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On the Sparre Andersen Transformation for Multidimensional Brownian Bridge

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Abstract. A family of law-preserving path transformations of d-dimensional Brownian bridge (pinned Brownian motion), $d \geq 1$, is constructed. This generalizes a result of one-dimensional cases obtained first by Embrechts, Rogers and Yor. Our approach and theirs are, however, completely different from each other.

1. Introduction and Statement of the Result

The study of law-preserving path transformations of one-dimensional Brownian motion is currently a popular subject. Karatzas-Shreve [7] and Bertoin [3] constructed a transformation connecting local minimum and excursions in half-lines. A generalization to a larger class of path transformations (Corollary 1.2 of the present paper) was recently obtained first by Embrechts-Rogers-Yor [5], whose approach is based on Brownian excursion theory. Also, the second author (Takaoka [9]; an English translated version is [10]) independently gives another proof of the same result; the proof is obtained by taking the continuous-time limit of the idea lying behind Richards' proof of Sparre Andersen's theorem on discrete-time processes (see Theorem 2.1 and its proof below).

In this paper, we give a further extension along the lines of [9] [10], even to multidimensional cases:

THEOREM 1.1. *Fix $d \in \boldsymbol{N}$, $A \in \mathcal{B}(\boldsymbol{R}^d)$ and $b \in \boldsymbol{R}^d$, where $\mathcal{B}(\boldsymbol{R}^d)$ is the Borel σ-algebra of $\boldsymbol{R}^d$. Let $(B_t)_{t\in[0,1]}$ be a d-dimensional Brownian bridge from 0 to b on a certain probability space $(\Omega, \mathcal{F}, P)$, i.e., a Brownian motion starting from the origin and conditioned to be at b at time 1. Let $(Z_t)_{t\in[0,1]}$ denote its time-reversed process:*

$$Z_t \stackrel{\text{def}}{=} b - B_{1-t} \quad \text{for } t \in [0,1].$$

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For $t \in [0,1]$, define:

$$\begin{aligned}
\Gamma_+(t) &\stackrel{\text{def}}{=} \int_0^t 1_A(B_s)ds; \\
\Gamma_-(t) &\stackrel{\text{def}}{=} t - \Gamma_+(t); \\
\Gamma_\pm^{-1}(t) &\stackrel{\text{def}}{=} \inf\{s \in [0,1] \mid \Gamma_\pm(s) \geq t \wedge \Gamma_\pm(1)\}; \\
Y_+(t) &\stackrel{\text{def}}{=} \int_{1-t}^1 1_A(b - Z_s)\, dZ_s; \\
Y_-(t) &\stackrel{\text{def}}{=} \int_{1-t}^1 1_{A^c}(b - Z_s)\, dZ_s; \\
B_\pm(t) &\stackrel{\text{def}}{=} Y_\pm(\Gamma_\pm^{-1}(t));
\end{aligned}$$

where we take the integrals $Y_\pm(t)$ with respect to the backward filtration, i.e., the filtration generated by Z rather than by B. Furthermore,

$$\tilde{B}_t \stackrel{\text{def}}{=} \begin{cases} B_+(\,\Gamma_+(1)\,) - B_+(\,\Gamma_+(1) - t\,), & \textit{if} \quad t \in [0, \Gamma_+(1)]; \\ B_+(\,\Gamma_+(1)\,) + B_-(\,t - \Gamma_+(1)\,), & \textit{if} \quad t \in (\Gamma_+(1), 1]. \end{cases}$$

Then we have

$$(\tilde{B}_t)_{t\in[0,1]} \stackrel{\text{(d)}}{=} (B_t)_{t\in[0,1]}.$$

REMARKS.

(i) Following [9] [10], we propose calling this transformation the **Sparre Andersen transformation** of $(B_t)_{t\in[0,1]}$ with respect to $A \in \mathcal{B}(\boldsymbol{R}^d)$, because, as we will see in more detail in Section 2, the starting point of our study is a combinatorial theorem of E. Sparre Andersen [1] on sums of exchangeable random variables.

(ii) Feller's proof [6] of Sparre Andersen's theorem has been used to derive some continuous-time properties, e.g. Bertoin [4]. It should be noted, however, that Richards' proof of Sparre Andersen's theorem, utilized in this paper (see Section 2), covers a wider variety of cases and thus offers a unified way of viewing the whole matter.

(iii) If $A = \emptyset$ then this transformation is the identity; if $A = \boldsymbol{R}^d$ it is the time-reversal transformation. Moreover, if we denote by $(\tilde{B}_t^A)_{t\in[0,1]}$ the resulting process with respect to A, then $\tilde{B}^A$ and $\tilde{B}^{A^c}$ are time reversals of each other for any $A \in \mathcal{B}(\boldsymbol{R}^d)$.

