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# *On $L^1$-Stability of Stationary Navier-Stokes Flows in $\mathbb{R}^n$*

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*Dedicated to the memory of Professor Peter Hess*

**Abstract.** Stability of stationary Navier-Stokes flows in $\mathbb{R}^n$, $n \geq 3$, is discussed in the function space $\boldsymbol{L}^1$ or $\boldsymbol{H}^1$ (Hardy space). It is shown that a stationary flow $\boldsymbol{w}$ is stable in $\boldsymbol{H}^1$ (resp. $\boldsymbol{L}^1$) if $\sup|x| \cdot |\boldsymbol{w}(x)| + \sup|x|^2|\nabla \boldsymbol{w}(x)|$ (resp. $\|\boldsymbol{w}\|_{(n,1)} + \|\nabla \boldsymbol{w}\|_{(n/2,1)}$) is small. Explicit decay rates of the form $O(t^{-\beta/2})$, $0 < \beta \leq 1$, are deduced for perturbations under additional assumptions on $\boldsymbol{w}$ and on initial data. The proofs of the results heavily rely on the theory of Hardy spaces $\boldsymbol{H}^p$ $(0 < p \leq 1)$ of Fefferman and Stein.

## 1. Introduction

This paper continues our previous study in [17] on the long time behavior of solutions to the incompressible Navier–Stokes equations on the entire space $\mathbb{R}^n$. We are concerned with stability properties of solutions to the stationary problem:

$$\text{(S)} \qquad \begin{aligned} -\Delta \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{w} &= \nabla \cdot \boldsymbol{F} - \nabla p \quad (x \in \mathbb{R}^n) \\ \nabla \cdot \boldsymbol{w} &= 0 \qquad\qquad\qquad (x \in \mathbb{R}^n) \\ \lim_{|x| \to \infty} \boldsymbol{w}(x) &= 0. \end{aligned}$$

Here, $\boldsymbol{w} = (w_1, \cdots, w_n)$ and $p$ denote, respectively, unknown velocity and pressure; the given external force is assumed to be of the form

$$\nabla \cdot \boldsymbol{F} = (f_1, \cdots, f_n), \qquad f_j = \sum_{k=1}^{n} \partial_k F_{kj}\,;$$

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and we use the notation:

$$\nabla = (\partial_1, \cdots, \partial_n), \qquad \partial_j = \partial/\partial x_j \quad (j = 1, \cdots, n),$$

$$\Delta \boldsymbol{w} = \sum_{j=1}^{n} \partial_j^2 \boldsymbol{w}, \qquad \boldsymbol{w} \cdot \nabla \boldsymbol{w} = \sum_{j=1}^{n} w_j \partial_j \boldsymbol{w}, \qquad \nabla \cdot \boldsymbol{w} = \sum_{j=1}^{n} \partial_j w_j.$$

When $n \geq 3$, we proved in [17] that problem (S) admits a smooth solution $\boldsymbol{w}$ satisfying

$$\boldsymbol{w} \in \boldsymbol{L}^n(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n), \qquad \nabla \boldsymbol{w} \in \boldsymbol{L}^{n/2}(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n), \tag{1.1}$$

provided that the tensor $\boldsymbol{F} = (F_{jk})$ is smooth and suitably small in $\boldsymbol{L}^{n/2}$. On the other hand, it is shown in [4, 19] that if $\boldsymbol{F}$ is smooth and satisfies

$$|F_{jk}| \leq C/(1+|x|)^{\mu-1}, \qquad |\nabla F_{jk}| \leq C/(1+|x|)^{\mu} \qquad \text{for some} \quad \mu \geq 3,$$

with a suitably small constant $C > 0$, then problem (S) has a smooth solution $\boldsymbol{w}$ such that

$$\begin{aligned} &|\boldsymbol{w}| \leq C/(1+|x|)^{\ell-2}, \\ &|\nabla \boldsymbol{w}| \leq C/(1+|x|)^{\ell-1} \qquad \text{with} \quad \ell = \min(n, \mu). \end{aligned} \tag{1.2}$$

A property closely related to (1.1) and (1.2) is :

$$\boldsymbol{w} \in \boldsymbol{L}^n_w(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n), \qquad \nabla \boldsymbol{w} \in \boldsymbol{L}^{n/2}_w(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n), \tag{1.2$'$}$$

where $L^p_w$ is the weak $L^p$ space [1, 28]. Solutions satisfying (1.2′) are constructed in [14].

In the case of the exterior problem, stability properties in various $\boldsymbol{L}^r$ spaces were discussed in [2, 4, 12, 14] for stationary flows $\boldsymbol{w}$ satisfying condition (1.1), (1.2) or (1.2′). In particular, it is proved in [2, 4] that if $1 < r \leq n/(n-1)$, then any initial perturbation given in $\boldsymbol{L}^r \cap \boldsymbol{L}^2$ evolves in time in all of the spaces $\boldsymbol{L}^q$, $r \leq q \leq 2$, with some definite decay rates if $r < q \leq 2$. It is possible to show the same result in the case of the entire space $\mathbb{R}^n$, $n \geq 3$.

The purpose of this paper is to discuss the stability properties of stationary flows $\boldsymbol{w}$ satisfying (1.1) or (1.2) on the entire space $\mathbb{R}^n$ and to examine what happens in the limiting case of $\boldsymbol{L}^1$. We proved in [17] that the rest state $\boldsymbol{w} = 0$ is stable under perturbations belonging to the Lebesgue space

$\boldsymbol{L}^1(\mathbb{R}^n)$ as well as the Hardy space $\boldsymbol{H}^1(\mathbb{R}^n)$. In this paper we extend these stability results to the case $\boldsymbol{w} \neq 0$, combining the methods of [2, 4, 17] and invoking the techniques given in [5] for manipulating functions in Hardy spaces.

We first consider a stationary flow $\boldsymbol{w}$ in $\mathbb{R}^n$ satisfying

$$(1.3) \qquad |\boldsymbol{w}| \leq C/(1+|x|), \qquad |\nabla \boldsymbol{w}| \leq C/(1+|x|)^2.$$

In order to estimate the size of $\boldsymbol{w}$, we employ the norms

$$(1.4) \qquad \|\boldsymbol{w}\| = \sup(|x| \cdot |\boldsymbol{w}(x)|), \qquad \|\nabla \boldsymbol{w}\| = \sup(|x|^2 |\nabla \boldsymbol{w}(x)|).$$

In Section 4 we show that a stationary flow $\boldsymbol{w}$ is stable under perturbations in the Hardy space $\boldsymbol{H}^1(\mathbb{R}^n)$ provided that $\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\|$ is small enough. Note that (1.2) implies (1.3); thus our stability result covers the class of stationary flows satisfying (1.2). The same type of result can also be obtained for flows satisfying (1.1) or (1.2′). Since this latter case is treated in almost the same way, we omit the details.

If $n \geq 4$ and $\mu > 3$, then (1.2) implies

$$(1.5) \quad \boldsymbol{w} \in \boldsymbol{L}^{(n,1)}(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n), \qquad \nabla \boldsymbol{w} \in \boldsymbol{L}^{(n/2,1)}(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n),$$

where $L^{(p,q)}$ is the Lorentz space [1, 28]. It is evident that conditon (1.5) is more stringent than conditon (1.1) or (1.3). In this connection, we show in Section 2 that if $n \geq 3$ and if $\boldsymbol{F}$ is smooth and belongs to $\boldsymbol{L}^1(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n)$, then (1.1) implies

$$(1.6) \quad \boldsymbol{w} \in \boldsymbol{L}_w^{n/(n-1)}(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n), \qquad \nabla \boldsymbol{w} \in \boldsymbol{L}_w^1(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n)\,;$$

and so in this case the flows $\boldsymbol{w}$ with property (1.1) satisfy (1.5). In Section 5 we shall deal with the stability problem of stationary flows $\boldsymbol{w}$ satisfying (1.5) and show that if the Lorentz norm $\|\boldsymbol{w}\|_{(n,1)} + \|\nabla \boldsymbol{w}\|_{(n/2,1)}$ is small enough, then $\boldsymbol{w}$ is stable under perturbations belonging to the Lebesgue space $\boldsymbol{L}^1(\mathbb{R}^n)$.

When $\boldsymbol{w} = 0$, we deduced in [17] explicit time-decay rates of the form $O(t^{-\beta/2})$, $0 < \beta \leq 1$, of $\boldsymbol{H}^1$ and $\boldsymbol{L}^1$ norms of the weak solutions corresponding to a specific class of initial data. In Section 6 we discuss the same problem in the case $\boldsymbol{w} \neq 0$ and deduce decay rates of the form $O(t^{-\beta/2})$,

$0 < \beta < 1$. Contrary to the case discussed in [17], it seems impossible to deduce the decay rate $O(t^{-1/2})$ if we assume merely (1.3) or (1.5).

To discuss the stability of stationary solutions, one needs careful analysis of the corresponding linearized problem. The bulk of this task is carried out in Section 3, by employing the theory of Hardy spaces as developed in [7, 26]. It is now well known that the Hardy space $\boldsymbol{H}^1(\mathbb{R}^n)$ is a good substitute for the Lebesgue space $\boldsymbol{L}^1(\mathbb{R}^n)$ with regard to the boundedness properties of various potential operators and singular integrals, and so the standard method of functional analysis can be effectively applied. Moreover, it is shown in [5] that the nonlinear term $\boldsymbol{w} \cdot \nabla \boldsymbol{w}$ of the Navier–Stokes equations belongs to the Hardy spaces $\boldsymbol{H}^p(\mathbb{R}^n)$, $n/(n+1) < p \le 1$, under suitable assumptions on velocity fields $\boldsymbol{w}$; and this suggests the possibility of effectively applying Hardy space theory to the mathematical treatment of the Navier–Stokes equations. We deal in Section 3 with a stationary flow $\boldsymbol{w}$ satisfying (1.3) and show that if the norm $\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\|$ is suitably small, then any solution of the linearized equation tends to zero in the Hardy space $\boldsymbol{H}^1_\sigma$ of solenoidal vector fields. We further deduce some rates of time-decay for the solutions in $\boldsymbol{H}^1_\sigma$, assuming in addition that the initial data are in the Hardy space $\boldsymbol{H}^p(\mathbb{R}^n)$, $n/(n+1) < p < 1$. Passing to the adjoint equation, we then find that the solutions of the adjoint equation decay in time with definite rates in the homogeneous Hölder spaces $C^\beta$, $0 < \beta < 1$, if the corresponding initial data belong to the space BMO of functions of bounded mean oscillation. These decay properties will then be applied in Sections 4 and 6 to show that a stationary flow $\boldsymbol{w}$ is stable under $\boldsymbol{H}^1$-perturbations provided that the norm $\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\|$ is suitably small, and that the perturbations decay in time in $\boldsymbol{H}^1_\sigma$ with definite rates if the initial data satisfy some additional condition.

To discuss the stability under perturbations belonging to $\boldsymbol{L}^1$, we have to assume (1.5) instead of (1.3). Assuming (1.5), we discuss the linearized problem in Section 5, prior to the discussion on stability for the nonlinear problem. Applying again the Hardy space techniques, we show that if the Lorentz norm $\|\boldsymbol{w}\|_{(n,1)} + \|\nabla \boldsymbol{w}\|_{(n/2,1)}$ is small, then any solutions of the linearized equation tend to zero in the Lebesgue space $\boldsymbol{L}^1_\sigma$ of solenoidal vector fields. Since condition (1.5) is stronger than (1.3), all the results obtained in Section 3 are also valid in this case; thus we can apply the same Hardy space techniques as in Section 3 to deduce definite decay rates for solutions of the adjoint of the linearized problem, and this decay result is then applied to get the desired stability results for the nonlinear problem in the space $\boldsymbol{L}^1_\sigma$.

