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Modular 3-Folds Obtained from Quaternion Unitary Groups of Degree 2

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Abstract. We consider three dimensional modular varieties obtained from quaternion unitary groups with level N. It is shown that these modular varieties are of general type if N is fully large.

§0. Introduction

In this paper we study modular 3-folds associated to quaternion unitary groups of degree 2. We recall some results about modular forms for quaternion unitary groups.

Let $\mathfrak{S}_2$ be the Siegel upper half space of degree 2, and $Sp_2(\mathbb{Z})$ the Siegel modular group of degree 2. Let $\Gamma_2(N)$ be the principal congruence subgroup of $Sp_2(\mathbb{Z})$ of level N. In [Y], Yamazaki studied Siegel modular forms with respect to $\Gamma_2(N)$ and obtained dimension formula for the space of cusp forms of weight ≥ 4 for $\Gamma_2(N)$, $N \geq 4$ by using the Riemann-Roch theorem. As a corollary, he proved that Siegel modular 3-fold associated to $\Gamma_2(N)$ is of general type if $N \geq 4$. Note that Y. Morita and U. Christian got the same dimension formula by using the Selberg trace formula.

In [A], Arakawa introduced quaternion unitary groups $\Gamma(N)$ (cf. §1) of degree 2 and studied modular forms with respect to such unitary groups. He obtained dimension formula for the space of cusp forms of weight ≥ 5 for $\Gamma(N)$, $N \geq 3$ by using the Selberg trace formula. His dimension formula is analogous to that of Yamazaki. Yamaguchi [Ya] also got the same result by using the Riemann-Roch theorem. Note that Hashimoto [H] obtained dimension formula for full modular group $\Gamma(1)$.

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The purpose of this paper is to prove an analogy of the result of Yamazaki [Y]. Namely, we shall prove the following:

THEOREM (Theorem 4.1, 4.6). *Let $Y(N)$ be a modular 3-fold associated to a quaternion unitary group $\Gamma(N)$. Then*

1. If N is fully large, then $Y(N)$ is of general type.

2. If the discriminant $d(\mathbf{B})$ of an indefinite division quaternion algebra $\mathbf{B}$ over $\mathbb{Q}$, which is used in order to define $Y(N)$, is fully large, then $Y(N)$ is of general type for $N \geq 3$.

Since the situation we here consider is $\mathbb{Q}$-rank 1 case, $\Gamma(N)$ has only 0-dimensional cusps. Hence our 3-folds are like Hilbert modular varieties in that they are obtained by adding 0-dimensional cusps to open 3-folds and desingularizing them. We shall apply methods for Hilbert modular varieties to our modular 3-folds. Let us explain the method for proving Theorem more precisely.

In [K], Knöller introduced defects for Hilbert modular cusp singularities. By using those, he proved that certain Hilbert modular 3-folds are of general type if discriminants related with these varieties are fully large. In [T], Tsuyumine proved that Hilbert modular varieites are of general type if associated discriminants are fully large. Both of these authors obtained some similar results independently. Their methods is essentially the same. In the process of intoducing defects for our singularities, we proceed the argument along the method of Knöller, referring also to that of Tsuyumine. We first define defects for our singularities as in [K]. Then we rewrite these defects in terms of the numbers of lattice points (Proposition 3.5). This result is an analogy of Theorem 2.4 in [K]. The corresponding result for Proposition 2 in [T] is Proposition 3.4.

The content of the paper is the following. In §1, we review some facts about quaternion unitary groups. In §2, we explain resolutions of cusp singularities. In §3, we define defects of cusps of $\Gamma(N)$. In §4, we prove our results by evaluating plurigenera $P_m(Y(N))$ from the below with the use of dimension formula of Arakawa. In §5, we give examples to Theorem 4.1 and Theorem 4.6.

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Notation. We denote by $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and $\mathbb{Q}_p$ the ring of rational integers, the field of rational, real, complex, and p-adic numbers, respectively. For a field K, $M_n(K)$ denotes the total matrix algebra over K of degree n. For a matrix $Z \in M_n(K)$, $\det(Z)$ denotes the determinant of Z, and $\mathrm{Im}(Z)$ denotes the imaginary part of Z if $K = \mathbb{C}$. For a matrix $A \in M_2(\mathbb{R})$, $A > 0$ means that A is positive definite. For a set S, $\#(S)$ stands for the cardinality of S. Put $\mathbb{R}_{\geq 0} := \{r \in \mathbb{R} \mid r \geq 0\}$.

§1. Quaternion unitary groups

In this section, we recall some facts about quaternion unitary groups needed later on. See Arakawa [A] and Hashimoto [H] for details.

