J. Math. Sci. Univ. Tokyo
4 (1997), 279–339.

On Simply Knotted Tori in S^4

By Akiko Shima

Abstract. Let T be a torus in S^4. If the singular set $\Gamma(T^*)$ of the projection T^* ($\subset S^3$) of T consists of three disjoint simple closed curves, then T can be moved to either the standard torus, the spun torus of the trefoil knot $T^0(L_3)$, the twist spun torus of the trefoil knot $T^3(L_3)$, or the torus obtained by attaching a handle to the spun 2-sphere of the trefoil knot, by an ambient isotopy of S^4.

1. Introduction

In [A], Aiso classified of simply knotted spheres with less than 6 crossing circles. In this paper we will study an embedded torus T in S^4. The author ([S1] and [S2]) previously showed the following: if the singular set $\Gamma(T^*)$ of the projection T^* ($\subset S^3$) of T consists of at most two disjoint simple closed curves, then T can be moved to the standard position by an ambient isotopy of S^4. Then if the singular set $\Gamma(T^*)$ consists of three disjoint simple closed curves, what can be said about the position of T? In this paper, the author answers this question.

Main Theorem 1 (Theorem 9.3). *Let T be a torus in S^4. If the singular set $\Gamma(T^*)$ consists of three disjoint simple closed curves, then T is ambient isotopic to one and only one of the following tori.*

(1) the standard torus,

(2) the spun torus of the trefoil knot $T^0(L_3)$,

(3) the twist spun torus of the trefoil knot $T^3(L_3)$, or

(4) the torus obtained by attaching a handle to the spun 2-sphere of the trefoil knot (Figure 1 (1)).

1991 *Mathematics Subject Classification.* Primary 57Q45; Secondary 57Q35..

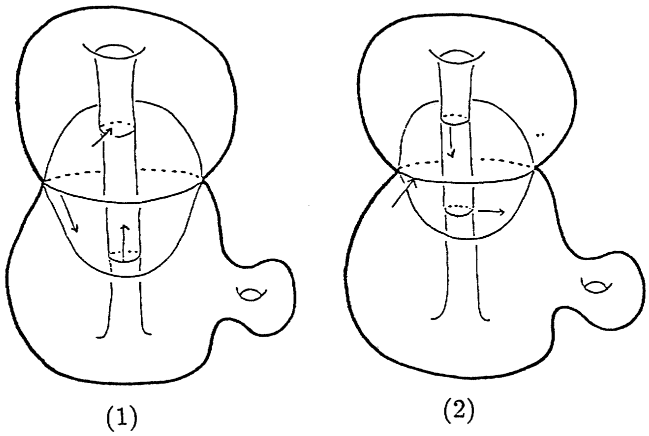


Figure 1

For the definition of $T^0(L_3)$ and $T^3(L_3)$, see Definition 2.1.

Let G_1 be the standard torus, $G_2 = T^0(L_3)$, $G_3 = T^3(L_3)$, and G_4 =the torus obtained by attaching a handle to the spun 2-sphere of the trefoil knot (Figure 1 (1)). Figure 1 (2) will be used in Section 9.

MAIN THEOREM 2 (Theorem 9.5). *If $i \neq j$, then G_i can not be moved to G_j by an ambient isotopy of S^4.*

We will work in the PL category. All submanifolds are assumed to be locally flat. Let S^n be the n-dimensional sphere, $\mathbb{R}^n = S^n \setminus \{\infty\}$ the n-dimensional Euclidean space, and $p : S^4 \setminus \{\infty\} \longrightarrow S^3 \setminus \{\infty\}$ the projection defined by $p(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$.

With every point P or subset F of $S^4 \setminus \{\infty\}$, we associate the point $P^* = p(P)$ or the subset $F^* = p(F)$ of $S^3 \setminus \{\infty\}$. A point P^* is an *i-th singular point*, if $|p^{-1}(P^*) \cap F| = i \geq 2$. We define $\Gamma(F^*)$ to be the set of all i-th singular points with $i \geqq 2$ and put $\Gamma(F) = p^{-1}(\Gamma(F^*)) \cap F$. Let $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 \leqq 1\}$ be the standard 3-ball in $\mathbb{R}^3$, and $P_i = B \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_i = 0\}$ for $i = 1, 2, 3$. Let F be a closed surface, and $f : F \longrightarrow S^3 \setminus \{\infty\}$ a map. We say that f is in *general position*, if for each element x of $f(F)$, there exist a regular neighborhood N of x in

$S^3 \setminus \{\infty\}$ and a homeomorphism $h : N \longrightarrow B$ such that N and h satisfy the following two conditions:

(1) Under h, $(N, N \cap f(F))$ is homeomorphic to either (B, P_1), $(B, P_1 \cup P_2)$, or $(B, P_1 \cup P_2 \cup P_3)$.

(2) Let R be a component of $f^{-1}(f(F) \cap N)$. There exists an integer i such that $h \circ f|R : R \longrightarrow P_i$ is a homeomorphism.

Note. Let F be a surface with $p|F$ in general position, $x \in \Gamma(F^*)$. If $|p^{-1}(x) \cap F| = 2$, then x is called a *double point*.

A solid torus V is said to be *standard* in S^3, if V is $N(K)$ where K is a trivial knot in S^3 and $N(K)$ is a regular neighborhood of K in S^3. And the torus $\partial V \subset S^3 \subset S^4$ is said to be a *standard torus* in S^4.

Let F be a surface in S^4, and $\Sigma_1, \ldots, \Sigma_m$ the closures of the components of $F \setminus \Gamma(F)$. Let γ^* be the intersection of two surfaces $A_i^* = \Sigma_i^* \cup \Sigma_{i+1}^*$ and $A_j^* = \Sigma_j^* \cup \Sigma_{j+1}^*$ (see Figure 2), and γ_i,γ_j the preimages of γ^* on A_i and A_j respectively. Let $u : S^4 \setminus \{\infty\} \longrightarrow \mathbb{R}$ be the height function defined by $u(x_1, x_2, x_3, x_4) = x_4$. If $u(\gamma_i) < u(\gamma_j)$ (i.e. $u(p_i) < u(p_j)$ for all points $p_i \in \gamma_i$ and $p_j \in \gamma_j$ with $p_i^* = p_j^*$), then we call A_i the *under surface* and A_j the *over surface* respectively. We use the notation in Figure 2 to represent the relation between the heights of two surfaces, where the small vector on the under surface indicates the orientation of the over surface (see [Y]).

All homology groups are with coefficients in $\mathbb{Z}$. We denote by $|A|$ the number of the components of A.

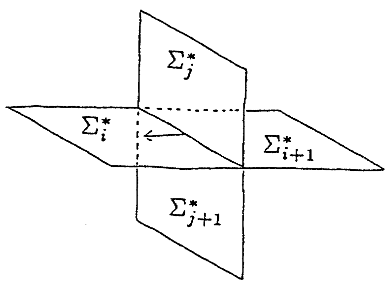


Figure 2

The paper is organized as follows. In Section 2, we define symmetry-spun tori $T^a(K_b)$ in S^4 and immersed tori $\alpha(a,b)$ in S^3. In Section 3, we consider solid tori and immersed surfaces in S^3. In Section 4, we find a necessary condition on singular sets $\Gamma(T)$. In Section 5, we study equivalence relations of knotted surfaces. In Section 6, we introduce a diagrammatic representation of singular sets $\Gamma(T)$. In Section 7, we study certain types of 2-complexes in S^3. In Section 8, we show that if each component of a singular set $\Gamma(T)$ is not contractible in T, then there exists a symmetry-spun torus which is ambient isotopic to T. In Section 9, we prove Main Theorem.

2. Definitions

Fix $\theta \in [0, 2\pi]$. Let $\mathbb{R}^3_\theta = \{(x, y\cos\theta, y\sin\theta, z) |\ (x,y,z) \in \mathbb{R}^3, y \geq 0\} \subset \mathbb{R}^4$. Then $\mathbb{R}^4 = \cup_{\theta=0}^{2\pi} \mathbb{R}^3_\theta$, and $p : S^4 \setminus \{\infty\} \cong \mathbb{R}^4 \longrightarrow S^3 \setminus \{\infty\} \cong \mathbb{R}^3$ is the projection with $p(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$. Let $r_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation map defined by

$$r_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x-2 \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Fix an integer b with $b \neq 0$. Let $q_b : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the map defined by

$$q_b(t\cos\theta + 2, t\sin\theta) = r_{b\theta}(t\cos\theta + 2, t\sin\theta)$$

where $0 \leq t$ and $\theta \in [0, 2\pi]$. Put $B^3 = \{(x, y, 0, z)\ ;\ (x-2)^2 + y^2 + z^2 \leq 1\}$ and $P = \{(2,0,0,z) |\ -1 \leq z \leq 1\}$. Let $K : S^1 \longrightarrow B^3 \setminus P$ be an embedding.

DEFINITION 2.1 (symmetry-spun tori [T]). Let $id : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the identity. Fix integers a, b with $b \neq 0$. We define a *symmetry-spun torus* $T^a(K_b)$ obtained from K as follows:

$$T^a(K_b) = \left\{ \begin{pmatrix} X \\ Y\cos\theta \\ Y\sin\theta \\ z \end{pmatrix} \;\middle|\; \begin{array}{c} 0 \leq \theta \leq 2\pi, \\ (x, y, 0, z) \in (q_b \times id)^{-1}(K(S^1)), \\ (X, Y) = r_{a\theta/b}(x, y) \end{array} \right\} \subset \mathbb{R}^4.$$

We denote by L and L' the knots in $B^3 \setminus P$ obtained by Figure 3.

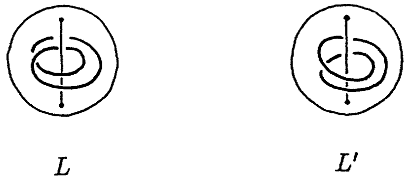


Figure 3

DEFINITION 2.2. Let $\alpha : J \longrightarrow p(B^3 \setminus P)$ be an immersion where J is a disjoint union of 1-spheres and intervals. Let $id : \mathbb{R} \longrightarrow \mathbb{R}$ be the identity. Fix integers a, b with $b \neq 0$. We define an immersed surface $\alpha(a, b)$ obtained by α in S^3 as follows:

$$\alpha(a,b) = \left\{ \begin{pmatrix} X \\ Y\cos\theta \\ Y\sin\theta \end{pmatrix} \;\middle|\; \begin{array}{c} 0 \leq \theta \leq 2\pi, \\ (x,y,0) \in (q_b \times id)^{-1}(Im\alpha), \\ (X,Y) = r_{a\theta/b}(x,y) \end{array} \right\} \subset \mathbb{R}^3.$$

Let $T_i : S^1 \longrightarrow p(B^3 \setminus P)$ be an immersion obtained by Figure 4 (i) for $i = 1, 2, 3, 4$. Let $T_5 : S^1 \cup [0,1] \longrightarrow p(B^3 \setminus P)$ be an immersion obtained by Figure 4 (5). In particular, we denote by $T_i(a, b)$ the immersed tori obtained from T_i ($i = 1, 2, 3, 4, 5$).

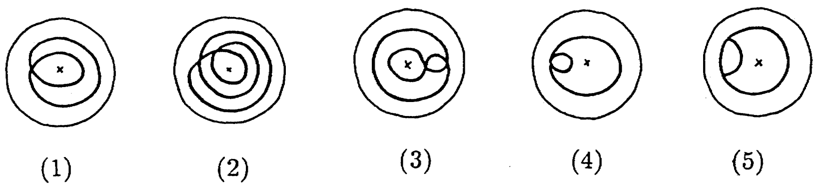


Figure 4

REMARK 2.3. (1) Let $T^a(K_b)$ be a symmetry-spun torus, and $\alpha = p \circ K : S^1 \longrightarrow p(B^3 \setminus P)$ an immersion. Then $(T^a(K_b))^* = \alpha(a, b)$.

(2) Let $\alpha(a, b)$ be an immersed torus obtained by an immersion $\alpha : S^1 \longrightarrow p(B^3 \setminus P)$. Then there exists an embedding $K : S^1 \longrightarrow B^3 \setminus P$ with $(T^a(K_b))^* = \alpha(a, b)$.

3. Solid tori and immersed surfaces in S^3

LEMMA 3.1 ([S1, Lemma 3.1]). *Let F be an oriented closed surface in S^4 with $p|F$ in general position. Let γ^* be a component of $\Gamma(F^*)$ which is a simple closed curve, and c_1, c_2 the components of $p^{-1}(\gamma^*) \cap F$. If γ^* satisfies one of the following conditions, then γ^* can be cancelled by an ambient isotopy of S^4.*

(1) There exist disks D_1, D_2 in F with $\partial D_i = c_i$, and $intD_i \cap \Gamma(F) = \phi$.

(2) There exist an annulus A in F and a solid torus V in S^3 such that $\partial A = c_1 \cup c_2$, $\partial V = A^*$, $intV \cap F^* = \phi$, and γ^* is a generator of $H_1(V) \cong \mathbb{Z}$.

(3) There exists an annulus A in F with $\partial A = c_1 \cup c_2$, $[c_i] = 1$ in $\pi_1(F)$, and $intA \cap \Gamma(F) = \phi$.

LEMMA 3.2 ([S1, Theorem 4.1] and [S2, Theorem 3.1]). *Let T be a torus in S^4. If the singular set $\Gamma(T^*)$ consists of at most two simple closed curves, then T can be moved to the standard position by an ambient isotopy of S^4.*

We will define the *D-surgery* to be the operation shown in Figure 5. If $\gamma^* \subset \Gamma(F^*)$ is the intersection curve in Figure 5, c_1 and c_2 are the preimages of γ^*, and c_1 bounds a disk D with $\Gamma(F) \cap intD = \phi$, then the *D-surgery* replaces the regular neighborhood of D^* by a pair of disks that do not intersect with F^*, and connects the centers of the disks by an arc the interior of which intersects F^* at one point at the center of D^*.

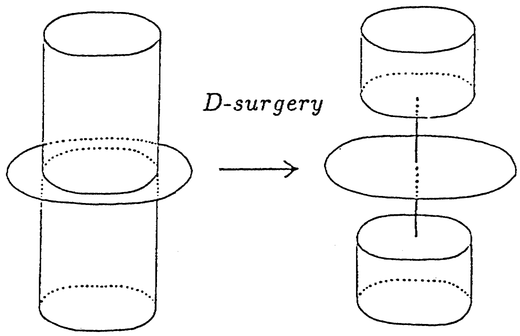


Figure 5

LEMMA 3.3. *Let F be an oriented closed surface with $p|F$ in general position. Let γ_1^*, γ_2^* be components of $\Gamma(F^*)$ which are simple closed curves. Suppose that c_1, c_2 are the components of $p^{-1}(\gamma_1^*) \cap F$, and d_1, d_2 are the components $p^{-1}(\gamma_2^*) \cap F$. If γ_1^*,γ_2^* satisfy the following condition (*), then γ_i^* can be cancelled by an ambient isotopy of S^4.*

(*) There exist disks D_1, D_2, D_3 in F with $\partial D_1 = c_1$, $\partial D_2 = c_2$, $\partial D_3 = d_1$, $D_1 \supset D_2 \cup D_3$, $D_2 \cap D_3 = \phi$, $intD_1 \cap \Gamma(F) = c_2 \cup d_1 \cup d_2$, $intD_2 \cap \Gamma(F) = \phi$, and $intD_3 \cap \Gamma(F) = \phi$.

PROOF. Suppose that γ_1^*,γ_2^* satisfy the above condition (*), then we distinguish three cases (see Figure 6).

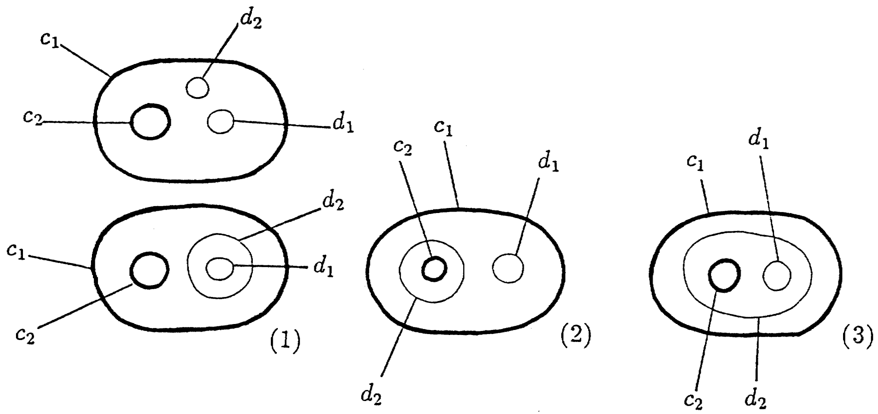


Figure 6

Case 1. By Lemma 3.1 (1) and (3), γ_2^* can be cancelled by an ambient isotopy of S^4.

Case 2. There exists a disk D_4 in D_1 with $\partial D_4 = d_2$. Let $F_1 = F \backslash intD_1$. Then $(F_1 \cup D_4)^*$ is as shown in Figure 7. The simple closed curve γ_1^* can be cancelled by an ambient isotopy of S^4.

Case 3. Perform *D-surgeries* along D_2, D_3. Then we obtain Figure 8 (1). The simple closed curves γ_1^* and γ_2^* can be cancelled by an ambient isotopy of S^4.

This completes the proof of Lemma 3.3. □

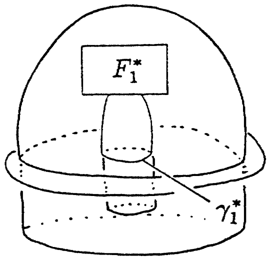


Figure 7

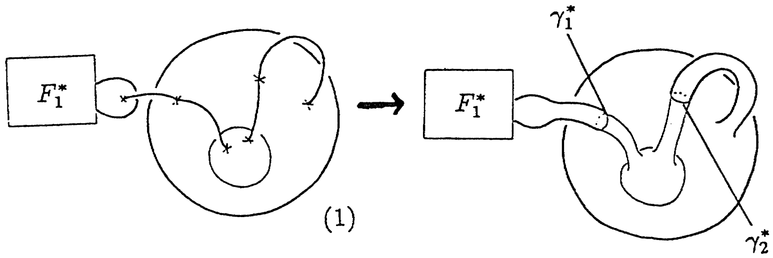


Figure 8

LEMMA 3.4. *Let F, $p|F$, γ_i^*, c_i, d_i $(i = 1, 2)$ be as above. If γ_1^*, γ_2^* and F satisfy one of the following conditions, then γ_1^* or γ_2^* can be cancelled by an ambient isotopy of S^4.*

(1) The surface F is a torus. There exist an annulus A and a disk D in F with $\partial D = c_1$, $\partial A = d_1 \cup d_2$, $intA \cap \Gamma(F) = \phi$, $[c_2] \neq 1 \in \pi_1(F)$, and $[d_i] \neq 1 \in \pi_1(F)$ $(i = 1, 2)$.

(2) There exist disks D_1, D_2 in F with $\partial D_1 = c_1$, $\partial D_2 = c_2$, $intD_1 \cap \Gamma(F) = \phi$, and $(F \setminus D_2) \cap \Gamma(F) = \phi$.

PROOF. Suppose that γ_1^*,γ_2^* satisfy (1). Then A^* is an embedded torus in S^3, and γ_2^* is a simple closed curve on A^*. Let V_1, V_2 be the closures of the components of $S^3 \setminus A^*$ with $V_1 \cup V_2 = S^3$, $\partial V_i = A^*$ $(i = 1, 2)$ and $V_1 \supset F^* \cup D^*$. By the solid torus theorem (see [R] p107), either V_1 or V_2 is a solid torus. Let B be an annulus in F with $\partial B = c_2 \cup d_j$ and $B \cap (c_1 \cup d_k) = \phi$ $(\{1, 2\} = \{j, k\})$. Put $D' = D \cup B$. In general, D'^* is a singular disk, and $(\partial D')^* = \gamma_2^*$. By Dehn's lemma, there exists a non-singular disk E with $intE \cap A^* = \phi$ and $\partial E = \gamma_2^*$. Suppose that V_1 is a solid torus. Then γ_2^* is a meridian of V_1. Moving F by some ambient isotopy without changing the topology of the singular set $\Gamma(F)$, we may assume that V_1 is standard. Then γ_2^* is a longitude of V_2, and γ_2^* can be cancelled by Lemma 3.1 (2). Suppose that V_2 is a solid torus. Then γ_2^* is a longitude of V_2. We have $\partial A = d_1 \cup d_2$, $\partial V_2 = A^*$, $intV_2 \cap F^* = \phi$, and $[\gamma_2^*] = \pm 1 \in H_1(V_2) \cong \mathbb{Z}$. Then we can cancel γ_2^* by Lemma 3.1 (2).

Suppose that γ_1^*,γ_2^* satisfy (2). Let $F_1 = F \setminus intD_2$. The singular surface $(F_1 \cup D_1)^*$ is as shown in Figure 9 (1). Moving F by an ambient isotopy of S^4 without changing the topology of the singular set $\Gamma(F)$, we may assume that $(F_1 \cup D_1)^*$ is contained in a regular neighborhood of D_1^* in S^3. Then the simple closed curve γ_1^* can be cancelled.

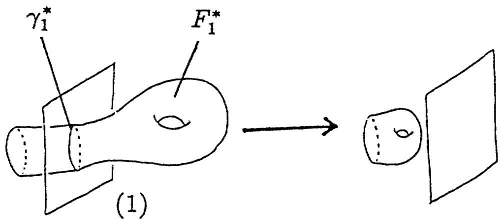


Figure 9

This completes the proof of Lemma 3.4. □

LEMMA 3.5. *Let F, $p|F$, γ_i^*, c_i, and d_i ($i = 1, 2$) be as above. Suppose that F is a torus. Consider the following conditions.*

(1) There exists an annulus A in F with $\partial A = c_1 \cup c_2$, $intA \cap \Gamma(F) = d_1$, $[d_1] = 1 \in \pi_1(A)$, and $[d_2] \neq 1 \in \pi_1(F)$.

(2) There exist disks D_1, D_2 in F with $\partial D_1 = c_1$, $\partial D_2 = c_2$, $intD_1 \cap \Gamma(F) = d_1$, $intD_2 \cap \Gamma(F) = \phi$, and $[d_2] \neq 1 \in \pi_1(F)$.