In the special case where $d = 1$ and $A = (a, \infty)$, $a \in \boldsymbol{R}$, then Tanaka's formula recovers the above mentioned result of Embrechts-Rogers-Yor [5]:

COROLLARY 1.2. *Let $(B_t)_{t \in [0,1]}$ be a one-dimensional standard Brownian motion starting from the origin on a certain probability space $(\Omega, \mathcal{F}, P)$. Fix $a \in \boldsymbol{R}$. For $t \in [0,1]$, define:*

$$
\begin{aligned}
\ell^a_t &\overset{\text{def}}{=} \text{its local time at } a \text{ up to time } t; \\
\Gamma^a_+(t) &\overset{\text{def}}{=} \int_0^t 1_{(a,\infty)}(B_s)\, ds; \\
\Gamma^a_-(t) &\overset{\text{def}}{=} t - \Gamma^a_+(t); \\
\{\Gamma^a_\pm\}^{-1}(t) &\overset{\text{def}}{=} \inf\{\, s \in [0,1] \mid \Gamma^a_\pm(s) \geq t \wedge \Gamma^a_\pm(1)\,\}; \\
Y^a_+(t) &\overset{\text{def}}{=} (B_t \vee a) - (a \vee 0) + \frac{\ell_t}{2}; \\
Y^a_-(t) &\overset{\text{def}}{=} (B_t \wedge a) - (a \wedge 0) - \frac{\ell_t}{2}; \\
B^a_\pm(t) &\overset{\text{def}}{=} Y^a_\pm(\{\Gamma^a_\pm\}^{-1}(t)).
\end{aligned}
$$

(Note that $B^a_\pm(0) = 0$ a.s.) Furthermore

$$
\tilde{B}^a_t \overset{\text{def}}{=} \begin{cases} B^a_+(\,\Gamma^a_+(1)\,) - B^a_+(\,\Gamma^a_+(1) - t\,), & \text{if } \ t \in [0, \Gamma^a_+(1)]; \\ B^a_+(\,\Gamma^a_+(1)\,) + B^a_-(\,t - \Gamma^a_+(1)\,), & \text{if } \ t \in (\Gamma^a_+(1), 1]. \end{cases}
$$

Then the process $(\tilde{B}^a_t)_{t \in [0,1]}$ is also a Brownian motion starting from the origin.

The rest of this paper is organized as follows. Section 2 explains the underlying discrete-time argument. In Section 3, we prove Theorem 1.1.

2. Path Transformation for Pinned Random Walk

As mentioned above in the Introduction, our starting point is Sparre Andersen's theorem:

THEOREM 2.1 (Sparre Andersen [1]). *Let $(S_k)_{k=0}^n$ be an arbitrary one-dimensional process starting from the origin and with exchangeable increments, i.e., the joint distribution of the n random variables*

$$S_1 - S_0,\ S_2 - S_1,\ \ldots,\ S_n - S_{n-1}$$

is symmetric with respect to the n *arguments. Then the two functionals*

$$(i) \quad \min\{k \in \{0, 1, \dots, n\};\, S_k = \max_{0 \le j \le n} S_j\},$$
$$(ii) \quad \sharp\{k \in \{1, \dots, n\};\, S_k > 0\}$$

are identically distributed.

Richards' proof (unpublished; see Baxter [2]) of Theorem 2.1 involves a path transformation of pinned random walk which we will use in the next section. The key idea is the following

LEMMA 2.2 (Richards; Baxter [2]). *(i) Fix* $d \in \boldsymbol{N}$ *and* $A \in \mathcal{B}(\boldsymbol{R}^d)$*. For each* $(x_1, x_2, \cdots, x_n) \in (\boldsymbol{R}^d)^n$*, form a new arrangement of the* x_k*'s by placing first in decreasing order of* k *the terms* x_k *for which* $s_k \in A$ *and then (afterwards) in increasing order of* k *the* x_k *for which* $s_k \notin A$*, where* $s_0 \stackrel{\text{def}}{=} 0$ *and* $s_k \stackrel{\text{def}}{=} \sum_{j=1}^k x_j$ *for* $k = 1, \dots, n$*. Denote this new arrangement by* $(\tilde{x}_1, \dots, \tilde{x}_n)$*. Then the transformation*

$$\begin{array}{rccc} \theta_A : & (\boldsymbol{R}^d)^n & \longrightarrow & (\boldsymbol{R}^d)^n \\ & (x_1, \cdots, x_n) & \longmapsto & (\tilde{x}_1, \dots, \tilde{x}_n) \end{array}$$

is one-to-one and onto. Furthermore, if μ *is an exchangeable measure on* $((\boldsymbol{R}^d)^n, \mathcal{B}((\boldsymbol{R}^d)^n))$*, i.e., if*

$$\forall \sigma \in \mathfrak{S}_n,\ \forall B \in \mathcal{B}((\boldsymbol{R}^d)^n), \qquad \mu[\sigma(B)] = \mu[B]$$

with $\mathfrak{S}_n$ *the symmetric group of order* n*, then*

$$\forall B \in \mathcal{B}((\boldsymbol{R}^d)^n), \qquad \mu[\theta_A(B)] = \mu[B].$$

(ii) Let $(S_k)_{k=0}^n$ *be a* d*-dimensional process starting from the origin and with exchangeable increments. Fix* $A \in \mathcal{B}(\boldsymbol{R}^d)$ *and define*

$$\begin{aligned} X_k &\stackrel{\text{def}}{=} S_k - S_{k-1} \qquad \text{for } k = 1, \cdots, n; \\ (\tilde{X}_1(\omega), \cdots, \tilde{X}_n(\omega)) &\stackrel{\text{def}}{=} \theta_A(X_1(\omega), \cdots, X_n(\omega)) \quad \text{for} \quad \omega \in \Omega; \\ \tilde{S}_0 &\stackrel{\text{def}}{=} 0; \\ \tilde{S}_k &\stackrel{\text{def}}{=} \sum_{j=1}^k \tilde{X}_j \qquad \text{for} \quad k = 1, \dots, n. \end{aligned}$$