As mentioned above, we cannot deduce an $L^1$ decay rate of the form $O(t^{-1/2})$ under the assumption (1.3) or (1.5). To see the situation more closely, we study in Section 7 the perturbations of stationary flows satisfying

$$\|\boldsymbol{w}\| < \frac{n-2}{2}, \qquad \boldsymbol{w} \in \boldsymbol{L}^2(\mathbb{R}^n). \tag{1.7}$$

We note that such flows $\boldsymbol{w}$ exist. A simple example is given by flows satisfying (1.2) with $\ell = n$ and $n \geq 5$. On the other hand, condition (1.6) implies $\boldsymbol{w} \in \boldsymbol{L}^2$, while the existence proof of [4, 19] gives stationary flows satisfying (1.3). It is easy to see that these two kinds of flows coincide provided that $\boldsymbol{F}$ is smooth, compactly supported and small in an appropriate sense. In Section 7 we shall show that if $\boldsymbol{w}$ satisfies (1.7), then there exist weak solutions of the perturbation equations which decay in $\boldsymbol{L}^1_\sigma$ like $t^{-1/2}$. It should be noticed that in this section we impose no assumptions on the derivatives $\nabla\boldsymbol{w}$, although we invoke the very strong condition: $\boldsymbol{w} \in \boldsymbol{L}^2$. The main tools of the proof are the bootstrap argument developed in [2, 3, 4, 11, 22, 23] for deducing $L^2$ decay rates and the fact that the first derivatives of the heat kernel belong to the Hardy space $\boldsymbol{H}^1(\mathbb{R}^n)$, with norm $\leq Ct^{-1/2}$. Our $L^2$ decay result given in Section 7 is just a part of the more general result of Grunau [9]. Contrary to the treatment in the preceding sections, we need no detailed analysis of the linearized operator. Section 7 is merely intended to give a special case where perturbations decay in $\boldsymbol{L}^1_\sigma$ like $t^{-1/2}$, and no generality is aimed at on conditions for $\boldsymbol{w}$ which ensure the same $L^1$ decay result.

## 2. A Result on Decay Properties of Stationary Flows

Hereafter $L^p_w(\mathbb{R}^n)$, $0 < p < \infty$, denotes the space of measurable functions $f$ on $\mathbb{R}^n$ satisfying

$$\|f\|_{p,w} \equiv \sup_{t>0} t|\{x : |f(x)| > t\}|^{1/p} < +\infty$$

with $|E|$ the $n$-dimensional Lebesgue measure of a measurable set $E$, and $\boldsymbol{L}^p_w$ denotes the $L^p_w$ space of vector-valued functions.

As stated in the Introduction, this section estabilshes the following

Theorem 2.1. *Let* $\boldsymbol{F} = (F_{jk})_{j,k=1}^n$ *be smooth, bounded and belong to* $\boldsymbol{L}^1(\mathbb{R}^n)$*, and let* $\boldsymbol{w}$ *be a solution of problem* (S) *satisfying* (1.1)*, i.e.,*

$$\boldsymbol{w} \in \boldsymbol{L}^n(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n), \qquad \nabla \boldsymbol{w} \in \boldsymbol{L}^{n/2}(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n). \tag{1.1}$$

*Then* $\boldsymbol{w}$ *satisfies* (1.6)*, i.e.,*

$$\boldsymbol{w} \in \boldsymbol{L}_w^{n/(n-1)}(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n), \qquad \nabla \boldsymbol{w} \in \boldsymbol{L}_w^1(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n). \tag{1.6}$$

Remarks. (i) The existence of $\boldsymbol{w}$ satisfying (1.1) is shown in [17] under the assumption that the function $\boldsymbol{F}$ is small in $\boldsymbol{L}^{n/2}$, $n \geq 3$. It is also proved in [17] that if $n = 3$, then under the assumption of Theorem 2.1 stated above, (1.1) implies (1.6). In what follows we show that the argument of [17] can be applied in all space dimensions $n \geq 3$.

(ii) The method of proof stated below stems from the recent studies on the exterior stationary problem in $\mathbb{R}^n$, $n \geq 3$, as given in [4, 13]. When $n = 3$, the properties (1.2) and (1.2′) are known to be optimal for exterior stationary flows; and moreover, one can show that property (1.1) implies (1.6). This last result can be proved in the same way as stated below, and the result itself improves [4, Theorem 2.5 (ii)]. The details are given in [18].

Proof of Theorem 2.1. We first note that (1.1) implies that the velocity $\boldsymbol{w}$ and the associated pressure $p$ are smooth on $\mathbb{R}^n$. This is easily deduced by the standard bootstrap argument based on the a-priori estimates for the linear Stokes system. Moreover, one also sees immediately that if (1.1) is valid, then $\boldsymbol{w}$ is written as the convolution integral :

$$\boldsymbol{w} = E \cdot (\nabla \cdot \boldsymbol{F} - \boldsymbol{w} \cdot \nabla \boldsymbol{w}) = (\nabla E) \cdot (\boldsymbol{F} - \boldsymbol{w} \otimes \boldsymbol{w}) \tag{2.1}$$

by means of the Stokes fundamental solution tensor $E = (E_{jk})_{j,k=1}^n$ with components

$$E_{jk}(x) = \frac{1}{2\omega_n}\left(\frac{\delta_{jk}}{n-2}|x|^{2-n} + \frac{x_j x_k}{|x|^n}\right). \tag{2.2}$$

Here $\omega_n$ is the area of the unit sphere $\{|x|=1\}$. An associated pressure $p$ is given by

$$p = Q \cdot (\nabla \cdot \boldsymbol{F} - \boldsymbol{w} \cdot \nabla \boldsymbol{w})$$

in terms of the vector function $Q = (Q_j)_{j=1}^n$ such that

$$Q_j(x) = \frac{x_j}{\omega_n |x|^n}. \tag{2.3}$$

More generally, for any fixed $R > 0$, Green's theorem applies to the domain $\Omega_R = \{|x| > R\}$ under our assumptions on $\boldsymbol{w}$, and we get for $x \in \Omega_R$,

$$\boldsymbol{w}(x) = (\nabla E) \cdot (\widetilde{\boldsymbol{F}} - \widetilde{\boldsymbol{w}} \otimes \widetilde{\boldsymbol{w}}) + \boldsymbol{w}_0, \tag{2.4}$$

where $\widetilde{\boldsymbol{F}} = \boldsymbol{F}$ in $\Omega_R$ and $\widetilde{\boldsymbol{F}} = 0$ outside $\Omega_R$, and

$$\boldsymbol{w}_0 = \int_{|y|=R} E \cdot (\boldsymbol{T}[\boldsymbol{w}, p] - \boldsymbol{w} \otimes \boldsymbol{w} + \boldsymbol{F}) \cdot \nu dS_y + \int_{|y|=R} \boldsymbol{w} \cdot \boldsymbol{T}[E, Q] \cdot \nu dS_y.$$

Here, $\nu$ is the unit outward normal to $\partial\Omega_R = \{|x| = R\}$, and $\boldsymbol{T}[\boldsymbol{w}, p] = (T_{jk}[\boldsymbol{w}, p])_{j,k=1}^n$, with

$$T_{jk}[\boldsymbol{w}, p] = -\delta_{jk} p + \partial_j w_k + \partial_k w_j.$$

Now, the first equation of (S) is written as

$$\nabla \cdot (\boldsymbol{T}[\boldsymbol{w}, p] - \boldsymbol{w} \otimes \boldsymbol{w} + \boldsymbol{F}) = 0$$

and so applying the divergence theorem in the bounded domain $\{|x| < R\}$ yields

$$\int_{|y|=R} (\boldsymbol{T}[\boldsymbol{w}, p] - \boldsymbol{w} \otimes \boldsymbol{w} + \boldsymbol{F}) \cdot \nu dS_y = 0. \tag{2.5}$$

Therefore, $\boldsymbol{w}_0$ is written as

$$\begin{aligned} \boldsymbol{w}_0 \quad &= \quad \int_{|y|=R} \widetilde{E} \cdot (\boldsymbol{T}[\boldsymbol{w}, p] - \boldsymbol{w} \otimes \boldsymbol{w} + \boldsymbol{F}) \cdot \nu dS_y \\ &\quad + \int_{|y|=R} \boldsymbol{w} \cdot \boldsymbol{T}[E, Q] \cdot \nu dS_y, \end{aligned} \tag{2.6}$$

where

$$\widetilde{E}(x, y) = E(x - y) - E(x) = \int_0^1 \frac{d}{dt} E(x - ty) dt.$$

Since $|\widetilde{E}(x,y)| \le C|x|^{1-n}$ and $|\nabla_x \widetilde{E}(x,y)| \le C|x|^{-n}$ for large $|x|$ uniformly in $y \in \partial\Omega_R$, and since the same estimates hold also for $\boldsymbol{T}[E,Q]$ and $\nabla_x \boldsymbol{T}[E,Q]$, it follows from (2.6) that

$$|\boldsymbol{w}_0(x)| = O(|x|^{1-n}) \quad \text{and} \quad |\nabla \boldsymbol{w}_0(x)| = O(|x|^{-n}) \qquad \text{as } |x| \to \infty.$$

Hence,

$$\text{(2.7)} \quad \boldsymbol{w}_0 \in \boldsymbol{L}_w^{n/(n-1)}(\Omega_R) \cap \boldsymbol{L}^\infty(\Omega_R), \qquad \nabla \boldsymbol{w}_0 \in \boldsymbol{L}_w^1(\Omega_R) \cap \boldsymbol{L}^\infty(\Omega_R).$$

Now, fix $n/(n-1) < r < n$, take a large $R > 0$ to be fixed later, and consider the linear iteration scheme in $\Omega_R$ :

$$\text{(2.8)} \quad \boldsymbol{v}_{k+1} = (\nabla E)\cdot(\widetilde{\boldsymbol{F}} - \widetilde{\boldsymbol{w}} \otimes \widetilde{\boldsymbol{v}_k}) + \boldsymbol{w}_0, \qquad \boldsymbol{v}_0 = \boldsymbol{w}_0, \qquad (k = 0,1,2,\ldots).$$

Applying the Hardy–Littlewood–Sobolev inequality [25, 26], we get from (2.7) and (2.8)

$$\begin{aligned} \|\boldsymbol{v}_{k+1}\|_{r,\Omega_R} &\le \|\boldsymbol{w}_0\|_{r,\Omega_R} + C_{r,n}(\|\boldsymbol{F}\|_{rn/(r+n),\Omega_R} + \|\boldsymbol{w}\|_{n,\Omega_R}\|\boldsymbol{v}_k\|_{r,\Omega_R}), \\ \|\boldsymbol{v}_{k+1}\|_{n,\Omega_R} &\le \|\boldsymbol{w}_0\|_{n,\Omega_R} + C_{n,n}(\|\boldsymbol{F}\|_{n/2,\Omega_R} + \|\boldsymbol{w}\|_{n,\Omega_R}\|\boldsymbol{v}_k\|_{n,\Omega_R}), \end{aligned}$$

and

$$\begin{aligned} \|\boldsymbol{v}_{k+1} - \boldsymbol{v}_k\|_{r,\Omega_R} &\le C_{r,n}\|\boldsymbol{w}\|_{n,\Omega_R}\|\boldsymbol{v}_k - \boldsymbol{v}_{k-1}\|_{r,\Omega_R}, \\ \|\boldsymbol{v}_{k+1} - \boldsymbol{v}_k\|_{n,\Omega_R} &\le C_{n,n}\|\boldsymbol{w}\|_{n,\Omega_R}\|\boldsymbol{v}_k - \boldsymbol{v}_{k-1}\|_{n,\Omega_R}. \end{aligned}$$

Here $\|\cdot\|_{q,\Omega_R}$ is the norm of $\boldsymbol{L}^q(\Omega_R)$. Since $C_{r,n}$ and $C_{n,n}$ are independent of $R > 0$, we can take $R > 0$ satisfying

$$C_{r,n}\|\boldsymbol{w}\|_{n,\Omega_R} \le 1/2 \quad \text{and} \quad C_{n,n}\|\boldsymbol{w}\|_{n,\Omega_R} \le 1/2,$$

and see that the sequence $\{\boldsymbol{v}_k\}$ converges in $\boldsymbol{L}^r(\Omega_R) \cap \boldsymbol{L}^n(\Omega_R)$ to a function $\boldsymbol{v}$ satisfying

$$\boldsymbol{v} = (\nabla E)\cdot(\widetilde{\boldsymbol{F}} - \widetilde{\boldsymbol{w}} \otimes \widetilde{\boldsymbol{v}}) + \boldsymbol{w}_0 \qquad \text{in } \Omega_R.$$

Subtracting this from (2.4), we have

$$\|\boldsymbol{w} - \boldsymbol{v}\|_{n,\Omega_R} \le \frac{1}{2}\|\boldsymbol{w} - \boldsymbol{v}\|_{n,\Omega_R},$$

and so $\boldsymbol{v} = \boldsymbol{w}$ in $\Omega_R$. Since $n/(n-1) < r < n$ was arbitrary and since $\boldsymbol{w}$ is bounded in $\mathbb{R}^n$, we conclude that $\boldsymbol{w}$ is in $\boldsymbol{L}^r(\mathbb{R}^n)$ for all $r$ with $n/(n-1) < r \leq \infty$.

