $\Gamma_{n,I}$ with the "congruence subgroups" $\Gamma(N)$ of Γ.

LEMMA 5.1 ([Bi1], Theorem 2). *The Torelli group of genus 2 is generated by the normal closure of* $(\theta_1\theta_2\theta_1)^4$.

PROPOSITION 5.2. *Under the same assumptions in Theorem 4.4, the group* $\Gamma_{n,I}$ *does not contain the Torelli group, hence any* $\Gamma(N)$.

PROOF. We see that 1-1 entry of $\rho_{n,I}((\theta_1\theta_2\theta_1)^4) = \eta^6$ which is not 1 modulo I by our assumptions. □

The geometrical interpretation of the above result is as follows.

Let $\mathcal{T}$ be the Teichmüller space of genus 2 and $\mathcal{M} = \mathcal{T}/\Gamma$ be the moduli space of compact Riemann surfaces of genus 2. Let $\mathcal{S}$ be the Siegel upper half space of degree 4 and $\mathcal{A} = \mathcal{S}/Sp_2(\mathbf{Z})$ be the moduli space of principally polarized abelian surfaces. The period map $\mathcal{T} \to \mathcal{S}$ is compatible with the actions of $\Gamma, Sp_2(\mathbf{Z})$ and θ, and thus we obtain the Torelli map $\mathcal{M} \longrightarrow \mathcal{A}$.

The Galois covering $\mathcal{A}_N = \mathcal{S}/Sp_2(\mathbf{Z}; N)$ over $\mathcal{A}$ with the Galois group $Sp_2(\mathbf{Z}/N\mathbf{Z})$ is the moduli space of principally polarized abelian surfaces with level N-structure. Then, Corollary 5.2 tells us that the spaces $\mathcal{T}/\Gamma_{n,I}$ give a family of Galois coverings over $\mathcal{M}$ with the Galois groups $SU_5(\mathcal{O}_K/I_K)$, which can not be obtained by the pull-back of any $\mathcal{A}_N$ via the Torelli map.

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