1.1. Modular groups

Let $\mathbf{B}$ be an indefinite division quaternion algebra over $\mathbb{Q}$, and $^- : \mathbf{B} \to \mathbf{B}$ $(a \mapsto \overline{a})$ the canonical involution of $\mathbf{B}$. For an element a in $\mathbf{B}$, we call $N(a) := a\overline{a}$ the norm of a, and $\mathrm{tr}(a) := a + \overline{a}$ the trace of a. Since $\mathbf{B}_\infty := \mathbf{B} \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$, we identify $\mathbf{B}_\infty$ with $M_2(\mathbb{R})$ by fixing an isomorphism. Let G be the $\mathbf{B}$-unitary group of degree 2. We put

$$G_{\mathbb{Q}} := \left\{ g \in M_2(\mathbf{B}) \mid g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\overline{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

where ${}^t\overline{g} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $G_{\mathbb{Q}}$ is $\mathbb{Q}$-rational points of G. Let $\mathfrak{O}$ be a maximal order of $\mathbf{B}$. For any basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{O}$ over $\mathbb{Z}$, put

$$d(\mathbf{B}) := |\det(\mathrm{tr}(u_i u_j))|^{\frac{1}{2}}.$$

This number is independent of the choice of $\mathfrak{O}$ and $\{u_1, u_2, u_3, u_4\}$. For a natural number N, set

$$\Gamma(N) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{Q}} \mid a - 1, b, c, d - 1 \in N\mathfrak{O} \right\}.$$

Let $\mathfrak{S}_2 := \{Z \in M_2(\mathbb{C}) \mid {}^tZ = Z, \quad \mathrm{Im}(Z) > 0\}$ be the Siegel upper half space of degree 2, and set

$$\mathfrak{H} := \left\{ Z \in M_2(\mathbb{C}) \mid ZJ^{-1} \in \mathfrak{S}_2 \right\}, \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For the group $G_{\mathbb{R}}$ of $\mathbb{R}$-rational points of G, we have

$$qG_{\mathbb{R}}q^{-1} = Sp_2(\mathbb{R}) := \left\{ g \in M_4(\mathbb{R}) \mid g \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\},$$

where $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, q := \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$. The group $G_{\mathbb{R}}$ acts on $\mathfrak{H}$ by $g\langle Z \rangle = (aZ+b)(cZ+d)^{-1}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{R}}$, $Z \in \mathfrak{H}$. Though two pairs $(G_{\mathbb{R}}, \mathfrak{H})$ and $(Sp_2(\mathbb{R}), \mathfrak{S}_2)$ are the same essentially, we here consider the pair $(G_{\mathbb{R}}, \mathfrak{H})$. Since the $\mathbb{Q}$-rank of $G_{\mathbb{Q}}$ is 1, $\Gamma(N)$ has only point cusps.

1.2. Modular forms

For any positive integer k, let $M_k(\Gamma(N))$ be the $\mathbb{C}$-vector space of modular forms of weight k with respect to $\Gamma(N)$. Namely, $M_k(\Gamma(N))$ is the space of holomorphic functions $f(Z)$ on $\mathfrak{H}$ satisfying

$$f(g\langle Z \rangle) = \det(cZ+d)^k f(Z) \quad \text{for all} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N).$$

An element $f(Z)$ in $M_k(\Gamma(N))$ is called a *cusp form* if $|f(Z)\det(\mathrm{Im}(Z))^{k/2}|$ is bounded on $\mathfrak{H}$. We denote by $S_k(\Gamma(N))$ the $\mathbb{C}$-vector space of cusp forms of weight k with respect to $\Gamma(N)$. The following theorem is a dimension formula by Arakawa.

THEOREM 1.1 (Arakawa [A], Theorem). *Assume $k \geq 5, N \geq 3$. Then we have*

$$\begin{aligned} &\dim_{\mathbb{C}} S_k(\Gamma(N)) \\ =&2^{-7}3^{-3}5^{-1}[\Gamma(1):\Gamma(N)](k-1)(k-\frac{3}{2})(k-2)\prod_{p|d(\mathbf{B})}(p-1)(p^2+1) \\ &+ 2^{-4}3^{-1}[\Gamma(1):\Gamma(N)]N^{-3}\prod_{p|d(\mathbf{B})}(p-1). \end{aligned}$$

1.3. Lattices

Let $\mathbf{B}^-$ be the set of pure quaternions in $\mathbf{B}$. We put

$$L := \mathfrak{O} \cap \mathbf{B}^-, \quad L^* := \{y \in \mathbf{B}^- \mid \operatorname{tr}(xy) \in \mathbb{Z} \quad \text{for all} \quad x \in L\}.$$

Then L and L^* are lattices of rank 3 in $\mathbf{B}^-$.

Lemma 1.2 (Arakawa [A], Lemma 1). *Let $\{v_1, v_2, v_3\}$ be any basis of L over $\mathbb{Z}$. Then we have*

$$\det(\operatorname{tr}(v_i v_j)) = -2d(\mathbf{B})^2.$$

For a rational prime number p, put $\mathbf{B}_p := \mathbf{B} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. If p does not divide $d(\mathbf{B})$, then $\mathbf{B}_p \cong M_2(\mathbb{Q}_p)$. If p divides $d(\mathbf{B})$, then $\mathbf{B}_p$ is a division quaternion algebra over $\mathbb{Q}_p$. Put $\mathfrak{O}_p := \mathfrak{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers. Then $\mathfrak{O}_p$ is a maximal order of $\mathbf{B}_p$. If p does not divide $d(\mathbf{B})$, then $\mathfrak{O}_p \cong M_2(\mathbb{Z}_p)$. If p divides $d(\mathbf{B})$, then $\mathfrak{O}_p$ is the unique maximal order of $\mathbf{B}_p$. Moreover, we set $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $L_p^* := L^* \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Then the following fact holds for L_p^*.

Lemma 1.3 (Arakawa).

$$L_p^* = \{y \in \mathbf{B}_p \mid \operatorname{tr}(y) = 0, \ \operatorname{tr}(yx) \in \mathbb{Z}_p \textit{ for all } x \in L_p\}.$$

From the proof of Lemma 1.2 and Lemma 1.3, we get the next lemma.