Now, we see that $\boldsymbol{w} \otimes \boldsymbol{w} \in \boldsymbol{L}^1(\mathbb{R}^n)$ ; so the Hardy–Littlewood–Sobolev inequality as applied to $L^1$ functions [25] gives

$$(\nabla E) \cdot (\boldsymbol{F} - \boldsymbol{w} \otimes \boldsymbol{w}) \in \boldsymbol{L}_w^{n/(n-1)}(\mathbb{R}^n).$$

Furthermore, since $\nabla^2 E$ is a Calderón–Zygmund kernel [25], it follows that

$$(\nabla^2 E) \cdot (\boldsymbol{F} - \boldsymbol{w} \otimes \boldsymbol{w}) \in \boldsymbol{L}_w^1(\mathbb{R}^n).$$

Hence, $\boldsymbol{w} \in \boldsymbol{L}_w^{n/(n-1)}(\mathbb{R}^n)$ and $\nabla \boldsymbol{w} \in \boldsymbol{L}_w^1(\mathbb{R}^n)$ by (2.4) and (2.7). The proof is complete. $\square$

## 3. The Linearized Problem

We begin with an abstract formulation of our perturbation problem :

$$\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{w} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{w} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \Delta \boldsymbol{u} - \nabla p \quad && (x \in \mathbb{R}^n, \quad t > 0) \\ &\nabla \cdot \boldsymbol{u} = 0 && (x \in \mathbb{R}^n, \quad t \geq 0) \\ &\boldsymbol{u}|_{t=0} = \boldsymbol{a}, \qquad \lim_{|x| \to \infty} \boldsymbol{u} = 0. \end{aligned} \tag{3.1}$$

Let $1 < r < \infty$. Using the Helmholtz decomposition

$$\boldsymbol{L}^r(\mathbb{R}^n) = \boldsymbol{L}_\sigma^r \oplus \boldsymbol{L}_\pi^r,$$

with

$$\boldsymbol{L}_\sigma^r = \{\boldsymbol{v} \in \boldsymbol{L}^r(\mathbb{R}^n) \ : \ \nabla \cdot \boldsymbol{v} = 0\}, \quad \boldsymbol{L}_\pi^r = \{\nabla p \in \boldsymbol{L}^r(\mathbb{R}^n) \ : \ p \in L_{\mathrm{loc}}^r(\mathbb{R}^n)\},$$

and the associated bounded projector $P = P_r$ onto $\boldsymbol{L}_\sigma^r$, we introduce the operators

$$A\boldsymbol{u} = -\Delta \boldsymbol{u} \quad \text{and} \quad B\boldsymbol{u} = P(\boldsymbol{w} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{w}) \qquad \text{in } \boldsymbol{L}_\sigma^r.$$

The formal adjoint $B^*$ of $B$ is given by

$$B^* \boldsymbol{u} = -P(\boldsymbol{w} \cdot \nabla \boldsymbol{u} + (\nabla \boldsymbol{u}) \cdot \boldsymbol{w}),$$

where $(\nabla \boldsymbol{u}) \cdot \boldsymbol{w}$ is the vector field with components

$$((\nabla \boldsymbol{u}) \cdot \boldsymbol{w})_j = (\partial_j \boldsymbol{u}) \cdot \boldsymbol{w} \qquad (j = 1, \cdots, n).$$

Note that $B^* \boldsymbol{u}$ does not involve the derivatives $\nabla \boldsymbol{w}$ of $\boldsymbol{w}$.

We next recall that the projector $P$ is written in the form

$$P\boldsymbol{u} = (I + R \otimes R) \cdot \boldsymbol{u} \tag{3.2}$$

in terms of the Riesz transforms $R = (R_1, \cdots, R_n)$ defined via the Fourier transform as

$$\widehat{R_j f}(\xi) \equiv \int e^{-ix\cdot\xi}(R_j f)(x)dx = \frac{i\xi_j}{|\xi|}\widehat{f}(\xi) \qquad (i = \sqrt{-1}, \quad j = 1, \cdots, n).$$

Since each $R_j$ is bounded on the Hardy space $\boldsymbol{H}^1(\mathbb{R}^n)$ (see [25]), in view of (3.2) we can deduce the Helmholtz decomposition of $\boldsymbol{H}^1(\mathbb{R}^n)$ :

$$\boldsymbol{H}^1(\mathbb{R}^n) = \boldsymbol{H}^1_\sigma \oplus \boldsymbol{H}^1_\pi \tag{3.3}$$

with

$$\boldsymbol{H}^1_\sigma = \{\boldsymbol{v} \in \boldsymbol{H}^1(\mathbb{R}^n) \ : \ \nabla \cdot \boldsymbol{v} = 0\}, \quad \boldsymbol{H}^1_\pi = \{\nabla p \ : \ p \in L^{(n/(n-1),1)}(\mathbb{R}^n)\}.$$

Here and hereafter $L^{(p,q)}$ is the Lorentz space [1]. We note that the space $\boldsymbol{C}^\infty_{0,\sigma}(\mathbb{R}^n)$ of compactly supported smooth solenoidal vector fields is dense in $\boldsymbol{L}^r_\sigma$ and in $\boldsymbol{H}^1_\sigma$. See [17] for the case of $\boldsymbol{H}^1_\sigma$ ; the case of $\boldsymbol{L}^r_\sigma$ is treated similarly.

Using these spaces and the linear operator

$$L = A + B, \tag{3.4}$$

we can formally write equation (3.1) in the form

$$\frac{d\boldsymbol{u}}{dt} + L\boldsymbol{u} + P(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = 0 \quad (t > 0), \qquad \boldsymbol{u}(0) = \boldsymbol{a}. \tag{3.5}$$

Problem (3.5) is then formally transformed into the integral equation

$$\boldsymbol{u}(t) = e^{-tL}\boldsymbol{a} - \int_0^t e^{-(t-\tau)L} P(\boldsymbol{u} \cdot \nabla \boldsymbol{u})(\tau) d\tau, \tag{3.6}$$

by employing the semigroup notation. As will be shown below, we can define the bounded analytic $C_0$ semigroup $\{e^{-tL}\}_{t\geq 0}$ generated by $-L$ in the spaces $\boldsymbol{L}^r_\sigma$, $1 < r < \infty$, and $\boldsymbol{H}^1_\sigma$.

Consider next the heat semigroup

$$(e^{-tA}\boldsymbol{a})(x) = \int E_t(x-y)\boldsymbol{a}(y)dy, \qquad E_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}.$$

As is well known, $\{e^{-tA}\}_{t\geq 0}$ is bounded-analytic in $\boldsymbol{L}^r_\sigma$, and for any fixed $\omega$ with $0 < \omega < \pi/2$ there is a constant $C_{r,\omega} > 0$ such that, for $j = 0, 1, 2$,

$$\|\nabla^j(\lambda + A)^{-1}\boldsymbol{u}\|_r \leq C_{r,\omega}\|\boldsymbol{u}\|_r/|\lambda|^{1-j/2} \qquad (|\arg\lambda| \leq \pi - \omega), \tag{3.7}$$

where $\|\cdot\|_r$ is the $L^r$-norm. Furthermore, we know (see [17]) that $\{e^{-tA}\}_{t\geq 0}$ is also bounded-analytic in $\boldsymbol{H}^1_\sigma$ and there is a constant $C_\omega > 0$ such that, for $j = 0, 1, 2$,

$$\|\nabla^j(\lambda + A)^{-1}\boldsymbol{u}\|_{H^1} \leq C_\omega\|\boldsymbol{u}\|_{H^1}/|\lambda|^{1-j/2} \qquad (|\arg\lambda| \leq \pi - \omega). \tag{3.8}$$

LEMMA 3.1. *Let $\boldsymbol{w}$ be a stationary flow satisfying* (1.3), *and so the norms $\|\boldsymbol{w}\|$ and $\|\nabla\boldsymbol{w}\|$ defined in* (1.4) *are finite.*

(i) *For $1 < r < n$ and $0 < \omega < \pi/2$ there is a constant $C = C_{r,\omega} > 0$ such that for $j = 0, 1$,*

$$\|\nabla^j(\lambda + A)^{-1}B\boldsymbol{u}\|_r \leq C\|\boldsymbol{w}\|\cdot\|\nabla\boldsymbol{u}\|_r/|\lambda|^{(1-j)/2} \qquad (|\arg\lambda| \leq \pi - \omega). \tag{3.9}$$

(ii) *For $1 < r < n$ we have the estimate*

$$\|B^*\boldsymbol{u}\|_r \leq C\|\boldsymbol{w}\|\cdot\|\nabla^2\boldsymbol{u}\|_r. \tag{3.10}$$

(iii) *For $n/(n-1) < r < \infty$ and $0 < \omega < \pi/2$ we have*

$$\|(\lambda + A)^{-1}B\boldsymbol{u}\|_r \leq C\|\boldsymbol{w}\|\cdot\|\boldsymbol{u}\|_r. \tag{3.11}$$

(iv) *We have*

$$\|\boldsymbol{w}\cdot\nabla\boldsymbol{u}\|_{H^1} \leq C\|\boldsymbol{w}\|\cdot\|\nabla^2\boldsymbol{u}\|_{H^1}, \tag{3.12}$$

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{H^1} \leq C\|\nabla\boldsymbol{w}\|\cdot\|\nabla^2\boldsymbol{u}\|_{H^1}, \tag{3.13}$$

*and therefore*

$$\|B\boldsymbol{u}\|_{H^1} \leq C(\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\|)\|\nabla^2 \boldsymbol{u}\|_{H^1}. \tag{3.14}$$

Proof. (i) First we show that

$$\left\| \frac{\boldsymbol{u}}{|x|} \right\|_r \leq C\|\nabla \boldsymbol{u}\|_r \qquad (1 < r < n). \tag{3.15}$$

Indeed, since $|x|^{-1} \in L^n_w(\mathbb{R}^n)$, the weak Hölder inequality as given in [4] yields

$$\left\| \frac{\boldsymbol{u}}{|x|} \right\|_{r,w} \leq C\|\boldsymbol{u}\|_{rn/(n-r),w} \leq C\|\nabla \boldsymbol{u}\|_r$$

whenever $1 < r < n$. Estimate (3.15) now follows from the Marcinkiewicz interpolation theorem [25]. Now we take an arbitrary $\varphi \in \boldsymbol{C}^\infty_{0,\sigma}$, to obtain for $j = 0, 1$,

$$\begin{aligned}
|\langle \nabla^j(\lambda + A)^{-1}B\boldsymbol{u}, \varphi\rangle| &= |\langle B\boldsymbol{u}, (\lambda + A)^{-1}\nabla^j\varphi\rangle| \\
&= |\langle \boldsymbol{w}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{w}, (\lambda + A)^{-1}\nabla^j\varphi\rangle| \\
&\leq |\langle \boldsymbol{u}, \boldsymbol{w}\cdot\nabla(\lambda + A)^{-1}\nabla^j\varphi\rangle| \\
&\quad + |\langle \boldsymbol{w}, \boldsymbol{u}\cdot\nabla(\lambda + A)^{-1}\nabla^j\varphi\rangle|.
\end{aligned}$$

By (3.14) we see that

$$\begin{aligned}
|\langle \boldsymbol{u}, \boldsymbol{w}\cdot\nabla(\lambda + A)^{-1}\nabla^j\varphi\rangle| &\leq \|\boldsymbol{w}\| \cdot \left\| \frac{\boldsymbol{u}}{|x|} \right\|_r \|\nabla(\lambda + A)^{-1}\nabla^j\varphi\|_{r'} \\
&\leq C\|\boldsymbol{w}\| \cdot \|\nabla \boldsymbol{u}\|_r \|\nabla^{1+j}(\lambda + A)^{-1}\varphi\|_{r'} \\
&\leq C\|\boldsymbol{w}\| \cdot \|\nabla \boldsymbol{u}\|_r \|\varphi\|_{r'}/|\lambda|^{(1-j)/2}
\end{aligned}$$

where $1/r' = 1 - 1/r$. Similarly we get

$$|\langle \boldsymbol{w}, \boldsymbol{u}\cdot\nabla(\lambda + A)^{-1}\nabla^j\varphi\rangle| \leq C\|\boldsymbol{w}\| \cdot \|\nabla \boldsymbol{u}\|_r \|\varphi\|_{r'}/|\lambda|^{(1-j)/2}.$$

Hence

$$|\langle \nabla^j(\lambda + A)^{-1}B\boldsymbol{u}, \varphi\rangle| \leq C\|\boldsymbol{w}\| \cdot \|\nabla \boldsymbol{u}\|_r \|\varphi\|_{r'}/|\lambda|^{(1-j)/2},$$

and this proves (3.9).

