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On Gross's Refined Class Number Formula for Elementary Abelian Extensions

By Joongul LEE

Abstract. In this paper we consider the conjecture of Gross on the special values of abelian L-functions when the Galois group G is an elementary abelian l-group. Under some restrictions, we prove that the conjecture holds when the class number of the base field is prime to l.

1. Introduction

Suppose L/K is an abelian extension of global fields and let $G = \mathrm{Gal}(L/K)$. In [3], B. Gross has conjectured a congruence relation involving the Stickelberger element in $\mathbb{Z}[G]$, class number of K and the generalized regulator. The relation can be thought of as a generalization of the classical class number formula which describes the leading term of the Taylor expansion of $\zeta_K(s)$ at $s = 0$ in terms of the class number and the regulator of K. In this paper we consider the case when G is an elementary abelian l-group. Our main result is Theorem 3, which states that the conjecture holds when the class number of K is prime to l and (when K contains a primitive l-th root of unity) T contains a place whose degree is prime to l. This improves the result that Gross obtained when G is cyclic of prime order (see [3]).

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2. The conjecture of Gross

Let L/K be an abelian extension of global fields with Galois group G. Let S be a finite non-empty set of places of K which contains all archimedean places and places ramified in L, and let T be a finite non-empty set of places

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of K which is disjoint from S. Let $n = |S| - 1$. For a finite place v of K, let $\mathbb{F}_v$ be the residue field of v.

For a complex character $\chi \in \widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$, the associated modified L-function is defined as

$$L_{S,T}(\chi, s) = \prod_{v \in T} (1 - \chi(g_v)\boldsymbol{N}v^{1-s}) \prod_{v \notin S} (1 - \chi(g_v)\boldsymbol{N}v^{-s})^{-1}, \tag{1}$$

where $g_v \in G$ is the Frobenius element for v.

The Fourier inversion formula tells us that there is a unique element $\theta_G \in \mathbb{C}[G]$ which satisfies

$$\chi(\theta_G) = L_{S,T}(\chi, 0) \tag{2}$$

for all $\chi \in \widehat{G}$. In fact, $\theta_G \in \mathbb{Z}[G]$ by works of Weil, Siegel, Deligne-Ribet and Cassou-Noguès (see [3] for more information).

Let Y be the free $\mathbb{Z}$-module generated by the places $v \in S$ and $X = \{\sum_{v \in S} a_v \cdot v \mid \sum a_v = 0\}$ the subgroup of elements of degree zero in Y. Let U_T denote the group of S-units which are congruent to $1 \pmod{T}$ (in other words, S-units which are congruent to $1 \pmod{v}$ for all $v \in T$). Then U_T is a free $\mathbb{Z}$-module of rank n if K is a function field, and to ensure that the same is true if K is a number field we require that T either contains places of different residue characteristics or contains a place v whose absolute ramification index e_v is strictly less than $(p-1)$, where p is the characteristic of $\mathbb{F}_v$. This assumption makes U_T a free $\mathbb{Z}$-module.

Let J denote the idele group of K, and $f : J \to G$ be the Artin reciprocity map. Let λ_G be the homomorphism

$$\begin{array}{rcl} \lambda_G : U_T & \longrightarrow & G \otimes X \\ \varepsilon & \longmapsto & \displaystyle\sum_S f(1, 1, \ldots, \varepsilon_v, \ldots, 1) \cdot v. \end{array} \tag{3}$$

We choose bases $\langle \varepsilon_1, \ldots, \varepsilon_n \rangle$ and $\langle x_1, \ldots, x_n \rangle$ for U_T and X. With respect to the chosen bases, we obtain an $n \times n$ matrix $((g_{ij}))$ for λ_G with entries in G.

Let $I \subset \mathbb{Z}[G]$ be the augmentation ideal, which is defined as the kernel of the ring homomorphism

$$\begin{array}{rcl} \mathbb{Z}[G] & \longrightarrow & \mathbb{Z} \\ g & \longmapsto & 1. \end{array} \tag{4}$$

It is well known that the map $g \mapsto g-1 \pmod{I^2}$ gives an isomorphism $G \cong I/I^2$ of abelian groups. We may therefore consider the matrix for λ_G as having entries $\eta_{ij} = g_{ij} - 1$ in I/I^2. We define

$$\begin{aligned} \det \lambda_G &= \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sign}(\sigma) \eta_{1\sigma(1)} \eta_{2\sigma(2)} \cdots \eta_{n\sigma(n)} \\ &\in I^n/I^{n+1}. \end{aligned} \tag{5}$$

Now we can state the main conjecture.