Lemma 1.4. *Let $\{u_1, u_2, u_3\}$ be any basis of L^* over $\mathbb{Z}$. Then we have*

$$\det(\operatorname{tr}(u_i u_j)) = -\frac{1}{2d(\mathbf{B})^2}.$$

Since the proof of Lemma 1.4 is similar to that of Lemma 1.2, we omit it.

1.4. Fourier expansions

Definition 1.5. Set

$$L_+^* := \{y \in L^* \mid yJ > 0\}, \qquad L_+ := \{x \in L \mid xJ^{-1} > 0\}.$$

By Lemma 4 and Lemma 5 in Arakawa [A], the number of cusps of $\Gamma(1)$ is 1. Hence any cusp λ of $\Gamma(N)$ has the form $\lambda = \rho\sigma^{-1}$ for some $M_\lambda := \begin{pmatrix} \rho & * \\ \sigma & * \end{pmatrix} \in \Gamma(1)$. Here we take 1, 0 as ρ, σ when $\lambda = \infty$. The matrix M_λ maps ∞ to λ. Set $W_\lambda = M_\lambda^{-1} Z$. We can regard Z (resp. W_λ) as the coordinate of some neighborhood of λ (resp. ∞).

Proposition 1.6 (Arakawa). *Notation being as above, each modular form $f(Z) \in M_k(\Gamma(N))$ has the following Fourier expansion*

$$\det(-\sigma Z + \rho)^k f(Z) = a_\lambda(0) + \sum_{t \in L_+^*} a_\lambda(t) e\left[\frac{1}{N}\operatorname{tr}(tW_\lambda)\right], \tag{1.6.1}$$

at each cusp λ. Here $e[\,\cdot\,] = \exp(2\pi i\,\cdot\,)$. In particular, $f(Z) \in S_k(\Gamma(N))$ is equivalent to $a_\lambda(0) = 0$ for any cusp λ.

For the proof of this proposition, we refer to §13 in Maass [M].

Let $\mathfrak{O}^\times$ be the group of units in $\mathfrak{O}$. For any element $\epsilon \in \mathfrak{O}^\times$ and $x \in L$, we have $\epsilon x \bar{\epsilon} \in L$. The lattice L^* also has this property. The Fourier coefficients $a(t)$ in the above Proposition satisfy

$$a_\lambda(\epsilon t \bar{\epsilon}) = (N\epsilon)^k a_\lambda(t) \tag{1.7}$$

for $\epsilon \in \mathfrak{O}^\times$.

§2. Resolutions of singularities

In this section we deal with resolutions of cusp singularities of $\overline{\Gamma(N) \setminus \mathfrak{H}}$. We refer to Oda [O] for fundamental facts about toric varieties. For more detailed information on this section, see Tsuchihashi [Ts]. Since our singularities are analogous to Hilbert modular cusp singularities, see also Ehlers [E] for cusp resolutions.

We assume $N \geq 3$ in this paper. Then $\Gamma(N)$ acts freely on $\mathfrak{H}$. Since the $\mathbb{Q}$-rank of $G_{\mathbb{Q}}$ is 1, rational boundary components of $\mathfrak{H}$ with respect to $\Gamma(N)$ are 0-dimensional. Adding a finite number of points to $\Gamma(N) \setminus \mathfrak{H}$, we get a normal compact algebraic variety $\overline{\Gamma(N) \setminus \mathfrak{H}}$, which is called the *Satake-Baily-Borel compactification* of $\Gamma(N) \setminus \mathfrak{H}$. The variety $\overline{\Gamma(N) \setminus \mathfrak{H}}$ has only cusp singularities. A toroidal compactification of $\Gamma(N) \setminus \mathfrak{H}$ gives a resolution of cusp singularities of $\overline{\Gamma(N) \setminus \mathfrak{H}}$.

Setting

$$V := \{x \in M_2(\mathbb{R}) \mid \operatorname{tr}(x) = 0\}, \qquad V_+ := \{x \in V \mid xJ^{-1} > 0\},$$

we have $\mathfrak{H} = V + iV_+$. The set V is a free $\mathbb{R}$-module of rank 3, and V_+ is an open convex cone in V. The set $\mathbb{R}_+ := \{r \in \mathbb{R} \mid r > 0\}$ acts on V_+ in the usual manner. Put $\overline{V_+} := V_+/\mathbb{R}_+$. The set $\overline{V_+}$ is isomorphic to a hyperbolic surface. Set $\mathfrak{O}_N^\times := \{\epsilon \in \mathfrak{O}^\times \mid \epsilon \equiv 1 \pmod{N\mathfrak{O}}\}$. The group $\mathfrak{O}_N^\times$ acts on V_+ by $x \mapsto \epsilon x \bar{\epsilon}$ $(x \in V_+, \epsilon \in \mathfrak{O}_N^\times)$. The group $\mathfrak{O}_N^\times$ also acts on L in the same way. We have $V = L \otimes_{\mathbb{Z}} \mathbb{R}$. If $N \geq 3$, then $\mathfrak{O}_N^\times$ acts properly discontinuously and freely on $\overline{V_+}$. We find that a pair $(V_+, \mathfrak{O}_N^\times)$ is a Tsuchihashi cusp singularity (cf. [Ts]).