(3) There exist disks D_1, D_2, D in F with $\partial D_1 = c_1$, $\partial D_2 = c_2$, $\partial D = d_2$, $intD_1 \cap \Gamma(F) = d_1$, $D \cap (D_1 \cup D_2) = D_i$ ($i = 1$ or 2), and $D_1 \cap D_2 = \phi$.

Then there does not exist such an embedded torus in S^4 that satisfies (1), (2) or (3).

PROOF. Suppose that there exists a torus F in S^4 which satisfies (1). Let $N(A), N(d_2)$ be regular neighborhoods of A, d_2 in F respectively. Let a_1, a_2 be the components of $\partial N(A)$, and a_3, a_4 the components of $\partial N(d_2)$.

(Case a) $F \setminus A$ is connected (see Figure 10 (a)).

By assumption, $F \setminus A \cup d_2$ consists of two components. Let G_1, G_2 be the components of $F \setminus int(N(A) \cup N(d_2))$ with $\partial G_1 = a_1 \cup a_3$ and $\partial G_2 = a_2 \cup a_4$. Then $(N(A) \cup N(d_2))^*$ is as shown in Figure 11. We connect a_1^* and a_3^* in $S^3 \setminus A^*$ by G_1^*. Then $a_1^*, a_2^*, a_3^*, a_4^*$ are as shown in Figure 11. But we cannot connect a_2^* and a_4^* in $S^3 \setminus A^*$ by G_2^*. This is a contradiction.

(Case b) $F \setminus A$ is not connected (see Figure 10 (b)).

By assumption, $F \setminus A$ consists of two components. Let G_1, G_2 be the components of $F \setminus int(N(A) \cup N(d_2))$ with $\partial G_1 = a_1$ and $\partial G_2 = a_2 \cup a_3 \cup a_4$. Then $(N(A) \cup N(d_2))^*$ is as shown in Figure 11. The curves a_2^*, a_3^* and a_4^* must be connected in $S^3 \setminus A^*$ by G_2^*. But we cannot connect a_2^*, a_3^* and a_4^* in $S^3 \setminus A^*$. This is a contradiction.

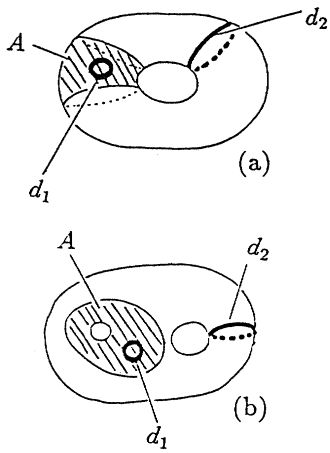


Figure 10

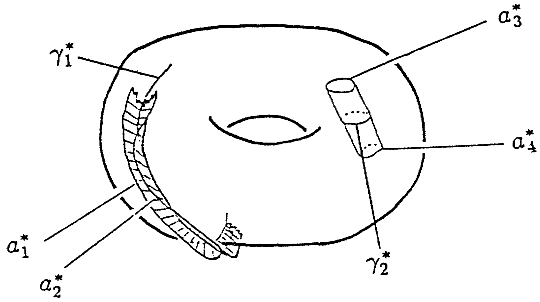


Figure 11

Suppose that there exists a torus F in S^4 which satisfies (2). See Figure 12. Let $N(D_1)$, $N(D_2)$, $N(d_2)$ be regular neighborhoods of D_1, D_2, d_2 in F, respectively. Let $F_1 = F \setminus int(N(D_1) \cup N(D_2) \cup N(d_2))$, and a_1, a_2, a_3, a_4

be the components of ∂F_1. Then $(N(D_1) \cup N(D_2) \cup N(d_2))^*$ is as shown in Figure 13. But we cannot connect a_1^*, a_2^*, a_3^* and a_4^* in $S^3 \setminus (D_1 \cup D_2)^*$ by F_1^*. This is a contradiction.

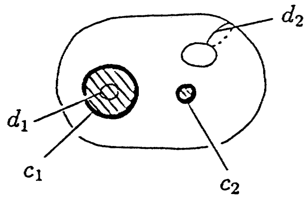


Figure 12

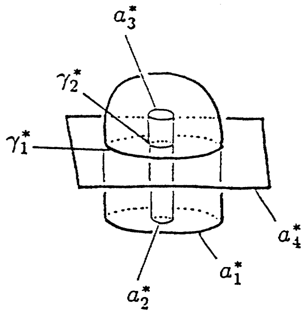


Figure 13

Suppose that there exists a torus F in S^4 which satisfies (3). See Figure 14. Let $N(D)$, $N(D_1)$, $N(D_2)$, $N(d_2)$ be regular neighborhoods of D, D_1, D_2, d_2 in F respectively. Put $F_1 = F \setminus int(N(D) \cup N(D_1) \cup N(D_2) \cup N(d_2))$ and $A = D \setminus int(N(D_i) \cup N(d_2))$. Let a_1, a_2 be the components of $\partial N(F_1)$, and a_3, a_4 the components of ∂A. Then $(N(D_1) \cup N(D_2) \cup N(d_2))^*$ is as shown in Figure 14. We connect a_1^*, a_2^* in $S^3 \setminus (D_1 \cup D_2)^*$ by F_1^*. Then $a_1^*, a_2^*, a_3^*, a_4^*$ are as shown in Figure 13. But we cannot connect a_3^*, a_4^* in $S^3 \setminus (D_1 \cup D_2)^*$ by A^*. This is a contradiction.

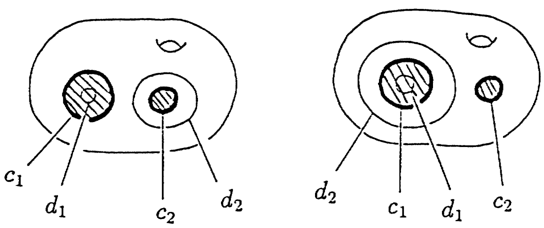


Figure 14

This completes the proof of Lemma 3.5. □

Let F be an embedded closed surface in S^4, and $x \in F^*$. We say that x is a *branch point* (also known as "Whitney's umbrella" or "a pinch point") if there exists a regular neighborhood N of x in S^3 such that $(N, N \cap F^*, x)$ is homeomorphic to $(B, Q, 0)$, and $p^{-1}(N \cap F^*) \cap F$ is homeomorphic to a 2-disk where $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 \leq 1\}$ and Q is a cone with vertex $0 = (0,0,0)$ of a figure eight in ∂B. We define *B-move* to be the operation shown in Figure 15. The symbol $\longleftrightarrow$ means that pictures pointed by an arrow are ambient isotopic. For canceling branch points, see [C-S] or [H-N].

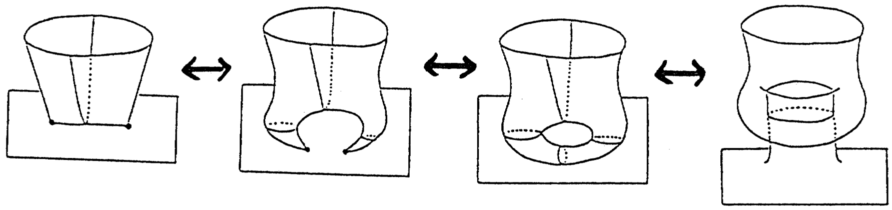


Figure 15

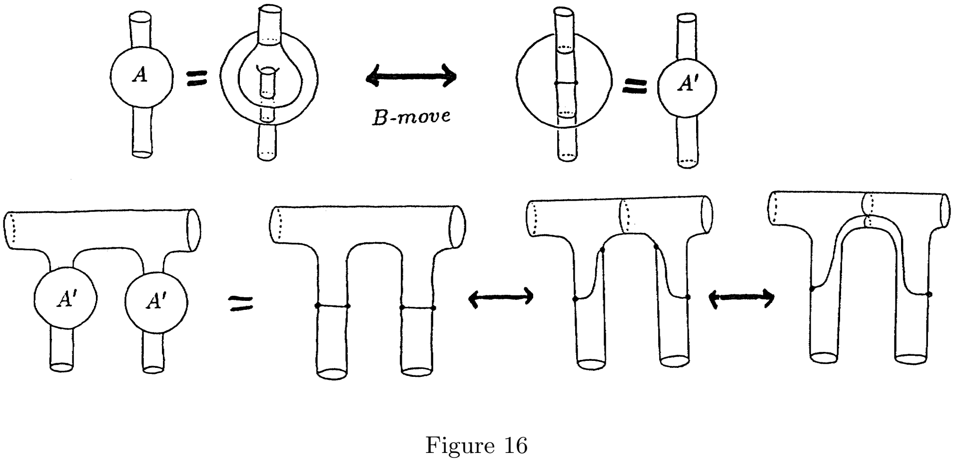


Figure 16

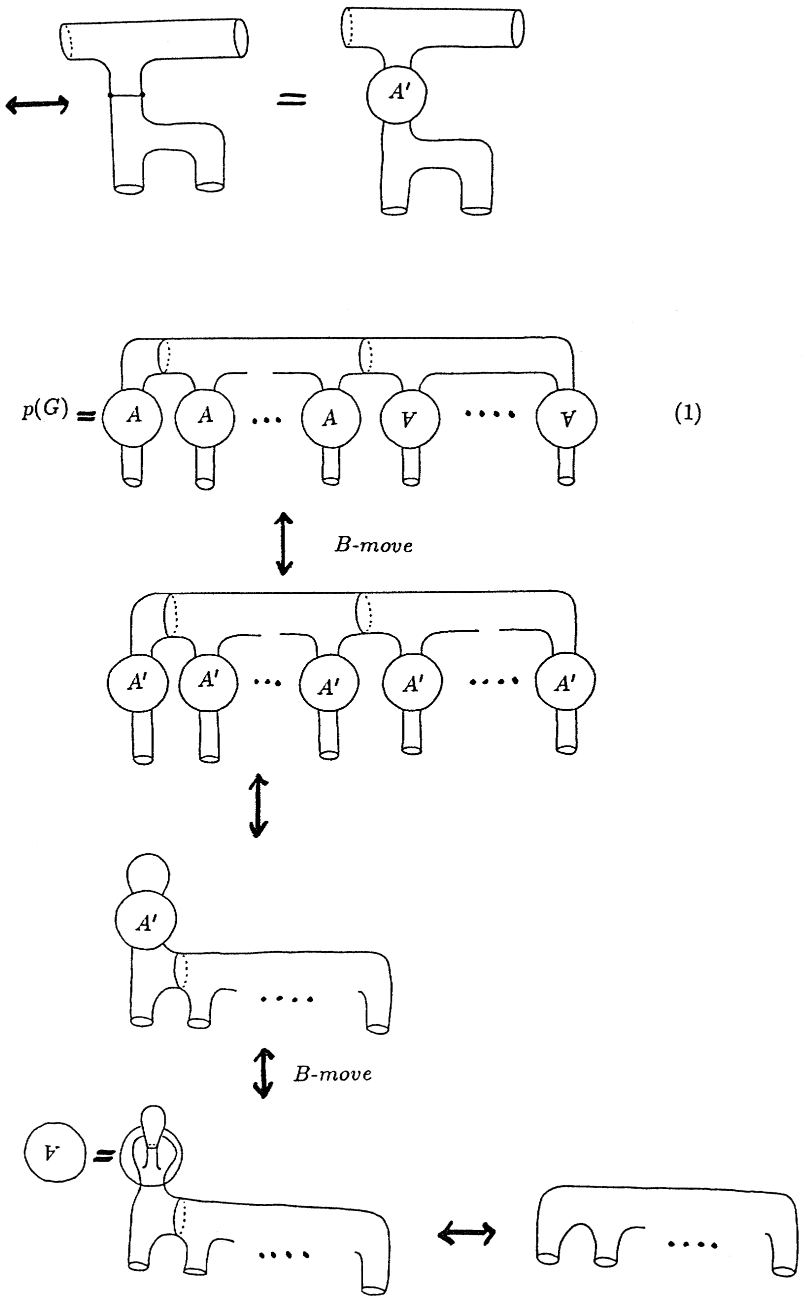


Figure 16 (continued)

LEMMA 3.6. *Let F be an oriented surface with $p|F$ in general position. Suppose that there exist components γ_i^* of $\Gamma(F^*)$ $(i = 1, 2, \dots, n)$ and a surface G in F such that γ_i^* is a simple closed curve of G^*, and $p(G)$ is Figure 16 (1). Then γ_i^* $(i = 1, 2, \dots, n)$ can be cancelled.*

PROOF. We obtain Figure 16 using *B-move*. In Figure 16, it is realized by an ambient isotopy only when the crossing information near the branch points matches. If the crossing information does not match, then we rotate one of the bottom handles by 180 degrees. Then an ambient isotopy can be realized. The simple closed curves γ_i^* can be cancelled. □

4. A necessary condition on singular sets

The following lemmas are generalizations of Aiso's lemmas. We use a technique of Aiso [A]. Let F be a torus or a 2-sphere in S^4 with $p|F$ in general position. In this section we assume that $\Gamma(F^*)$ consists only of double points. Let γ^* be a component of $\Gamma(F^*)$, and c_1, c_2 the components of $p^{-1}(\gamma^*) \cap F$. Suppose that c_1 and c_2 are situated on F as shown by one of the figures of Figure 17. Take an embedded arc α in F such that α joins c_1 and c_2, $\alpha \cap (c_1 \cup c_2) = \partial\alpha$, and $int\alpha$ is transverse to $\Gamma(F)$. Suppose that α satisfies that either $|c \cap \alpha| = 0$ or 1 for any component c of $\Gamma(F)$ with $c \neq c_i$ $(i = 1, 2)$.

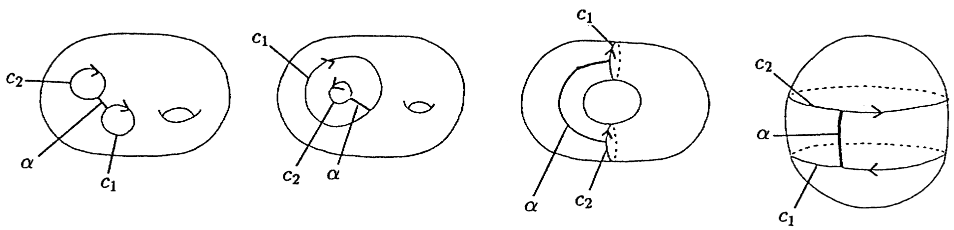


Figure 17

Let $\Gamma = \{\gamma^*$ is a component of $\Gamma(F^*)$; $p^{-1}(\gamma^*) \cap F$ is one of the figures of Figure 17$\}$. We define a function $\rho : \Gamma \longrightarrow \mathbb{Z}_2$ as follows:

$$\rho(\gamma^*) = \begin{cases} 1 & \text{if } |\alpha \cap \Gamma(F)| \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

We will show that ρ is well defined.

FACT 4.1. *Let F be a closed surface in S^4 with $p|F$ in general position. Let c be a simple closed curve in S^3 such that c is transverse to F^* and $c \cap \Gamma(F^*) = \phi$. Then $|c \cap F^*|$ is even.*

Let A_γ be the closure of the component of $F \setminus (c_1 \cup c_2)$ with $A_\gamma \supset \alpha$. The orientations of c_1, c_2 are the induced orientations from A_γ.

LEMMA 4.2. *If $\rho(\gamma^*) = 0$, then the orientation of c_1^* is opposite to the orientation of c_2^*. If $\rho(\gamma^*) = 1$, then the orientation of c_1^* is the same as the orientation of c_2^*.*

PROOF. We will make a simple closed curve $\tilde{\alpha}$ in S^3 such that $\tilde{\alpha}$ is transverse to F^* and $\tilde{\alpha} \cap \Gamma(F^*) = \phi$. We may assume that α^* is a simple closed curve in S^3. We want to make α^* by pushing out α to the positive normal direction induced by an orientation of F.

If A_γ^* is an orientable singular surface, then we construct $\tilde{\alpha}$ as in Figure 18 (1), and in this case the orientation of c_1^* is opposite to the orientation of c_2^*. We have $|\tilde{\alpha} \cap F^*| \equiv |\alpha \cap \Gamma(F)| \equiv \rho(\gamma^*) \pmod 2$. By Fact 4.1, $\rho(\gamma^*) = 0$.

If A_γ^* is a non-orientable singular surface, then we construct $\tilde{\alpha}$ as in Figure 18 (2). In this case, the orientation of c_1^* is the same as the orientation of c_2^*, and we have $|\tilde{\alpha} \cap F^*| \equiv |\alpha \cap \Gamma(F)| + 1 \equiv \rho(\gamma^*) + 1 \pmod 2$. By Fact 4.1, $\rho(\gamma^*) = 1$. This completes the proof of Lemma 4.2. $\square$

Obviously Lemma 4.2 implies that ρ is well defined.

LEMMA 4.3. *If $\rho(\gamma^*) = 1$, then there exists a component μ^* of $\Gamma(F^*)$ with $|d_1 \cap \alpha| = 1$, $|d_2 \cap \alpha| = 0$, and $d_1 \cup d_2 \subset A_\gamma$, where d_1, d_2 are the components of $p^{-1}(\mu^*) \cap F$.*

PROOF. Let μ^* be a component of $\Gamma(F^*)$ with $\mu^* \neq \gamma^*$, and d_1, d_2 the components of $p^{-1}(\mu^*) \cap F$. Let $N(d_i)$ be a regular neighborhood of d_i in F, and $\beta, \tilde{\beta}$ subarcs in $\alpha, \tilde{\alpha}$, respectively such that $\beta \cap \Gamma(F) = \beta \cap d_1 = \{\text{one point}\}$, $\tilde{\beta} \cap N(d_2)^* = \{\text{one point}\}$, and $\tilde{\beta}$ is obtained from β^* pushing it slightly in the positive normal direction induced from the orientation of F. If $d_2 \subset A_\gamma$, then $\tilde{\beta} \cap A_\gamma^* = \tilde{\beta} \cap N(d_2)^* = \{\text{one point}\}$. If $d_2 \cap A_\gamma = \phi$, then $N(d_2) \cap A_\gamma = \phi$ and $\tilde{\beta} \cap A_\gamma^* = \phi$. Let Ω_1 be the set of components μ^* of

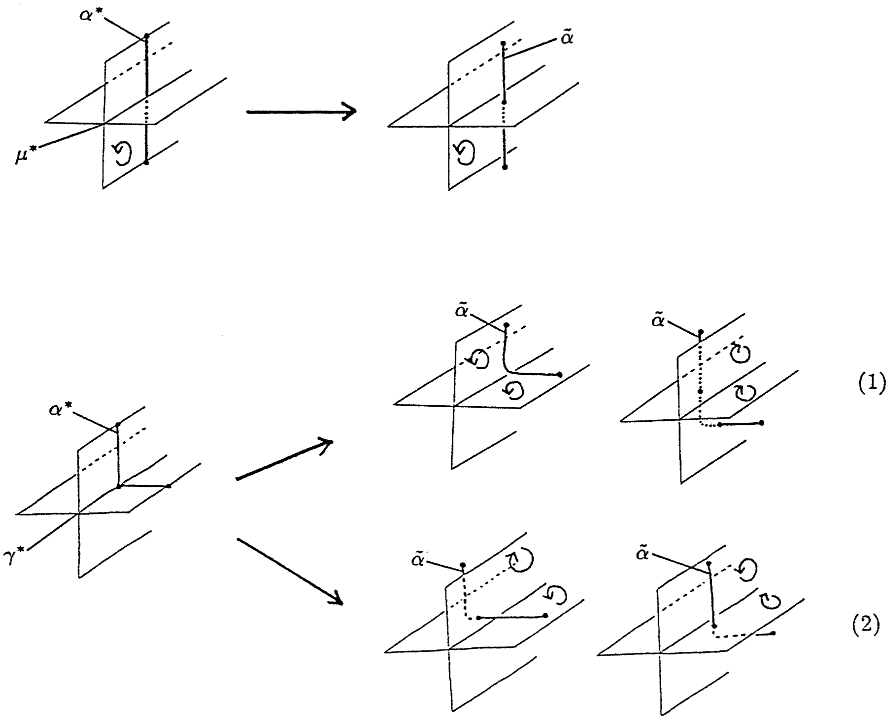


Figure 18

$\Gamma(F^*)$ with $|d_1 \cap \alpha| = 1$, $|d_2 \cap \alpha| = 0$ and $d_1 \cup d_2 \subset A_\gamma$. We will show $\Omega_1 \neq \phi$. Let Ω_2 be the set of components μ^* of $\Gamma(F^*)$ with $|d_i \cap \alpha| = 1$ ($i = 1$ and 2). We use a simple closed curve $\tilde{\alpha}$ in Lemma 4.2. We have $|\tilde{\alpha} \cap A^*_\gamma| \equiv |\Omega_2| \times 2 + |\Omega_1| + 1$. By Fact 4.1, $|\tilde{\alpha} \cap A^*_\gamma|$ is even. Therefore $|\Omega_1| \equiv 1 (\mathrm{mod}\ 2)$, and $\Omega_1 \neq \phi$. This completes the proof of Lemma 4.3. □

COROLLARY 4.4. *Let F be a 2-sphere or a torus with $p|F$ in general position. Let γ_1^*, γ_2^* be components of $\Gamma(F^*)$ which are simple closed curves, c_1, c_2 the components of $p^{-1}(\gamma_1^*) \cap F$, and d_1, d_2 the components $p^{-1}(\gamma_2^*) \cap F$. Consider the following condition: there exists an annulus A with $\partial A = c_1 \cup c_2$, $intA \cap \Gamma(F) = d_1$, and $[d_1] \neq 1 \in \pi_1(A)$.*

Then there does not exist such a surface satisfying this condition.