(ii) By definition of $B^*$ and (3.15) we easily obtain

$$\|B^*\boldsymbol{u}\|_r \le C\|\boldsymbol{w}\| \cdot \left\| \frac{\nabla \boldsymbol{u}}{|x|} \right\|_r \le C\|\boldsymbol{w}\| \cdot \|\nabla^2 \boldsymbol{u}\|_r \qquad (1 < r < n),$$

which proves (3.10).

(iii) Estimate (3.10) implies that if $1 < r < n$, then

$$\|B^*(\lambda + A)^{-1}\boldsymbol{u}\|_r \le C\|\boldsymbol{w}\| \cdot \|\nabla^2(\lambda + A)^{-1}\boldsymbol{u}\|_r \le C\|\boldsymbol{w}\| \cdot \|\boldsymbol{u}\|_r.$$

Estimate (3.11) is now obtained by duality.

(iv) Estimate (3.14) is obtained by combining (3.12) and (3.13) with the boundedness of the projector $P$ on $\boldsymbol{H}^1$. So we need only prove (3.12) and (3.13). To do this, it suffices to prove the following estimates (3.16) and (3.17), which are essentially due to [5] :

$$\|\boldsymbol{w} \cdot \nabla \boldsymbol{u}\|_{H^1} \le C\|\boldsymbol{w}\| \cdot \|\nabla \boldsymbol{u}\|_{(n/(n-1),1)}, \tag{3.16}$$

$$\|\boldsymbol{u} \cdot \nabla \boldsymbol{w}\|_{H^1} \le C\|\nabla \boldsymbol{w}\| \cdot \|\boldsymbol{u}\|_{(n/(n-2),1)}. \tag{3.17}$$

(Here and in what follows $\|\cdot\|_{(p,q)}$ is the $L^{(p,q)}$-norm.) Estimates (3.12) and (3.13) are then deduced via the Sobolev inequalities ([6, 10, 15]) :

$$\|\nabla \boldsymbol{u}\|_{(n/(n-1),1)} \le C\|\nabla^2 \boldsymbol{u}\|_{H^1}, \qquad \|\boldsymbol{u}\|_{(n/(n-2),1)} \le C\|\nabla^2 \boldsymbol{u}\|_{H^1}. \tag{3.18}$$

We shall prove only (3.16) ; (3.17) is proved similarly. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be supported in the unit ball $\{|x| \le 1\}$ and satisfy $\int \varphi dx = 1$. Since $\nabla \cdot \boldsymbol{w} = 0$, we have $\boldsymbol{w} \cdot \nabla \boldsymbol{u} = \nabla \cdot (\boldsymbol{w} \otimes (\boldsymbol{u} - \boldsymbol{c}))$ for any constant vector $\boldsymbol{c}$. Integration by parts thus gives

$$(\varphi_t * (\boldsymbol{w} \cdot \nabla \boldsymbol{u}))(x) = \frac{1}{t^{n+1}} \int (\nabla\varphi)((x-y)/t)\, [\boldsymbol{w}(y) \otimes (\boldsymbol{u}(y) - \overline{\boldsymbol{u}}_t)]\, dy$$

where $\overline{\boldsymbol{u}}_t = |B_t(x)|^{-1} \int_{B_t(x)} \boldsymbol{u}(y) dy$ is the average of $\boldsymbol{u}$ over the open ball $B_t(x)$ with radius $t$ centered at $x$. We take $\alpha > 0$ and $\beta > 0$ so that

$$1/\alpha + 1/\beta = 1 + 1/n \quad \text{and} \quad 1/\beta^* \equiv 1/\beta - 1/n > 0,$$

and apply the Hölder and the Poincaré-Sobolev inequalities, to get

$$|(\varphi_t * (\boldsymbol{w} \cdot \nabla \boldsymbol{u}))(x)| \ \le \ \frac{C}{t^{n+1}} \int_{B_t(x)} |\boldsymbol{w}(y)| \cdot |\boldsymbol{u}(y) - \overline{\boldsymbol{u}}_t| dy$$

$$
\begin{aligned}
&\le \frac{C}{t^{n+1}} \left( \int_{B_t(x)} |\boldsymbol{w}|^\alpha dy \right)^{1/\alpha} \\
&\quad \cdot \left( \int_{B_t(x)} |\boldsymbol{u} - \overline{\boldsymbol{u}}_t|^{\beta^*} dy \right)^{1/\beta^*} \\
&\le C \left( \frac{1}{|B_t|} \int_{B_t(x)} |\boldsymbol{w}|^\alpha dy \right)^{1/\alpha} \\
&\quad \cdot \left( \frac{1}{|B_t|} \int_{B_t(x)} |\nabla \boldsymbol{u}|^\beta dy \right)^{1/\beta} \\
&\le C\|\boldsymbol{w}\| M(|y|^{-\alpha})^{1/\alpha} M(|\nabla \boldsymbol{u}|^\beta)^{1/\beta}
\end{aligned}
$$

where $M(|f|)$ stands for the Hardy–Littlewood maximal function of $|f|$ ([25, 26]). Now we fix $\alpha$ and $\beta$ such that $1 < \alpha < n$ and $1 < \beta < n/(n-1)$, and apply the duality relation :

$$
\begin{aligned}
&\|M(|y|^{-\alpha})^{1/\alpha} M(|\nabla \boldsymbol{u}|^\beta)^{1/\beta}\|_1 \\
&\qquad \le \|M(|y|^{-\alpha})^{1/\alpha}\|_{n,w} \|M(|\nabla \boldsymbol{u}|^\beta)^{1/\beta}\|_{(n/(n-1),1)}.
\end{aligned}
$$

Since $\|M(|y|^{-\alpha})^{1/\alpha}\|_{n,w} < +\infty$ and since $\|M(|\nabla \boldsymbol{u}|^\beta)^{1/\beta}\|_{(n/(n-1),1)} \le C\|\nabla \boldsymbol{u}\|_{(n/(n-1),1)}$ by the Hardy–Littlewood maximal theorem ([25, 26]), we get $\sup_{t>0} |\varphi_t * (\boldsymbol{w} \cdot \nabla \boldsymbol{u})| \in L^1$ and

$$
\|\boldsymbol{w} \cdot \nabla \boldsymbol{u}\|_{H^1} \equiv \left\| \sup_{t>0} |\varphi_t * (\boldsymbol{w} \cdot \nabla \boldsymbol{u})| \right\|_1 \le C\|\boldsymbol{w}\| \cdot \|\nabla \boldsymbol{u}\|_{(n/(n-1),1)}.
$$

This proves (3.16). □

REMARKS. (i) When $\boldsymbol{w}$ satisfies (1.1), one can show that if $1 < r < n$, then

$$
\|\nabla^j (\lambda + A)^{-1} B\boldsymbol{u}\|_r \le C\|\boldsymbol{w}\|_n \|\nabla \boldsymbol{u}\|_r / |\lambda|^{(1-j)/2} \qquad (|\arg \lambda| \le \pi - \omega)
$$

for $j = 0, 1$, and

$$
\|B^* \boldsymbol{u}\|_r \le C\|\boldsymbol{w}\|_n \|\nabla^2 \boldsymbol{u}\|_r.
$$

It is also easy to show that if $n/(n-1) < r < \infty$, then

$$
\|(\lambda + A)^{-1} B\boldsymbol{u}\|_r \le C\|\boldsymbol{w}\|_n \|\boldsymbol{u}\|_r.
$$

These are all easy exercises of the Hölder and Sobolev inequalities and duality argument.

(ii) One can also show that

$$\|\boldsymbol{w}\cdot\nabla\boldsymbol{u}\|_{H^1}\le C\|\boldsymbol{w}\|_n\|\nabla^2\boldsymbol{u}\|_{H^1},\qquad \|\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{H^1}\le C\|\nabla\boldsymbol{w}\|_{n/2}\|\nabla^2\boldsymbol{u}\|_{H^1}.$$

Indeed, for instance the first estimate is deduced via (3.17) if we show

$$\|\boldsymbol{w}\cdot\nabla\boldsymbol{u}\|_{H^1}\le C\|\boldsymbol{w}\|_n\|\nabla\boldsymbol{u}\|_{(n/(n-1),1)}.$$

To deduce this last estimate we proceed as in the proof of (3.15) to get

$$|\varphi_t*(\boldsymbol{w}\cdot\nabla\boldsymbol{u})|\le CM(|\boldsymbol{w}|^\alpha)^{1/\alpha}M(|\nabla\boldsymbol{u}|^\beta)^{1/\beta}.$$

Applying the Hölder and the maximal inequalities, we obtain

$$\begin{aligned}&\|M(|\boldsymbol{w}|^\alpha)^{1/\alpha}M(|\nabla\boldsymbol{u}|^\beta)^{1/\beta}\|_1\\ &\quad\le C\|M(|\boldsymbol{w}|^\alpha)^{1/\alpha}\|_n\|M(|\nabla\boldsymbol{u}|^\beta)^{1/\beta}\|_{n/(n-1)}\\ &\quad\le C\|\boldsymbol{w}\|_n\|\nabla\boldsymbol{u}\|_{n/(n-1)}\le C\|\boldsymbol{w}\|_n\|\nabla\boldsymbol{u}\|_{(n/(n-1),1)}.\end{aligned}$$

We thus conclude that Lemma 3.1 is also valid in this case even if we replace $\|\boldsymbol{w}\|$ and $\|\nabla\boldsymbol{w}\|$ by $\|\boldsymbol{w}\|_n$ and $\|\nabla\boldsymbol{w}\|_{n/2}$, respectively.

(iii) In (i) and (ii) above, one can also replace condition (1.3) with (1.2′). In the case of (i), the details are given in [4]. For (ii), we have only to apply the duality relation ([1]) :

$$\|fg\|_1\le\|f\|_{p',w}\|g\|_{(p,1)}\qquad(1<p<\infty,\quad p'=p/(p-1)),\tag{3.19}$$

in deducing assertions corresponding to Lemma 3.1 (iii).

Lemma 3.2. *Let $1<r<n$ and $0<\omega<\pi/2$.*

(i) *There is a constant $\mu=\mu(r,\omega)>0$ such that if*

$$\|\boldsymbol{w}\|\le\mu,$$

*then for all $\lambda\in\mathbb{C}\setminus 0$ with $|\arg\lambda|\le\pi-\omega$, we have*

$$\begin{aligned}\|\nabla^j(\lambda+L)^{-1}\boldsymbol{u}\|_r&\le C\|\boldsymbol{u}\|_r/|\lambda|^{1-j/2}\qquad(j=0,1)\\ \|\nabla^j(\lambda+L^*)^{-1}\boldsymbol{u}\|_r&\le C\|\boldsymbol{u}\|_r/|\lambda|^{1-j/2}\qquad(j=0,1,2).\end{aligned}\tag{3.20}$$

(ii) *There is a constant* $\mu' = \mu'(\omega) > 0$ *such that if*

$$\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\| \leq \mu'$$

*then for all* $\lambda \in \mathbb{C} \setminus 0$ *with* $|\arg \lambda| \leq \pi - \omega$, *we have*

$$\|\nabla^j (\lambda + L)^{-1} \boldsymbol{u}\|_{H^1} \leq C \|\boldsymbol{u}\|_{H^1} / |\lambda|^{1-j/2} \qquad (j = 0, 1, 2). \tag{3.21}$$

(iii) *Let* $1 < r < \infty$. *There is a constant* $\mu'' = \mu''(r, \omega) > 0$ *such that if*

$$\|\boldsymbol{w}\| \leq \mu'',$$

*then for all* $\lambda \in \mathbb{C} \setminus 0$ *with* $|\arg \lambda| \leq \pi - \omega$, *we have*

$$\begin{aligned} \|(\lambda + L)^{-1} \boldsymbol{u}\|_r &\leq C \|\boldsymbol{u}\|_r / |\lambda| \\ \|(\lambda + L^*)^{-1} \boldsymbol{u}\|_r &\leq C \|\boldsymbol{u}\|_r / |\lambda|. \end{aligned} \tag{3.22}$$

(iv) *Under the assumption of* (ii), *the operator* $L$ *is injective on* $\boldsymbol{H}^1_\sigma$.