There exists a fan Σ of V such that

(i) $\displaystyle\bigcup_{\sigma \in \Sigma - \{0\}} (\sigma - \{0\}) = V_+$,

(ii) for any compact subset K of V_+, $\#\{\sigma \in \Sigma \mid \sigma \cap K \neq \phi\} < \infty$,

(iii) Σ is $\mathfrak{O}_N^\times$-invariant,

(iv) $\mathfrak{O}_N^\times$ acts freely on $\Sigma - \{0\}$, and

(v) $\#\left((\Sigma - \{0\})/\mathfrak{O}_N^\times\right) < \infty$.

Moreover, taking a $\mathfrak{O}_N^\times$-invariant subdivision of Σ, we may suppose the following conditions:

(vi) for any $\sigma, \tau \in \Sigma$, $\#\{\epsilon \in \mathfrak{O}_N^\times \mid \epsilon \cdot \sigma \cap \tau \neq \{0\}\} \leq 1$, and

(v) each $\sigma \in \Sigma$ is nonsingular, i.e., σ is spanned by a part of a $\mathbb{Z}$-basis of L.

For a positive integer $k (\leq 3)$, let Σ_k be the set of k-dimensional cones in Σ. For any element τ of Σ_1, $v(\tau)$ stands for the primitive element of L with $\mathbb{R}_{\geq 0} v(\tau) = \tau$. Since each element of Σ is nonsingular, every element $\sigma \in \Sigma$ has k one-dimensional faces. For any $\sigma \in \Sigma_k$, S_σ denotes the $(k-1)$-simplex in V spanned by $\{v(\tau) \mid \tau \in \Sigma_1, \tau \text{ is a face of } \sigma\}$. Put $\widetilde{K} := \{S_\sigma \mid \sigma \in \Sigma - \{0\}\}$. Then $|\widetilde{K}| := \displaystyle\bigcup_{\sigma \in \Sigma - \{0\}} S_\sigma$ is isomorphic to $\overline{V_+}$ by a natural map $V_+ \to \overline{V_+}$, and $\widetilde{K}$ gives a triangulation of $\overline{V_+}$. The group $\mathfrak{O}_N^\times$ acts on freely on $\widetilde{K}$ by (iv). Set $K := \widetilde{K}/\mathfrak{O}_N^\times$. Then K gives a triangulation of 2-dimensional compact topological manifold $\overline{V_+}/\mathfrak{O}_N^\times$. This triangulation describes a resolution of a cusp singularity of $\overline{\Gamma(N) \setminus \mathfrak{H}}$.

§3. Defects

In this section we define defects of cusps for $\Gamma(N)$.

Let Ω be the sheaf of the canonical differential forms. Since

$$\mathfrak{H} = \left\{ Z = X + iY \in M_2(\mathbb{C}) \mid \operatorname{tr}(Z) = 0,\ YJ^{-1} > 0 \right\},$$

we use the coordinate $\begin{pmatrix} z_1 & z_2 \\ z_3 & -z_1 \end{pmatrix} \in \mathfrak{H}$. Let $\omega = dz_1 \wedge dz_2 \wedge dz_3$ be the standard volume element. For $f \in S_{3m}(\Gamma(N))$, $f\omega^{\otimes m}$ is $\Gamma(N)$-invariant. Assume $N \geq 3$. Then $\Gamma(N)$ is torsion-free. Hence $f\omega^{\otimes m}$ becomes a section of $H^0(\Gamma(N) \setminus \mathfrak{H}, \Omega^{\otimes m})$. Let $Y(N)$ be a smooth compactification of $\Gamma(N) \setminus \mathfrak{H}$ obtained by using a fan Σ in the previous section. We would like to know the extendability of $f\omega^{\otimes m}$ to a section of $H^0(Y(N), \Omega^{\otimes m})$.

Let λ be a cusp singularity of $\overline{\Gamma(N) \setminus \mathfrak{H}}$. We define an invariant which measures the obstruction for extending sections of multi-canonical system of degree m ($m \geq 1$). Let $\pi : Y(N) \to \overline{\Gamma(N) \setminus \mathfrak{H}}$ be a cusp resolution associated to Σ. We set

$$d_m(\lambda) := \dim_{\mathbb{C}} \varinjlim_{\lambda \in U} H^0(\pi^{-1}(U - \lambda), \Omega^{\otimes m}) / H^0(\pi^{-1}(U), \Omega^{\otimes m}),$$

where U runs over a fundamental system of neighbourhoods of λ. This number $d_m(\lambda)$ is called the *m-th defect* of λ.