PROOF. Take an arc α on A as in Figure 19. We have that $|\alpha \cap \Gamma(F)|$ is

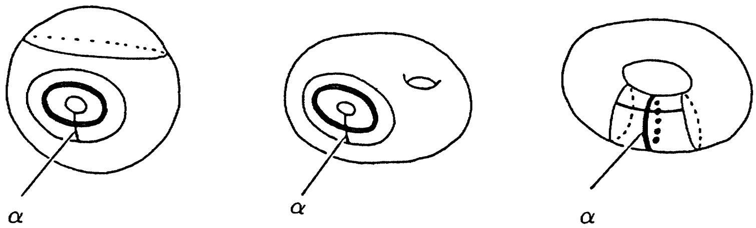


Figure 19

odd, and $\rho(\gamma^*) = 1$. But there does not exist a component of $\Gamma(F^*)$ which satisfies the condition of Lemma 4.3. Therefore there does not exist such a surface satisfying this condition. $\square$

5. Equivalence relations of knotted surfaces

We will define two equivalence relations. We will show that they are the same relations. Let F be a closed surface, f_1, $f_2 : F \longrightarrow S^4$ embeddings and $F_i = f_i(F)$. We say that F_1 is *ambient isotopic* to F_2, if there exists a level preserving homeomorphism $H : S^4 \times I \longrightarrow S^4 \times I$ with $h_0 = id$ and $h_1(F_1) = F_2$. Two knotted surfaces (S^4, F_1) and (S^4, F_2) are *equivalent* if there exists a homeomorphism $f : (S^4, F_1) \longrightarrow (S^4, F_2)$ preserving the orientation of S^4. We let the notation $(S^4, F_1) \cong (S^4, F_2)$ stand for this equivalence.

FACT 5.1 ([G, Theorem 1]). *Let $f : S^4 \longrightarrow S^4$ be an orientation preserving homeomorphism, then f is ambient isotopic to an identity map.*

LEMMA 5.2. *The following conditions are equivalent.*
(1) F_1 is ambient isotopic to F_2.
(2) $(S^4, F_1) \cong (S^4, F_2)$.

PROOF. If F_1 is ambient isotopic to F_2, then $(S^4, F_1) \cong (S^4, F_2)$ by definition. Suppose that $(S^4, F_1) \cong (S^4, F_2)$. There exists a homeomorphism $f : S^4 \longrightarrow S^4$ preserving the orientation of S^4 with $f(F_1) = F_2$. By Fact 5.1, there exists an isotopy $H : S^4 \times I \longrightarrow S^4 \times I$ with $h_0 = f$ and $h_1 = id$. Let $H^{-1} : S^4 \times I \longrightarrow S^4 \times I$ $((x,t) \longmapsto (h_{1-t}(x), t))$. Using H^{-1}, we can show that F_1 is ambient isotopic to F_2. $\square$

Let T be a torus in S^4. If there exists an immersion $\alpha : S^1 \longrightarrow p(B \setminus P)$ with $T^* = \alpha(a, b)$, $(a, b) = 1$, and $b \neq 0$, then there exists a knot $\tilde{\alpha}$ in $B^3 \setminus P$ such that $(T^a(\tilde{\alpha}_b))^* = \alpha(a, b)$, and T is ambient isotopic to $T^a(\tilde{\alpha}_b)$ (see Remark 2.3).

LEMMA 5.3. *The above torus T is ambient isotopic to $T^0(\tilde{\alpha}_1)$ or $T^1(\tilde{\alpha}_1)$.*

PROOF. By [T, Theorem 8], $(S^4, T^a(\tilde{\alpha}_b)) \cong (S^4, T^0(\tilde{\alpha}_1))$ or $(S^4, T^1(\tilde{\alpha}_1))$. Now Lemma 5.3 follows from Lemma 5.2. □

6. A diagrammatic representation of $\Gamma(T)$

Notes ([S1, Lemma 2.3] and [S1, Lemma 2.4]).

(1) If F is an oriented closed surface in S^4 with $p|F$ in general position, then $F \setminus \Gamma(F)$ is divided into some regions. Thus we can color each region black or white so that adjacent regions have different colors.

(2) Let F, $p|F$ be as above, and γ^* a component of $\Gamma(F^*)$. If γ^* is a simple closed curve, then $p^{-1}(\gamma^*) \cap F$ consists of two disjoint simple closed curves.

Let T be a torus in S^4 with $p|T$ in general position. In this section we assume that $\Gamma(T^*)$ consists only of double points. By Notes (2), $\Gamma(T)$ consists of even disjoint simple closed curves on T. By Notes (1), then $\Gamma(T)$ satisfies that

(**) the regions of $T \setminus \Gamma(T)$ can be colored black or white so that adjacent regions have different colors.

We will consider a certain diagrammatic representation of $\Gamma(T)$. Let c_t (t=1, 2, ..., $2n$) be disjoint simple closed curves in T. Put $C(T) = \cup_{t=1}^{2n} c_t$. Then $T \setminus C(T)$ is divided into several regions. Suppose that $C(T)$ satisfies (**). Let Σ_1, ..., Σ_m be the closures of the components of $T \setminus C(T)$. Then Σ_i is either a *torus* with holes or a 2-*sphere* with holes. For a simple closed curve c_t, there exists a unique pair (Σ_i, Σ_j) with $\Sigma_i \cap \Sigma_j = \partial\Sigma_i \cap \partial\Sigma_j \supset c_t$, and $i \neq j$. We construct the graph $G_{C(T)}$ as follows. The vertices are in one to one correspondence to the regions $\{\Sigma_i\}$, and the edges are in one to one correspondence to the simple closed curves $\{c_t\}$. If a simple closed curve c_t is a component of $\partial\Sigma_i \cap \partial\Sigma_j$, we connect the vertices $v(\Sigma_i)$ and $v(\Sigma_j)$ by the edge $e(c_t)$. In particular, if Σ_i is a torus with holes, then we call $v(\Sigma_i)$

the *special vertex* and we denote it by $\bigstar$. Let $|G|$ be the union of all edges of a graph G. Let G be a connected graph having an even number of edges. For an arbitrary graph we call it the *special vertex* when we specify one of the vertices. When a vertex is not specified, we say that a graph does not have the special vertex. If G satisfies one of the following two conditions, we call G an *S-graph*.

(1) The graph G does not have the special vertex. There exists a unique 1-cycle C such that C consists of an even number of edges and $G \setminus C$ is a collection of trees ($G \setminus C$ consists of all vertices and edges which are not contained in $|C|$).

(2) The graph G is a collection of trees having the special vertex.

It is easy to prove the following lemmas.

LEMMA 6.1. *Let $C(T) = \cup c_t$ and $\tilde{C}(T) = \cup \tilde{c}_s$ be two systems of disjoint simple closed curves on T satisfying the condition (**).*

(1) If there exists a homeomorphism $f : T \longrightarrow T$ with $f(C(T)) = \tilde{C}(T)$, then $G_{C(T)}$ and $G_{\tilde{C}(T)}$ are the same S-graphs.

(2) If G is an S-graph, then there exist disjoint simple closed curves $C(T) = \cup c_t$ on T such that G and $G_{C(T)}$ are the same graphs.

LEMMA 6.2. *Let $C(T) = \cup_{t=1}^{6} c_t$ be a system of six disjoint simple closed curves on T satisfying (**). Then the graphs $G_{C(T)}$ are enumerated in Figure 20.*

Suppose that $\Gamma(T^*)$ consists of three disjoint simple closed curves. By Lemma 3.1 (1) and (3), we may assume the following conditions.

(1) A number of disks in $\{\Sigma_i\}$ is at most three.

(2) If Σ_i is an annulus and $[c_t] = [c_s] = 1 \in \pi_1(T)$ where c_t, c_s are the components of $\partial\Sigma_i$, then $c_t^* \cap c_s^* = \phi$.

If $\Gamma(T)$ does not satisfy the above conditions (1) or (2), then the torus T is ambient isotopic to the standard torus.

LEMMA 6.3. *All the possible configurations of $\Gamma(T)$ that satisfy the above conditions (1) and (2) are listed in Figure 21.*

7. 2-complexes consisting of annuli in S^3

We will study certain types of 2-complexes in S^3. Put B=$\{(x, y, z) \in \mathbb{R}^3 | x^2+y^2+z^2 \leq 1\}$, Q_1=$\{(x, y, z) \in \mathbb{R}^3 | \ x^2+y^2 \leq 1, z = 0\}$, Q_2=$\{(x, y, z) \in$

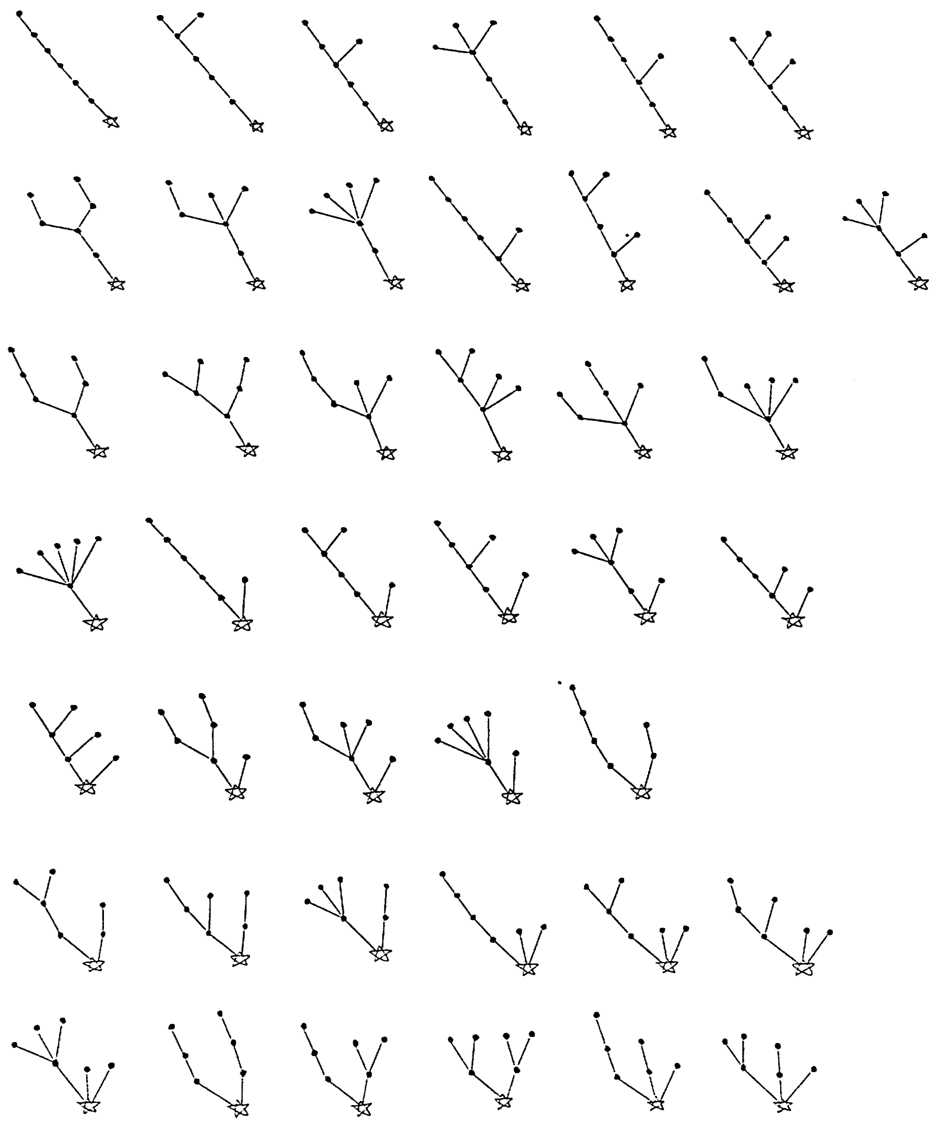


Figure 20

$\mathbb{R}^3|\ (x^2+y^2 \le 1, z=0)$ or $(x^2+z^2 \le 1, z \ge 0, y=0)\}$, and Q_3=$\{(x,y,z) \in \mathbb{R}^3|\ (x^2+y^2 \le 1, z=0)$ or $(x^2+z^2 \le 1, y=0)\}$.

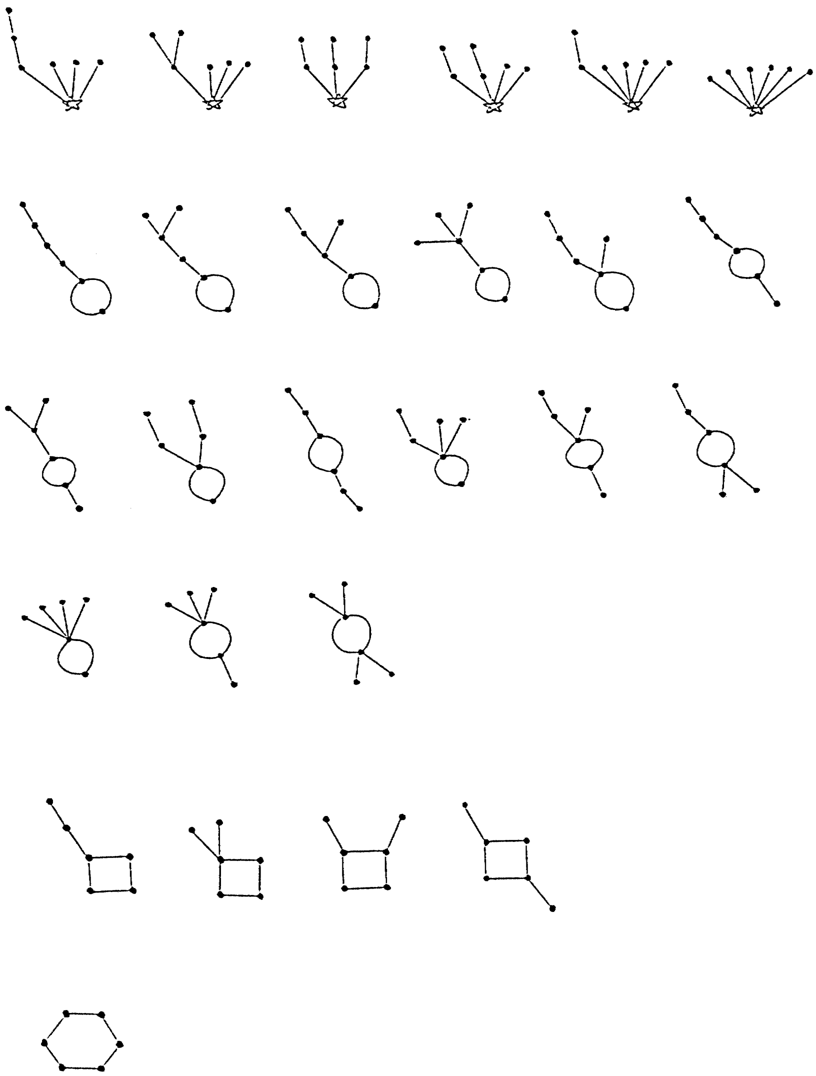


Figure 20 (continued)

DEFINITION 7.1. Let K be a 2-complex in S^3. We call K a *2-complex consisting of annuli in* S^3 if K satisfies the following five conditions:

(1) K is connected.

(2) For each point x of K there exists a regular neighborhood N of x

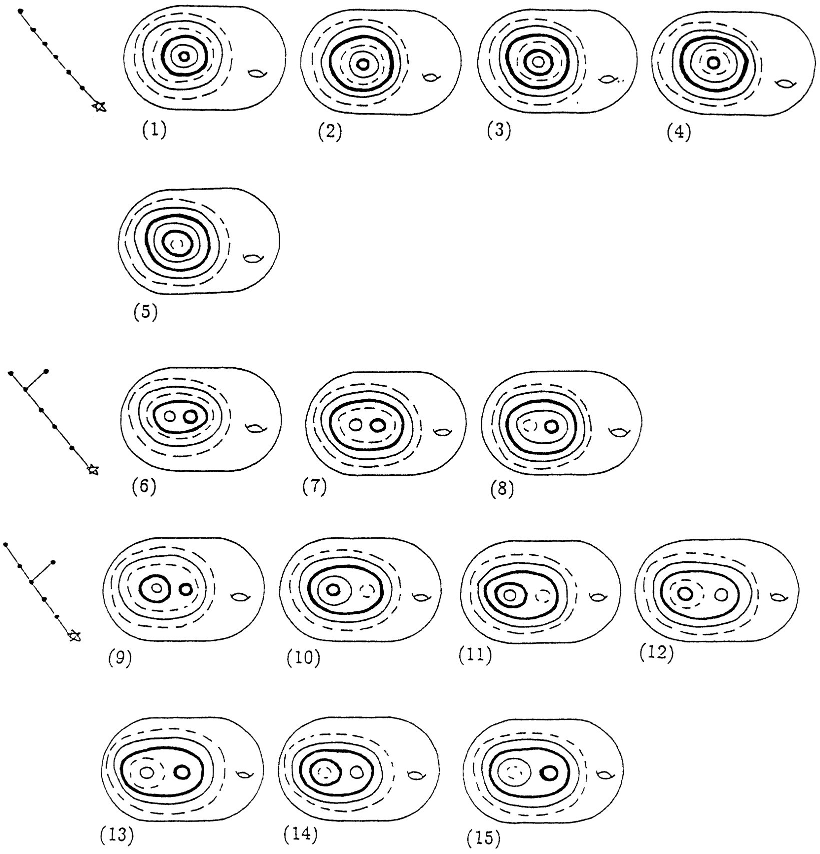


Figure 21

in S^3 such that $(x, N \cap K, N)$ is homeomorphic to either $((0,0,0), Q_1, B)$, $((0,0,0), Q_2, B)$ or $((0,0,0), Q_3, B)$.
Let $S(K)$ be the set of all points x of K having a regular neighborhood N of x such that $(x, N \cap K, N)$ is homeomorphic to $((0,0,0), Q_3, B)$. Let $S'(K)$ be the set of all points x of K having a regular neighborhood N of x such that $(x, N \cap K, N)$ is homeomorphic to $((0,0,0), Q_2, B)$.

(3) $K \setminus (S(K) \cup S'(K))$ consists of finite open annuli.

Let C be a component of $S(K) \cup S'(K)$, and $N(C)$ a regular neighborhood of C in S^3.

(4) If $C \subset S(K)$, then $\partial N(C) \cap K$ consists of two or four components.

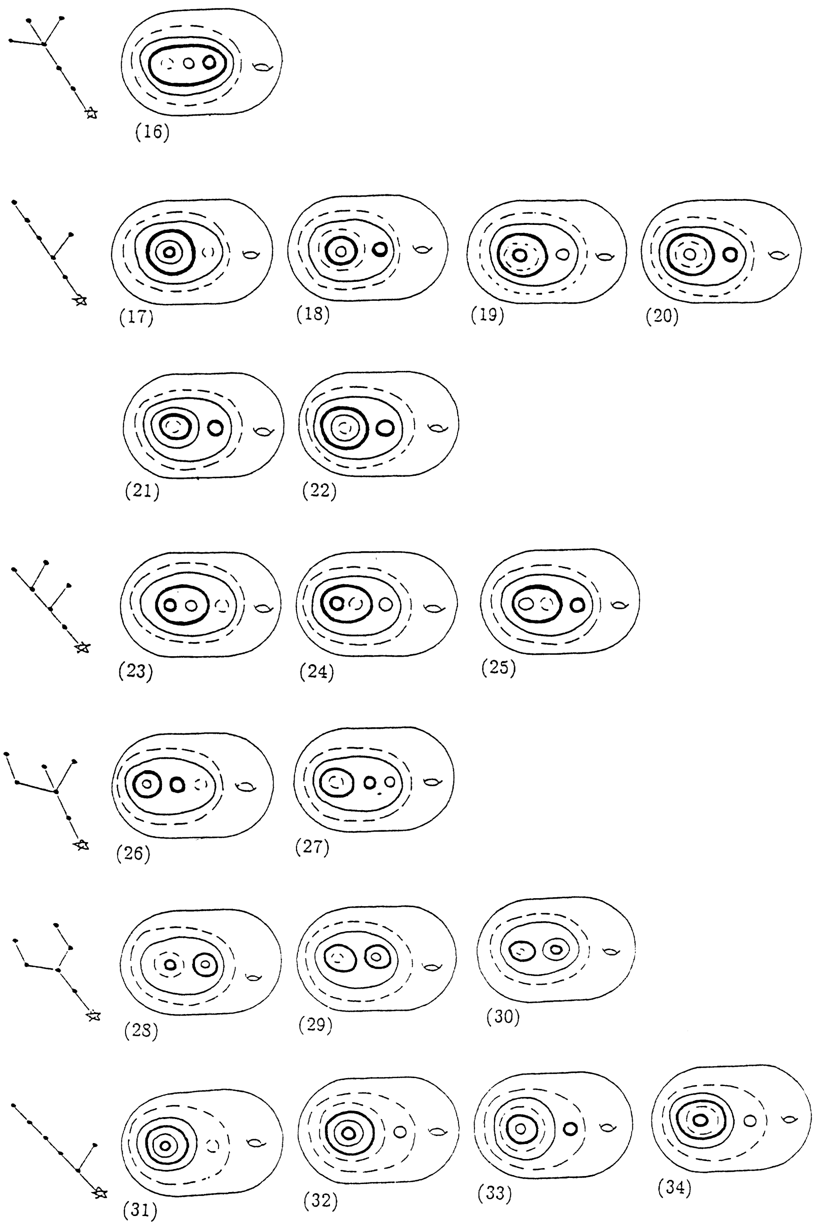
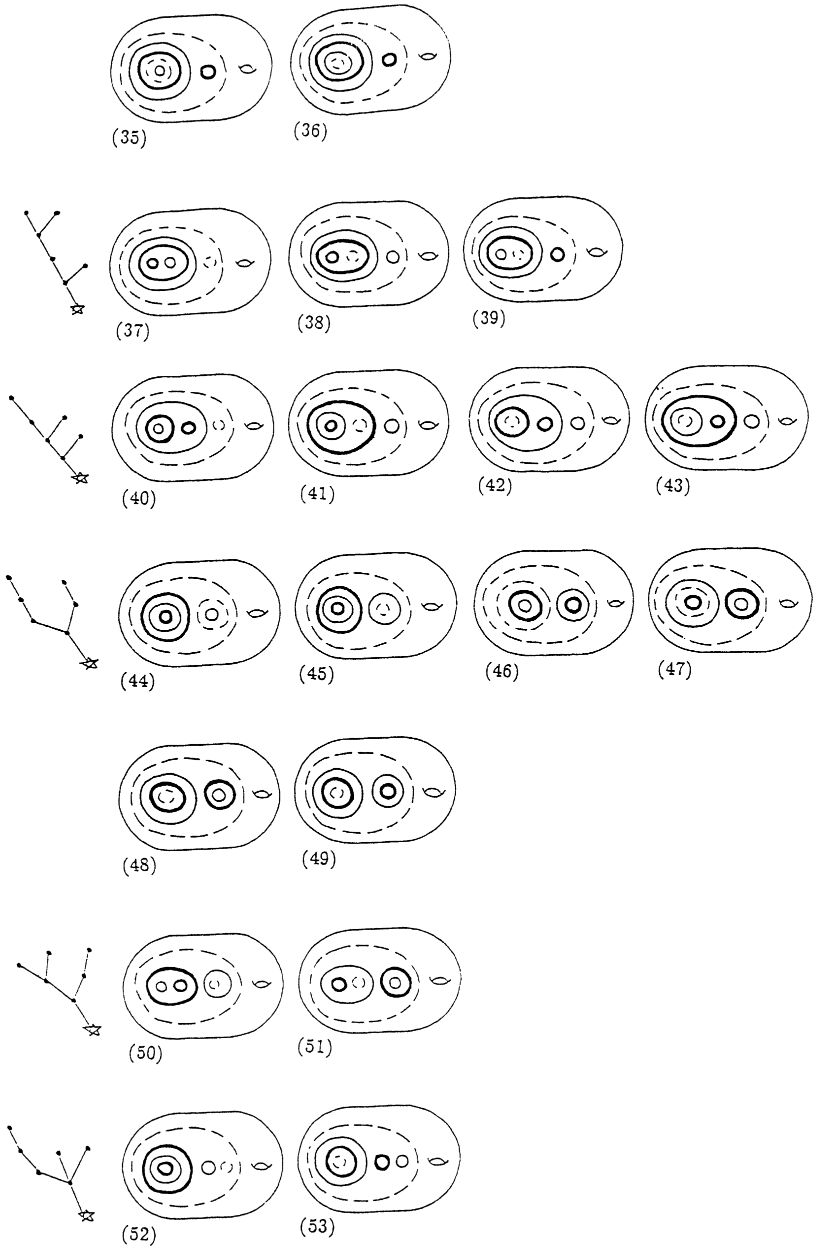