Then

$$(S_k)_{k=0}^n \overset{\text{(d)}}{=} (\tilde{S}_k)_{k=0}^n.$$

Proof of Lemma 2.2. (i) It is straightforward to check that θ_A is one-to-one and onto. Next, let $\Lambda \overset{\text{def}}{=} \{0,1\}^n$. For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Lambda$ define

$$C_\lambda \overset{\text{def}}{=} \{(x_1, \cdots, x_n) \in (\boldsymbol{R}^d)^n \mid s_k \in A \ \text{ if } \lambda_k = 1, \\ s_k \notin A \ \text{ if } \lambda_k = 0, \ k = 1, 2, \cdots, n\}.$$

Then it is clear that

$$\begin{aligned} C_\lambda \cap C_{\lambda'} &= \emptyset \quad \text{if} \quad \lambda \neq \lambda', \\ \bigcup_{\lambda \in \Lambda} C_\lambda &= (\boldsymbol{R}^d)^n. \end{aligned}$$

In addition, for each $\lambda \in \Lambda$ there exists a $\sigma_\lambda \in \mathfrak{S}_n$ such that

$$\forall B \subset (\boldsymbol{R}^d)^n, \quad \theta_A(B \cap C_\lambda) = \sigma_\lambda(B \cap C_\lambda).$$

Therefore, for any $B \in \mathcal{B}((\boldsymbol{R}^d)^n)$:

$$\begin{aligned} \mu[\theta_A(B)] &= \sum_{\lambda \in \Lambda} \mu[\theta_A(B \cap C_\lambda)] \\ &= \sum_{\lambda} \mu[\sigma_\lambda(B \cap C_\lambda)] \\ &= \sum_{\lambda} \mu\,[\, B \cap C_\lambda \,] \quad \text{by exchangeability} \\ &= \mu\,[B]. \end{aligned}$$

(ii) This is an immediate consequence of (i). □

Remarks. (i) We can apply the above argument to all d-dimensional random walks and pinned random walks, $d \geq 1$.

(ii) We propose that $(\tilde{S}_k)_{k=0}^n$ be called the *Sparre Andersen transformation* of $(S_k)_{k=0}^n$ with respect to $A \in \mathcal{B}(\boldsymbol{R}^d)$.

Proof of Theorem 2.1 (Richards). If $d = 1$ and $A = (0, \infty)$, then $(S_k)_{k=0}^n$and its transformation $(\tilde{S}_k)_{k=0}^n$ have the following relation:

$$\sharp\{k \in \{1, \ldots, n\};\, S_k > 0\,\} \;=\; \min\{k \in \{0, 1, \ldots, n\};\, \tilde{S}_k = \max_{0 \leq j \leq n} \tilde{S}_j\} \quad \text{a.s.}$$

The proof of Theorem 2.1 is therefore complete. □

Finally, a little closer look at this transformation immediately gives the following property, the proof of which we omit.

PROPOSITION 2.3. *With the notations of Lemma 2.2(ii) assumed, define for* $k = 0, \dots, n$:

$$\begin{aligned}
\Gamma_+^S(k) &\stackrel{\text{def}}{=} \sharp\{j \in \{1,\dots,k\};\, S_j \in A\}; \qquad (\,\Gamma_+^S(0) \stackrel{\text{def}}{=} 0\,) \\
\Gamma_-^S(k) &\stackrel{\text{def}}{=} k - \Gamma_+^S(k); \\
\{\Gamma_\pm^S\}^{-1}(k) &\stackrel{\text{def}}{=} \min\{j \in \{0,1,\dots,n\};\, \Gamma_\pm^S(j) \ge k \wedge \Gamma_\pm^S(n)\}; \\
Y_+^S(k) &\stackrel{\text{def}}{=} \sum_{j=1}^k 1_A(S_j)\,(S_j - S_{j-1}); \\
Y_-^S(k) &\stackrel{\text{def}}{=} \sum_{j=1}^k 1_{A^c}(S_j)\,(S_j - S_{j-1}); \\
S_\pm(k) &\stackrel{\text{def}}{=} Y_\pm^S(\{\Gamma_\pm^S\}^{-1}(k)).
\end{aligned}$$

Then, a.s.,

$$\tilde{S}_k = \begin{cases} S_+(\Gamma_+^S(n)) - S_+(\Gamma_+^S(n) - k), & \text{if } \ 0 \le k \le \Gamma_+^S(n); \\ S_+(\Gamma_+^S(n)) + S_-(k - \Gamma_+^S(n)), & \text{if } \ \Gamma_+^S(n) < k \le n. \end{cases}$$

3. Proof of the Main Theorem

The idea of the proof of Theorem 1.1 is to show that our path transformation of Brownian bridge is the continuous-time limit of the Sparre Andersen transformation for pinned random walk. A quite similar method was employed in the proof of the main theorem of [9] [10] (Corollary 1.2 of the present paper). It should be noted, however, that the way we approximate the paths of $(B_t)_{t\in[0,1]}$ with pinned random walk here is not the same as in [9] [10].

DEFINITION 3.1. Let

$$S_k^{(n)} \stackrel{\text{def}}{=} B(\tfrac{k}{2^n}) \qquad \text{for} \quad k = 0, 1, \cdots, 2^n, \quad n \in \boldsymbol{N}.$$

Clearly $(S_k^{(n)})_{k=0}^{2^n}$ is a d-dimensional random walk pinned at b. Also, let $(\tilde{S}_k^{(n)})_{k=0}^{2^n}$ denote its Sparre Andersen transformation with respect to A.