Take a $\mathbb{Z}$-basis $\alpha_1, \alpha_2, \alpha_3$ of L such that $\mathbb{R}_{\geq 0}\alpha_1 + \mathbb{R}_{\geq 0}\alpha_2 + \mathbb{R}_{\geq 0}\alpha_3$ is an element of Σ. We can find a $\mathbb{Z}$-basis $\beta_1, \beta_2, \beta_3$ of L^* such that $\operatorname{tr}(\alpha_i\beta_j) = \delta_{ij}$ (Kronecker's delta). Each element t of L^* can be written as $t = t_1\beta_1 + t_2\beta_2 + t_3\beta_3$ ($t_i \in \mathbb{Z}$). We change the coordinates as

$$x_j = e\left[\frac{1}{N}\operatorname{tr}(\beta_j Z)\right], \quad j = 1, 2, 3, \tag{3.1}$$

where Z is the coordinate of $\mathfrak{H}$ as described at the top of this section. If f has the Fourier expansion as in Proposition 1.6, then we have

$$
\begin{aligned}
f\omega^{\otimes m} =& f(dz_1 \wedge dz_2 \wedge dz_3)^{\otimes m} \\
=& \left(\frac{2\pi i}{N}\right)^{-3m} \cdot 2^{-m} \cdot \delta(L^*)^{-m} \\
& \cdot \Big(\sum_{t \in L_+^*} a_\infty(t) x_1^{t_1} x_2^{t_2} x_3^{t_3}\Big)(x_1 x_2 x_3)^{-m} \\
& \times (dx_1 \wedge dx_2 \wedge dx_3)^{\otimes m},
\end{aligned}
\tag{3.2}
$$

where

$$
\delta(L^*) := \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}, \qquad \beta_j = \begin{pmatrix} \lambda_j & \mu_j \\ \nu_j & -\lambda_j \end{pmatrix} \ (j = 1, 2, 3).
$$

This $\delta(L^*)$ is uniquely determined by L^* up to the sign $\pm$.

Definition 3.3. For any positive integer m, we define

$$
\Lambda_m(\infty) := \{y \in L_+^* \mid \operatorname{tr}(yx) \leq m \text{ for some } x \in L_+\}.
$$

Let $R_{3m}(\Gamma(N))$ be the space of all cusp forms $f \in S_{3m}(\Gamma(N))$ such that in (1.6.1) $a_\lambda(t) = 0$ holds for any cusp λ of $\Gamma(N)$ and for any element t of $\Lambda_{m-1}(\infty)$. Let $P_m(Y(N)) = \dim_{\mathbb{C}} H^0(Y(N), \Omega^{\otimes m})$ be the m-th plurigenus of $Y(N)$.

Proposition 3.4. *Assume $N \geq 3$. For any positive integer m, we have $P_m(Y(N)) = \dim_{\mathbb{C}} R_{3m}(\Gamma(N))$.*

Proof. Take an element f of $S_{3m}(\Gamma(N))$. Since $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$, each cusp of $\Gamma(N)$ has the same properties as the cusp ∞ (note that $\Gamma(1)$ has only one cusp ∞ from Lemma 4 and Lemma 5 in Arakawa [A]). Hence we only consider the extendability of $f\omega^{\otimes m}$ to the resolution of ∞ (for other cusps, moving each cusp to ∞ by using some element of $\Gamma(1)$, each cusp case reduces to ∞ case). Let $x \in Y(N) - \Gamma(N) \setminus \mathfrak{H}$ be a point on the resolution of ∞. We take a sufficiently small neighbourhood U_x at x. Then the coordinate (x_1, x_2, x_3) on U_x is expressed as (3.1) for some $\mathbb{Z}$-basis $\alpha_1, \alpha_2, \alpha_3$ of L such that $\alpha_i \in L_+$ $(i = 1, 2, 3)$ ($\mathbb{R}_{\geq 0}\alpha_1 + \mathbb{R}_{\geq 0}\alpha_2 + \mathbb{R}_{\geq 0}\alpha_3$ forms a polyhedral cone in L_+, which is appeared in the resolution of cusp singularities of $\overline{\Gamma(N) \setminus \mathfrak{H}}$ in the previous section) and its dual $\mathbb{Z}$-basis $\beta_1, \beta_2, \beta_3$ of L^*. Then $f\omega^{\otimes m}$ can be written as (3.2) on U_x. Hence for any point x, $f\omega^{\otimes m}|U_x$ becomes a section of $H^0(U_x, \Omega^{\otimes m})$ if and only if in (1.6.1) each $t \in L_+^*$ such that $a_\lambda(t) \neq 0$ satisfies $\mathrm{tr}(tv) \geq m$ for all $v \in L_+$. This condition is equivalent to $f \in R_{3m}(\Gamma(N))$. $\Box$

PROPOSITION 3.5. *For any positive integer m, we have*

$$d_m(\infty) = \#(\Lambda_m(\infty)/\sim),$$

where we write $y_1 \sim y_2$ when $y_1 = \epsilon y_2 \bar{\epsilon}$ holds for some element ϵ in $\mathfrak{O}_N^\times$.

PROOF. For any element $t \in L_t^*$, put

$$P_t(Z) := \sum_{\epsilon \in \mathfrak{O}_N^\times} e\left[\frac{1}{N}\,\mathrm{tr}(\epsilon t \bar{\epsilon} Z)\right].$$

Then $P_t(Z)$ is absolutely convergent over $\mathfrak{H}$. Since $N \geq 3$, the norm of any element of $\mathfrak{O}_N^\times$ is 1. From (1.7), each Fourier coefficient $a_\infty(t)$ in (1.6.1) satisfies $a_\infty(\epsilon t \bar{\epsilon}) = a_\infty(t)$ for $\epsilon \in \mathfrak{O}_N^\times$. Hence by Proposition 1.6, $S_{3m}(\Gamma(N))$ is generated by $P_t(Z)$ $(t \in L_+^*)$. We have $P_{\epsilon t \bar{\epsilon}}(Z) = P_t(Z)$ for $\epsilon \in \mathfrak{O}_N^\times$. Combining these facts with the proof of Proposition 3.4, we get the proposition. $\Box$

As we saw above, the m-th defect $d_m(\infty)$ of ∞ measures the extendability of $f\omega^{\otimes m}$ to a section of $H^0(Y(N), \Omega^{\otimes m})$.