Figure 21 (continued)



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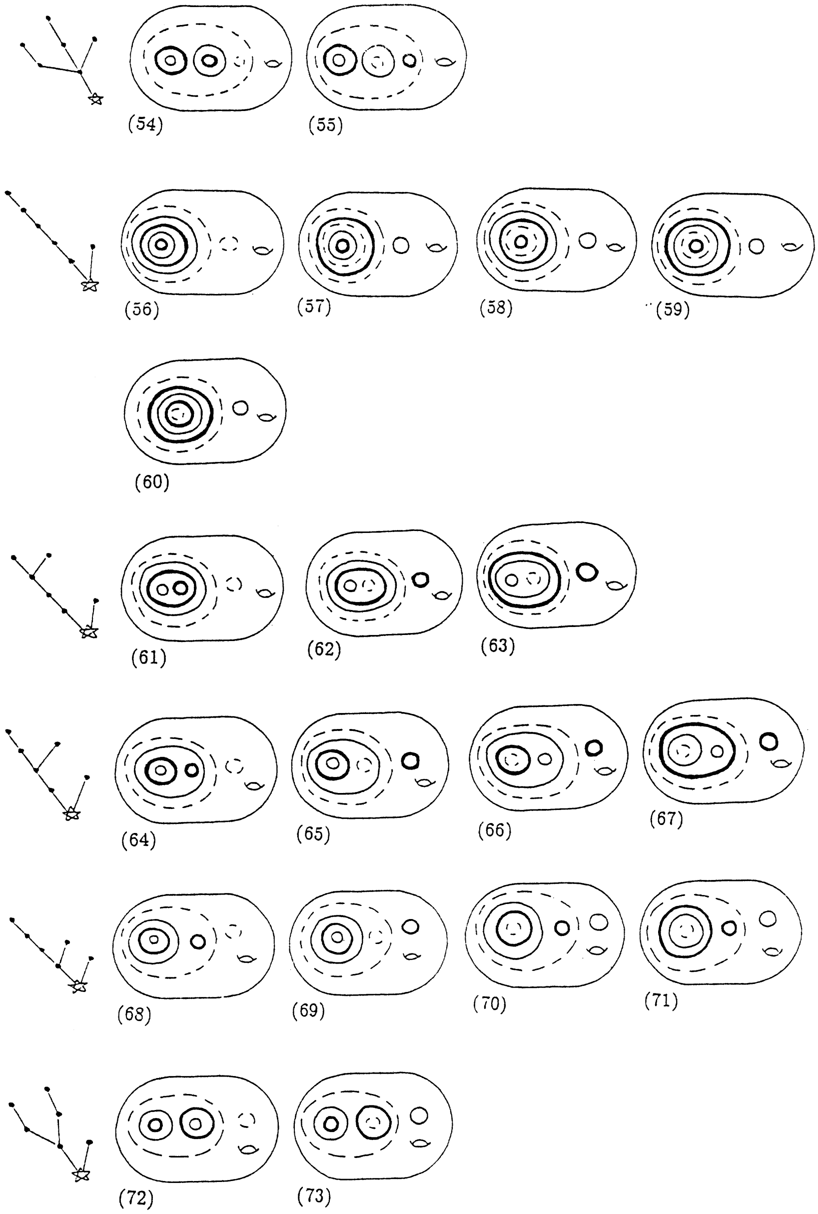


Figure 21 (continued)

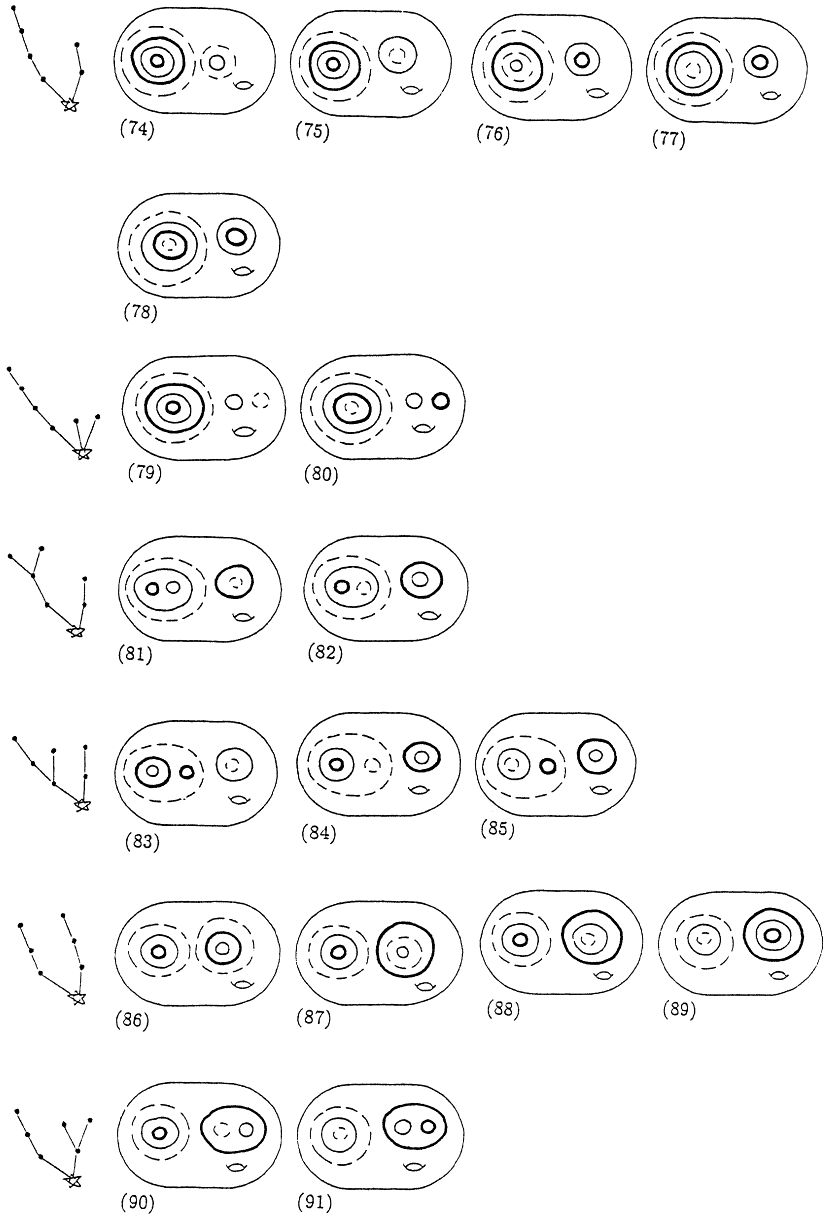
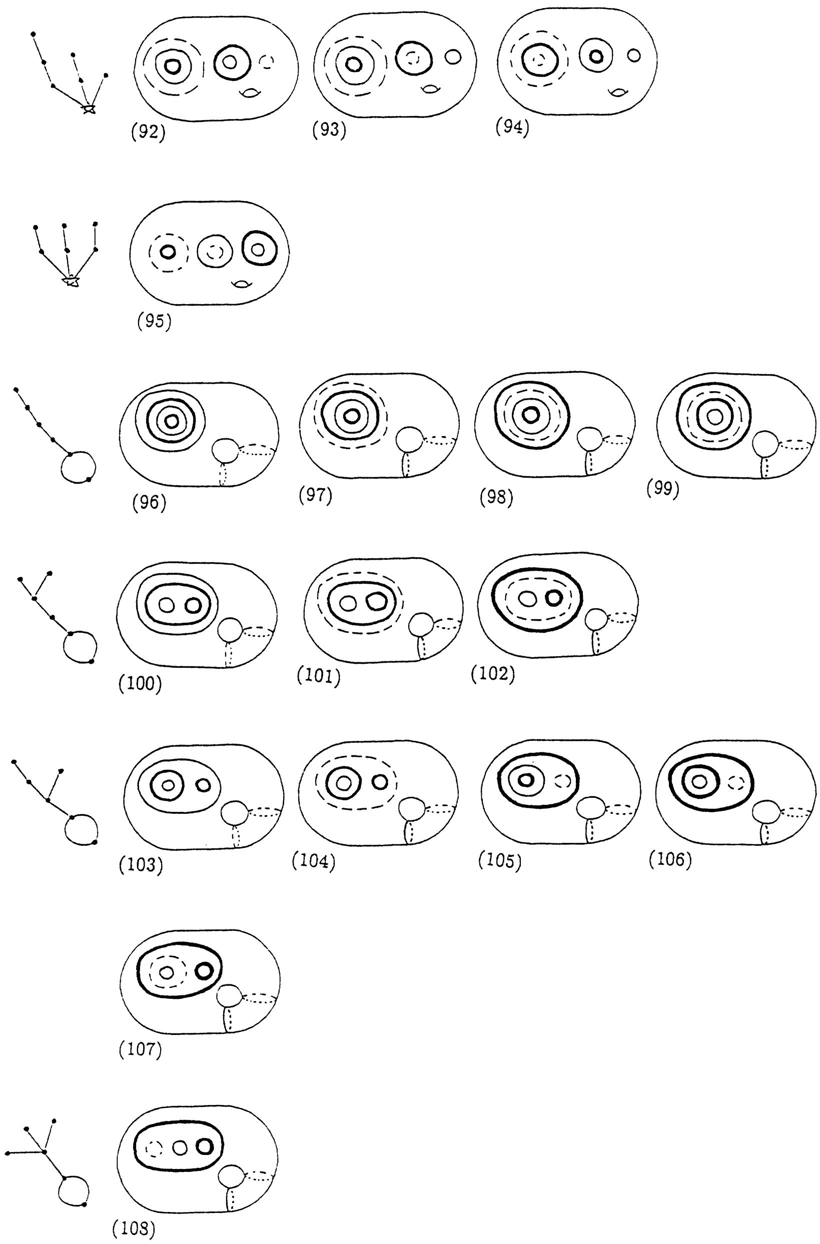


Figure 21 (continued)



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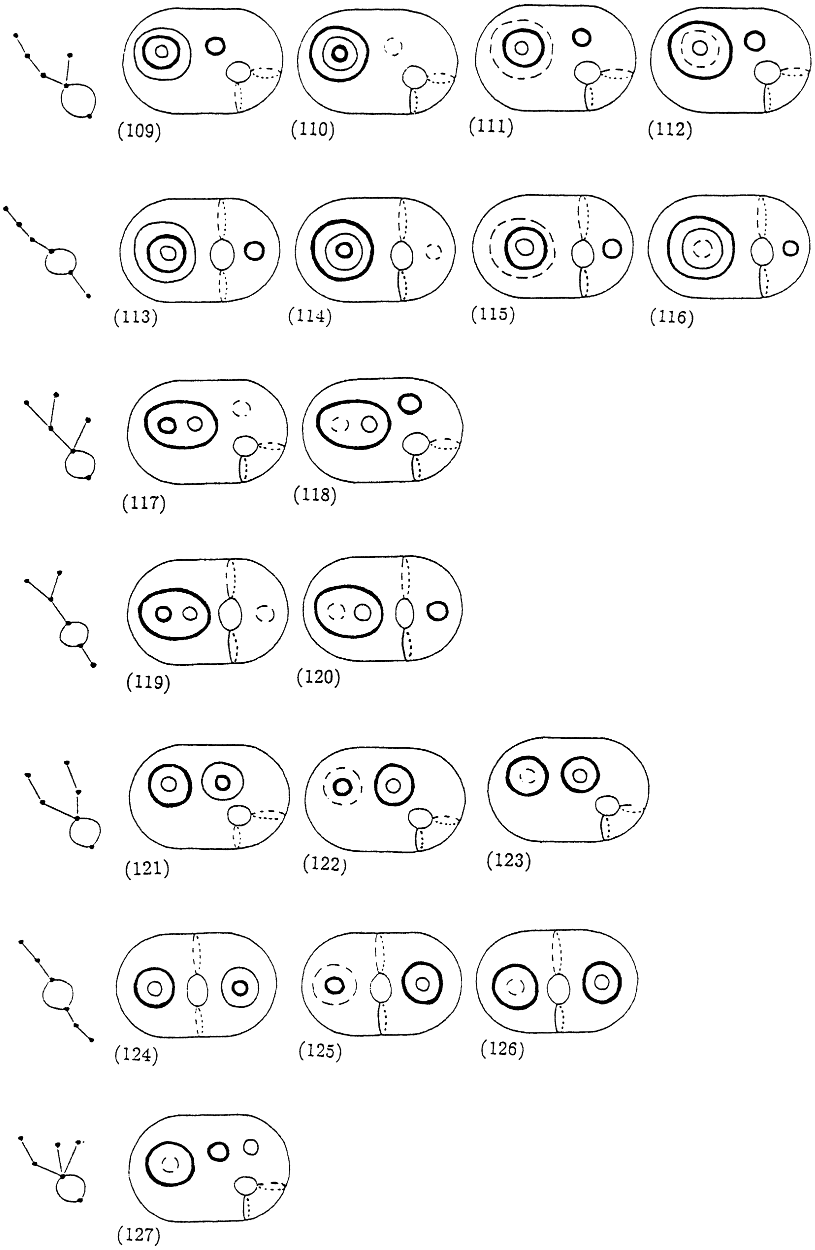

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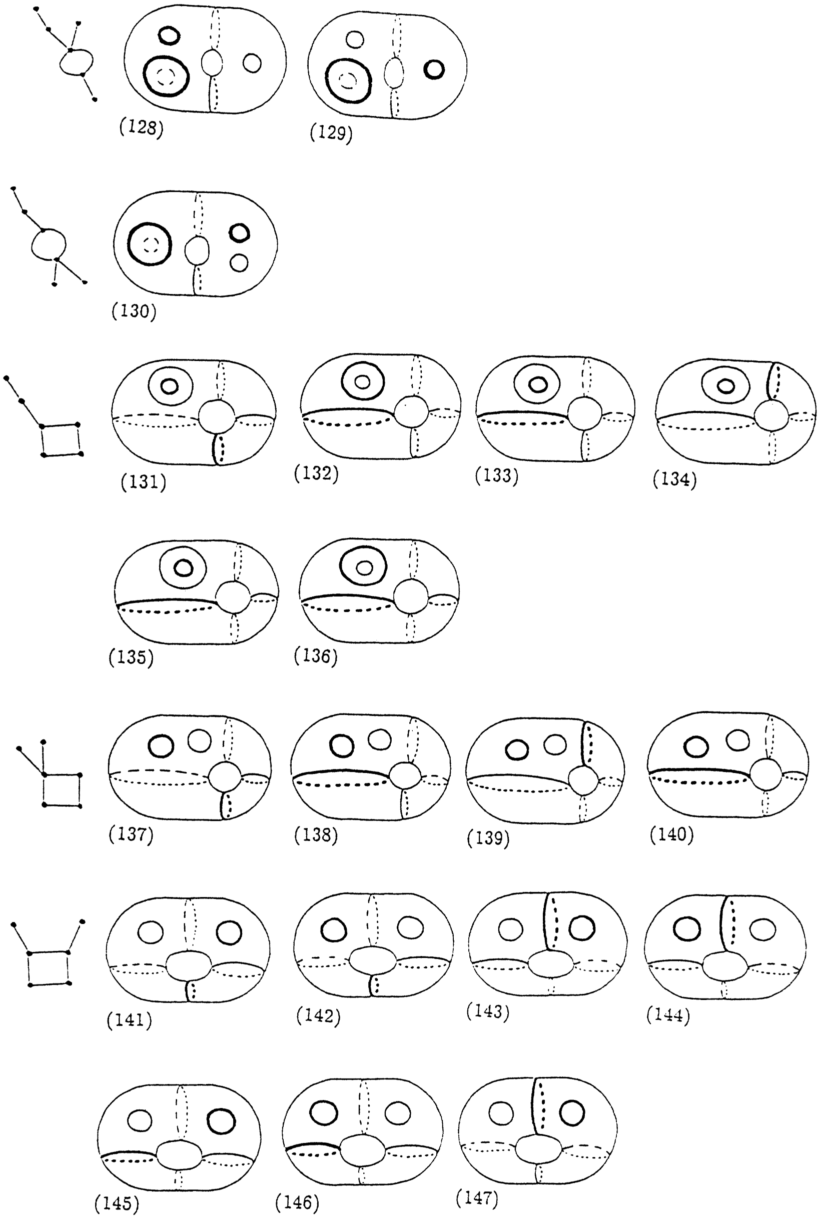


Figure 21 (continued)

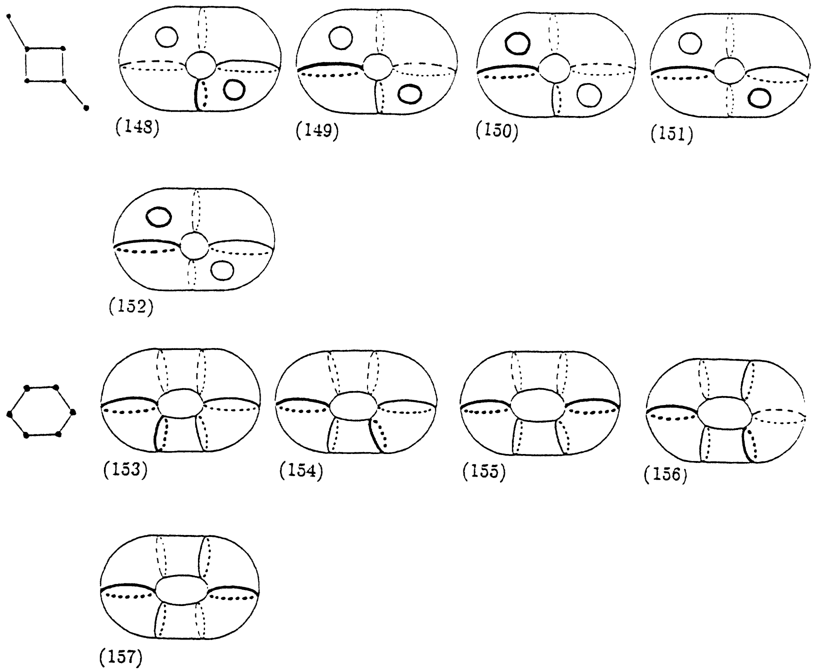


Figure 21 (continued)

If $C \subset S'(K)$, then $\partial N(C) \cap K$ consists of three components.

(5) $K \setminus R$ is connected for all components R of $K \setminus (S(K) \cup S'(K))$.

Suppose $C \subset S(K)$. If $N(C) \cap K$ is two immersed annuli, then we call C an *A-curve*. If $N(C) \cap K$ is two immersed Möbius bands, then we call C an *M-curve*.

Lemma 7.2. *Let K be a 2-complex consisting of annuli such that $S(K)$ consists only of A-curves.*

(1) The closure of each component of $K \setminus (S(K) \cup S'(K))$ is either an annulus or a torus.

(2) Let C, C' be components of $S(K) \cup S'(K)$, then C and C' are homologous in $H_1(K)$ for certain orientations.

Proof. We will prove (1). Let B be the closure of a component of $K \setminus (S(K) \cup S'(K))$. Suppose that B is a Klein bottle. There exists a component C of $S(K)$ with $C \subset B$. Then C is an M-curve. This is a contradiction.

We will prove (2). There exists an annulus B in K with $\partial B = C \cup C'$. Then $[C] = [C'] \in H_1(K)$. □

Let K be a 2-complex consisting of annuli such that $S(K)$ consists only of A-curves. Let $V_1, V_2, \ldots, V_k$ be solid tori in S^3, and $\mathfrak{V} = \{V_1, V_2, \ldots, V_k\}$. We say that $\mathfrak{V}$ is *a solid tori sequence for* K if $\mathfrak{V}$ satisfies the following two conditions:

(1) $\partial V_j \subset K$ for all j $(j = 1, 2, \ldots, k)$.

(2) If $j \neq s$, then $V_j \cap V_s = \partial V_j \cap \partial V_s$ is either one simple closed curve, an annulus or empty.