PROOF. (i) We prove (3.20) with the aid of the (formal) expansions

$$\begin{aligned} \nabla^j (\lambda + L)^{-1} &= \nabla^j (\lambda + A)^{-1} \sum_{k=0}^\infty [-B(\lambda + A)^{-1}]^k \\ &= \nabla^j \sum_{k=0}^\infty [-(\lambda + A)^{-1} B]^k (\lambda + A)^{-1} \qquad (j = 0, 1), \\ \nabla^j (\lambda + L^*)^{-1} &= \nabla^j (\lambda + A)^{-1} \sum_{k=0}^\infty [-B^*(\lambda + A)^{-1}]^k \qquad (j = 0, 1, 2). \end{aligned} \tag{3.23}$$

Assuming that $\|\boldsymbol{w}\|$ is small, we see from (3.9) that

$$\begin{aligned} \|\nabla^j (\lambda + L)^{-1} \boldsymbol{u}\|_r &\leq \left( \sum_{k=0}^\infty [C \|\boldsymbol{w}\|]^k \right) |\lambda|^{(j-1)/2} \|\nabla (\lambda + A)^{-1} \boldsymbol{u}\|_r \\ &\leq \left( \sum_{k=0}^\infty [C \|\boldsymbol{w}\|]^k \right) \|\boldsymbol{u}\|_r |\lambda|^{-1+j/2} \\ &= C \|\boldsymbol{u}\|_r / |\lambda|^{1-j/2} \end{aligned}$$

which shows (3.20) for $L$. To deduce (3.20) for $L^*$, we note that

$$\|B^* (\lambda + A)^{-1} \boldsymbol{u}\|_r \leq C \|\boldsymbol{w}\| \cdot \|\nabla^2 (\lambda + A)^{-1} \boldsymbol{u}\|_r \leq C \|\boldsymbol{w}\| \cdot \|\boldsymbol{u}\|_r.$$

Thus, if $\|\boldsymbol{w}\|$ is sufficiently small, we see from (3.23) that

$$\begin{aligned} \|\nabla^j(\lambda+L^*)^{-1}\boldsymbol{u}\|_r &\leq \|\nabla^j(\lambda+A)^{-1}\|\cdot\left(\sum_{k=0}^{\infty}[C\|\boldsymbol{w}\|]^k\right)\|\boldsymbol{u}\|_r \\ &\leq C\|\boldsymbol{u}\|_r/|\lambda|^{1-j/2} \end{aligned}$$

for $j=0,1,2$. This proves (i).

(ii) We again invoke (3.23) to see that for $j=0,1,2$,

$$\begin{aligned} &\|\nabla^j(\lambda+L)^{-1}\boldsymbol{u}\|_{H^1} \\ &\quad\leq \|\nabla^j(\lambda+A)^{-1}\|\left(\sum_{k=0}^{\infty}[C(\|\boldsymbol{w}\|+\|\nabla\boldsymbol{w}\|)]^k\right)\|\nabla^2(\lambda+A)^{-1}\boldsymbol{u}\|_{H^1} \\ &\quad\leq C\|\boldsymbol{u}\|_{H^1}/|\lambda|^{1-j/2}. \end{aligned}$$

This proves (ii).

(iii) From (i) we obtain

$$\|(\lambda+L)^{-1}\boldsymbol{u}\|_r \leq C\|\boldsymbol{u}\|_r/|\lambda| \qquad (1<r<n) \tag{3.24}$$

and

$$\|(\lambda+L^*)^{-1}\boldsymbol{u}\|_r \leq C\|\boldsymbol{u}\|_r/|\lambda| \qquad (1<r<n). \tag{3.25}$$

Applying a duality argument to (3.25) gives

$$\|(\lambda+L)^{-1}\boldsymbol{u}\|_r \leq C\|\boldsymbol{u}\|_r/|\lambda| \qquad (n/(n-1)<r<\infty). \tag{3.26}$$

Since $n/(n-1)<n$ because $n\geq 3$, the result follows from (3.24) and (3.26).

(iv) From (3.21) with $j=2$ we get

$$C_1\|\nabla^2\boldsymbol{u}\|_{H^1} \leq \|L\boldsymbol{u}\|_{H^1} \leq C_2\|\nabla^2\boldsymbol{u}\|_{H^1} \qquad \text{for all } \boldsymbol{u}\in D(L).$$

The injectivity of $L$ then follows immediately. The proof is complete. □

Lemmas 3.1 and 3.2 can be refined into a form which involves some $\boldsymbol{H}^p$-norms, $0<p<1$.

LEMMA 3.3. *Let $n/(n+1)<p<1$. Then we have*

$$\begin{aligned} \|\boldsymbol{w}\cdot\nabla\boldsymbol{u}\|_{H^p} &\leq C\|\boldsymbol{w}\|\cdot\|\nabla^2\boldsymbol{u}\|_{H^p}, \\ \|\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{H^p} &\leq C\|\nabla\boldsymbol{w}\|\cdot\|\nabla^2\boldsymbol{u}\|_{H^p} \end{aligned} \tag{3.27}$$

*and therefore*

$$\|B\boldsymbol{u}\|_{H^p} \le C(\|\boldsymbol{w}\|^p + \|\nabla \boldsymbol{w}\|^p)^{1/p}\|\nabla^2 \boldsymbol{u}\|_{H^p}. \tag{3.28}$$

PROOF. As in the proof of Lemma 3.1, we use

$$\sup_{t>0} |\varphi_t * (\boldsymbol{w}\cdot\nabla\boldsymbol{u})| \le C\|\boldsymbol{w}\| M(|y|^{-\alpha})^{1/\alpha} M(|\nabla\boldsymbol{u}|^\beta)^{1/\beta}$$

where

$$0<\alpha<n, \quad 0<\beta<\frac{pn}{n-p}, \quad \text{and} \quad \frac{1}{\alpha}+\frac{1}{\beta}=1+\frac{1}{n}.$$

To estimate the right-hand side, we apply the duality relation (3.19) and the fact that

$$\|f\|^t_{(p,q)} = \|\,|f|^t\|_{(p/t,q/t)}, \qquad (0<p<\infty,\ \ 0<q\le\infty,\ \ 0<t<\infty),$$

which is easily deduced from the definition of the Lorentz-norms (note that $\|\cdot\|_{p,w} = \|\cdot\|_{(p,\infty)}$). We then obtain

$$\begin{aligned} &\|M(|y|^{-\alpha})^{1/\alpha} M(|\nabla\boldsymbol{u}|^\beta)^{1/\beta}\|_p \\ &\quad \le \|M(|y|^{-\alpha})^{1/\alpha})\|_{n,w} \|M(|\nabla\boldsymbol{u}|^\beta)^{1/\beta}\|_{(pn/(n-p),p)} \\ &\quad \le C\|\nabla\boldsymbol{u}\|_{(pn/(n-p),p)}. \end{aligned}$$

The first estimate of (3.27) now follows from the Sobolev inequality ([6, 10, 15])

$$\|\nabla\boldsymbol{u}\|_{(pn/(n-p),p)} \le C\|\nabla^2\boldsymbol{u}\|_{H^p}.$$

The second estimate is deduced similarly. Finally, (3.28) immediately follows from (3.27), the boundedness of $P$ on $\boldsymbol{H}^p(\mathbb{R}^n)$ and the fact that $\|\cdot\|^p_{H^p}$ satisfies the triangle inequality. The proof is complete. □

LEMMA 3.4. *Let $n/(n+1)<p<1$ and $0<\omega<\pi/2$. Then there is a constant $c>0$ such that if*

$$\|\boldsymbol{w}\| + \|\nabla\boldsymbol{w}\| \le c,$$

*then*

$$\|(\lambda+L)^{-1}\boldsymbol{u}\|_{H^1} \le C\|\boldsymbol{u}\|_{H^p}/|\lambda|^{1-n(1/p-1)/2} \qquad (|\arg\lambda| \le \pi-\omega). \tag{3.29}$$

PROOF. We apply the Mikhlin multiplier theorem [21; 25, p. 237] in $H^p$ spaces to see that

$$\|A^{n(1/p-1)/2}(e^{i\theta}+A)^{-1}\boldsymbol{u}\|_{H^p} \le C_\omega \|\boldsymbol{u}\|_{H^p}$$

for all $\theta$ with $|\theta| \le \pi - \omega$. Substituting $\boldsymbol{u}_t(x) = \boldsymbol{u}(x/\sqrt{t})$ in the above estimate and writing $\lambda = te^{i\theta}$, we obtain

$$\|A^{n(1/p-1)/2}(\lambda+A)^{-1}\boldsymbol{u}\|_{H^p} \le C\|\boldsymbol{u}\|_{H^p}/|\lambda|^{1-n(1/p-1)/2}.$$

Applying the Hardy–Littlewood–Sobolev inequality ([6, 10, 15]), we obtain

$$\begin{aligned} \|(\lambda+A)^{-1}\boldsymbol{u}\|_{H^1} &\le C\|A^{n(1/p-1)/2}(\lambda+A)^{-1}\boldsymbol{u}\|_{H^p} \\ &\le C\|\boldsymbol{u}\|_{H^p}/|\lambda|^{1-n(1/p-1)/2}. \end{aligned} \tag{3.30}$$

Similarly, we have

$$\|\nabla^j(\lambda+A)^{-1}\boldsymbol{u}\|_{H^p} \le C\|\boldsymbol{u}\|_{H^p}/|\lambda|^{1-j/2} \qquad (|\arg\lambda| \le \pi-\omega, \quad j=0,1,2). \tag{3.31}$$

The result is obtained from (3.28), (3.30) and (3.31), via the expansion (3.23), as follows :

$$\begin{aligned} \|(\lambda+L)^{-1}\boldsymbol{u}\|_{H^1} &\le \sum_{k=0}^{\infty} \|(\lambda+A)^{-1}[-B(\lambda+A)^{-1}]^k\boldsymbol{u}\|_{H^1} \\ &\le C|\lambda|^{n(1/p-1)/2-1} \sum_{k=0}^{\infty} \|[B(\lambda+A)^{-1}]^k\boldsymbol{u}\|_{H^p} \\ &\le C|\lambda|^{n(1/p-1)/2-1} \\ &\quad \cdot \left(\sum_{k=0}^{\infty}[C(\|\boldsymbol{w}\|^p+\|\nabla\boldsymbol{w}\|^p)^{1/p}]^k\right) \|\nabla^2(\lambda+A)^{-1}\boldsymbol{u}\|_{H^p} \\ &\le C\|\boldsymbol{u}\|_{H^p}/|\lambda|^{1-n(1/p-1)/2}. \end{aligned}$$

The proof is complete. □

To state the next result, we recall that (see [26, 28]) the space VMO is defined to be the closure of $C_c^\infty(\mathbb{R}^n)$ in the space BMO, and that we have the duality relations [7, 26, 28] :

$$\boldsymbol{H}^1(\mathbb{R}^n)^* = \mathrm{BMO}, \qquad (\mathrm{VMO})^* = \boldsymbol{H}^1(\mathbb{R}^n).$$

Here and hereafter we write the quotient Banach space $\mathrm{BMO}/\mathbb{R}^n$ simply as BMO; so the (vector-valued) BMO functions $f$ and $g$ are identified if and only if $f - g$ is a constant vector. The operator $P$ then defines a bounded projector on both BMO and VMO, and so in particular we have the Helmholtz decompositon ([17]):

$$\mathrm{VMO} = \mathrm{VMO}_\sigma \oplus \mathrm{VMO}_\pi$$

with

$$\mathrm{VMO}_\sigma = \{\boldsymbol{u} \in \mathrm{VMO} \ : \ \nabla \cdot \boldsymbol{u} = 0\}, \qquad \mathrm{VMO}_\pi = \{\nabla p \in \mathrm{VMO} \ : \ p \in \mathcal{C}_0^1\}.$$

Here $\mathcal{C}^\alpha$ denotes the homogeneous Hölder–Zygmund space of order $\alpha$ ([28]) and $\mathcal{C}_0^\alpha$ is the $\mathcal{C}^\alpha$-closure of $C_c^\infty(\mathbb{R}^n)$. Furthermore, $\boldsymbol{C}_{0,\sigma}^\infty(\mathbb{R}^n)$ is dense in $\mathrm{VMO}_\sigma$ ([17]), and we have

$$\boldsymbol{H}_\sigma^1 = (\mathrm{VMO}_\sigma)^*, \qquad \boldsymbol{H}_\pi^1 = (\mathrm{VMO}_\pi)^*. \tag{3.32}$$

On the other hand, it is true that

$$\begin{aligned} (\boldsymbol{H}_\sigma^1)^* &= \mathrm{BMO}_\sigma = \{\boldsymbol{u} \in \mathrm{BMO} \ : \ \nabla \cdot \boldsymbol{u} = 0\}, \\ (\boldsymbol{H}_\pi^1)^* &= \mathrm{BMO}_\pi = \{\nabla p \in \mathrm{BMO} \ : \ p \in \mathcal{C}^1\}, \end{aligned}$$

but $\boldsymbol{C}_{0,\sigma}^\infty(\mathbb{R}^n)$ is *not* dense in $\mathrm{BMO}_\sigma$.