Definition 3.2. For $t \in [0,1]$, define:

$$\begin{aligned}
\Gamma_+^{(n)}(t) &\stackrel{\text{def}}{=} \int_0^t 1_A\left(B(\tfrac{[2^n s+1]}{2^n})\right) ds; \\
\Gamma_-^{(n)}(t) &\stackrel{\text{def}}{=} t - \Gamma_+^{(n)}(t); \\
\{\Gamma_\pm^{(n)}\}^{-1}(t) &\stackrel{\text{def}}{=} \inf\{\, s \in [0,1] \mid \Gamma_\pm^{(n)}(s) \geq t \wedge \Gamma_\pm^{(n)}(1)\,\}; \\
Y_+^{(n)}(t) &\stackrel{\text{def}}{=} \sum_{k=1}^{2^n} 1_A\left(B(\tfrac{k}{2^n})\right)\left(B(t \wedge \tfrac{k}{2^n}) - B(t \wedge \tfrac{k-1}{2^n})\right); \\
Y_-^{(n)}(t) &\stackrel{\text{def}}{=} \sum_{k=1}^{2^n} 1_{A^c}\left(B(\tfrac{k}{2^n})\right)\left(B(t \wedge \tfrac{k}{2^n}) - B(t \wedge \tfrac{k-1}{2^n})\right); \\
B_\pm^{(n)}(t) &\stackrel{\text{def}}{=} Y_\pm^{(n)}(\{\Gamma_\pm^{(n)}\}^{-1}(t)).
\end{aligned}$$

Furthermore,

$$\tilde{B}_t^{(n)} \stackrel{\text{def}}{=} \begin{cases} B_+^{(n)}(\,\Gamma_+^{(n)}(1)\,) \;-\; B_+^{(n)}(\,\Gamma_+^{(n)}(1) - t\,), & \text{if} \quad t \in [0, \Gamma_+^{(n)}(1)]; \\ B_+^{(n)}(\,\Gamma_+^{(n)}(1)\,) \;+\; B_-^{(n)}(\,t - \Gamma_+^{(n)}(1)\,), & \text{if} \quad t \in (\Gamma_+^{(n)}(1), 1]. \end{cases}$$

Proposition 3.3. *We have*

$$\tilde{S}_k^{(n)} = \tilde{B}^{(n)}(\tfrac{k}{2^n}) \qquad k = 0, 1, \cdots, 2^n, \quad a.s.,$$

Proof. This is a straightforward consequence of Proposition 2.3 and the following fact:

$$\frac{1}{2^n}\,\sharp\{j \in \{1, \cdots, k\} \mid S_j^{(n)} \in A\} = \Gamma_+^{(n)}(\tfrac{k}{2^n}), \qquad k = 0, 1, \cdots, 2^n. \ \square$$

We shall use the next lemma to prove Proposition 3.5 below.

Lemma 3.4. *For any bounded Borel measurable function $f : \boldsymbol{R}^d \to \boldsymbol{R}$ we have:*

$$\forall t \in (0,1), \quad \lim_{\epsilon \downarrow 0} E\,[|f(B_{t+\epsilon}) - f(B_t)|\,|\,\mathcal{F}_t] = 0 \quad a.s.,$$

where $(\mathcal{F}_t)_{t\in[0,1]}$ *is the filtration generated by* B.

PROOF. Let $p(t;x,y)$ denote the transition density function of d-dimensional Brownian motion. Then, for any fixed $t \in (0,1)$ and $x \in \boldsymbol{R}^d$ there exists a positive constant $C_{x,t}$ such that

$$\begin{aligned} P[\,B_{t+\epsilon} \in dy \,|\, B_t = x\,] \,/\, dy \;&=\; \frac{p(\epsilon; x, y)\, p(1-t-\epsilon;\, y, b)}{p(1-t;\, x, b)} \\ &\le\; C_{x,t}\, p(\epsilon; x, y) \end{aligned}$$

for all $y \in \boldsymbol{R}^d$ and all sufficiently small $\epsilon > 0$. Consequently, for each $a > 0$ we have

$$\begin{aligned} &E\,[\,|f(B_{t+\epsilon}) - f(B_t)|\,|\,B_t = x\,] \\ &\le\; C_{x,t}\left\{ (2\pi\epsilon)^{-d/2} \int_{|y-x|<a\sqrt{\epsilon}} |f(y) - f(x)|\, dy \right. \\ &\qquad\qquad \left. +\, 2\,||f||_\infty \int_{|y-x|\ge a\sqrt{\epsilon}} p(\epsilon; x, y)\, dy \right\} \\ &=\; C_{x,t}\left\{ (2\pi\epsilon)^{-d/2} \int_{|y-x|<a\sqrt{\epsilon}} |f(y) - f(x)|\, dy \right. \\ &\qquad\qquad \left. +\, 2\,||f||_\infty \int_{|y|\ge a} p(1; 0, y)\, dy \right\}. \end{aligned}$$

Furthermore, the Lebesgue differentiation theorem (see e.g. Stroock [8] §5.3) states that, for Lebesgue-almost every $x \in \boldsymbol{R}^d$:

$$\lim_{\epsilon\downarrow 0} \epsilon^{-d} \int_{|y-x|<\epsilon} |f(y) - f(x)|\, dy = 0,$$

and hence

$$\overline{\lim_{\epsilon\downarrow 0}}\, E\,[\,|f(B_{t+\epsilon}) - f(B_t)|\,|\,B_t = x\,] \;\le\; 2\,C_{x,t}\,||f||_\infty \int_{|y|\ge a} p(1; 0, y)\, dy.$$