Conjecture 1 (Gross). $\theta_G \equiv m \cdot \det \lambda_G \pmod{I^{n+1}}$.

Here $m = \pm h_{S,T}$ is the modified class number of the S-integers of K and the sign depends on the choice of ordered bases of X and U_T (see [3]).

We summarize some basic facts on Conjecture 1.

Proposition 2. *(a) Suppose $S \subset S'$ and $T \subset T'$. If Conjecture 1 holds for the set S and T, it holds for S' and T'.*
(b) Suppose H is a subgroup of G. The natural map $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$ maps θ_G and $\det \lambda_G$ to $\theta_{G/H}$ and $\det \lambda_{G/H}$ respectively. Hence Conjecture 1 holds for G/H if it holds for G.
(c) Conjecture 1 holds for G if and only if it holds for all its p-Sylow quotients.
(d) If S contains a place v that splits completely in L, then $\theta_G \equiv m \cdot \det \lambda_G \equiv 0 \pmod{I^{n+1}}$.
(e) If $n = 0$ then $\det \lambda_G = 1, m = h_{S,T}, I^n/I^{n+1} = \mathbb{Z}$, and conjecture 1 holds because it is equivalent to the classical class number formula.

See [3, 8] for (a) and (b). (c) was pointed out by J. Tate. For (d) we note that the Euler factor for v is zero, so $\theta_G = 0$, and also the row of the matrix of λ_G which correspond to v is zero and hence $\det \lambda_G \equiv 0 \pmod{I^{n+1}}$. (e) follows from the definitions of the related quantities.

In [3], B. Gross proved that the Conjecture 1 holds when S consists of the archimedean places of K. He also treated the case when $G \cong \mathbb{Z}/l\mathbb{Z}$ is cyclic of prime order. In this case, $I^n/I^{n+1} \cong \mathbb{Z}/l\mathbb{Z}$ for $n \geq 1$, and Gross proved that his conjecture is true up to an element of $(\mathbb{Z}/l\mathbb{Z})^*$, in the sense that θ_G always belongs to I^n (hence we are comparing two elements in I^n/I^{n+1}) and that $\theta_G \in I^{n+1}$ if and only if $m \cdot \det \lambda_G \in I^{n+1}$. In [9],

M. Yamagishi treated the case when $K = \mathbb{Q}$ and got some partial result, and N. Aoki proved that the conjecture is true for $K = \mathbb{Q}$ in [1]. D. Hayes proved a refined version of the Stark conjecture (conjectured by Gross) for function fields in [4], which implies Conjecture 1 for $n = 1$. In [6], K.-S. Tan proved the case when K is a function field of characteristic p and G is a p-group.

3. The main theorem

Let l be a prime. Our goal is to prove the following theorem.

THEOREM 3. *Suppose G is an elementary abelian l-group. If K is a function field suppose also that h_K, the number of divisor classes of degree 0 of K, is prime to l, and, in case K contains a primitive l-th root of unity, that T contains a place whose degree is prime to l. Then conjecture 1 holds.*

If K is a number field, the existence of the archimedean places assures that Conjecture 1 is true when $l \geq 3$ since the archimedean places split completely in L, and when $l = 2$ Conjecture 1 follows from the work of Gross and corollary 5 below. Therefore we may assume that K is a function field. Also, since Tan proved Conjecture 1 for p-groups ([6]), we may assume that l is different from the characteristic of K. Hence we will be dealing only with tame ramification. Also we may assume that T consists of a single place whose degree is prime to l if K contains a primitive l-th root of unity, via proposition 2.

Let $S = \{v_0, v_1, \ldots, v_n\}$, $n = |S| - 1$, and $T = \{v_T\}$. Let K_S be the maximal extension of K unramified outside of S whose Galois group is an elementary abelian l-group. Let $G_S = \mathrm{Gal}(K_S/K)$, and for $i = 0, \ldots, n$, let $I_i \subset G_S$ be the inertia group of v_i. Let D_T be the decomposition group of v_T. Notice that I_i is cyclic because K_S/K has only tame ramification, and that D_T is also cyclic because v_T is unramified in K_S and its residue field is finite. It follows from proposition 2 that we may assume that $n \geq 1$. We can also assume without loss of generality that $L = K_S$, and that all the places in S are ramified in K_S.