Let $\mathfrak{V} = \{V_1, V_2, \ldots, V_k\}$ be a solid tori sequence for K. Let c_j be a component of $S(K) \cup S'(K)$ with $c_j \subset \partial V_j$. Let n be the minimal number of intersection points of c_j and a meridional disk of the solid torus V_j. For a solid torus V_j we define $n(V_j)$ as follows.

$$n(V_j) = \begin{cases} n & \text{if } n \geq 1, \\ 0 & \text{if } n = 0,\ V_j \text{ is non-standard}, \\ \infty & \text{if } n = 0,\ V_j \text{ is standard}. \end{cases}$$

We will construct the graph $G(\mathfrak{V})$ obtained by $\mathfrak{V}$ as follows. The vertices are in one to one correspondence with the solid tori $\{V_j\}$, and the edges are in one to one correspondence with the set $\{V_j \cap V_s \neq \phi\}$. If $V_j \cap V_s \neq \phi$, then we connect the vertices $v(V_j)$ and $v(V_s)$ by the edge e_{js}.

Definition 7.3. Let $\mathfrak{V} = \{V_1, V_2, \ldots, V_k\}$ be a solid tori sequence for K and i an integer with $1 \leq i \leq k$. We say that $(\mathfrak{V}, i)$ is *good*, if $(\mathfrak{V}, i)$ satisfies the following four conditions:

(1) $G(\mathfrak{V})$ is a connected tree.

(2) There exists a vertex $v(V_1)$ of $G(\mathfrak{V})$ such that $n(V_j)$ equals one for all solid tori V_j with $V_j \neq V_1$.

(3) If $j \neq s$, then $V_j \cap V_s$ is either one simple closed curve or empty.

(4) If B is an annulus with $B \subset K$ and $(\cup\mathfrak{V}) \cap B = \partial B$, then $\partial B \subset \partial V_i$.

We say that $v(V_1)$ is the *special vertex*. And we say that $\mathfrak{V}$ is *good* if $\mathfrak{V}$ satisfies (1), (2), (3), and (4') there does not exist an annulus with $B \subset K$ and $(\cup\mathfrak{V}) \cap B = \partial B$.

If $V_j \cap V_s$ is one simple closed curve, let N_{js} be a regular neighborhood of $V_j \cap V_s$ in S^3. If $V_j \cap V_s = \phi$, let $N_{js} = \phi$. If $V_j \cap V_s$ is an annulus, let $N_{js} = V_j \cap V_s$.

LEMMA 7.4 ([S1, Lemma 2.2] and [S1, Theorem 4.1]). *Let $\mathfrak{V} = \{V_1, V_2\}$ be a solid tori sequence for a 2-complex consisting of annuli. Put $V = V_1 \cup V_2 \cup N_{12}$.*

(1) If V is a solid torus, then $n(V_1) = 1$ or $n(V_2) = 1$.

(2) If V is not a solid torus, then $n(V_1) > 1$, $n(V_2) > 1$, and V_1, V_2 are standard solid tori in S^3.

PROOF. We will prove (2). Suppose that $V_1 \cap V_2$ is an annulus. Let $V_3 = S^3 \setminus int(V_1 \cup V_2)$. Then V_3 is a solid torus by the solid torus theorem (see [R] p107). The set $V_1 \cap V_2 \cap V_3$ consists of two disjoint simple closed curves. Let c be a component of $V_1 \cap V_2 \cap V_3$. Let l_i be a preferred longitude of ∂V_i, and m_i a meridian of ∂V_i. For certain orientations, we denote by $c = p_i l_i + q_i m_i \in H_1(\partial V_i)$ (i=1,2 or 3) where (p_i, q_i) is a pair of relatively prime integers. By van Kampen's theorem, we have $\pi_1(V_i \cup V_j) \cong < l_i, l_j | \, l_i^{p_i} = l_j^{p_j} >$. We get

$$H_1(V_i \cup V_j) \cong \begin{cases} \mathbb{Z} & \text{if } (p_i, p_j) = 1 \\ \mathbb{Z} \oplus \mathbb{Z}_{|d|} & \text{if } (p_i, p_j) = d \neq 1 \\ \mathbb{Z} \oplus \mathbb{Z}_{|p_s|} & p_k = 0,\, p_s \neq 0, \{k, s\} = \{i, j\} \\ \mathbb{Z} \oplus \mathbb{Z} & p_i = p_j = 0 \end{cases}$$

Since $V_i \cup V_j$ is the complement of an open regular neighborhood of some knot, $H_1(V_i \cup V_j) \cong \mathbb{Z}$. Hence we have to consider the following cases:

(2-i) $p_i \neq 0, p_j \neq 0, (p_i, p_j) = 1$, or

(2-ii) $p_k = 0, p_s = \pm 1, \{k, s\} = \{i, j\}$.

We will prove (2-i). We construct a Seifert fibration on S^3 in which each solid torus V_i has c as a fiber. If $|p_i| \neq 1$ for all i, then there are three exceptional fibers. But we can show that in any Seifert fibration of the 3-sphere, there are at most two exceptional fibers (see [J-S] p 181). This is a contradiction. Hence there exists an integer k with $p_k = \pm 1$. We have $\pi_1(V_i \cup V_k) \cong < l_i, l_k | \, l_i^{p_i} = l_k^{\pm 1} > \cong \mathbb{Z}$. Therefore V_j is a standard solid torus where $j \neq i, k$. Similarly, V_i is a standard solid torus.

It is easy to prove (2-ii) (see [S1, Lemma 2.2]). If $V_1 \cap V_2$ is a simple closed curve, then we can prove in a similar way as above.

We will prove (1). We compute $H_1(V_1 \cup V_2)$ in a similar way to Case (2). If $n(V_1) \neq 1$ and $n(V_2) \neq 1$, then V is the complement of an open regular neighborhood of a non trivial torus-knot in S^3. Therefore V is not a solid torus. This is a contradiction. This completes the proof of Lemma 7.4. □

LEMMA 7.5. *Let $\mathfrak{V} = \{V_1, V_2, \ldots, V_k\}$ be a good solid tori sequence for K such that $v(V_1)$ is the special vertex. Let c_1 be a component of $S(K) \cup S'(K)$ with $c_1 \subset \partial V_1$, and $V = (\cup \mathfrak{V}) \cup (\cup N_{js})$. Then V is a solid torus, and $[c_1] = \pm n(V_1) \in H_1(V)$.*

PROOF. We will prove by induction on the number k of solid tori. By definition of a 2-complex consisting of annuli, $S(K) \cup S'(K) \neq \phi$. If $k = 1$, we have $[c_1] = \pm n(V_1)$ by definition of $n(V_1)$. If $k = 2$, then Lemma 7.5 follows from Lemma 7.4. Suppose that Lemma 7.5 is true for $k < s$. We will show it for $k = s$. Since $G(\mathfrak{V})$ is a tree, we may assume that V_s is a solid torus of $\mathfrak{V}$ such that $v(V_s)$ is an endpoint of $G(\mathfrak{V})$ and is not the special vertex. Then $c = V_s \cap (\cup_{j=1}^{s-1} V_j)$ is one simple closed curve on ∂V_s. Since $v(V_s)$ is not the special vertex, $n(V_s) = 1$ and $[c] = \pm 1 \in H_1(V_s) \cong \mathbb{Z}$. By the inductive assumption, $V' = (\cup_{j=1}^{s-1} V_j) \cup (\cup_{i \neq s, j \neq s} N_{ij})$ is a solid torus. Since $(\cup \mathfrak{V}) \cap K$ is a 2-complex consisting of annuli, c and c_1 are homologous in $\cup \mathfrak{V}$ by Lemma 7.2. Then $[c] = [c_1] = \pm n(V_1) \in H_1(V') \cong \mathbb{Z}$. Since $[c] = \pm 1 \in H_1(V_s) \cong \mathbb{Z}$, then $V' \cup V_s \cup N(c)$ is a solid torus and $[c_1] = \pm n(V_1) \in H_1(V' \cup V_s \cup N(c)) = H_1(V) \cong \mathbb{Z}$ where $N(c)$ is a regular neighborhood of c in S^3. This completes the proof of Lemma 7.5. □

REMARK 7.6. (1) Let $(\mathfrak{V}, i)$ be a good solid tori sequence for K with $\cup \mathfrak{V} \not\supset K$. Then there exists an embedded surface $B \subset K$ in S^3 such that B is either an annulus or a torus, and

$$B \cap (\cup \mathfrak{V}) = \begin{cases} \text{one simple closed curve} & \text{if } B \text{ is a torus,} \\ B \cap \partial V_i = \partial B & \text{if } B \text{ is an annulus.} \end{cases}$$

(2) Let $\mathfrak{V}$ be a good solid tori sequence for K with $\cup \mathfrak{V} \not\supset K$. Then there exists an embedded torus $B \subset K$ in S^3 such that $B \cap (\cup \mathfrak{V})$ is one simple closed curve.

LEMMA 7.7. *Let $(\mathfrak{V}, i)$ be a good solid tori sequence for K with $\cup\mathfrak{V} \not\supset K$. Then there exists a good solid tori sequence $(\mathfrak{W}, q)$ such that the number of the components of $K \setminus (S(K) \cup S'(K))$ in $\cup\mathfrak{V}$ is smaller than the number of the components of $K \setminus (S(K) \cup S'(K))$ in $\cup\mathfrak{W}$.*

PROOF. Let $v(V_1)$ be the special vertex of $G(\mathfrak{V})$ and $\mathfrak{V} = \{V_1, V_2, \dots, V_k\}$. By Remark 7.6 (1), there exists an embedded surface $B \subset K$ in S^3 such that B is either an annulus or a torus, and

$$B \cap (\cup\mathfrak{V}) = \begin{cases} \text{one simple closed curve} & \text{if } B \text{ is a torus,} \\ B \cap \partial V_i = \partial B & \text{if } B \text{ is an annulus.} \end{cases}$$

Case 1) There exists an annulus B with $B \cap (\cup\mathfrak{V}) = B \cap \partial V_i = \partial B$ and $B \subset K$.

Let B' be the closure of a component of $\partial V_i \setminus B$. Then $B \cup B'$ is a torus. By the solid torus theorem ([R] p107), there exists a solid torus V in S^3 with $\partial V = B \cup B'$. Then $V \cap V_i$ is an annulus, or $V_i \subset V$. If $V_i \subset V$, then $\partial V \cap \partial V_i$ is a boundary parallel annulus in V, $\overline{V \setminus V_i}$ is a solid torus, and $V_i \cap (\overline{V \setminus V_i})$ is an annulus. We may assume that $V \cap V_i$ is an annulus. Let c be a component of ∂B, $\mathfrak{V}'_1 = \{V_{i_1}, \dots, V_{i_{t-1}}\}$ the subset of $\mathfrak{V}$ with $V_{i_j} \not\subset V \cup V_i$ $(1 \leq j < t)$, $V_{i_t} = V_i$, and $\mathfrak{V}'_2 = \{V_{i_{t+1}}, \dots, V_{i_k}\}$ the subset of $\mathfrak{V}$ with $V_{i_j} \subset V$ $(t < j \leq k)$. Let N_{js} be a regular neighborhood of $V_j \cap V_s$ in S^3 if $V_j \cap V_s \neq \phi$. Put $\mathfrak{V}_j = \mathfrak{V}'_j \cup \{V_i\}$ for $j = 1, 2$. Then $G(\mathfrak{V}_1)$ and $G(\mathfrak{V}_2)$ are connected subgraphs of $G(\mathfrak{V})$. Let $N_1 = (\cup\mathfrak{V}_1) \cup (\cup_{1 \leq j,s \leq t} N_{i_j i_s})$ and $N_2 = (\cup\mathfrak{V}_2) \cup (\cup_{t \leq j,s \leq k} N_{i_j i_s})$. By Lemma 7.5, N_1 and N_2 are solid tori.

(1-i) $V \cup N_1$ is not a solid torus.

By Lemma 7.4, then $[c] \neq \pm 1 \in H_1(N_1)$, $[c] \neq \pm 1 \in H_1(V)$, and V is standard. Since the curve c is homologous to a curve on $(S(K) \cup S'(K)) \cap \partial V_1$ in N_1, then $G(\mathfrak{V}_1)$ contains the special vertex $v(V_1)$ of $G(\mathfrak{V})$ and $n(V_1) \neq 1$. Let $W_j = V_{i_j}$ $(t < j \leq k)$, $W_t = S^3 \setminus intV$ and $\mathfrak{W} = \{W_t, \dots, W_k\}$. Then $(\mathfrak{W}, t)$ is a good solid tori sequence for K and $\cup\mathfrak{V} \subset \cup\mathfrak{W}$.

(1-ii) $V \cup N_1$ is a solid torus.

Suppose $[c] = \pm 1 \in H_1(V) \cong \mathbb{Z}$. Let $W_j = V_{i_j}$ $(1 \leq j < t)$, $W_t = V_i \cup V$, and $\mathfrak{W} = \{W_1, \dots, W_t\}$. Then $(\mathfrak{W}, t)$ is a good solid tori sequence for K, and $\cup\mathfrak{V} \subset \cup\mathfrak{W}$.

Suppose $[c] \neq \pm 1 \in H_1(V) \cong \mathbb{Z}$. If $[c] \neq \pm 1 \in H_1(N_1)$, then $N_1 \cup V$ is not a solid tours by Lemma 7.4. This is a contradiction. We have $[c] = \pm 1 \in H_1(N_1)$ and $n(V_{i_j}) = 1$ $(1 \leq j < t)$. Let $W_j = V_{i_j}$ $(1 \leq j < t)$, $W_t = V_i \cup V$, and $\mathfrak{W} = \{W_1, W_2, \ldots, W_t\}$. Then $(\mathfrak{W}, t)$ is a good solid tori sequence, $\cup\mathfrak{V} \subset \cup\mathfrak{W}$, and $v(W_t)$ is the special vertex of $G(\mathfrak{W})$.

Case 2) There does not exist an annulus B with $B \cap (\cup\mathfrak{V}) = B \cap \partial V_i = \partial B$ and $B \subset K$.

Then B is a torus in K such that $B \cap (\cup\mathfrak{V})$ is one simple closed curve. By the solid torus theorem ([R] p107), there exists a solid torus V with $\partial V = B$. Let N_{js} be a regular neighborhood of $V_j \cap V_s$ in S^3, if $V_j \cap V_s \neq \phi$. Let $N = (\cup\mathfrak{V}) \cup (\cup N_{js})$. By Lemma 7.5, N is a solid torus. Let $c = B \cap (\cup\mathfrak{V})$, then c is one simple closed curve. If $V \supset \cup\mathfrak{V}$, then let $W_1 = V$, and $\mathfrak{W} = \{W_1\}$. Then $(\mathfrak{W}, 1)$ is a good solid tori sequence, and $\cup\mathfrak{V} \subset \cup\mathfrak{W}$. We may assume that $c = V \cap (\cup\mathfrak{V})$. Let $N(c)$ be a regular neighborhood of c in S^3. By Lemma 7.5, $[c] = \pm n(V_1) \in H_1(N)$.

(2-i) $V \cup N \cup N(c)$ is not a solid torus.

By Lemma 7.4 (2), $S^3 \setminus intV$ is a solid torus. Let $W_1 = S^3 \setminus intV$, and $\mathfrak{W} = \{W_1\}$. Then $(\mathfrak{W}, 1)$ is a good solid tori sequence, and $\cup\mathfrak{V} \subset \cup\mathfrak{W}$.

(2-ii) $V \cup N \cup N(c)$ is a solid torus.

Suppose $[c] = \pm 1 \in H_1(V) \cong \mathbb{Z}$. Let $V_{k+1} = V$, and $\mathfrak{W} = \mathfrak{V} \cup \{V_{k+1}\}$. Then $(\mathfrak{W}, k+1)$ is a good solid tori sequence, and $\cup\mathfrak{V} \subset \cup\mathfrak{W}$.

Suppose $[c] \neq \pm 1 \in H_1(V) \cong \mathbb{Z}$. We have that $\{V, N\}$ is a good solid tori sequence for $\partial V \cup \partial N$, $\partial V \cap \partial N = c$, $n(V) \neq 1$, and $n(N) = n(V_1)$. By Lemma 7.4 (1), $n(N) = 1$. Hence $n(V_1) = 1$. Let $W_1 = V$, $W_{k+1} = V_1$, $W_j = V_j$ $(j = 2, \ldots, k)$, and $\mathfrak{W} = \{W_1, W_2, \ldots, W_{k+1}\}$. Then $(\mathfrak{W}, 1)$ is a good solid tori sequence, $\cup\mathfrak{V} \subset \cup\mathfrak{W}$, and $v(W_1)$ is the special vertex of $G(\mathfrak{W})$. $\square$

PROPOSITION 7.8. *Let K be a 2-complex consisting of annuli, then there exists a good solid tori sequence $\mathfrak{V}$ for K with $\cup\mathfrak{V} \supset K$.*

PROOF. Let B be a torus in K. By the solid tours theorem, there exists a solid torus V_1 with $\partial V_1 = B$. Put $\mathfrak{V}_1 = \{V_1\}$. Then $(\mathfrak{V}_1, 1)$ is a good solid tori sequence for K. By Lemma 7.7, there exist good solid tori sequences $(\mathfrak{V}_2, q_2), \ldots, (\mathfrak{V}_i, q_i), \ldots$ with $\cup\mathfrak{V}_1 \subset \cup\mathfrak{V}_2 \subset \cdots \subset \cup\mathfrak{V}_i \subset \cdots$. Since K is a finite 2-complex, there exists an integer n with $K \subset \cup\mathfrak{V}_n$. $\square$

Lemma 7.9. *Let $\mathfrak{V}$ be a good solid tori sequence for K with $\cup\mathfrak{V} \supset K$, and ($n(V_1) = 0$ or $n(V_1) = \infty$). If B is a non-standard torus in S^3 with $B \subset K$, then $B \subset V_1$.*

Proof. By definition of good, there exists a solid torus $V_i \in \mathfrak{V}$ with $B \subset V_i$. Suppose $i \neq 1$. Let V be a 3-manifold in V_i with $\partial V = B$. By the solid torus theorem, V is either a solid torus or the complement of an open regular neighborhood of a non-trivial knot. Since $n(V_i) = 1$, all component of $(S(K) \cup S'(K)) \cap B$ are not zero in $H_1(V_i)$. Hence V is a solid torus. Since $n(V_1) = 0$ or $n(V_1) = \infty$, there exists a meridional disk D' in V_1 with $\partial D' \subset S(K) \cup S'(K)$. Since K is connected, there exists an annulus A' in V_i such that $A' \cap V$ is a simple closed curve of $\partial A'$ on ∂V, and $A' \cap \partial V_i$ is a simple closed curve of $\partial A'$ on ∂V_i. Since K is connected, there exists an annulus A in $\cup\mathfrak{V}$ with $\partial A = \partial D' \cup (A' \cap \partial V_i)$. Put $D = A' \cup D' \cup A$. Then D is a disk in S^3 with $V \cap D = \partial D$. We have that V is a standard solid torus, and B is a standard tours. This is a contradiction. □

Definition 7.10. Let K be a 2-complex consisting of annuli. We say that K is *super* if there exists a good solid tori sequence $\mathfrak{V}$ for K such that $\cup\mathfrak{V} \supset K$, $n(V_1) = \infty$, and all embedded tori in K are standard tori.

Proposition 7.11. *Let $\mathfrak{V}$ be a good solid tori sequence for K with $\cup\mathfrak{V} \supset K$ and $n(V_1) = \infty$. If there exists a non-standard torus B in S^3 with $B \subset K$, then there exists a good solid tori sequence $\mathfrak{W}$ for K with $\cup\mathfrak{W} \supset K$ and $n(W_1) = 0$.*

Proof. Let $\mathfrak{B} = \{B \mid B$ is a non-standard torus in S^3 with $B \subset K\}$. By assumption, $\mathfrak{B} \neq \phi$. Let $B_1, B_2 \in \mathfrak{B}$. By the solid torus theorem, there exist solid tori X_1, X_2 with $\partial X_i = B_i$. If $X_1 \subset X_2$, then we define $B_1 \leq B_2$. Let B be a maximal torus in $\mathfrak{B}$, and W_1 a solid torus in S^3 whose boundary is B. Put $\mathfrak{W}_1 = \{W_1\}$. Then $(\mathfrak{W}_1, 1)$ is a good solid tori sequence for K, and $n(W_1) = 0$. Suppose $K \not\subset \mathfrak{W}_1$. By Remark 7.6 (1), there exists an embedded surface $B_2 \subset K$ in S^3 such that B_2 is either an annulus or a torus, and

$$B_2 \cap (\cup\mathfrak{W}_1) = \begin{cases} \text{one simple closed curve} & \text{if } B_2 \text{ is a torus,} \\ B_2 \cap \partial W_1 = \partial B_2 & \text{if } B_2 \text{ is an annulus.} \end{cases}$$

By the solid torus theorem, there exists a solid torus V_2 in S^3 such that $B_2 \subset \partial V_2 \subset K$, and $(\cup\mathfrak{W}_1) \cap V_2$ is either one simple closed curve or an annulus. Since B is maximal in $\mathfrak{B}$, then ∂V_2 is standard and $(\cup\mathfrak{W}_1) \cap \partial V_2$ is one simple closed curve. We have that $(\cup\mathfrak{W}_1) \cap V_2$ is one simple closed curve, and $n(V_2) = 1$. Put $W_2 = V_2$, and $\mathfrak{W}_2 = \mathfrak{W}_1 \cup \{W_2\}$. Then $(\mathfrak{W}_2, 2)$ is a good solid tori sequence for K, $\cup\mathfrak{W}_1 \subset \cup\mathfrak{W}_2$, and $n(W_1) = 0$. Suppose $K \not\subset \mathfrak{W}_2$. In a similar way as above, there exists a solid torus V_3 in S^3 such that $B_2 \subset \partial V_3 \subset K$, $n(V_3) = 1$ and $(\cup\mathfrak{W}_2) \cap V_3$ is either one simple closed curve or an annulus. If $(\cup\mathfrak{W}_2) \cap V_3$ is one simple closed curve, put $W_3 = V_3$, $\mathfrak{W}_3 = \mathfrak{W}_2 \cup \{W_3\}$, and $q_3 = 3$. If $(\cup\mathfrak{W}_2) \cap V_3$ is an annulus, put $W_3 = W_2 \cap V_3$, $\mathfrak{W}_3 = \{W_1, W_3\}$, and $q_3 = 3$. Then $(\mathfrak{W}_3, q_3)$ is a good solid tori sequence for K, $\cup\mathfrak{W}_2 \subset \cup\mathfrak{W}_3$, and $n(W_1) = 0$. In a similar way as above, there exist good solid tori sequences $(\mathfrak{W}_4, q_4), \dots, (\mathfrak{W}_i, q_i), \dots$ with $W_1 \in \mathfrak{W}_i$ for all i, $n(W_1) = 0$, and $\cup\mathfrak{W}_1 \subset \cup\mathfrak{W}_2 \subset \cdots \subset \cup\mathfrak{W}_i \subset \cdots$. Since K is a finite 2-complex, there exists an integer n with $K \subset \cup\mathfrak{W}_n$. We complete the proof of Proposition 7.11. □

LEMMA 7.12. *Let K be a 2-complex consisting of annuli. If K is super, then there exists a good solid tori sequence $\mathfrak{W}$ for K with $\cup\mathfrak{W} \supset K$, and $n(W_i) = 1$ for all associated vertices of $G(\mathfrak{W})$.*

PROOF. Since K is super, there exists a good solid tori sequence $\mathfrak{V}$ for K with $\cup\mathfrak{V} \supset K$, $n(V_1) = \infty$. Put $W_1 = S^3 \setminus V_1$ and $\mathfrak{W}_1 = \{W_1\}$. Then $(\mathfrak{W}_1, 1)$ is a good solid tori sequence for K and $n(W_1) = 1$. In a similar way to Proposition 7.11 there exists a good solid tori sequence $\mathfrak{W}$ for K with $\cup\mathfrak{W} \supset K$ and $n(W_i) = 1$ for all $W_i \in \mathfrak{W}$. We complete the proof of Lemma 7.12. □

LEMMA 7.13 ([S1, Lemma 2.1], [S1, Theorem 4.1], and [S2, Lemma 2.1]). *Let $\{V\}$ be a solid tori sequence for K such that $K \subset V$, and the number of the closures of the components of $K \setminus (S(K) \cup S'(K))$ is two. Let W be the closure of a component of $V \setminus K$ with $W \not\supset \partial V$, and $C = S(K) \cup S'(K)$.*

(1) If $[C] \neq 0$ in $H_1(V)$, then W is a solid torus, and K can be moved to either $T_1(a,b)$, $T_4(a,b)$ or $T_5(a,b)$ by an ambient isotopy of S^3 where (a,b)=1, $b = n(V)$, and T_1, T_4, T_5 are immersions obtained from Figure 4.