Since a can be made arbitrarily large, we conclude

$$\lim_{\epsilon\downarrow 0} E\,[\,|f(B_{t+\epsilon}) - f(B_t)|\,|\,B_t = x\,] = 0. \;\square$$

Proposition 3.5. *The following assertions hold:*

$$(i) \qquad \lim_{n\uparrow\infty} E\left[\sup_{t\in[0,1]} |\Gamma_\pm^{(n)}(t) - \Gamma_\pm(t)|\right] = 0,$$

$$(ii) \qquad \lim_{n\uparrow\infty} E\left[\sup_{t\in[0,1]} |Y_\pm^{(n)}(t) - Y_\pm(t)|\right] = 0.$$

Consequently, there exists a subsequence $(n_k)_{k=1}^\infty$ *such that*

$$\begin{aligned}\lim_{k\uparrow\infty} \sup_{t\in[0,1]} |\Gamma_\pm^{(n_k)}(t) - \Gamma_\pm(t)| &= 0 \quad a.s.,\\ \lim_{k\uparrow\infty} \sup_{t\in[0,1]} |Y_\pm^{(n_k)}(t) - Y_\pm(t)| &= 0 \quad a.s.\end{aligned}$$

Proof. (i) It holds that

$$\begin{aligned}&\overline{\lim_{n\uparrow\infty}}\, E\left[\sup_{t\in[0,1]} |\Gamma_+^{(n)}(t) - \Gamma_+(t)|\right]\\ &= \overline{\lim_{n\uparrow\infty}}\, E\left[\sup_{t\in[0,1]} \left|\int_0^t \left\{1_A\left(B(\tfrac{[2^n s+1]}{2^n})\right) - 1_A(B_s)\right\} ds\right|\right]\\ &\le \overline{\lim_{n\uparrow\infty}}\, E\left[\int_0^1 \left|1_A\left(B(\tfrac{[2^n s+1]}{2^n})\right) - 1_A(B_s)\right| ds\right]\\ &\le \int_0^1 ds\, \overline{\lim_{n\uparrow\infty}}\, E\left[\left|1_A\left(B(\tfrac{[2^n s+1]}{2^n})\right) - 1_A(B_s)\right|\right]\\ &= 0 \qquad \text{by Lemma 3.4.}\end{aligned}$$

(ii) We have

$$\begin{aligned}Y_+^{(n)}(t) &= \sum_{k=1}^{2^n} 1_A\left(b - Z(\tfrac{2^n-k}{2^n})\right)\\ &\quad \cdot\left\{Z\left((1-t)\vee \tfrac{(2^n-k)+1}{2^n}\right) - Z\left((1-t)\vee\tfrac{2^n-k}{2^n}\right)\right\}\\ &= \sum_{k=0}^{2^n-1} 1_A\left(b - Z(\tfrac{k}{2^n})\right)\\ &\quad \cdot\left\{Z\left((1-t)\vee \tfrac{k+1}{2^n}\right) - Z\left((1-t)\vee\tfrac{k}{2^n}\right)\right\} \quad \text{by reindexing}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{2^n-1} 1_A\left(b - Z(\tfrac{k}{2^n})\right)\left\{Z(\tfrac{k+1}{2^n}) - Z(\tfrac{k}{2^n})\right\} \\
&\quad - \sum_{k=0}^{2^n-1} 1_A\left(b - Z(\tfrac{k}{2^n})\right) \\
&\quad \cdot\left\{Z\left((1-t)\wedge\tfrac{k+1}{2^n}\right) - Z\left((1-t)\wedge\tfrac{k}{2^n}\right)\right\} \\
&= \int_0^1 1_A\left(b - Z(\tfrac{[2^n s]}{2^n})\right) dZ_s - \int_0^{1-t} 1_A\left(b - Z(\tfrac{[2^n s]}{2^n})\right) dZ_s \\
&= \int_{1-t}^1 1_A\left(b - Z(\tfrac{[2^n s]}{2^n})\right) dZ_s,
\end{aligned}
$$

where the third equality holds since

$$
Z\left((1-t)\vee\tfrac{k}{2^n}\right) + Z\left((1-t)\wedge\tfrac{k}{2^n}\right) = Z(1-t) + Z(\tfrac{k}{2^n}).
$$

Moreover, there exists a $\mathcal{G}$-Brownian motion $(W_t)_{t\in[0,1]}$ such that

$$
dZ_t = dW_t + \frac{b - Z_t}{1-t}dt,
$$

where $\mathcal{G} = (\mathcal{G}_t)_{t\in[0,1]}$ is the filtration generated by Z. It follows that

$$
\begin{aligned}
&E\left[\sup_{t\in[0,1]} |Y_+^{(n)}(t) - Y_+(t)|\right] \\
&= E\left[\sup_{t\in[0,1]}\left|\int_{1-t}^1 \left\{1_A\left(b - Z(\tfrac{[2^n s]}{2^n})\right) - 1_A(b - Z_s)\right\}dZ_s\right|\right] \\
&\le 2\,E\left[\sup_{t\in[0,1]}\left|\int_0^t \left\{1_A\left(b - Z(\tfrac{[2^n s]}{2^n})\right) - 1_A(b - Z_s)\right\}dZ_s\right|\right] \\
&\le 2\,E\left[\sup_{t\in[0,1]}\left|\int_0^t \left\{1_A\left(b - Z(\tfrac{[2^n s]}{2^n})\right) - 1_A(b - Z_s)\right\}dW_s\right|\right] \\
&\quad +2\,E\left[\sup_{t\in[0,1]}\left|\int_0^t \left\{1_A\left(b - Z(\tfrac{[2^n s]}{2^n})\right) - 1_A(b - Z_s)\right\}\frac{b - Z_s}{1-s}ds\right|\right] \\
&\le 2\,C\,E\left[\left\{\int_0^1 \left|1_A\left(b - Z(\tfrac{[2^n s]}{2^n})\right) - 1_A(b - Z_s)\right|^2 ds\right\}^{\frac{1}{2}}\right] \\
&\quad +2\,E\left[\int_0^1 \left|1_A\left(b - Z(\tfrac{[2^n s]}{2^n})\right) - 1_A(b - Z_s)\right|\frac{|\,b - Z_s\,|}{1-s}ds\right]
\end{aligned}
$$