Here is our strategy for proving Theorem 3. We first discuss the structure of I^n/I^{n+1}, and we find a homogeneous polynomial f of degree n with coefficients in $\mathbb{F}_l$ which may be viewed as a function on $\widehat{G_S}$ with values in $\mathbb{F}_l$, such that the validity of the conjecture is equivalent to the vanishing of f on $\widehat{G_S}$. Next we study the structure of G_S in section 5., and we show that $I_0, \ldots, I_n$ generate a subgroup of G_S of rank n or $n+1$, depending on whether K contains a primitive l-th root of unity or not. We also show that $I_0, \ldots, I_n, D_T$ generate a subgroup of G_S of rank $n+1$ when K contains a primitive l-th root of unity. In section 6. we prove that if a polynomial function on $\widehat{G_S}$ vanishes on $n+1$ linearly independent subspaces of codimension 1 and its degree is bounded by n, then it must vanish on $\widehat{G_S}$. It turns out that this is exactly what we need in order to make the induction on $n = |S| - 1$ work, and the induction is carried out in section 7..

4. The structure of I^n/I^{n+1}

Choose a primitive l-th root of unity $\zeta_l \in \mathbb{C}^*$, and let $\lambda = \zeta_l - 1$. (λ) is a prime ideal in $\mathbb{Z}[\zeta_l]$ whose residue field is isomorphic to $\mathbb{Z}/l\mathbb{Z}$, and we have $(l) = (\lambda)^{l-1}$. Also note that a character $\chi \in \widehat{G}$ can be extended by linearity to a ring homomorphism $\chi : \mathbb{Z}[G] \longrightarrow \mathbb{C}$.

Lemma 4 (Passi-Vermani). *Suppose G is an elementary abelian l-group. If $\xi \in I$, then, for each integer $k \geq 1$, $\xi \in I^k$ if and only if $\lambda^k \mid \chi(\xi)$ for every complex character $\chi \in \widehat{G}$.*

Proof. See [3] for the case when $G \cong \mathbb{Z}/l\mathbb{Z}$, and [5, 7] for elementary abelian case. □

As we discussed in section 2., Gross proved that both θ_G and $m \cdot \det \lambda_G$ are in I^n when G is cyclic of prime order, which, together with Lemma 4, implies that both θ_G and $m \cdot \det \lambda_G$ are in I^n when G is an elementary abelian group.

Corollary 5. *Suppose $G = \mathrm{Gal}(L/K)$ is an elementary abelian l-group. Then the conjecture holds for L/K if and only if it holds for L'/K for all cyclic subextensions L'/K of L/K.*

Proof. Set $\xi = \theta_G - m \cdot \det \lambda_G$ and apply lemma 4. □

Let $N = \dim_{\mathbb{F}_l} \widehat{G} - 1$ and choose a basis $\{\chi_0, \dots, \chi_N\}$ of $\widehat{G}$. In general, we have

$$\zeta_l^m - 1 = (\zeta_l - 1)(\zeta_l^{m-1} + \dots + 1) \equiv m(\zeta_l - 1) \pmod{\lambda^2}. \tag{6}$$

Hence, given $\chi = \prod_{i=0}^N {\chi_i}^{m_i} \in \widehat{G}$ and $\sigma \in G$, we may write

$$\chi(\sigma - 1) = \zeta_l^{\sum a_i m_i} - 1 \equiv \sum a_i m_i \cdot \lambda \pmod{\lambda^2}, \tag{7}$$

where $a_i \in \mathbb{F}_l$ is defined by $\chi_i(\sigma) = \zeta_l^{a_i}$.

If $\xi \in I^n$, then since ξ can be written as a linear combination of $\prod_{j=1}^n (\tau_j - 1)$ where $\tau_j \in G$ for all j, we have

$$\chi(\xi) \equiv p(m_0, \dots, m_N) \cdot \lambda^n \pmod{\lambda^{n+1}}, \tag{8}$$

where $p(X_0, \dots, X_N) \in \mathbb{F}_l[X_0, \dots, X_N]$ is a homogeneous polynomial of degree n. We can see from Lemma 4 that $\xi \in I^{n+1}$ if and only if $p = 0$ as a function on $\widehat{G}$.