(2) If $[C] = 0$ in $H_1(V)$, then W is the complement of an open regular neighborhood of some knot in S^3.

(3) If $[C]=0$ *in* $H_1(V)$ *and* $[C]=0$ *in* $H_1(W)$*, then* W *is a solid torus.*

Proof. We will prove (1). Suppose $C \subset S'(K)$. Then the closures of the components of $K \setminus C$ are an annulus and a torus. Let B be a properly embedded annulus into V with $[b_i] \neq 0$ in $H_1(V)$ where b_0, b_1 are the components of ∂B, then B is a boundary parallel annulus in V (see [S1, Lemma 2.1]). There exists a meridional disk D of V such that $D \cap K$ is as shown in Figure 22 (1). Therefore K is ambient isotopic to $T_5(a,b)$ where $(a,b)=1, b=n(V)$. And W is a solid tours.

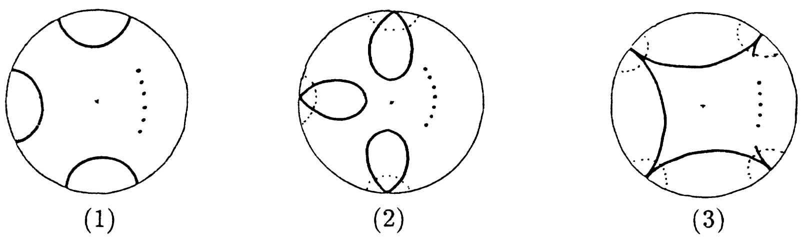


Figure 22

Suppose $C \subset S(K)$. Then the closures of the components of $K \setminus C$ are two tori. By assumption, $[C] \neq 0$ in $H_1(V)$. Let N be a regular neighborhood of C in V, $B = \overline{\partial N \cap intV}$ an annulus, and b_0, b_1 the components of ∂B. Since $[C] \neq 0$ in $H_1(V)$, $[b_i] \neq 0$ in $H_1(V)$. Then B is a boundary parallel annulus in V. Let $X = \overline{V \setminus N}$, and $B' = \overline{K \setminus (\partial V \cup N)}$. Then X is a solid torus, and B' is an annulus. Let b_0', b_1' be the components of $\partial B'$, then $[b_i'] \neq 0$ in $H_1(X)$. Then B' is a boundary parallel annulus in X. There exists a meridional disk E of V such that $E \cap V$ is as shown in Figure 22 (2) or (3). We can show that K is ambient isotopic to $T_1(a,b)$ or $T_4(a,b)$ where $(a,b)=1, b=n(V)$ (see [S1, Theorem 4.1]). We have that W is a solid torus. This completes the proof of (1).

We will prove (2) and (3). Suppose $[C]=0 \in H_1(V)$. Let D be a disk in V with $D \cap \partial W = \partial D$ (see [S2, Lemma 2.1]). Using the disk D, we can prove that W is a solid torus, or W is the complement of an open regular neighborhood of some knot in S^3. $\square$

LEMMA 7.14. *Let $\{V\}$ be a good solid tori sequence for K with $K \subset V$. If $n(V) \neq 0$ and $n(V) \neq \infty$, then there exists an immersion $\alpha : J \longrightarrow p(B^3 \setminus P)$ such that K can be moved to $\alpha(a, b)$ by an ambient isotopy of S^3 where $(a, b) = 1$, $b = n(V)$, and J is a disjoint union of 1-spheres and intervals.*

PROOF. We will prove by induction on the number m of the components of $S(K) \cup S'(K)$. Suppose $m = 1$. By Lemma 7.13, the intersection of a meridional disk of V and K is as shown in Figure 23 (1). And α is defined as in Figure 23 (2). Let c be a component of $(S(K) \cup S'(K)) \cap \partial V$, m a meridian on ∂V, and l a preferred longitude on ∂V. For certain orientations, we denote by $c = am + n(V)l$ with $(a, n(V)) = 1$. Then K is ambient isotopic to $\alpha(a, n(V))$ in S^3.

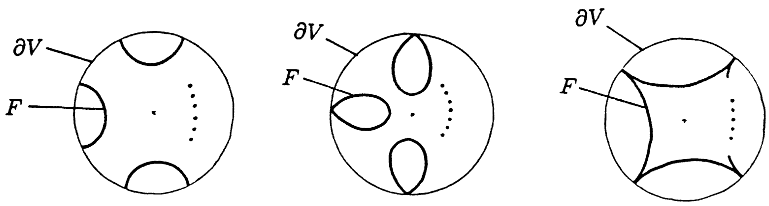


Figure 23 (1)

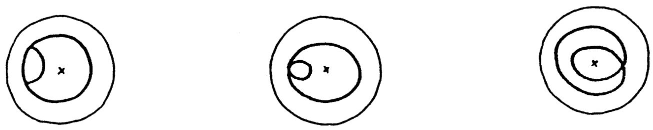


Figure 23 (2)

Suppose that Lemma 7.14 is true for $m < k$. We will show it for $m = k$. We take an embedded surface $F \subset K$ such that F is either an annulus or a torus, and

$$F \cap \partial V = \begin{cases} \text{one simple closed curve} & \text{if } F \text{ is a torus,} \\ \partial F & \text{if } F \text{ is an annulus.} \end{cases}$$

Let N be a regular neighborhood of $F \cap \partial V$ in V, and V_1, V_2 the closures of the components of $V \setminus F$. Then $\overline{V_i \setminus N}$ is a solid torus ($i = 1, 2$). Let $B = \overline{V_1 \setminus N} \cap \overline{V_2 \setminus N}$. Then B is an annulus. Let c be a component of ∂B, m_i a meridian on $\partial(V_i \setminus N)$, and l_i a preferred longitude on $\partial(V_i \setminus N)$ for $i = 1, 2$. For certain orientations, we denote $c = a_1 m_1 + n(V) l_1$ and $c = a_2 m_2 + l_2$ with $(a_1, n(V)) = 1$. Put $b_1 = n(V)$, $b_2 = 1$. By the inductive assumption, there exist immersions $\alpha_j : J_j \longrightarrow p(B^3 \setminus P)$ such that $\overline{V_j \setminus N} \cap K$ is moved to $\alpha_j(a_j, b_j)$ by an ambient isotopy of S^3 where $(a_j, b_j) = 1$, $b_j \neq 0$ ($j = 1, 2$), and J_j is a disjoint union of 1-spheres and intervals. Then the intersection of F and a meridional disk of V is as shown in Figure 24 (1). And the intersection of K and a meridional disk of V is as shown in Figure 24 (2). Therefore there exists an immersion $\alpha : J \longrightarrow p(B^3 \setminus P)$ such that K is moved to $\alpha(a_1, b_1)$ by an ambient isotopy of S^3 where $(a_1, b_1) = 1, b_1 = n(V)$, and J is a disjoint union of 1-spheres and intervals (see Figure 24 (3)). Thus Lemma 7.14 is proved by induction on m. □

Proposition 7.15. *Let $\mathfrak{V}$ be a good solid tori sequence for K with $\cup\mathfrak{V} \supset K$ where $v(V_1)$ is the special vertex. If $n(V_1) \neq 0$ and $n(V_1) \neq \infty$, then there exists an immersion $\alpha : J \longrightarrow p(B^3 \setminus P)$ such that $\alpha(a, b)$ can be moved to K by an ambient isotopy of S^3 where $(a, b) = 1$, $b = n(V_1)$, and J is a disjoint union of 1-spheres and intervals.*

Proof. Let $\mathfrak{V} = \{V_1, \ldots, V_k\}$. We will prove Proposition 7.15 by induction on the number k of solid tori V_i. If $k = 1$, this follows from by Lemma 7.14. Suppose that Proposition 7.15 is true for $m < k$. We will show it for $m = k$. We may assume that $v(V_m)$ is an endpoint of $G(\mathfrak{V})$ and is not the special vertex. Then $c = V_m \cap (\cup_{j=1}^{m-1} V_j)$ is one simple closed curve. Let $N = (\cup\mathfrak{V}) \cup (\cup N_{ij})$, and $N' = (\cup_{j=1}^{m-1} V_j) \cup (\cup_{i \neq m, j \neq m} N_{ij})$ where N_{ij} is a regular neighborhood of $V_i \cap V_j$ in S^3. By Lemma 7.5, then N and N' are solid tori. By the inductive assumption, there exist immersions $\alpha_j : J_j \longrightarrow p(B^3 \setminus P)$ such that $N' \cap K$ is ambient isotopic to $\alpha_1(a_1, b_1)$ in S^3 and $V_m \cap K$ is ambient isotopic to $\alpha_2(a_2, b_2)$ in S^3 where $(a_j, b_j) = 1$ (j=1,2), $b_1 = n(V_1)$, $b_2 = n(V_m) = 1$, and J_j is a disjoint union of 1-spheres and intervals. The intersection of K and a meridional disk of V_m is as shown in Figure 25 (1). And the intersection K and a meridional disk of N is as shown in Figure 25 (1). Therefore there exists an immersion

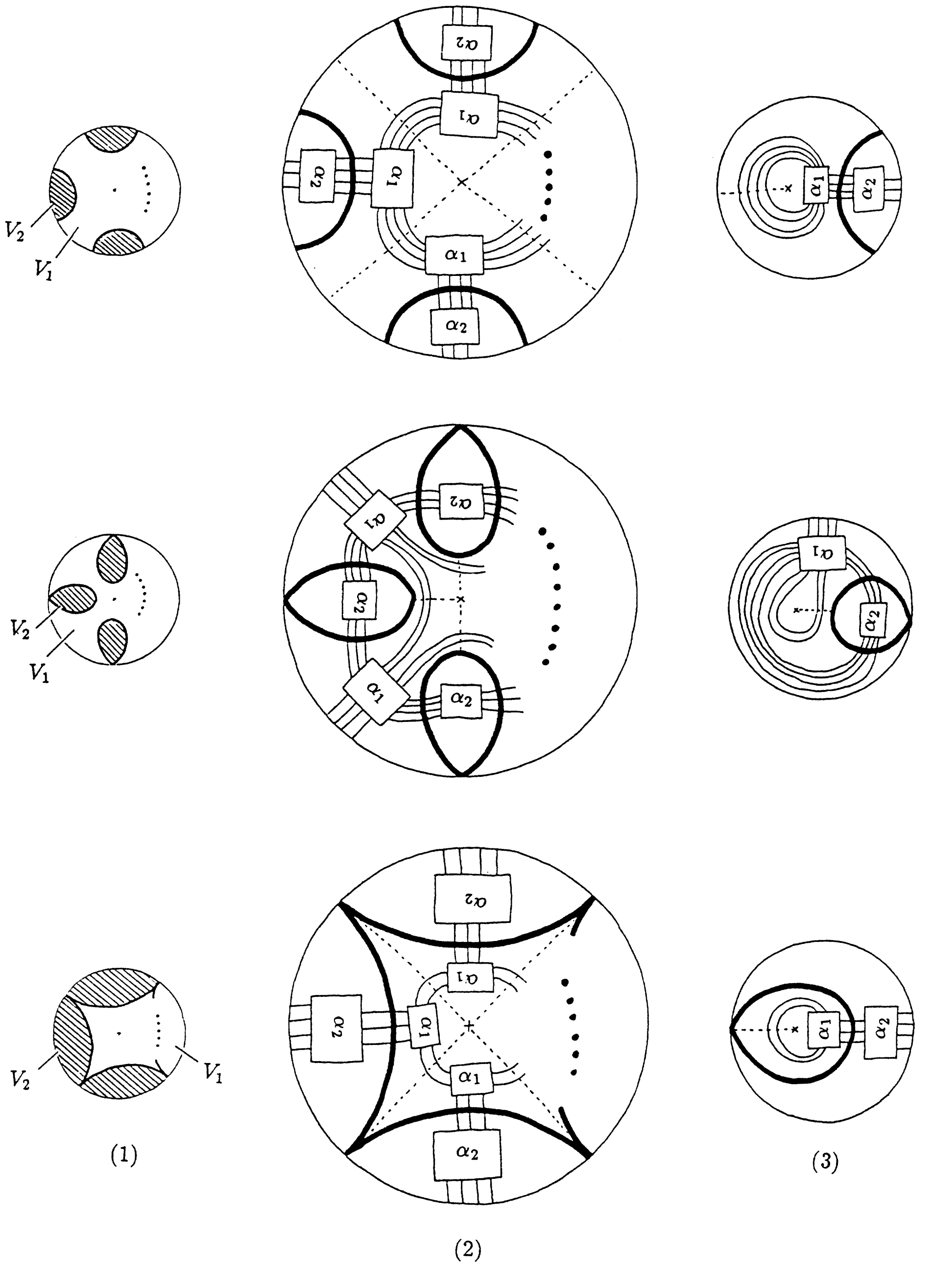
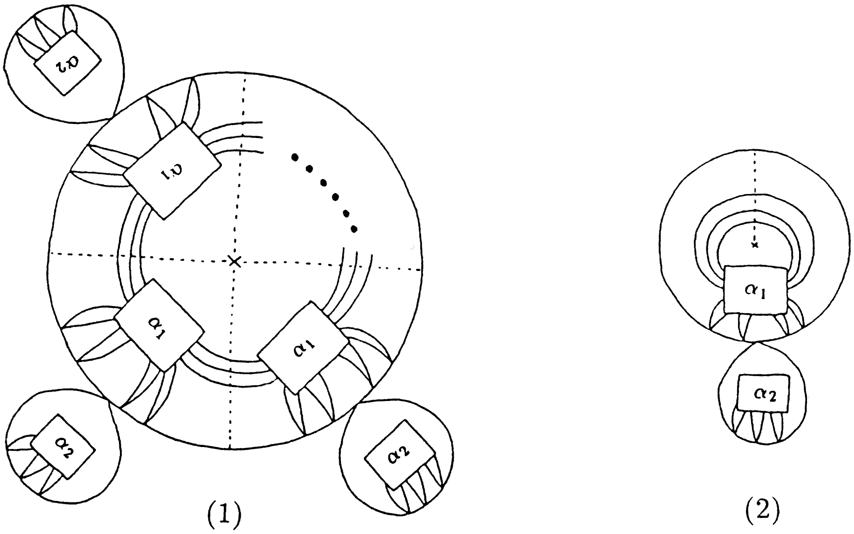


Figure 24



(1) (2)

Figure 25

$\alpha : J \longrightarrow p(B^3 \setminus P)$ such that $N \cap K$ is ambient isotopic to $\alpha(a_1, b_1)$ in S^3 where $b = n(V_1)$ and J is a disjoint union of 1-spheres and intervals (see Figure 25 (2)). Thus Proposition 7.15 is proved by induction on k. $\square$

8. Symmetry-spun tori

Let T be a torus in S^4 with $p|T$ in general position. In this section we assume that $\Gamma(T^*)$ consists only of double points. We will show that if all components of $\Gamma(T)$ are not contractible in T, then there exists a symmetry-spun torus $T^a(K_b)$ in S^4 which is ambient isotopic to T (see Proposition 8.3).

LEMMA 8.1. *Let T be a torus in S^4 such that $\Gamma(T^*)$ consists only of double points, and each component of $\Gamma(T)$ is not contractible in T. Let $\mathfrak{V}$ be a good solid tori sequence for T^* with $T^* \subset \cup\mathfrak{V}$. If $n(V_1) = 0$ or $n(V_1) = \infty$, then there exist a torus T' in S^4 and a good solid tori sequence $\mathfrak{W}$ for the 2-complex T'^* such that T is ambient isotopic to T' in S^4, T'^* is super, and the number of components of $\Gamma(T'^*)$ equals the number of components of $\Gamma(T^*)$.*

PROOF. Let m be the number of the elements of $\{B|\ B$ is a non-standard torus in S^3 with $B \subset T^*\}$. We can move T by some ambient isotopy of S^4 so that for the resulting T we have $m = 0$. We may assume that $v(V_1) = 0$ by Proposition 7.11. We will show that m can be reduced. Let $N(\cup\mathfrak{V})$ be a regular neighborhood of $\cup\mathfrak{V}$ in S^3. Then $N(\cup\mathfrak{V})$ is a solid torus by Lemma 7.5. Since $n(V_1) = 0$, $N(\cup\mathfrak{V})$ is a non-standard solid torus in S^3. Move T by an ambient isotopy of S^4 so that $N(\cup\mathfrak{V})$ is standard. Let T' be a torus obtained from T as above, and V_i' a solid torus obtained from V_i as above. Suppose that there exists an embedded torus $B \subset T^*$ in S^3 such that B' is a non-standard torus which is obtained from B as above. By Lemma 7.9, $B \subset V_1$ and $B' \subset V_1'$. Let Y' be the closure of the component of $S^3 \setminus B'$ such that Y' is a solid torus. If $Y' \not\subset V_1'$, then B is a non-standard torus (see Figure 26 (1)). If $Y' \subset V_1'$ (see Figure 26 (2)), there exists an annulus A in $T^{'*}$ with $A \cap (Y' \cup \partial V_1') = \partial A$, $a_1 \subset \partial Y'$, and $a_2 \subset \partial V_1'$ where a_1, a_2 are the components of ∂A. Let $N(A)$ be a regular neighborhood of A in $\overline{V_1' \setminus Y'}$. We have that $\overline{V_1' \setminus N(A)}$ is a solid tours, and a_1 is homologous zero in $H_1(Y')$. Let D'' be a meridional disk in Y' with $\partial D'' = a_1$, and $D' = A \cup D''$. Then D' is a meridional disk in V_1' such that $D' \cap Y'$ is a meridional disk of Y'. Let D be a meridional disk of V_1 such that D' is obtained from D as above. Using D, we have that B is a non-standard torus. If B is a standard torus in T^*, then B' is a standard torus obtained from B as above. Hence m is reduced. Therefore we may assume $m = 0$. This completes the proof of Lemma 8.1. $\square$

LEMMA 8.2. *Let T be a torus as above. Then there exist a torus T' in S^4 and a good solid tori sequence $\mathfrak{V}$ for $T^{'*}$ such that T is ambient isotopic to T' in S^4, $T^{'*} \subset \cup\mathfrak{V}$, $n(V_1) \neq 0$, $n(V_1) \neq \infty$, and the number of components of $\Gamma(T^{'*})$ equals the number of components of $\Gamma(T^*)$ where $v(V_1)$ is the special vertex of $G(\mathfrak{V})$.*

PROOF. By Proposition 7.8, there exists a good solid tori sequence $\mathfrak{V}$ for T^* with $T^* \subset \cup\mathfrak{V}$. If $n(V_1) = 0$ or $n(V_1) = \infty$, then by Lemma 8.1 there exists a torus T' in S^4 such that $T^{'*}$ is super, and T' is ambient isotopic to T. By Lemma 7.12, there exists a good solid tori sequence $\mathfrak{W}$ for $T^{'*}$ such that T' and $\mathfrak{W}$ satisfy the above condition by Lemma 8.1. This completes the proof of Lemma 8.2. $\square$

PROPOSITION 8.3. *Let T be as above. Then there exists a symmetry-*

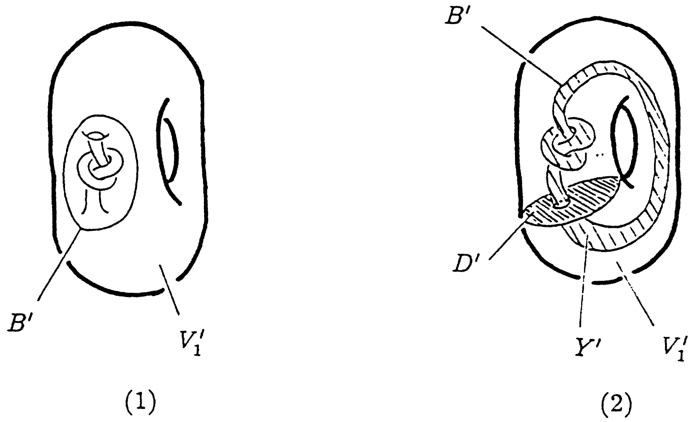


Figure 26

spun torus $T^a(K_b)$ in S^4 such that $T^a(K_b)$ is ambient isotopic to T with $(a,b)=1$, $b\neq 0$, and the number of components of $\Gamma((T^a(K_b))^*)$ equals the number of components of $\Gamma(T^*)$.