§4. Results

Put $N(L_+) := \min\{N(x) \mid x \in L_+\}$.
The following is our main result.

THEOREM 4.1. *Assume $N \geq 3$. If*

$$N^3 > \frac{\sqrt{2} \cdot 2^7 \cdot 3 \cdot 5\pi}{N(L_+)^{3/2} d(\mathbf{B}) \prod_{p|d(\mathbf{B})}(1 + \frac{1}{p^2})}, \tag{4.1.1}$$

then $Y(N)$ is of general type. Namely, $Y(N)$ is of general type if N is fully large.

PROOF OF THEOREM 4.1.

We have the following inequality:

$$P_m(Y(N)) = \dim_{\mathbb{C}} R_{3m}(\Gamma(N)) \geq \dim_{\mathbb{C}} S_{3m}(\Gamma(N)) - \sum_{\lambda} d_m(\lambda). \tag{4.2}$$

Here the sum $\sum$ runs over cusp singularities of $\Gamma(N)$. By Lemma 4 and Lemma 5 in Arakawa [A], the number of cusps of $\Gamma(1)$ is 1. The fixed subgroup of the cusp ∞ in $\Gamma(1)$ is

$$\Gamma(1)_\infty := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid c = 0 \right\}.$$

Since $[\Gamma(1)_\infty : \Gamma(1)_\infty \cap \Gamma(N)] = [\mathfrak{O}^\times : \mathfrak{O}_N^\times]N^3$, the number of cusps for $\Gamma(N)$ is $[\Gamma(1) : \Gamma(N)]/([\mathfrak{O}^\times : \mathfrak{O}_N^\times]N^3)$. Since $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$, each cusp of $\Gamma(N)$ has the same property as the cusp ∞. Hence by (4.2), we have

$$P_m(Y(N)) \geq \dim_{\mathbb{C}} S_{3m}(\Gamma(N)) - \frac{[\Gamma(1) : \Gamma(N)]}{[\mathfrak{O}^\times : \mathfrak{O}_N^\times]N^3} \cdot d_m(\infty). \tag{4.3}$$

LEMMA 4.4. *Let y be an element of $\Lambda_m(\infty)$. If $\mathrm{tr}(yx) \leq m$ for some $x \in L_+$, then $N(y)N(x) \leq 9m^2$.*

PROOF. It suffices to show $N(y)N(x) \leq 9\,\mathrm{tr}(yx)^2$ for $y \in L_+^*, x \in L_+$. Since $\mathbf{B} \subset M_2(\mathbb{R})$, y and x can be expressed as $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{pmatrix}, x = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}$, respectively. We have

$$\begin{gathered} y_2 < 0,\ y_3 > 0,\ -y_2y_3 - y_1^2 > 0, \\ x_2 > 0,\ x_3 < 0,\ -x_2x_3 - x_1^2 > 0, \end{gathered} \tag{4.4.1}$$

by definitions of L_+^* and L_+. Hence

$$\begin{aligned}
&9\,\mathrm{tr}(yx)^2 - N(y)N(x)\\
&= 9(x_3y_2 + 2x_1y_1 + x_2y_3)^2 - (y_1^2 + y_2y_3)(x_1^2 + x_2x_3)\\
&= A_1 + A_2 + A_3,
\end{aligned}$$

where we set

$$\begin{aligned}
A_1 :=&(x_2x_3y_2y_3 - x_1^2y_1^2),\\
A_2 :=&(-x_1^2y_2y_3) + (-x_2x_3y_1^2) - 2x_1^2y_1^2,\\
A_3 :=&(2x_1y_1 + x_3y_2)^2 + (2x_1y_1 + x_2y_3)^2\\
&+ 8(x_3y_2 + 2x_1y_1 + x_2y_3)^2 - 2x_1^2y_1^2.
\end{aligned}$$

Since $x_2x_3y_2y_3 - x_1^2y_1^2 > 0$ by (4.4.1), $A_1 > 0$. By (4.4.1), we have $-x_1^2y_2y_3 \geq 0$, $-x_2x_3y_1^2 \geq 0$. Using (4.4.1) again, $A_2 \geq 0$ holds. We consider A_3. In the case $x_1y_1 \geq 0$, it is clear that $A_3 \geq 0$. We consider the case $x_1y_1 < 0$. We now assume that $(2x_1y_1+x_3y_2)^2$, $(2x_1y_1+x_2y_3)^2$, and $(x_3y_2+2x_1y_1+x_2y_3)^2$ are smaller than $x_1^2y_1^2/4$. Then we have

$$(4.4.2)\qquad -\frac{3}{2}x_1y_1 < x_3y_2 < -\frac{5}{2}x_1y_1,$$

$$(4.4.3)\qquad -\frac{3}{2}x_1y_1 < x_2y_3 < -\frac{5}{2}x_1y_1,$$

$$(4.4.4)\qquad -\frac{3}{2}x_1y_1 < x_3y_2 + x_2y_3 < -\frac{5}{2}x_1y_1.$$

By (4.4.2) and (4.4.3), $-3x_1y_1 < x_3y_2 + x_2y_3 < -5x_1y_1$. This inequality contradicts (4.4.4). Hence one of numbers $(2x_1y_1 + x_3y_2)^2$, $(2x_1y_1 + x_2y_3)^2$, and $(x_3y_2 + 2x_1y_1 + x_2y_3)^2$ is not smaller than $x_1^2y_1^2/4$.