For $\chi \in \widehat{G}$, define

$$f(\chi) = \frac{\chi(\theta_G - m \cdot \det \lambda_G)}{\lambda^n} \pmod{\lambda}. \tag{9}$$

The above argument shows that f can be represented by a homogeneous polynomial of degree n. Let K_χ be the fixed field of $\ker \chi$. Then $f(\chi) = 0$ if and only if the conjecture holds for K_χ/K with respect to S and T.

We also note that if K contains an l-th root of unity and T contains a place v that splits completely in L, then the modifying Euler factor for v is $(1 - \boldsymbol{N}v)$ which is divisible by l. Since $l \cdot \xi \in I^{m+(l-1)}$ whenever $\xi \in I^m$, which follows from lemma 4, θ_G will be in I^{n+1}. With the work of Gross, that implies $m \cdot \det \lambda_G \in I^{n+1}$. As a result, Conjecture 1 holds trivially (in the sense that the conjecture becomes $0 = 0$) when K contains an l-th root of unity and T contains a place that splits completely in L.

5. The structure of G_S

In this section, we study the structure of G_S and the inertia groups of S in G_S using class field theory. (reference:[2])

Let $\mathbb{F}_q$ be the exact field of constants of K. For each place v of K, let K_v be the completion of K at v, U_v the set of local units in K_v, and $U_v^1 \subset U_v$ the local units which are congruent to $1 \pmod v$.

Let J_0 be the set of ideles of degree 0. It is easy to see that J is (non-canonically) isomorphic to $\mathbb{Z} \times J_0$, because K is known to have a divisor (not necessarily prime) of degree 1.

There is an exact sequence

$$(10) \quad 0 \to (\prod_{v \in S} \mathbb{F}_v^*)/\mathbb{F}_q^* \to J/K^* \cdot \prod_{v \notin S} U_v \cdot \prod_{v \in S} U_v^1 \to J/K^* \cdot \prod_v U_v \to 0.$$

If we let K_{unr} be the maximal unramified abelian extension of K, and K'_S the maximal abelian extension of K unramified outside of S and tamely ramified in S, then $J/K^* \cdot \prod_{v \notin S} U_v \cdot \prod_{v \in S} U_v^1$ and $J/K^* \cdot \prod_v U_v$ have dense images in $\mathrm{Gal}(K'_S/K)$ and $\mathrm{Gal}(K_{unr}/K)$ respectively, via the Artin reciprocity map.

Observe that $J/K^* \cdot \prod_v U_v$ is isomorphic to $\mathbb{Z} \times H$, where $H = J_0/K^* \cdot \prod_v U_v$ and since we assume that $h_K = |H|$ is not divisible by l, we have $(\mathbb{Z} \times H) \otimes \mathbb{Z}/l\mathbb{Z} = \mathbb{Z}/l\mathbb{Z}$ and $\mathrm{Tor}(\mathbb{Z} \times H, \mathbb{Z}/l\mathbb{Z}) = 0$. Hence tensoring the exact sequence with $\mathbb{Z}/l\mathbb{Z}$ preserves the exactness;

$$(11) \quad 0 \to (\prod_{v \in S} \mathbb{F}_v^*/\mathbb{F}_v^{*l})/\widetilde{\mathbb{F}_q^*} \to J/J^l \cdot K^* \cdot \prod_{v \notin S} U_v \cdot \prod_{v \in S} U_v^1 \to \mathbb{Z}/l\mathbb{Z} \to 0,$$

where $\widetilde{\mathbb{F}_q^*}$ is the image of $\mathbb{F}_q^*$ in $\prod_{v \in S} \mathbb{F}_v^*/\mathbb{F}_v^{*l}$. Class field theory tells us that G_S is isomorphic to the middle term of the exact sequence, hence G_S is isomorphic to $\mathbb{Z}/l\mathbb{Z} \times (\prod_{v \in S} \mathbb{F}_v^*/\mathbb{F}_v^{*l})/\widetilde{\mathbb{F}_q^*}$ and I_i is the image of $\mathbb{F}_{v_i}^*/\mathbb{F}_{v_i}^{*\,l}$ in G_S.