Combining Lemmas 3.2–3.4 and the above duality results with the standard semigroup theory, we easily obtain

COROLLARY 3.5. (i) *For each $1 < r < n$ there is a constant $c_r = c_r(n) > 0$ such that if*

$$\|\boldsymbol{w}\| \le c_r,$$

*then both $\{e^{-tL}\}_{t\ge 0}$ and $\{e^{-tL^*}\}_{t\ge 0}$ define bounded analytic $C_0$ semigroups on $\boldsymbol{L}_\sigma^r$ and we have*

$$\begin{aligned} \|\nabla^j e^{-tL}\boldsymbol{a}\|_r &\le Ct^{-j/2}\|\boldsymbol{a}\|_r \qquad (j = 0, 1), \\ \|\nabla^j e^{-tL^*}\boldsymbol{a}\|_r &\le Ct^{-j/2}\|\boldsymbol{a}\|_r \qquad (j = 0, 1, 2). \end{aligned}$$

(ii) *For each $1 < r < \infty$ there is a constant $c_r' = c_r'(n) > 0$ such that if*

$$\|\boldsymbol{w}\| \le c_r',$$

*then both* $\{e^{-tL}\}_{t\geq 0}$ *and* $\{e^{-tL^*}\}_{t\geq 0}$ *define bounded analytic* $C_0$ *semigroups on* $\boldsymbol{L}^r_\sigma$.

(iii) *There is a constant* $c_n > 0$ *such that if*

$$\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\| \leq c_n,$$

*then* $\{e^{-tL}\}_{t\geq 0}$ *and* $\{e^{-tL^*}\}_{t\geq 0}$ *define, respectively, bounded analytic* $C_0$ *semigroups on* $\boldsymbol{H}^1_\sigma$ *and* $\mathrm{VMO}_\sigma$. *Hence, we have*

$$\|Le^{-tL}\boldsymbol{a}\|_{H^1} \leq Ct^{-1}\|\boldsymbol{a}\|_{H^1}, \quad \|L^*e^{-tL^*}\boldsymbol{a}\|_{\mathrm{BMO}} \leq Ct^{-1}\|\boldsymbol{a}\|_{\mathrm{BMO}} \quad \text{for all } t > 0. \tag{3.33}$$

*Furthermore, for* $n/(n+1) < p < 1$ *there is a constant* $c_p \leq c_n$ *such that if*

$$\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\| \leq c_p,$$

*then*

$$\|e^{-tL}\boldsymbol{a}\|_{H^1} \leq Ct^{-n(1/p-1)/2}\|\boldsymbol{a}\|_{H^p}. \tag{3.34}$$

(iv) *Under the assumption of* (iii), *we have*

$$\|\nabla^j e^{-tL}\boldsymbol{a}\|_{H^1} \leq Ct^{-j/2}\|\boldsymbol{a}\|_{H^1} \qquad \text{for } j = 0, 1, 2. \tag{3.35}$$

See [20, 27] for basic results on the theory of analytic semigroups. We here notice only that the operators $e^{-tL}$ and $e^{-tL^*}$ are defined, respectively, by the Dunford integrals :

$$e^{-tL} = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\lambda + L)^{-1}d\lambda, \quad e^{-tL^*} = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\lambda + L^*)^{-1}d\lambda. \tag{3.36}$$

Here, the path of integration $\Gamma = \Gamma_+ \cup \Gamma_0 \cup \Gamma_-$ is defined by

$$\Gamma_\pm = \{re^{\pm i\theta} \;:\; t^{-1} \leq r < \infty\}, \quad \Gamma_0 = \{t^{-1}e^{i\varphi} \;:\; -\theta \leq \varphi \leq \theta\},$$

for an arbitrarily fixed $\theta$ such that $\pi/2 < \theta < \pi - \omega$. Assertions (i) and (ii) are well known (see, *e.g.*, [4]) and follow from Lemma 3.2 (i) and (iii), respectively, via (3.36). For (iii), the strong continuity of the semigroups $\{e^{-tL}\}_{t\geq 0}$ and $\{e^{-tL^*}\}_{t\geq 0}$ follows from the fact that the space $\boldsymbol{C}^\infty_{0,\sigma}(\mathbb{R}^n)$ is contained in $D(L)$ and in $D(L^*)$, respectively, and so $D(L)$ is dense in $\boldsymbol{H}^1_\sigma$

and $D(L^*)$ is dense in $\mathrm{VMO}_\sigma$. Estimates (3.35) are deduced via (3.36) from (3.21); and estimate (3.34) follows via (3.36) from (3.29).

COROLLARY 3.6. *Under the assumption of Corollary* 3.5 (iii), *we have*

$$\lim_{t\to\infty} \|e^{-tL}\boldsymbol{a}\|_{H^1} = 0 \qquad \text{for all } \ \boldsymbol{a} \in \boldsymbol{H}^1_\sigma. \tag{3.37}$$

PROOF. It suffices to show that the range $R(L)$ of the operator $L$ is dense in $\boldsymbol{H}^1_\sigma$. Indeed, observe first that if $\boldsymbol{a} \in R(L)$ with $\boldsymbol{a} = L\boldsymbol{b}$ for some $\boldsymbol{b} \in D(L)$, then (3.33) gives

$$\|e^{-tL}\boldsymbol{a}\|_{H^1} = \|Le^{-tL}\boldsymbol{b}\|_{H^1} \le Ct^{-1}\|\boldsymbol{b}\|_{H^1} \to 0 \qquad \text{as } \ t \to \infty.$$

Suppose next that $R(L)$ is dense in $\boldsymbol{H}^1_\sigma$. For any $\boldsymbol{a} \in \boldsymbol{H}^1_\sigma$ and any $\varepsilon > 0$, we can find $\boldsymbol{b} \in R(L)$ such that $\|\boldsymbol{a} - \boldsymbol{b}\|_{H^1} < \varepsilon$. Hence, estimate (3.35) with $j = 0$ gives

$$\begin{aligned}\|e^{-tL}\boldsymbol{a}\|_{H^1} &\le \|e^{-tL}\boldsymbol{b}\|_{H^1} + \|e^{-tL}(\boldsymbol{a}-\boldsymbol{b})\|_{H^1} \\ &\le \|e^{-tL}\boldsymbol{b}\|_{H^1} + C\|\boldsymbol{a}-\boldsymbol{b}\|_{H^1} \le \|e^{-tL}\boldsymbol{b}\|_{H^1} + C\varepsilon\end{aligned}$$

with $C > 0$ independent of $\varepsilon > 0$, and so

$$\limsup_{t\to\infty} \|e^{-tL}\boldsymbol{a}\|_{H^1} \le C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves (3.32).

To show $\overline{R(L)} = \boldsymbol{H}^1_\sigma$, we recall that estimates (3.14) and (3.20) together imply

$$C_1\|\nabla^2\boldsymbol{u}\|_{H^1} \le \|L\boldsymbol{u}\|_{H^1} \le C_2\|\nabla^2\boldsymbol{u}\|_{H^1} \qquad \text{for } \ \boldsymbol{u} \in D(L). \tag{3.38}$$

Let $\widehat{D}$ be the completion of $D(L)$ in the norm $\|\nabla^2\boldsymbol{u}\|_{H^1}$. Then, we see by an easy calculation using the Sobolev inequalities (3.18) as well as (3.38) that

$$\widehat{D} = \{\boldsymbol{u} \in \boldsymbol{L}^{(n/(n-2),1)}_\sigma \ : \ \ \nabla\boldsymbol{u} \in \boldsymbol{L}^{(n/(n-1),1)}_\sigma, \ \ \nabla^2\boldsymbol{u} \in \boldsymbol{H}^1_\sigma\} \tag{3.39}$$

and (3.38) is valid for all $\boldsymbol{u} \in \widehat{D}$. This implies that $L$ defines an isomorhism between the Banach space $\widehat{D}$ and a closed subspace of $\boldsymbol{H}^1_\sigma$. Thus, if $L$ :

$\widehat{D} \to \boldsymbol{H}^1_\sigma$ is surjective, it follows that the range of the original operator $L$ is dense in $\boldsymbol{H}^1_\sigma$. Therefore, we need only show the solvability (in $\widehat{D}$) of the equation

$$Lu = f \in \boldsymbol{H}^1_\sigma, \tag{3.40}$$

assuming that $\boldsymbol{w}$ is sufficiently small. We rewrite (3.40) in the form

$$\boldsymbol{u} = T\boldsymbol{u} \equiv A^{-1}(\boldsymbol{f} - B\boldsymbol{u}),$$

where $A^{-1}$ stands for the convolution with the Stokes fundamental solution tensor $E = (E_{jk})$ as given in (2.2). Since the Riesz transforms are bounded on $\boldsymbol{H}^1(\mathbb{R}^n)$, the operator $\nabla^2 A^{-1}$ is bounded from $\boldsymbol{H}^1_\sigma$ to $\boldsymbol{H}^1(\mathbb{R}^n)$. Thus, direct calculation using (3.14) gives

$$\|\nabla^2 T\boldsymbol{u}\|_{H^1} \le C_3\|\boldsymbol{f}\|_{H^1} + C_4(\|\boldsymbol{w}\| + \|\nabla\boldsymbol{w}\|)\|\nabla^2\boldsymbol{u}\|_{H^1},$$

and so the affine map $T$ is bounded from $\widehat{D}$ to itself. The same calculation shows

$$\|\nabla^2(T\boldsymbol{u} - T\boldsymbol{v})\|_{H^1} \le C_4(\|\boldsymbol{w}\| + \|\nabla\boldsymbol{w}\|)\|\nabla^2(\boldsymbol{u} - \boldsymbol{v})\|_{H^1},$$

and therefore $T$ defines a contraction map on the space $\widehat{D}$ provided $\boldsymbol{w}$ is sufficiently small. The solvability of (3.40) in $\widehat{D}$ is thus proved for all $\boldsymbol{f} \in \boldsymbol{H}^1_\sigma$. This proves Corollary 3.6. □

Corollary 3.7. *Under the assumption of Corollary* 3.6*, we have*

$$\lim_{t\to\infty} \|e^{-tL^*}\boldsymbol{a}\|_{\mathrm{VMO}} = 0 \qquad \text{for all } \boldsymbol{a} \in \mathrm{VMO}_\sigma. \tag{3.41}$$

*Furthermore, $L^*$ is injective on* $\mathrm{VMO}_\sigma$.