$$= 2\,C\,E\left[\left\{\int_0^1 \left|1_A\left(B(\tfrac{[2^n s+1]}{2^n})\right) - 1_A(B_s)\right| ds\right\}^{\frac{1}{2}}\right]$$
$$+2\,E\left[\int_0^1 \left|1_A\left(B(\tfrac{[2^n s+1]}{2^n})\right) - 1_A(B_s)\right| \frac{|\,B_s\,|}{s} ds\right]$$

with C the constant appearing in the Burkholder-Davis-Gundy inequality. The same reasoning as in (i) then leads to the desired property. □

The following lemma is needed to prove Proposition 3.7 below.

Lemma 3.6. *Let*

$$\mathcal{S} \stackrel{\text{def}}{=} \left\{ (y,\gamma) \in C\big([0,1];\boldsymbol{R}^d\big) \times C\big([0,1];\boldsymbol{R}\big) \;\middle|\; \begin{array}{l} \gamma \text{ non-decreasing, } \gamma(0)=0,\ \gamma(1)\le 1, \\ y(s)=y(t) \text{ if } \gamma(s)=\gamma(t),\ s<t \end{array} \right\}$$

equipped with the metric induced by the sup norm. Define $\Phi : \mathcal{S} \to C([0,1];\,\boldsymbol{R}^d)$ *by*

$$\Phi(y,\gamma) \stackrel{\text{def}}{=} y(\gamma^{-1}(\cdot)),$$

where

$$\gamma^{-1}(t) \stackrel{\text{def}}{=} \inf\{\, s \in [0,1] \mid \gamma(s) \ge t \wedge \gamma(1)\,\}$$

for $t \in [0,1]$. *Then* Φ *is a continuous mapping.*

Proof. We divide the proof into two steps.

Step 1. We first show that

$$\forall (y,\gamma) \in \mathcal{S}, \quad \Phi(y,\gamma) \in C([0,1];\boldsymbol{R}^d),$$

i.e., for any sequence $(t_n)_{n=1}^{\infty} \subset [0,1]$ with $t_n \to t$

$$\lim_{n\uparrow\infty} \Phi(y,\gamma)(t_n) = \Phi(y,\gamma)(t).$$

We will prove this only for the case where $(t_n)_n$ is non-increasing: the other cases can be proved similarly.

Since we have assumed that $(t_n)_n$ is non-increasing, $\left(\gamma^{-1}(t_n)\right)_n$ is also non-increasing and so $\lim_{n\uparrow\infty} \gamma^{-1}(t_n)$ exists. In addition, it is easy to verify that

$$\begin{aligned} \gamma(\gamma^{-1}(t_n)) &= t_n \wedge \gamma(1), \\ \gamma(\gamma^{-1}(t)) &= t \wedge \gamma(1). \end{aligned}$$

Therefore,

$$\begin{aligned}\gamma(\lim_{n\uparrow\infty}\gamma^{-1}(t_n)) &= \lim_{n\uparrow\infty}\gamma(\gamma^{-1}(t_n))\\ &= \lim_{n\uparrow\infty}(t_n\wedge\gamma(1))\\ &= t\wedge\gamma(1)\\ &= \gamma(\gamma^{-1}(t))\end{aligned}$$

and thus, by definition of $\mathcal{S}$ we have

$$\begin{aligned}y(\gamma^{-1}(t)) &= y(\lim_{n\uparrow\infty}\gamma^{-1}(t_n))\\ &= \lim_{n\uparrow\infty}y(\gamma^{-1}(t_n)).\end{aligned}$$

Step 2. Next we prove that Φ is a continuous mapping. Let $((y_n,\gamma_n))_{n=1}^{\infty}\subset\mathcal{S}$ and $(y_\infty,\gamma_\infty)\in\mathcal{S}$ be such that

$$\lim_{n\uparrow\infty}(y_n,\gamma_n)=(y_\infty,\gamma_\infty)\quad\text{in }\mathcal{S}.$$

What we wish to show is:

$$\overline{\lim_{n\uparrow\infty}}\sup_{t\in[0,1]}|y_n(\gamma_n^{-1}(t))-y_\infty(\gamma_\infty^{-1}(t))|=0.$$

It is easy to see that

$$\sup_{t\in[0,1]}|(t\wedge\gamma_n(1))-(t\wedge\gamma_\infty(1))|\to 0\qquad(n\uparrow\infty)$$

and also

$$\begin{aligned}\sup_{t\in[0,1]}|(t\wedge\gamma_n(1))-\gamma_\infty(\gamma_n^{-1}(t))| &= \sup_{t\in[0,1]}\left|\gamma_n(\gamma_n^{-1}(t))-\gamma_\infty(\gamma_n^{-1}(t))\right|\\ &\le \sup_{t\in[0,1]}|\gamma_n(t)-\gamma_\infty(t)|\\ &\to 0\qquad(n\uparrow\infty),\end{aligned}$$

which combine to yield

$$\lim_{n\uparrow\infty}\sup_{t\in[0,1]}|(t\wedge\gamma_\infty(1))-\gamma_\infty(\gamma_n^{-1}(t))|=0. \tag{3.1}$$