(i) *The case* $(2x_1y_1 + x_3y_2)^2 \geq x_1^2y_1^2/4$:

Since $x_3y_2 > 0$, $x_3y_2 \geq -5x_1y_1/2$. Using (4.4.3), $-4x_1y_1 < x_3y_2 + x_2y_3$. This inequality contradicts (4.4.4). Hence $(x_3y_2 + 2x_1y_1 + x_2y_3)^2 \geq x_1^2y_1^2/4$ holds. Thus we have $A_3 \geq 0$.

(ii) *The case* $(2x_1y_1 + x_2y_3)^2 \geq x_1^2y_1^2/4$:

As in the case (i), we have $A_3 \geq 0$.

(iii) *The case* $(x_3y_2 + 2x_1y_1 + x_2y_3)^2 \geq x_1^2y_1^2/4$:

We can easily see $A_3 \geq 0$. $\square$

Thus we have

$$d_m(\infty) \leq \# \left(\left\{ y \in L_+^* \mid N(y) \leq \frac{9m^2}{N(L_+)} \right\} / \sim \right),$$

where $\sim$ is the same relation defined above. It suffices to evaluate the right hand side of the above inequality. This cardinality is not larger than

$$c_m := \# \left(\left\{ y \in L^* \mid 0 < N(y) \leq \frac{9m^2}{N(L_+)} \right\} / \sim \right).$$

We would like to compute c_m.

Let $\{e_1, e_2, e_3\}$ be a basis of L^* over $\mathbb{Z}$. Any element x in V can be written as $x = x_1e_1 + x_2e_2 + x_3e_3$ $(x_1, x_2, x_3 \in \mathbb{R})$. Let $\mathfrak{F}$ be fundamental domain of $\mathfrak{O}_N^\times$ in $\{x \in V \mid 0 < N(x) \leq 1\}$ under the action $x \to \epsilon x \overline{\epsilon}$ $(x \in V, \epsilon \in \mathfrak{O}_N^\times)$.

LEMMA 4.5. *For any positive real number r, we put*

$$b_r := \# \left(\{ y \in L^* \mid 0 < N(y) \leq r \} / \sim \right).$$

Then

$$b_r \leq \left(\int_{\mathfrak{F}} dx_1 dx_2 dx_3 \right) r^{\frac{3}{2}} + \alpha r^{\frac{3}{2}}$$

holds for any small positive real number α if r is large enough.

PROOF. The number b_r is expressed as

$$b_r = \#(\{ y \in L^* \mid 0 < N(yr^{-\frac{1}{2}}) \leq 1\} / \sim).$$

Such element $yr^{-\frac{1}{2}}$ in $\mathbf{B}^-$ is considered as a lattice point with width $r^{-\frac{1}{2}}$ concerning the coordinate (x_1, x_2, x_3). Hence we have $\lim_{r \to \infty} b_r r^{-\frac{3}{2}} = \int_{\mathfrak{F}} dx_1 dx_2 dx_3$. $\square$

We next calculate $I := \int_{\mathfrak{F}} dx_1 dx_2 dx_3$. Set

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\{E_{11}, E_{12}, E_{21}\}$ is a basis of V over $\mathbb{R}$. Any element x in V is expressed as $x = x_{11}E_{11} + x_{12}E_{12} + x_{21}E_{21}$ $(x_{11}, x_{12}, x_{21} \in \mathbb{R})$. Since $\det(\mathrm{tr}(e_ie_j)) = -\frac{1}{2d(\mathbf{B})^2}$ (Lemma 1.4), we have $I = \sqrt{2}d(\mathbf{B}) \int_{\mathfrak{F}} dx_{11}dx_{12}dx_{21}$. Put

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} J^{-1} = t \begin{pmatrix} u & xu^{-1} \\ 0 & u^{-1} \end{pmatrix} {}^t \begin{pmatrix} u & xu^{-1} \\ 0 & u^{-1} \end{pmatrix}.$$

Here x, u and t satisfy $x \in \mathbb{R}, u > 0$, and $0 < t \leq 1$, respectively. Moreover, set $y = u^2$. Then we have

$$\begin{aligned} \int_{\mathfrak{F}} dx_{11}dx_{12}dx_{21} =& 2\int_0^1 t^2dt \int_{\mathfrak{O}_N^\times \backslash H} \frac{dxdy}{y^2} \\ =& [\mathfrak{O}^\times : \mathfrak{O}_N^\times] \int_0^1 t^2dt \int_{\mathfrak{O}^1 \backslash H} \frac{dxdy}{y^2} \\ =& \frac{\pi}{9}[\mathfrak{O}^\times : \mathfrak{O}_N^\times] \prod_{p|d(\mathbf{B})} (p-1), \end{aligned}$$

where $\mathfrak{O}^1$ is the set of norm 1 units of $\mathfrak{O}^\times$, and H is the complex upper half plane. The groups $\mathfrak{O}_N^\times$ and $\mathfrak{O}^1$ act on H by using $\mathfrak{O}^1 \hookrightarrow SL_2(\mathbb{R})$. Note that $\mathfrak{O}_N^\times \subset \mathfrak{O}^1$ because of $N \geq 3$.