PROOF. By Proposition 7.15 and Lemma 8.2, there exist an immersion $\alpha : J \longrightarrow p(B^3\setminus P)$ such that $\alpha(a,b)$ is a projection of a torus T' in S^4 such that T' is ambient isotopic to T where J is a disjoint union of 1-spheres and intervals. By Remark 2.3, there exists a link K in $B^3\setminus P$ such that $T^a(K_b)$ is ambient isotopic to T. If the number of components of K were greater than one, then $T^a(K_b)$ would consist of more than one component (see [T]). Therefore K is a knot. This completes the proof of Proposition 8.3. □

9. Main Theorem

Let $p : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the projection with $p(x,y,z,w) = (x,y,z)$, $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ the map defined by $f(x,y,z,w) = (x,y,z,-w)$, $g : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ the map defined by $g(x,y,z,w) = (x,y,-z,w)$, $K : S^1 \longrightarrow B^3$ an embedding, and $\overline{K} = f\circ K$.

LEMMA 9.1. *$(f\circ g)T^n(K_1) = T^{-n}(\overline{K}_1)$ for all $n\in\mathbb{Z}$.*

PROOF. Let $(x, y, 0, z) \in (q_1 \times id)^{-1}(K(S^1))$, and $(X, Y) = r_{n\theta}(x, y)$. We have

$$(X, Y) = r_{n\theta}(x, y) = r_{(-n)(-\theta)}(x, y), \text{ and}$$

$$\begin{aligned}(f \circ g)(X, Y\cos\theta, Y\sin\theta, z) &= (X, Y\cos\theta, -Y\sin\theta, -z) \\ &= (X, Y\cos(-\theta), Y\sin(-\theta), -z).\end{aligned}$$

Therefore

$$\begin{aligned}(f \circ g)T^n(K_1) &= \left\{ \begin{pmatrix} X \\ Y\cos\theta \\ Y\sin\theta \\ -z \end{pmatrix} \middle| \begin{array}{c} -2\pi \le \theta \le 0 \\ (x, y, 0, z) \in (q_1 \times id)^{-1}(K(S^1)) \\ (X, Y) = r_{-n\theta}(x, y) \end{array} \right\} \\ &= T^{-n}(\overline{K}_1). \ \square\end{aligned}$$

Let F_i be a torus in S^4 obtained by Figure 1 (i) $(i = 1, 2)$. Let F be an embedded oriented surface in S^4, $-F$ the surface having the opposite orientation of F. If F is ambient isotopic to $-F$ preserving the orientation of F and S^4, we call F *invertible*.

PROPOSITION 9.2.
(1) $T^0(L_3)$ is ambient isotopic to $T^0(L_3')$.
(2) $T^3(L_3)$ is ambient isotopic to $T^3(L_3')$.
(3) F_1 is ambient isotopic to F_2.
Moreover $T^0(L_3)$, $T^3(L_3)$ and F_1 are invertible.

PROOF. One can prove (1) and (2) by Lemma 9.1. We will prove (3). We have

$$(f \circ g)(x, y\cos\theta, y\sin\theta, z) = (x, y\cos(-\theta), y\sin(-\theta), -z).$$

Then $f \circ g$ is an orientation preserving homeomorphism, and $\mathbb{R}^3_\theta \cap F_1$ and $f \circ g(\mathbb{R}^3_\theta \cap F_1) = \mathbb{R}^3_{-\theta} \cap F_2$ are as shown in Figure 27. By Figure 27 and Lemma 5.2, we can prove (3).

If F is $T^0(L_3)$ or $T^3(L_3)$, then F is invertible by the fact that a trefoil knot is invertible. We can prove that F_1 is invertible by Figure 28. $\square$

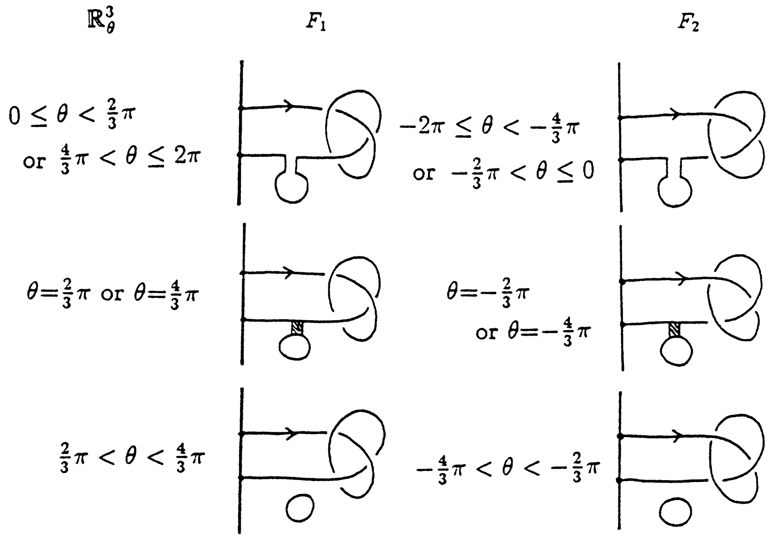


Figure 27

THEOREM 9.3. *Let T be a torus in S^4. If the singular set $\Gamma(T^*)$ consists of three disjoint simple closed curves, then T is ambient isotopic to one and only one of the following tori.*

(1) the standard torus,

(2) the spun torus of the trefoil knot $T^0(L_3)$ (for L, see Figure 3),

(3) the twist spun torus of the trefoil knot $T^3(L_3)$, or

(4) the torus obtained by attaching a handle to the spun 2-sphere of the trefoil knot (see Figure 1 (1)).

PROOF. By Lemma 6.3, we will investigate the 157 cases of configurations of $\Gamma(T)$ (see Figure 21).

If the configuration of $\Gamma(T)$ is either (61), (64), (100) or (103), then by Lemma 3.3 T can be moved so that the resulting $\Gamma(T^*)$ consists of one simple closed curve. We can apply Lemma 3.2. Therefore T is ambient isotopic to the standard torus.

If the configuration of $\Gamma(T)$ is either (132), (133), (134), (138), (139), (143), (144), (149) or (150), then by Lemma 3.4 (1) T can be moved so that the resulting $\Gamma(T^*)$ consists of two simple closed curves. We can apply Lemma 3.2. Therefore T is ambient isotopic to the standard torus.

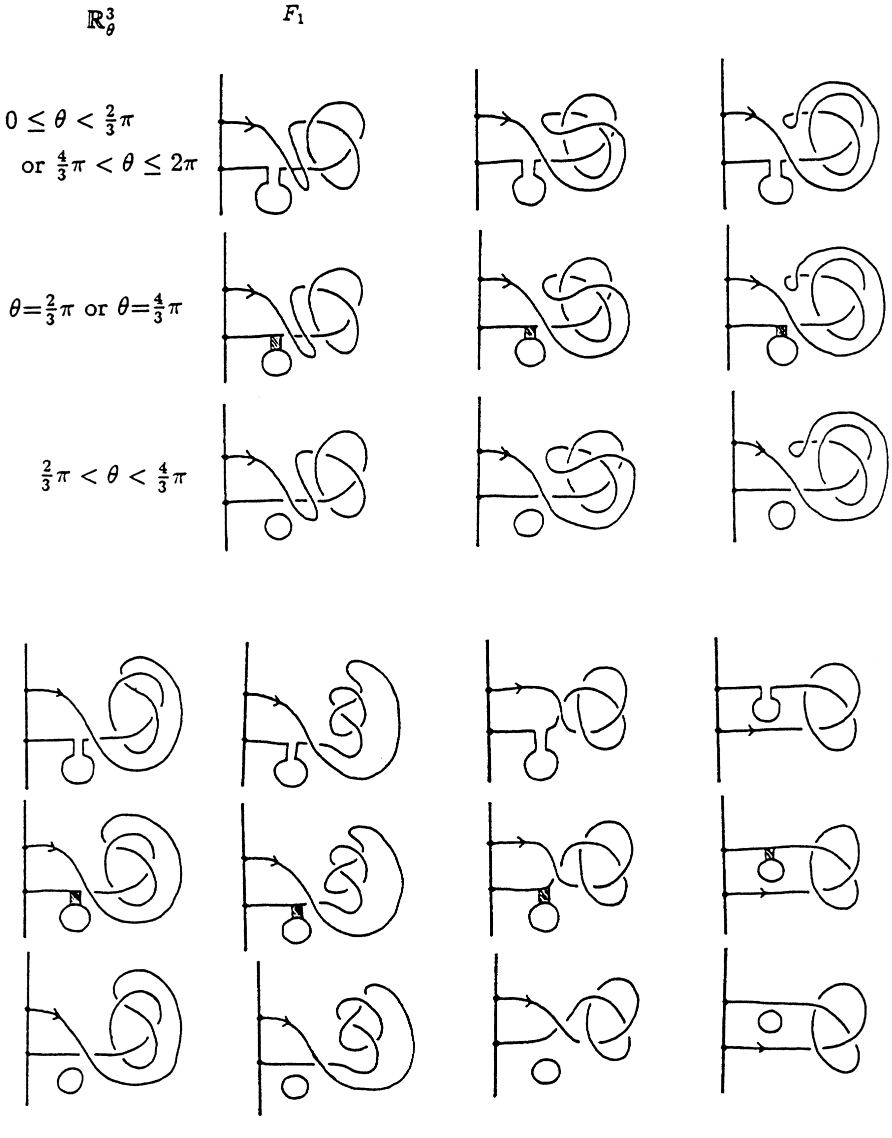


Figure 28

If the configuration of $\Gamma(T)$ is either (8), (10), (11), (14), (15), (16), (17), (21), (22), (23), (24), (25), (26), (27), (29), (30), (31), (36), (37), (38), (39), (40), (41), (42), (43), (45), (48), (49), (50), (51), (52), (53), (54) or

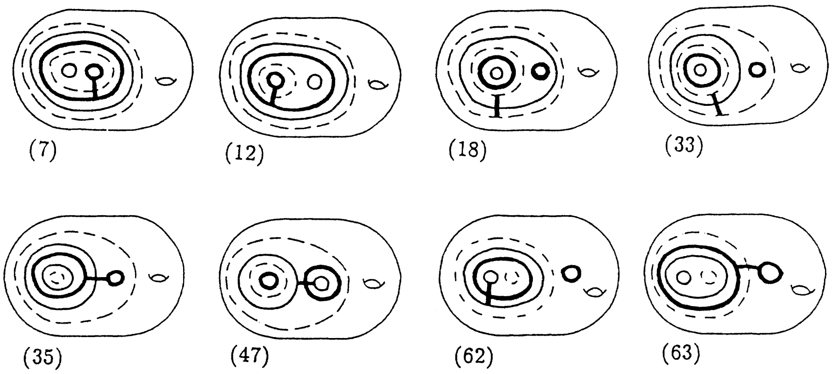


Figure 29

(55), then by Lemma 3.4 (2) T can be moved so that the resulting $\Gamma(T^*)$ consists of two simple closed curves. We can apply Lemma 3.2. Therefore T is ambient isotopic to the standard torus.

If the configuration of $\Gamma(T)$ is either (101), (106), (117), (119), (141), (142) or (148), then by Lemma 3.5 (1) there does not exist a torus with such a singular set.

If the configuration of $\Gamma(T)$ is either (104), (111), (115), (122), (125), (127) or (128), then by Lemma 3.5 (2) there does not exist a torus with such a singular set.

If the configuration of $\Gamma(T)$ is either (46), (66), (71), (72), (84), (86), (92), (121) or (124), then by Lemma 3.5 (3) there does not exist a torus with such a singular set.

If the configuration of $\Gamma(T)$ is either (1), (3), (4), (5), (6), (9), (19), (32), (34), (44), (56), (57), (58), (60), (68), (69), (74), (75), (76), (79), (89), (91), (94), (96), (97), (99), (109), (110), (113), (114), (135), (136), (140), (147), (154) or (156), then by Corollary 4.4 there does not exist a torus with such a singular set.

If the configuration of $\Gamma(T)$ is either (7), (12), (18), (33), (35), (47), (62), (63), (65), (67), (70), (78), (80), (81), (82), (83), (84), (85), (87), (90), (93), (102), (105), (116), (120), (126), (129) or (130), then by Lemma 4.3 there does not exist a torus with such a singular set. Take an arc α for each of the cases shown in Figure 29.

Case (98).

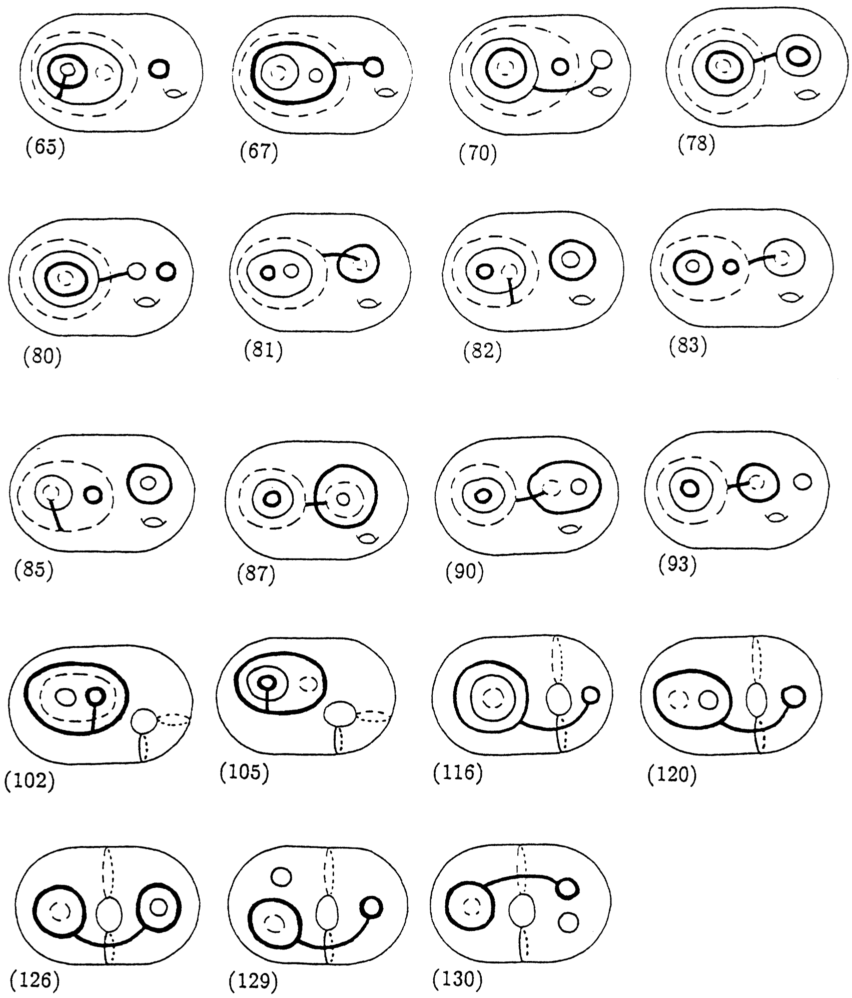


Figure 29 (continued)

Perform *D-surgeries* along the disks (see Figure 30 (1)). We obtain an embedded 2-sphere S, an embedded torus S', and an arc β in S^3 such that $S \cap S'$ consists of disjoint two simple closed curves, $\beta \cap S$ consists of two points, and $\beta \cap S'$ consists of one endpoint of β. By the Schoënflies Theorem ([R] p 34), S is the boundary of a 3-ball. By the solid torus Theorem ([R] p 107), S' is the boundary of a solid torus. Then $S \cup S' \cup \beta$ is as shown in

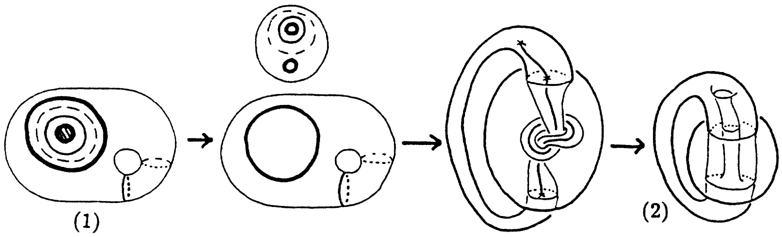


Figure 30

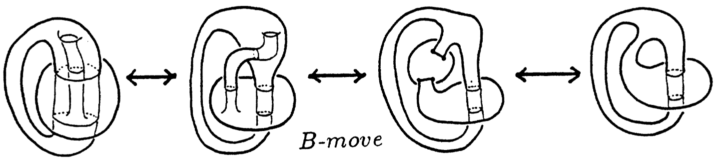


Figure 31

Figure 30 (2). The trefoil in the ball just represents the fact that the torus may be knotted. Also the arc inside may have a different knotting. These knottings can be untied by an ambient isotopy. We can check in Figure 31 that T can be moved to the standard position by an ambient isotopy.

Case (107).

Perform *D-surgeries* along the disks (see Figure 32 (1)). We obtain T^* in a similar way to Case (98) (see Figure 32 (2)). We can check in Figure 33 that T can be moved to the standard position by an ambient isotopy.

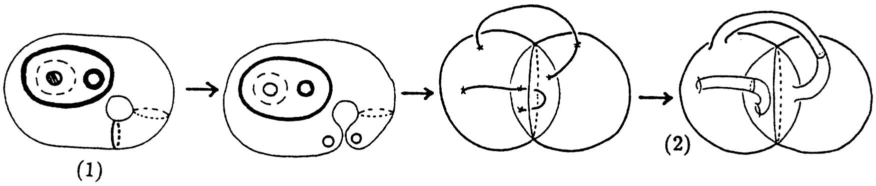


Figure 32

Case (112).

Perform *D-surgeries* along the disks (see Figure 34 (1)). We obtain T^* in a similar way to Case (98) (see Figure 34 (2)). We can check in Figure 35 that T can be moved to the standard position by an ambient isotopy.

Cases (108), (118), (123) and (137).

Perform *D-surgeries* along the disks (see Figure 36 (1)). We obtain T^* in a similar way to Case (98) (see Figure 36 (2)). We can check in Figure 37 that T can be moved to the standard position by an ambient isotopy.

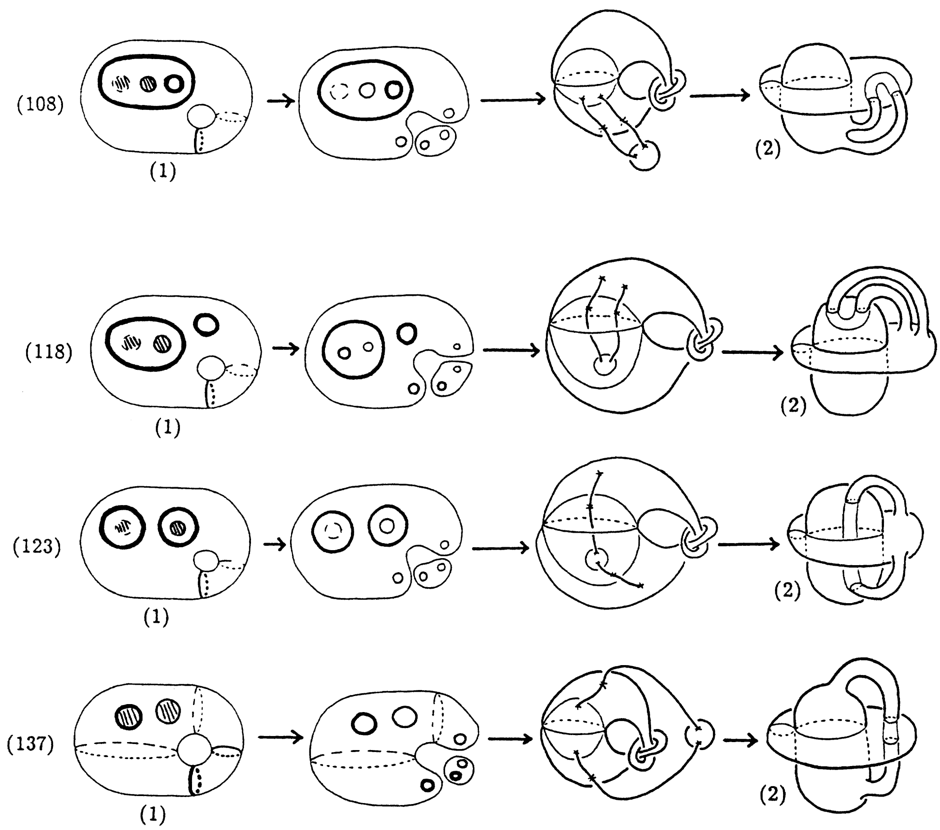


Figure 36

Case (131).