Furthermore, since

$$\gamma_\infty(\gamma_\infty^{-1}(\gamma_\infty(t))) = \gamma_\infty(t), \qquad t \in [0,1],$$

we have, by definition of $\mathcal{S}$,

$$y_\infty(\gamma_\infty^{-1}(\gamma_\infty(t))) = y_\infty(t), \qquad t \in [0,1]. \tag{3.2}$$

It follows that

$$\begin{aligned}
&\sup_{t\in[0,1]} |y_n(\gamma_n^{-1}(t)) - y_\infty(\gamma_\infty^{-1}(t))| \\
&\quad \le \sup_{t\in[0,1]} |y_n(\gamma_n^{-1}(t)) - y_\infty(\gamma_n^{-1}(t))| \\
&\qquad + \sup_{t\in[0,1]} |y_\infty(\gamma_n^{-1}(t)) - y_\infty(\gamma_\infty^{-1}(t))| \\
&\quad \le \sup_{t\in[0,1]} |y_n(t) - y_\infty(t)| \\
&\qquad + \sup_{t\in[0,1]} \left| y_\infty\left(\gamma_\infty^{-1}\left(\gamma_\infty(\gamma_n^{-1}(t))\right)\right) - y_\infty(\gamma_\infty^{-1}(t))\right| \qquad \text{by (3.2)}
\end{aligned}$$

and therefore

$$\begin{aligned}
&\overline{\lim_{n\uparrow\infty}} \sup_{t\in[0,1]} |y_n(\gamma_n^{-1}(t)) - y_\infty(\gamma_\infty^{-1}(t))| \\
&\quad \le \overline{\lim_{n\uparrow\infty}} \sup_{t\in[0,1]} \left| y_\infty\left(\gamma_\infty^{-1}\left(\gamma_\infty(\gamma_n^{-1}(t))\right)\right) - y_\infty(\gamma_\infty^{-1}(t))\right| \\
&\quad = 0 \qquad \text{by (3.1)},
\end{aligned}$$

which completes the proof of Lemma 3.6. □

PROPOSITION 3.7. *(i)* $(B_\pm(t))_{t\in[0,1]}$ *have continuous paths a.s.* *(ii) For the subsequence* $(n_k)_{k=1}^\infty$ *in Proposition 3.5,*

$$\lim_{k\uparrow\infty} \sup_{t\in[0,1]} |B_\pm^{(n_k)}(t) - B_\pm(t)| = 0 \quad a.s.$$

PROOF. We see that, for almost all $\omega \in \Omega$ and all $n \in \boldsymbol{N}$,

$$\begin{aligned}
(Y_\pm^{(n)}(\omega,t),\, \Gamma_\pm^{(n)}(\omega,t))_{t\in[0,1]} &\in \mathcal{S}, \\
(Y_\pm(\omega,t),\, \Gamma_\pm(\omega,t))_{t\in[0,1]} &\in \mathcal{S}.
\end{aligned}$$

The desired properties then follow from Proposition 3.5 and Lemma 3.6. □

Proposition 3.8. *(i) $(\tilde{B}_t)_{t\in[0,1]}$ has continuous paths a.s.*
(ii) For the subsequence $(n_k)_k$ in Propositions 3.5 and 3.7 we have:

$$\lim_{k\uparrow\infty} \sup_{t\in[0,1]} |\tilde{B}_t^{(n_k)} - \tilde{B}_t| = 0 \quad a.s.$$

Proof. (i) The first assertion follows immediately from Proposition 3.7(i).

(ii) For the sake of brevity we introduce the following notations:

$$\begin{aligned} \alpha_n &\stackrel{\text{def}}{=} |\Gamma_+^{(n)}(1) - \Gamma_+(1)|; \\ \beta_n &\stackrel{\text{def}}{=} 1 - |\Gamma_+^{(n)}(1) - \Gamma_+(1)|. \end{aligned}$$

Then, for $\alpha_n \le t \le \Gamma_+^{(n)}(1)$ we have

$$\begin{aligned} &\left|\tilde{B}_t^{(n)} - \tilde{B}\left(t + \Gamma_+(1) - \Gamma_+^{(n)}(1)\right)\right| \\ &\quad = \left|\left\{B_+^{(n)}(\Gamma_+^{(n)}(1)) - B_+^{(n)}(\Gamma_+^{(n)}(1) - t)\right\}\right. \\ &\qquad \left. - \left\{B_+(\Gamma_+(1)) - B_+(\Gamma_+^{(n)}(1) - t)\right\}\right| \\ &\quad \le \left|B_+^{(n)}(\Gamma_+^{(n)}(1)) - B_+(\Gamma_+(1))\right| \\ &\qquad + \left|B_+^{(n)}(\Gamma_+^{(n)}(1) - t) - B_+(\Gamma_+^{(n)}(1) - t)\right| \\ &\quad \le \left|Y_+^{(n)}(1) - Y_+(1)\right| + \sup_{t\in[0,1]} \left|B_+^{(n)}(t) - B_+(t)\right|. \end{aligned}$$