Proof. The injectivity of $L^*$ follows by duality from the fact that $R(L)$ is dense in $\boldsymbol{H}^1_\sigma$. On the other hand, since

$$\|e^{-tL^*}L^*\boldsymbol{b}\|_{\mathrm{VMO}} = \|L^*e^{-tL^*}\boldsymbol{b}\|_{\mathrm{VMO}} \le Ct^{-1}\|\boldsymbol{b}\|_{\mathrm{VMO}} \to 0 \qquad \text{as } t \to \infty,$$

it suffices to show that $R(L^*)$ is dense in $\mathrm{VMO}_\sigma$ in order to deduce (3.41). To see this density property, suppose that $\boldsymbol{a} \in \boldsymbol{H}^1_\sigma = (\mathrm{VMO}_\sigma)^*$ satisfies

$$\langle \boldsymbol{a}, L^*\boldsymbol{b}\rangle = 0 \qquad \text{for all } \boldsymbol{b} \in D(L^*).$$

Then, for all $\boldsymbol{b} \in D(L^*)$ we have

$$0 = \langle \boldsymbol{a}, L^*(I + L^*)^{-1}\boldsymbol{b}\rangle = \langle \boldsymbol{a}, (I + L^*)^{-1}L^*\boldsymbol{b}\rangle = \langle (I + L)^{-1}\boldsymbol{a}, L^*\boldsymbol{b}\rangle.$$

Since $(I + L)^{-1}\boldsymbol{a}$ is in $D(L)$, we get $L(I + L)^{-1}\boldsymbol{a} = 0$. The injectivity of $L$ then implies $(I + L)^{-1}\boldsymbol{a} = 0$, and so $\boldsymbol{a} = 0$. Thus, $R(L^*)$ is dense in $\mathrm{VMO}_\sigma$ by the Hahn–Banach theorem. The proof is complete. □

We conclude this section with the following, which is obtained from Corollary 3.5 via the Sobolev embedding, duality, complex interpolation and the semigroup property.

COROLLARY 3.8. (i) *For $1 \le r < \infty$ there is a constant $c > 0$ such that if*

$$\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\| \le c,$$

*then we have the following: When $1 < r < \infty$, both $\{e^{-tL}\}_{t\ge 0}$ and $\{e^{-tL^*}\}_{t\ge 0}$ define bounded analytic $C_0$ semigroups on $\boldsymbol{L}^r_\sigma$. If $r = 1$, then $\{e^{-tL}\}_{t\ge 0}$ and $\{e^{-tL^*}\}_{t\ge 0}$ define bounded analytic $C_0$ semigroups on $\boldsymbol{H}^1_\sigma$ and $\mathrm{VMO}_\sigma$, respectively.*

(ii) *For $1 < r \le q < \infty$ there is a constant $c' > 0$ such that if*

$$\|\boldsymbol{w}\| \le c',$$

*then*

$$\begin{aligned} \|e^{-tL}\boldsymbol{a}\|_q &\le Ct^{-(n/r-n/q)/2}\|\boldsymbol{a}\|_r \\ \|e^{-tL^*}\boldsymbol{a}\|_q &\le Ct^{-(n/r-n/q)/2}\|\boldsymbol{a}\|_r. \end{aligned}$$

(iii) *The estimate in* (ii) *for $\{e^{-tL}\}_{t\ge 0}$ holds for $r = 1$ if we replace $\|\cdot\|_r$ by $\|\cdot\|_{H^1}$.*

(iv) *For $1 < r < \infty$ there is a constant $c'' > 0$ such that if*

$$\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\| \le c'',$$

*then*

$$\|e^{-tL^*}\boldsymbol{a}\|_{\mathrm{BMO}} \le Ct^{-n/2r}\|\boldsymbol{a}\|_r.$$

*This estimate holds for $r = 1$ if we replace $\|\cdot\|_r$ by $\|\cdot\|_{H^1}$.*

REMARKS. (i) Using Corollary 3.8 and the fact that $\boldsymbol{C}_{0,\sigma}^{\infty}(\mathbb{R}^n)$ is dense in $\boldsymbol{L}_\sigma^r$ for all $1 < r < \infty$, we see that if $\|\boldsymbol{w}\|$ is sufficiently small depending on $r$, then

$$\lim_{t\to\infty} \|e^{-tL}\boldsymbol{a}\|_r = 0 \qquad \text{for all } \ \boldsymbol{a} \in \boldsymbol{L}_\sigma^r \ \text{ and } \ 1 < r < \infty. \tag{3.42}$$

The same is true of the operator $L^*$. Consequently, the operators $L$ and $L^*$ are both injective and have dense ranges in all $\boldsymbol{L}_\sigma^r$. Indeed, if $L\boldsymbol{a} = 0$ or $L^*\boldsymbol{a} = 0$, then $e^{-tL}\boldsymbol{a} = \boldsymbol{a}$ or $e^{-tL^*}\boldsymbol{a} = \boldsymbol{a}$ ; and so

$$\|\boldsymbol{a}\|_r = \lim_{t\to\infty} \|e^{-tL}\boldsymbol{a}\|_r = 0 \qquad \text{or} \qquad \|\boldsymbol{a}\|_r = \lim_{t\to\infty} \|e^{-tL^*}\boldsymbol{a}\|_r = 0.$$

Hence, $L$ and $L^*$ are both injective, and so by duality they have dense ranges.

(ii) All the results of this section remain valid if we replace (1.3) with (1.1) or (1.2′). Indeed, the whole results are based on Lemmas 3.1 and 3.3 ; and the conclusions of these lemmas remain valid if we replace $\|\boldsymbol{w}\|$ and $\|\nabla\boldsymbol{w}\|$ by $\|\boldsymbol{w}\|_{n,w}$ and $\|\nabla\boldsymbol{w}\|_{n/2,w}$, respectively. This fact will be freely used in Section 5 in discussing $L^1$ stability of stationary flows $\boldsymbol{w}$ satisfying condition (1.5), which is clearly stronger than (1.2′).

(iii) Obviously, we need no assumptions on the size of the derivatives $\nabla\boldsymbol{w}$ to deduce Corollary 3.8 (i) for $1 < r < \infty$.

## 4. Stability in $\boldsymbol{H}_\sigma^1$

We first introduce the standard notion of weak solution of problem (3.6), which is essentially due to Masuda [16]. Let $\boldsymbol{a} \in \boldsymbol{L}_\sigma^2$. A weakly continuous function $\boldsymbol{u} : [0,\infty) \to \boldsymbol{L}_\sigma^2$ is called a *weak solution* of problem (3.6) if

$$\boldsymbol{u}(0) = \boldsymbol{a}\,; \qquad \boldsymbol{u} \in L^\infty(0,T : \boldsymbol{L}_\sigma^2) \quad \text{and} \quad \nabla\boldsymbol{u} \in L^2(0,T : \boldsymbol{L}^2)$$

for all $0 < T < \infty$; and if the identity

$$\begin{aligned} &\langle \boldsymbol{u}(t), \varphi(t)\rangle - \langle \boldsymbol{u}(s), \varphi(s)\rangle + \int_s^t \langle \nabla\boldsymbol{u}, \nabla\varphi\rangle d\tau \\ &\qquad = \int_s^t \langle \boldsymbol{u}, \varphi'\rangle d\tau - \int_s^t \langle B\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}, \varphi\rangle d\tau \end{aligned} \tag{4.1}$$

holds for all $0 \le s \le t$ and all $\varphi$ such that

$$\varphi \in \boldsymbol{C}^1([s,t] : \boldsymbol{L}_\sigma^2) \cap \boldsymbol{C}([s,t] : \boldsymbol{L}_\sigma^n) \quad \text{and} \quad \nabla\varphi \in \boldsymbol{C}([s,t] : \boldsymbol{L}^2).$$

The condition $\varphi \in \boldsymbol{L}^n_\sigma$ is needed in the case $n \geq 5$ for the nonlinear term in (4.1) to make sense. The existence of a global-in-time weak solution for an arbitrary $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma$ is now well known. But, the uniqueness and the regularity of weak solutions still remain open.

In this section we shall prove the following, which is one of the main results of this paper.

THEOREM 4.1. *There is a constant $\ell > 0$ such that if*

$$\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\| \leq \ell, \tag{4.2}$$

*then for each $\boldsymbol{a} \in \boldsymbol{H}^1_\sigma \cap \boldsymbol{L}^2_\sigma$ there is a weak solution $\boldsymbol{u}$ of* (3.6) *satisfying*

$$\boldsymbol{u}(t) \in \boldsymbol{H}^1_\sigma \qquad \textit{for all} \ \ t \geq 0, \tag{4.3}$$

*and*

$$\lim_{t\to\infty} \|\boldsymbol{u}(t)\|_{H^1} = 0. \tag{4.4}$$

PROOF. Given an $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma$, the approximation method as given in [11, 22, 23] provides a weak solution $\boldsymbol{u}$ satisfying the energy inequality

$$\text{(E)} \quad \|\boldsymbol{u}(t)\|_2^2 + 2\int_0^t (\|\nabla \boldsymbol{u}\|_2^2 + \langle \boldsymbol{u} \cdot \nabla \boldsymbol{w}, \boldsymbol{u}\rangle) d\tau \leq \|\boldsymbol{a}\|_2^2 \qquad \text{for all} \ \ t \geq 0.$$

The Poincaré–Sobolev inequality

$$\left\| \frac{\boldsymbol{u}}{|x|} \right\|_2 \leq \frac{2}{n-2} \|\nabla \boldsymbol{u}\|_2$$

implies

$$|\langle \boldsymbol{u} \cdot \nabla \boldsymbol{w}, \boldsymbol{u}\rangle| = |\langle \boldsymbol{w}, \boldsymbol{u} \cdot \nabla \boldsymbol{u}\rangle| \leq \|\boldsymbol{w}\| \cdot \|\nabla \boldsymbol{u}\|_2 \left\| \frac{\boldsymbol{u}}{|x|} \right\|_2 \leq \frac{2}{n-2} \|\boldsymbol{w}\| \cdot \|\nabla \boldsymbol{u}\|_2^2,$$

and so we have

$$\|\nabla \boldsymbol{u}\|_2^2 + \langle \boldsymbol{u} \cdot \nabla \boldsymbol{w}, \boldsymbol{u}\rangle \geq \left(1 - \frac{2}{n-2}\|\boldsymbol{w}\|\right) \|\nabla \boldsymbol{u}\|_2^2.$$

Therefore, if $\|\boldsymbol{w}\| < (n-2)/2$, then the energy inequality (E) gives

$$\|\boldsymbol{u}(t)\|_2 \leq \|\boldsymbol{a}\|_2, \qquad \int_0^\infty \|\nabla \boldsymbol{u}\|_2^2 d\tau \leq C\|\boldsymbol{a}\|_2^2. \tag{4.5}$$

On the other hand, if $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{H}^1_\sigma$, then Corollary 3.8 gives

$$\|e^{-tL}\boldsymbol{a}\|_2 \leq Ct^{-n/4}\|\boldsymbol{a}\|_{H^1}.$$

Furthermore, denoting by $E_\lambda$ the spectral measure associated to the positive self-adjoint operator $A$ in $\boldsymbol{L}^2_\sigma$, we see by Corollary 3.8 that

$$\begin{aligned}\|e^{-tL^*}E_\lambda \boldsymbol{v}\|_{\mathrm{BMO}} &\leq Ct^{-1/2}\|E_\lambda \boldsymbol{v}\|_n \\ &\leq Ct^{-1/2}\|A^{(n-2)/4}E_\lambda \boldsymbol{v}\|_2 \leq Ct^{-1/2}\lambda^{(n-2)/4}\|\boldsymbol{v}\|_2,\end{aligned}$$

and so

$$\begin{aligned}|\langle E_\lambda e^{-(t-\tau)L}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u}),\psi\rangle| &= |\langle \boldsymbol{u}\cdot\nabla\boldsymbol{u}, e^{-(t-\tau)L^*}E_\lambda\psi\rangle| \\ &\leq C\|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{H^1}\|e^{-(t-\tau)L^*}E_\lambda\psi\|_{\mathrm{BMO}} \\ &\leq C(t-\tau)^{-1/2}\lambda^{(n-2)/4}\|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2\|\psi\|_2\end{aligned}$$

for $\psi \in \boldsymbol{C}^\infty_{0,\sigma}(\mathbb{R}^n)$. Here we have applied the following estimate, due to [5] :

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{H^1} \leq C\|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2.$$