We return to the first stage. Using the result obtained above, we have

$$c_m = b_{9m^2/N(L_+)} \leq \left(\frac{3\sqrt{2}\pi d(\mathbf{B})[\mathfrak{O}^\times : \mathfrak{O}_N^\times]}{N(L_+)^{\frac{3}{2}}} \prod_{p|d(\mathbf{B})} (p-1) \right) m^3 + \frac{\alpha m^3}{N(L_+)^{\frac{3}{2}}}.$$

By applying this evaluation and dimension formula of Arakawa to (4.3), the proof of Theorem 4.1 is completed. □

By Theorem 4.1, we can easily see the following:

THEOREM 4.6. *Assume $N \geq 3$. If*

$$d(\mathbf{B}) > \frac{\sqrt{2} \cdot 2^7 \cdot 5\pi}{3^2 N(L_+)^{3/2} \prod_{p|d(\mathbf{B})} \left(1 + \frac{1}{p^2}\right)},$$

then $Y(N)$ is of general type. Namely, $Y(N)$ is of general type if $d(\mathbf{B})$ is fully large.

§5. Examples

In this section, we give examples of 3-folds $Y(N)$ of general type. We first quote the result of Ibukiyama from [I]:

THEOREM 5.1 (Ibukiyama [I]). *Let $p_1, p_2, \cdots, p_{2r}$ be distinct prime numbers, and set $m = p_1 \cdots p_{2r}$. Take a prime number q satisfying $q \equiv 5 \pmod 8$ and $\left(\frac{q}{p_i}\right) = -1$ for any prime number $p_i \neq 2$ (Here $(-)$ is the Legendre symbol). Then we can show that*

$$\mathbf{B} = \mathbb{Q} + \mathbb{Q}\alpha + \mathbb{Q}\beta + \mathbb{Q}\alpha\beta, \quad \alpha^2 = m,\ \beta^2 = q,\ \alpha\beta = -\beta\alpha$$

is an indefinite division quaternion algebra over $\mathbb{Q}$ with discriminant $d(\mathbf{B}) = m$ and that for any rational integer a such that $a^2 \equiv m \pmod q$,

$$\mathfrak{O} = \mathbb{Z} + \mathbb{Z}\frac{1+\beta}{2} + \mathbb{Z}\frac{\alpha(1+\beta)}{2} + \mathbb{Z}\frac{(a+\alpha)\beta}{q}$$

is a maximal order of $\mathbf{B}$.

When the discriminant $d(\mathbf{B})$ of $\mathbf{B}$ is small, we must take a large natural number N in order to get a 3-fold $Y(N)$ of general type. If $d(\mathbf{B})$ is fully large, then we can get a 3-fold $Y(N)$ of general type with small level N by Theorem 4.6. Using the above theorem, we give such examples.

Example 5.2. (1) We put

$$\mathbf{B} = \mathbb{Q} + \mathbb{Q}\alpha + \mathbb{Q}\beta + \mathbb{Q}\alpha\beta, \quad \alpha^2 = 6,\ \beta^2 = 5,\ \alpha\beta = -\beta\alpha.$$

Then $\mathbf{B}$ is an indefinite division quaternion algebra over $\mathbb{Q}$ with $d(\mathbf{B}) = 6$, and

$$\mathfrak{O} = \mathbb{Z} + \mathbb{Z}\frac{1+\beta}{2} + \mathbb{Z}\frac{\alpha(1+\beta)}{2} + \mathbb{Z}\frac{(1+\alpha)\beta}{5}$$

is a maximal order of $\mathbf{B}$. We can easily see that

$$L = \mathbb{Z}\beta + \mathbb{Z}\frac{\alpha(1+\beta)}{2} + \mathbb{Z}\frac{(1+\alpha)\beta}{5}$$

and that $N(L_+) \geq 1/10$. If $N \geq 32$, then the inequality (4.1.1) holds. Threfore $Y(N)$ is of general type for $N \geq 32$.

(2) We put

$$\mathbf{B} = \mathbb{Q} + \mathbb{Q}\alpha + \mathbb{Q}\beta + \mathbb{Q}\alpha\beta, \quad \alpha^2 = 213486,\ \beta^2 = 5,\ \alpha\beta = -\beta\alpha.$$

Then $\mathbf{B}$ is an indefinite division quaternion algebra over $\mathbb{Q}$ with $d(\mathbf{B}) = 2 \cdot 3 \cdot 7 \cdot 13 \cdot 17 \cdot 23 = 213486$, and

$$\mathfrak{O} = \mathbb{Z} + \mathbb{Z}\frac{1+\beta}{2} + \mathbb{Z}\frac{\alpha(1+\beta)}{2} + \mathbb{Z}\frac{(4+\alpha)\beta}{5}$$

is a maximal order of $\mathbf{B}$. We can easily see that

$$L = \mathbb{Z}\beta + \mathbb{Z}\frac{\alpha(1+\beta)}{2} + \mathbb{Z}\frac{(4+\alpha)\beta}{5}$$

and that $N(L_+) \geq 1/10$. If $N \geq 3$, then the inequality (4.1.1) holds. Therefore $Y(N)$ is of general type for $N \geq 3$.

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