Similarly, for $\Gamma_+^{(n)}(1) \le t \le \beta_n$,

$$\begin{aligned} \left|\tilde{B}_t^{(n)} - \tilde{B}\left(t + \Gamma_+(1) - \Gamma_+^{(n)}(1)\right)\right| \le \; &\left|Y_+^{(n)}(1) - Y_+(1)\right| \\ &+ \sup_{t\in[0,1]} \left|B_-^{(n)}(t) - B_-(t)\right|. \end{aligned}$$

It follows that

$$\sup_{\alpha_n \le t \le \beta_n} \left|\tilde{B}_t^{(n)} - \tilde{B}\left(t + \Gamma_+(1) - \Gamma_+^{(n)}(1)\right)\right|$$

$$\le \left|Y_+^{(n)}(1) - Y_+(1)\right| + \sup_{t\in[0,1]} \left|B_+^{(n)}(t) - B_+(t)\right|$$
$$+ \sup_{t\in[0,1]} \left|B_-^{(n)}(t) - B_-(t)\right| \quad \text{a.s.}$$

and hence by Propositions 3.5 and 3.7(ii)

$$\lim_{k\uparrow\infty} \sup_{\alpha_{n_k}\le t\le\beta_{n_k}} \left|\tilde{B}_t^{(n_k)} - \tilde{B}\left(t + \Gamma_+(1) - \Gamma_+^{(n_k)}(1)\right)\right| = 0 \quad \text{a.s.} \tag{3.3}$$

This implies

$$\begin{aligned}
&\overline{\lim_{k\uparrow\infty}} \sup_{t\in[0,1]} |\tilde{B}_t^{(n_k)} - \tilde{B}_t| \\
&\le \overline{\lim_{k\uparrow\infty}} \sup_{0\le t\le\alpha_{n_k}} |\tilde{B}_t^{(n_k)} - \tilde{B}_t| + \overline{\lim_{k\uparrow\infty}} \sup_{\beta_{n_k}\le t\le 1} |\tilde{B}_t^{(n_k)} - \tilde{B}_t| \\
&\quad + \overline{\lim_{k\uparrow\infty}} \sup_{\alpha_{n_k}\le t\le\beta_{n_k}} |\tilde{B}_t^{(n_k)} - \tilde{B}_t| \\
&= \overline{\lim_{k\uparrow\infty}} \sup_{\alpha_{n_k}\le t\le\beta_{n_k}} |\tilde{B}_t^{(n_k)} - \tilde{B}_t| \qquad \text{by Proposition 3.5} \\
&\le \overline{\lim_{k\uparrow\infty}} \sup_{\alpha_{n_k}\le t\le\beta_{n_k}} \left|\tilde{B}_t^{(n_k)} - \tilde{B}\left(t + \Gamma_+(1) - \Gamma_+^{(n_k)}(1)\right)\right| \\
&\quad + \overline{\lim_{k\uparrow\infty}} \sup_{\alpha_{n_k}\le t\le\beta_{n_k}} \left|\tilde{B}\left(t + \Gamma_+(1) - \Gamma_+^{(n_k)}(1)\right) - \tilde{B}_t\right| \\
&= 0 \quad \text{a.s.} \quad \text{by (3.3)} \ \square
\end{aligned}$$

We are now in a position to prove our main theorem.

Proof of Theorem 1.1. It is clear that

$$\lim_{n\uparrow\infty} \sup_{t\in[0,1]} \left|S_{[2^n t]}^{(n)} - B_t\right| = 0 \quad \text{a.s.}$$

Also, we have

$$\begin{aligned}
\sup_{t\in[0,1]} \left|\tilde{S}_{[2^n t]}^{(n)} - \tilde{B}_t\right| &= \sup_{t\in[0,1]} \left|\tilde{B}^{(n)}(\tfrac{[2^n t]}{2^n}) - \tilde{B}_t\right| \quad \text{by Proposition 3.3} \\
&\le \sup_{t\in[0,1]} \left|\tilde{B}^{(n)}(\tfrac{[2^n t]}{2^n}) - \tilde{B}(\tfrac{[2^n t]}{2^n})\right| + \sup_{t\in[0,1]} \left|\tilde{B}(\tfrac{[2^n t]}{2^n}) - \tilde{B}_t\right| \\
&\le \sup_{t\in[0,1]} \left|\tilde{B}_t^{(n)} - \tilde{B}_t\right| + \sup_{t\in[0,1]} \left|\tilde{B}(\tfrac{[2^n t]}{2^n}) - \tilde{B}_t\right| \quad \text{a.s.,}
\end{aligned}$$

which in turn implies that

$$\lim_{k\uparrow\infty}\sup_{t\in[0,1]}\left|\tilde{S}^{(n_k)}_{[2^{n_k}t]}-\tilde{B}_t\right|=0\quad\text{a.s.}$$

with $(n_k)_k$ the subsequence in Proposition 3.8. The argument in Section 2 shows

$$\forall n\in\boldsymbol{N},\qquad \left(\tilde{S}^{(n)}_{[2^n t]}\right)_{t\in[0,1]}\overset{(\mathrm{d})}{=}\left(S^{(n)}_{[2^n t]}\right)_{t\in[0,1]},$$

and therefore, for any $m\in\boldsymbol{N}$ and $0\le t_1<t_2<\dots t_m\le 1$,

$$\left(\tilde{B}_{t_1},\tilde{B}_{t_2},\dots,\tilde{B}_{t_m}\right)\overset{(\mathrm{d})}{=}\left(B_{t_1},B_{t_2},\dots,B_{t_m}\right).$$

This and the path continuity of $(\tilde{B}_t)_{t\in[0,1]}$ (see Proposition 3.8(i)) complete the proof. □

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