If we look at the map

$$(12) \qquad \mathbb{F}_q^* \hookrightarrow \prod_{v \in S} \mathbb{F}_v^* \to \prod_{v \in S} \mathbb{F}_v^*/\mathbb{F}_v^{*l},$$

then since $\mathbb{F}_q^*$ is cyclic and $\prod_{v \in S} \mathbb{F}_v^*/\mathbb{F}_v^{*l}$ is killed by l, $\widetilde{\mathbb{F}_q^*}$ is either 0 or cyclic of order l. It is clear that $\widetilde{\mathbb{F}_q^*} = 0$ when $q \not\equiv 1 \pmod l$. When $q \equiv 1 \pmod l$, we can see, for example by using Kummer theory, that $\mathbb{F}_q^*$ is contained in $(\mathbb{F}_v^*)^l$ if and only if $\deg v$ is divisible by l. Hence $\widetilde{\mathbb{F}_q^*}$ is non-trivial only when $q \equiv 1 \pmod l$ and there is a place $v \in S$ such that l does not divide $\deg v$.

For each $i = 0, \ldots, n$, let $\sigma_i \in G_S$ be a generator of I_i, and σ_T a generator of D_T. When $\widetilde{\mathbb{F}_q^*} = 0$, $\{\sigma_i\}_{i=0}^n$ are linearly independent, viewing G_S as a vector space over $\mathbb{F}_l$, and $\dim_{\mathbb{F}_l} G_S = n+2$. On the other hand, when $\widetilde{\mathbb{F}_q^*} \neq 0$, it gives a non-trivial linear relation among σ_j's for j such that $l \nmid \deg v_j$, and hence $\dim_{\mathbb{F}_l} G_S = n+1$. As we have seen before, this case happens only when K contains a primitive l-th root of unity and there is a place in S whose degree is prime to l. In that case, we may assume that l does not divide $\deg v_0$, then $\{\sigma_i\}_{i=1}^n$ are linearly independent. Furthermore, with the assumption $l \nmid \deg v_T$, v_T does not split completely in $K \cdot \mathbb{F}_{q^l}$, which is the maximal unramified extension in K_S by class field theory and the assumption $l \nmid h$. Hence $\sigma_T \notin \langle \sigma_0, \ldots, \sigma_n \rangle$, which implies that $\{\sigma_T, \sigma_1, \ldots, \sigma_n\}$ are linearly independent. Hence we have proved the following theorem.

THEOREM 6. *(a) If K does not contain a primitive l-th root of unity, then the inertia groups of places in S are linearly independent in G_S.*
(b) If K contains a primitive l-th root of unity, then the inertia groups of places in S generate a subgroup of G_S of rank at least n, and the decomposition group of v_T is not contained in the subgroup as long as $\deg v_T$ *is prime to l.*

REMARK. This argument shows that the assumption on T is necessary only when $\widetilde{\mathbb{F}_q^*}$ is non-trivial, i.e. when K contains an l-th root of unity and S contains a place whose degree is prime to l.

6. Functions on the $\mathbb{F}_l$-vector space

Let V be a $\mathbb{F}_l$-vector space of dimension $N+1$. Choose a basis $\{w_0, \ldots, w_N\}$ of V, and for $i = 0, \ldots, N$ define $X_i \in \mathrm{Hom}(V, \mathbb{F}_l)$ by $X_i(w_j) = \delta_{ij}$. We may view a polynomial $f \in \mathbb{F}_l[X_0, \ldots, X_N]$ as a function on V via the above identification.

The goal of this section is to prove the following theorem, which will be used in proving Theorem 3.

THEOREM 7. *Suppose $f \in \mathbb{F}_l[X_0, \ldots, X_N]$ is a polynomial of degree $\leq n$, which we view as a function on V, and $\{V_i\}_{i=0}^n$ are $n+1$ linearly independent subspaces of codimension 1 in V. If f vanishes on V_i for all i, then f vanishes on V.*

Definition. We say that a polynomial $p(X_0,\dots,X_N) \in \mathbb{F}_l[X_0,\dots,X_N]$ is *reduced* if for each X_i, $\deg_{X_i} p(X_0,\dots,X_N) < l$.