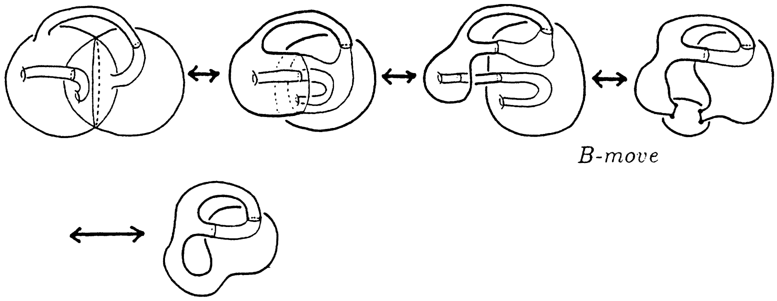


Figure 33

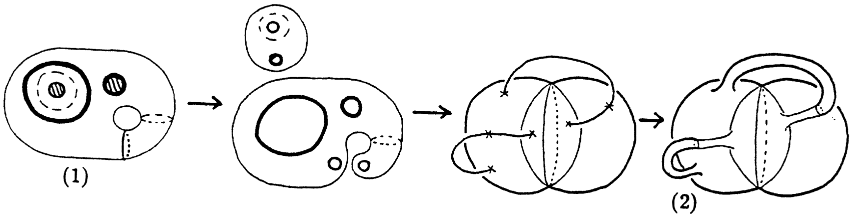


Figure 34

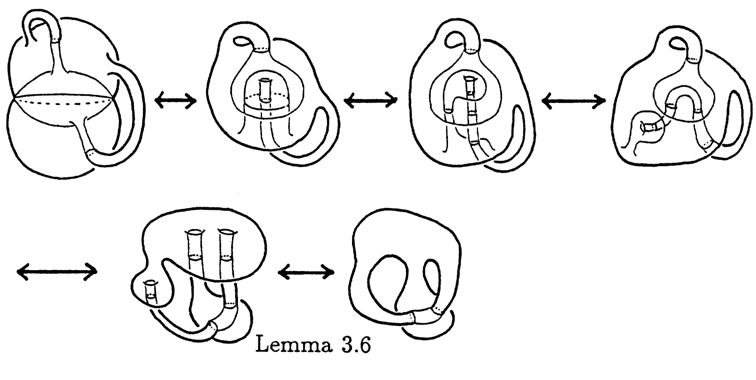


Figure 35

Perform *D-surgeries* along the disks (see Figure 38 (1)). Then we get an arc α and an immersed 2-sphere S^* having the singular set of two simple

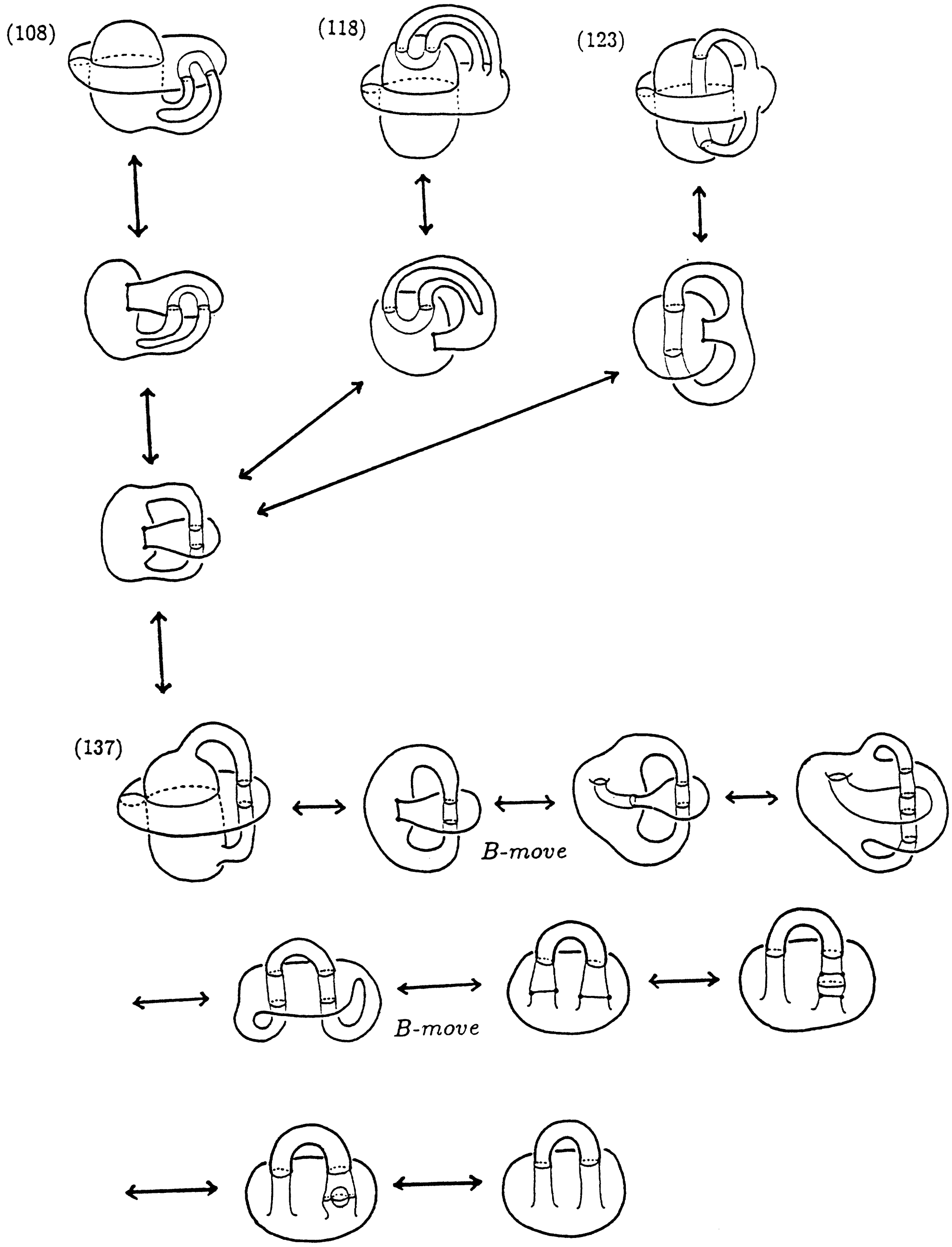


Figure 37

closed curves. An immersed 2-sphere having the singular set of one simple closed curve is as shown in Figure 38 (2). The arc α and the immersed 2-sphere S^* are as shown in Figure 38 (3). We obtain T^* in a similar way to Case (98) (see Figure 38 (4)). We can check in Figure 39 that T can be moved to the standard position by an ambient isotopy.

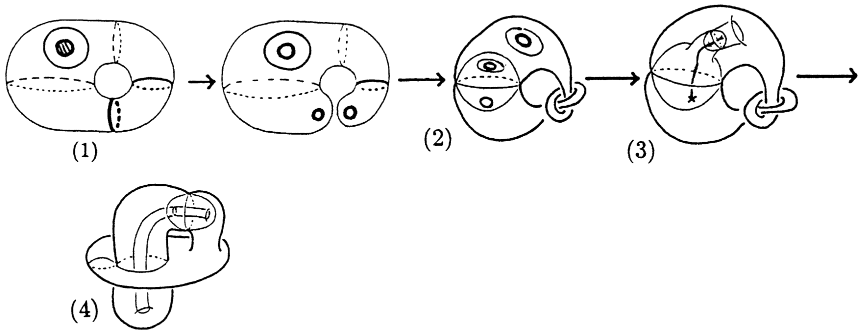


Figure 38

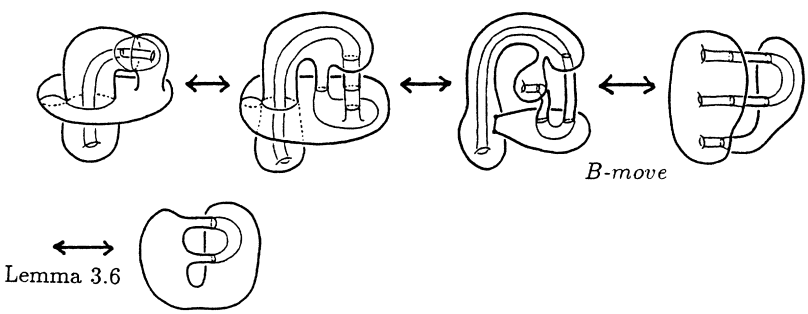


Figure 39

Case (146).

Perform *D-surgeries* along the disks (see Figure 40 (1)). We obtain T^* in a similar way to Case (98) (see Figure 40 (2)). We can check in Figure 41 that T can be moved the standard position by an ambient isotopy.

Case (152).

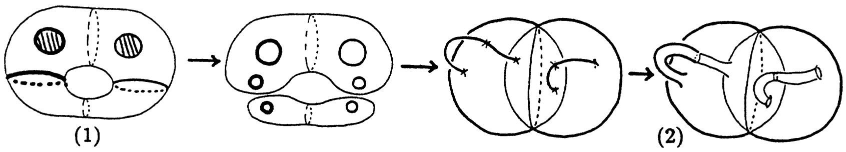


Figure 40

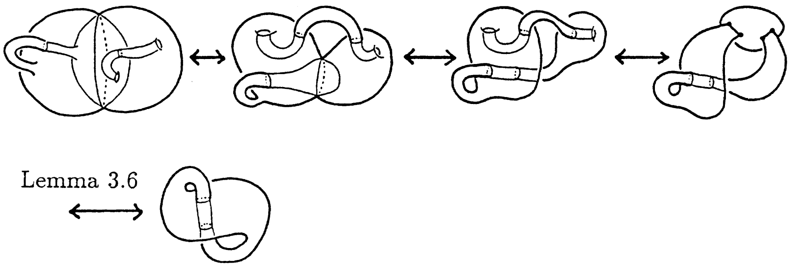


Figure 41

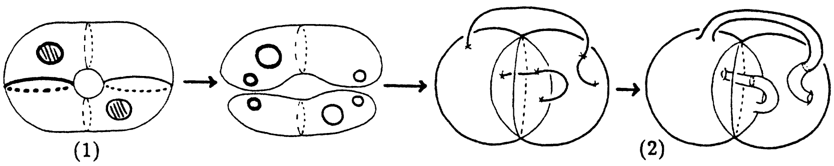


Figure 42

Perform *D-surgeries* along the disks (see Figure 42 (1)). We obtain T^* in a similar way to Case (98) (see Figure 42 (2)). We can check in Figure 43 that T can be moved the standard position by an ambient isotopy.

Cases (145) and (151).

We will show that there does not exist a torus having a singular set of these types. Suppose that there exists a torus with one of these types of singular sets. Perform *D-surgeries* along the disks (see Figure 44 (1)). Then we obtain an arc and an immersed 2-sphere having the singular set of two double closed curves. We see the singularity of the 2-sphere as in

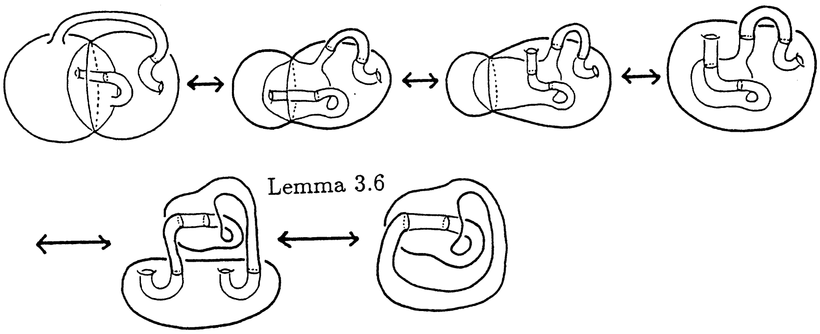


Figure 43

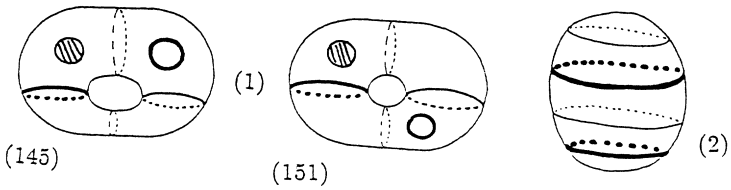


Figure 44

Figure 44 (2). By Corollary 4.4, there does not exist such a singularity of the 2-sphere. This is a contradiction.

Cases (2), (59), (77) and (88).

We will show that T is ambient isotopic to the torus obtained by attaching a handle to the spun 2-sphere of the trefoil knot (see Figure 1 (1),(2)). Perform *D-surgeries* along the disks (see Figure 45). Then we obtain T^* in a similar way to Case (98). We see the attaching handles in Figure 46 in respective cases.

Cases (13), (20), (28), (73) and (95).

We will show that there does not exist the torus having a singular set of these types. Suppose that there exists a torus with one of these types of singular sets. Perform *D-surgeries* along the disks (see Figure 47). Then we obtain arcs and embedded closed surfaces. But we cannot connect arcs. This is a contradiction.

Cases (153), (155) and (157).

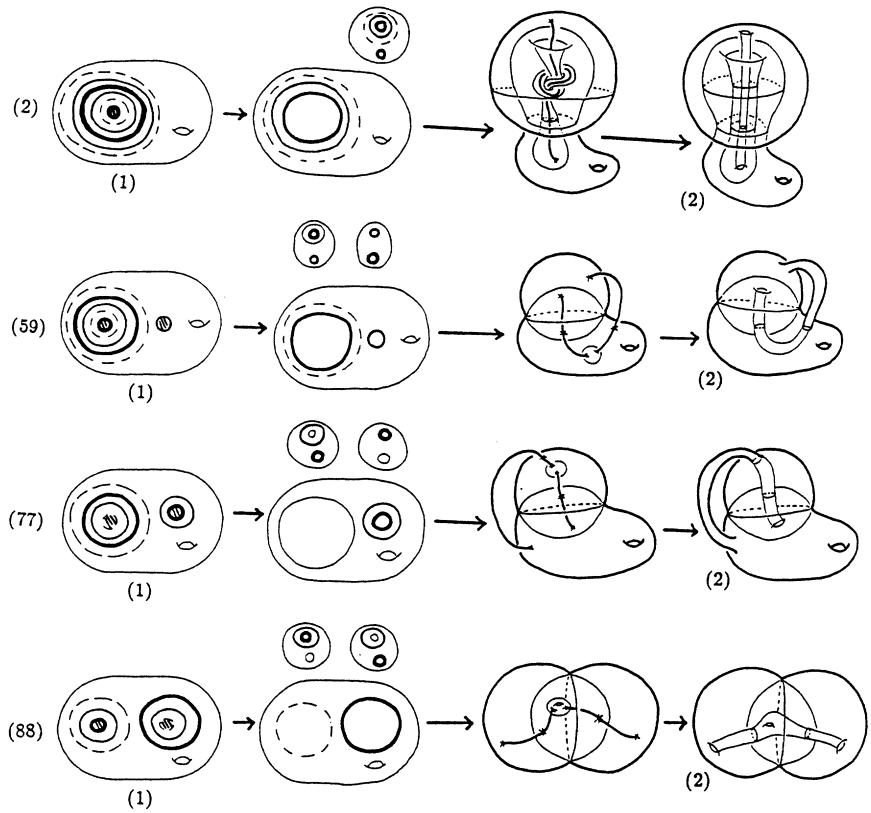


Figure 45

By Proposition 8.3, there exists a symmetry-spun torus $T^a(K_b)$ such that $T^a(K_b)$ is ambient isotopic to T, and $\Gamma((T^a(K_b))^*)$ consists of three disjoint simple closed curves. By Lemma 5.3, $T^a(K_b)$ is ambient isotopic to $T^0(K_1)$ or $T^1(K_1)$. Since $\Gamma(T^*)$ consists of three disjoint simple closed curves, we may assume K is the trefoil knot. Therefore T is ambient isotopic to $T^0(L_3)$, $T^0(L_3')$, $T^3(L_3)$ or $T^3(L_3')$. By Proposition 9.2, T is ambient isotopic to $T^0(L_3)$ or $T^3(L_3)$.

This completes the proof of Theorem 9.3. □

Let G_1 be the standard torus, $G_2 = T^0(L_3)$, $G_3 = T^3(L_3)$, and $G_4 = F_1$.

REMARKS. (1) $\pi_1(S^4 \setminus G_1) \cong \mathbb{Z}$.

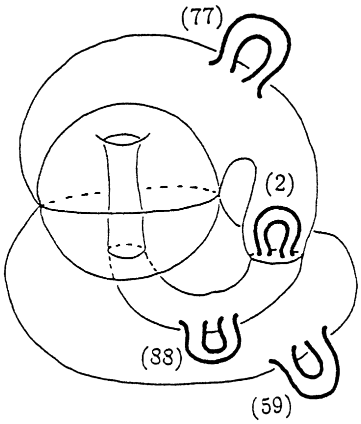


Figure 46

(2) If $i = 2, 3$ or 4, then $\pi_1(S^4 \setminus G_i) \cong \pi_1(S^3 \setminus \text{trefoil knot})$ (see [Y]).

Let $N(G_i)$ be a regular neighborhood of G_i in S^4 and $X_i = S^4 \setminus intN(G_i)$. Let $f_{i*} : \pi_1(\partial X_i) \longrightarrow \pi_1(X_i)$ be the map induced by the inclusion map $\partial X_i \longrightarrow X_i$.

LEMMA 9.4. *If $i = 2$ or 3, then $kerf_{i*} \cong \mathbb{Z}$. And if $i = 4$, then $kerf_{4*}$ contains the subset which is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.*

PROOF. Let $B^3 = \{(x, y, 0, z) \in \mathbb{R}^4; (x-2)^2 + y^2 + z^2 \leq 1\}$. We will show that if $i = 2$ or 3, then $kerf_{i*} \cong \mathbb{Z}$. We use a technique in [B]. Let

$$B^3 \times S^1 = \left\{ \begin{pmatrix} x \\ y\cos\theta \\ y\sin\theta \\ z \end{pmatrix} \;\middle|\; (x, y, 0, z) \in B^3, \quad 0 \leq \theta \leq 2\pi \right\}.$$

Then $S^4 \setminus int(B^3 \times S^1)$ is homeomorphic to $S^2 \times D^2$ where D^2 is a 2-disk. Fix an integer n. Let $\eta_n : \partial(B^3 \times S^1) \longrightarrow \partial(B^3 \times S^1)$ be a map defined as follows:

$$\eta_n(x, y\cos\theta, y\sin\theta, z) = (X, Y\cos\theta, Y\sin\theta, z)$$

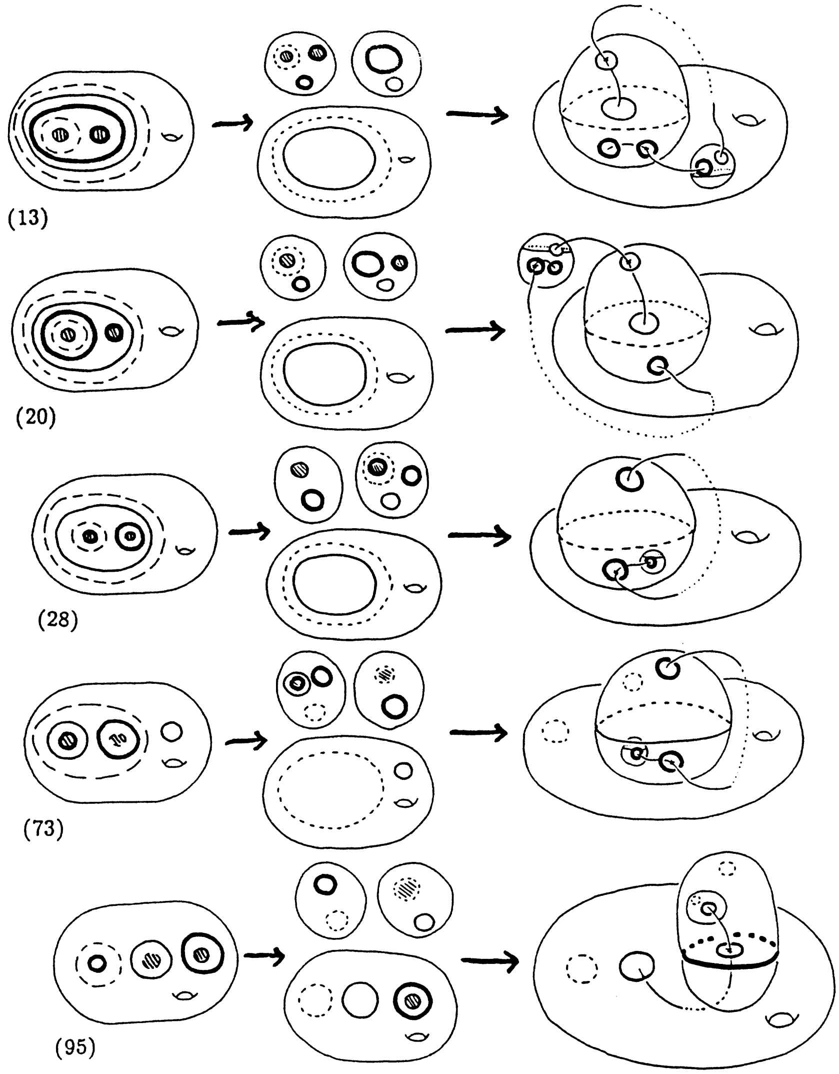


Figure 47

where $(x, y, 0, z) \in B^3$ and $(X, Y) = r_{n\theta}(x, y)$. In particular η_0 is the identity map. Let (B^3, K) denote a trefoil knot K in the 3-ball B^3. Then

$$(S^4, G_2) = (B^3 \times S^1, G_2) \cup_{\eta_0} (S^4 \setminus int(B^3 \times S^1)),$$

$$
\begin{aligned}
&= (B^3, K) \times S^1 \cup_{\eta_0} (S^4 \setminus int(B^3 \times S^1)), \text{ and} \\
(S^4, G_3) &\cong (B^3 \times S^1, G_2) \cup_{\eta_1} (S^4 \setminus int(B^3 \times S^1) \\
&\cong (B^3, K) \times S^1 \cup_{\eta_1} (S^4 \setminus int(B^3 \times S^1)).
\end{aligned}
$$

Let $N(K)$ be a regular neighborhood of K in B^3. We have $\pi_1((B^3 \setminus int(N(K))) \times S^1) \cong \pi_1(B^3 \setminus int(N(K))) \oplus \mathbb{Z}$. By Van Kampen's theorem, the effect of gluing $S^4 \setminus int(B^3 \times S^1)$ to $(B^3 \setminus int(N(K))) \times S^1$ is to kill the $\mathbb{Z}$ summands. Hence the inclusion map $B^3 \setminus int(N(K)) \longrightarrow X_i$ induces an isomorphism of fundamental groups. Since $\pi_1(\partial N(K))$ injects into $\pi_1(B^3 \setminus int(N(K)))$, we see $ker f_{i*} \cong \mathbb{Z}$.

If $i = 4$, then the proof is obvious. This completes the proof of Lemma 9.4. □

THEOREM 9.5. *If $i \neq j$, then G_i cannot be moved to G_j by an ambient isotopy of S^4.*

PROOF. Using Lemma 9.4, we have that G_i ($i = 2$ or 3) cannot be moved to G_4 by an ambient isotopy of S^4. By [B, Theorem 1.1], We can show that G_2 cannot be moved to G_3 by an ambient isotopy of S^4. □

Acknowledgements. The author would like to express her sincere gratitude to Professor Yukio Matsumoto for his valuable advice and encouragement.

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(Received March 7, 1996)
(Revised November 29, 1996)

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