Lemma 8. *Every function on V with values in $\mathbb{F}_l$ can be uniquely expressed as a reduced polynomial in $\mathbb{F}_l[X_0,\dots,X_N]$.*

Proof. This is a well-known result, and we give a short proof here.

Observe that for $a_i \in \mathbb{F}_l$, $i = 0,\dots,N$, we have

$$\prod_{i=0}^{N}(1-(x_i-a_i)^{l-1}) = \begin{cases} 1 & \text{if } x_i = a_i \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

By taking linear combination, we see that any function on V can be represented by a reduced polynomial. Uniqueness follows from counting such polynomials. $\square$

For each polynomial $p(X_0,\dots,X_N) \in \mathbb{F}_l[X_0,\dots,X_N]$, we can associate the reduced polynomial $p_r(X_0,\dots,X_N)$ of $p(X_0,\dots,X_N)$, which is reduced and defines the same function on V as $p(X_0,\dots,X_N)$. We can get $p_r(X_0,\dots,X_N)$ from $p(X_0,\dots,X_N)$ by using the relations $X_i^l = X_i$ for all i to replace X_i^m by $X_i^{m-(l-1)}$ until $m < l$. Notice that for each i, $\deg_{X_i} p_r \le \deg_{X_i} p$, and hence $\deg p_r \le \deg p$.

Lemma 9. *Suppose $p(X_0,\dots,X_N)$ is a reduced polynomial. If $p(0,x_1,\dots,x_N) = 0$ for all $(x_1,\dots,x_N) \in {\mathbb{F}_l}^N$, then $X_0 \mid p(X_0,\dots,X_N)$.*

Proof. Write $p(X_0,\dots,X_N) = X_0 \cdot q(X_0,\dots,X_N) + r(X_1,\dots,X_N)$. For all $(x_1,\dots,x_N) \in {\mathbb{F}_l}^N$, $r(x_1,\dots,x_N) = p(0,x_1,\dots,x_N) - 0 \cdot q(0,x_1,\dots,x_N) = 0$. Since r is also reduced, we conclude that $r = 0$. $\square$

Proof of Theorem 7. By change of coordinates, we may assume that for each $i = 0,\dots,n$, V_i is given by the equation $X_i = 0$. Let f_r be the reduced polynomial of f. According to Lemma 9, f_r is divisible by X_i for all i and since we have unique factorization, it follows that f_r is divisible by $\prod_{i=0}^{n} X_i$. Since we have $\deg f_r \le \deg f \le n$, it follows that $f_r = 0$, and hence f vanishes on V. $\square$

7. The induction step

We prove Theorem 3 by induction on n. When $n = 0$, the conjecture holds as noted in Proposition 2.

Suppose $n \geq 2$. For each $i = 0, \ldots, n$, let $S_i = S \setminus \{v_i\}$. Then $\widehat{G_{S_i}}$ is the orthogonal of I_i, hence is a subspace of codimension 1 in $\widehat{G_S}$ since we assumed that v_i is ramified in K_S. Note that v_i is unramified in K_{S_i}.

By induction we can assume that for all $i = 0, \ldots, n$, Conjecture 1 holds for K_{S_i}/K with respect to S_i and T. Then Proposition 2 shows that Conjecture 1 holds for K_{S_i}/K with respect to S and T. Hence $f|_{\widehat{G_{S_i}}} = 0$. When K does not contain a primitive l-th root of unity, it follows from Theorem 6 and Theorem 7 that $f = 0$ on $\widehat{G_S}$.

If K contains a primitive l-th root of unity, we let $G_T = G_S/D_T$. Then the place v_T will split completely in K_χ for all $\chi \in \widehat{G_T}$ which implies, as we discussed at the end of section 4., that we have $f|_{\widehat{G_T}} = 0$. Again, it follows from Theorem 6 and Theorem 7 that $f = 0$ on $\widehat{G_S}$, and hence $\theta_{G_S} \equiv m \cdot \det \lambda_{G_S} \pmod{I^{n+1}}$ in all cases. $\square$

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Department of Mathematics
The Ohio State University
Columbus, OH 43210
U.S.A.
E-mail: jlee@math.ohio-state.edu