Since $(\boldsymbol{L}^2_\sigma)^* = \boldsymbol{L}^2_\sigma$, we thus obtain

$$\|E_\lambda e^{-(t-\tau)L}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_2 \leq C(t-\tau)^{-1/2}\lambda^{(n-2)/4}\|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2.$$

Therefore, the argument in [3] shows that if $\boldsymbol{w}$ is sufficiently small in the sense of (4.2), then for any $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{H}^1_\sigma$ there is a weak solution $\boldsymbol{u}$ satisfying energy inequality (E) such that

$$\|\boldsymbol{u}(t)\|_2 \leq C(1+t)^{-n/4}. \tag{4.6}$$

A detailed proof of (4.6) will be given also in Section 6. We will now show that this solution $\boldsymbol{u}$ satisfies (4.3) and (4.4). By the definition of weak solution we have

$$\langle \boldsymbol{u}(t),\psi\rangle = \langle e^{-(t-s)L}\boldsymbol{u}(s),\psi\rangle - \int_s^t \langle \boldsymbol{u}\cdot\nabla\boldsymbol{u}, e^{-(t-\tau)L^*}\psi\rangle d\tau$$

for all $\psi \in \boldsymbol{C}^\infty_{0,\sigma}(\mathbb{R}^n)$ and $0 \leq s \leq t$. This is easily derived by setting $\varphi(\tau) = e^{-(t-\tau)L^*}\psi$ in (4.1). Due to the boundedness of the semigroup $\{e^{-tL^*}\}_{t\geq 0}$ on $\mathrm{VMO}_\sigma$, we see that

$$\begin{aligned}|\langle \boldsymbol{u}(t),\psi\rangle| &\leq \left(\|e^{-(t-s)L}\boldsymbol{u}(s)\|_{H^1} + \int_s^t \|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{H^1}d\tau\right)\|\psi\|_{\mathrm{VMO}} \\ &\leq \left(\|e^{-(t-s)L}\boldsymbol{u}(s)\|_{H^1} + C\int_s^t \|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2 d\tau\right)\|\psi\|_{\mathrm{VMO}}.\end{aligned}$$

Since $(\mathrm{VMO}_\sigma)^* = \boldsymbol{H}^1_\sigma$ and since $\boldsymbol{C}^\infty_{0,\sigma}(\mathbb{R}^n)$ is dense in $\mathrm{VMO}_\sigma$, taking $s=0$ we obtain

$$\begin{aligned}\|\boldsymbol{u}(t)\|_{H^1} &\leq \|e^{-tL}\boldsymbol{a}\|_{H^1} + C\int_0^t \|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2 d\tau \\ &\leq \|e^{-tL}\boldsymbol{a}\|_{H^1} + C\left(\int_0^t \|\boldsymbol{u}\|_2^2 d\tau\right)^{1/2}\left(\int_0^t \|\nabla\boldsymbol{u}\|_2^2 d\tau\right)^{1/2} < \infty,\end{aligned}$$

and this proves (4.3).

We can now deduce (4.4). A similar estimate using (4.6) gives, for all $0 \leq s \leq t$,

$$\begin{aligned}\|\boldsymbol{u}(t)\|_{H^1} &\leq \|e^{-(t-s)L}\boldsymbol{u}(s)\|_{H^1} + C\left(\int_s^t \|\boldsymbol{u}\|_2^2 d\tau\right)^{1/2}\left(\int_s^t \|\nabla\boldsymbol{u}\|_2^2 d\tau\right)^{1/2} \\ &\leq \|e^{-(t-s)L}\boldsymbol{u}(s)\|_{H^1} \\ &\quad + C\left(\int_0^\infty (1+\tau)^{-n/2} d\tau\right)^{1/2}\left(\int_s^\infty \|\nabla\boldsymbol{u}\|_2^2 d\tau\right)^{1/2} \\ &= \|e^{-(t-s)L}\boldsymbol{u}(s)\|_{H^1} + C'\left(\int_s^\infty \|\nabla\boldsymbol{u}\|_2^2 d\tau\right)^{1/2},\end{aligned} \tag{4.7}$$

since $n/2 > 1$ (recall that $n \geq 3$).

Now let $\varepsilon > 0$. By (4.5) there is an $s > 0$ so that the last term of (4.7) is $< \varepsilon$. Hence,

$$\|\boldsymbol{u}(t)\|_{H^1} \leq \|e^{-(t-s)L}\boldsymbol{u}(s)\|_{H^1} + \varepsilon \qquad \text{for all } \ t \ \text{ with } \ t > s.$$

Applying (3.31) thus gives

$$\limsup_{t\to\infty} \|\boldsymbol{u}(t)\|_{H^1} \leq \lim_{t\to\infty} \|e^{-(t-s)L}\boldsymbol{u}(s)\|_{H^1} + \varepsilon = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves (4.4). The proof is complete. □

## 5. Stability in $\boldsymbol{L}^1_\sigma$

In this section we consider the stationary flows $\boldsymbol{w}$ satisfying (1.5), *i.e.*,

$$\boldsymbol{w} \in \boldsymbol{L}^{(n,1)}(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n), \qquad \nabla\boldsymbol{w} \in \boldsymbol{L}^{(n/2,1)}(\mathbb{R}^n) \cap \boldsymbol{L}^\infty(\mathbb{R}^n), \tag{5.1}$$

and show that $\boldsymbol{w}$ is stable under perturbations from the space

$$\boldsymbol{L}^1_\sigma = \{\boldsymbol{u} \in \boldsymbol{L}^1(\mathbb{R}^n) \ : \ \nabla \cdot \boldsymbol{u} = 0\}$$

provided $\|\boldsymbol{w}\|_{(n,1)}+\|\nabla\boldsymbol{w}\|_{(n/2,1)}$ is sufficiently small. Since $\boldsymbol{L}^{(r,q)}\subset\boldsymbol{L}^{(r,\infty)}=\boldsymbol{L}^r_w$ for all $q\le\infty$, and since the results of Section 3 are all valid with $\|\boldsymbol{w}\|$ and $\|\nabla\boldsymbol{w}\|$ replaced by $\|\boldsymbol{w}\|_{n,w}$ and $\|\nabla\boldsymbol{w}\|_{n/2,w}$, respectively, we can freely use the results of Section 3.

Although the idea of the proof is basically the same as that given in the previous sections, we need some modifications of estimates given in Lemmas 3.1 and 3.3 in order to discuss the time-evolution of perturbations not in $\boldsymbol{H}^1_\sigma$ but in $\boldsymbol{L}^1_\sigma$. These modifications are given in the following, in which the property (5.1) plays a decisive role.

LEMMA 5.1. *Let $n/(n+1)<p\le 1$. Then we have the estimates*

$$\begin{aligned}\|B\boldsymbol{u}\|_1 &\le C(\|\boldsymbol{w}\|_{(n,1)}+\|\nabla\boldsymbol{w}\|_{(n/2,1)})\|A\boldsymbol{u}\|_1,\\ \|B\boldsymbol{u}\|_{H^p} &\le C(\|\boldsymbol{w}\|^p_{(n,1)}+\|\nabla\boldsymbol{w}\|^p_{(n/2,1)})^{1/p}\|\nabla^2\boldsymbol{u}\|_{H^p}.\end{aligned}\tag{5.2}$$

PROOF. To prove the first estimate, it suffices to show the following two estimates

$$\|\boldsymbol{w}\cdot\nabla\boldsymbol{u}\|_{H^1}\le C\|\boldsymbol{w}\|_{(n,1)}\|A\boldsymbol{u}\|_1,\tag{5.3}$$

$$\|\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{H^1}\le C\|\nabla\boldsymbol{w}\|_{(n/2,1)}\|A\boldsymbol{u}\|_1.\tag{5.4}$$

Indeed, (5.3), (5.4) and the boundedness of $P$ on $\boldsymbol{H}^1(\mathbb{R}^n)$ together imply that

$$\begin{aligned}\|B\boldsymbol{u}\|_1 &\le C\|B\boldsymbol{u}\|_{H^1}\le C(\|\boldsymbol{w}\cdot\nabla\boldsymbol{u}\|_{H^1}+\|\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{H^1})\\ &\le C(\|\boldsymbol{w}\|_{(n,1)}+\|\nabla\boldsymbol{w}\|_{(n/2,1)})\|A\boldsymbol{u}\|_1,\end{aligned}$$

and we get the first estimate of (5.2). To show (5.3), we proceed as in the proof of Lemma 3.1 to get the estimate

$$|\varphi_t*(\boldsymbol{w}\cdot\nabla\boldsymbol{u})|\le CM(|\boldsymbol{w}|^\alpha)^{1/\alpha}M(|\nabla\boldsymbol{u}|^\beta)^{1/\beta},$$

where $0<\alpha<n$, $0<\beta<n/(n-1)$, and $1/\alpha+1/\beta=1+1/n$. We apply (3.19) to get

$$\|M(|\boldsymbol{w}|^\alpha)^{1/\alpha}M(|\nabla\boldsymbol{u}|^\beta)^{1/\beta}\|_1\le\|M(|\boldsymbol{w}|^\alpha)^{1/\alpha}\|_{(n,1)}\|M(|\nabla\boldsymbol{u}|^\beta)^{1/\beta}\|_{n/(n-1),w}.$$

Since

$$\|M(|\boldsymbol{w}|^\alpha)^{1/\alpha}\|_{(n,1)} \le C\|\boldsymbol{w}\|_{(n,1)} \quad \text{and}$$
$$\|M(|\nabla \boldsymbol{u}|^\beta)^{1/\beta}\|_{n/(n-1),w} \le C\|\nabla \boldsymbol{u}\|_{n/(n-1),w},$$

it follows that

$$\|\boldsymbol{w}\cdot\nabla \boldsymbol{u}\|_{H^1} \le C\|\boldsymbol{w}\|_{(n,1)}\|\nabla \boldsymbol{u}\|_{n/(n-1),w}.$$

Estimate (5.3) now follows via the Sobolev inequality [25]

$$\|\nabla \boldsymbol{u}\|_{n/(n-1),w} \le C\|A\boldsymbol{u}\|_1.$$

Estimate (5.4) is similarly deduced with the aid of the Sobolev inequality [25]

$$\|\boldsymbol{u}\|_{n/(n-2),w} \le C\|A\boldsymbol{u}\|_1.$$

Finally, the second estimate of (5.2) is obtained as in Section 3, since

$$\|\boldsymbol{u}\|_{q,w} \le \|\boldsymbol{u}\|_{(q,p)} \le C\|\nabla^2 \boldsymbol{u}\|_{H^p} \quad \text{and}$$
$$\|\nabla \boldsymbol{u}\|_{r,w} \le \|\nabla \boldsymbol{u}\|_{(r,p)} \le C\|\nabla^2 \boldsymbol{u}\|_{H^p}$$

for $1/q = 1/p - 2/n$ and $1/r = 1/p - 1/n$, respectively. The proof is complete. □

COROLLARY 5.2. (i) *For any $0 < \omega < \pi/2$ there is a constant $\eta = \eta(\omega, n) > 0$ such that if*

$$\|\boldsymbol{w}\|_{(n,1)} + \|\nabla \boldsymbol{w}\|_{(n/2,1)} \le \eta,$$

*then there is a constant $C > 0$ such that, for all $\lambda \in \mathbb{C}\backslash 0$ with $|\arg\lambda| \le \pi-\omega$,*

$$\|A^{j/2}(\lambda + L)^{-1}\boldsymbol{u}\|_1 \le C\|\boldsymbol{u}\|_1/|\lambda|^{1-j/2} \qquad (j = 0, 1, 2). \tag{5.5}$$

*Hence, $\{e^{-tL}\}_{t\ge 0}$ defines a bounded analytic $C_0$ semigroup on $\boldsymbol{L}^1_\sigma$ such that*

$$\|A^{j/2}e^{-tL}\boldsymbol{a}\|_1 \le Ct^{-j/2}\|\boldsymbol{a}\|_1 \qquad (j = 0, 1, 2). \tag{5.6}$$

(ii) *Under the assumption of* (i), *$L$ is injective and has a dense range in $\boldsymbol{L}^1_\sigma$, and*

$$\lim_{t\to\infty} \|e^{-tL}\boldsymbol{a}\|_1 = 0 \qquad \textit{for all } \boldsymbol{a} \in \boldsymbol{L}^1_\sigma. \tag{5.7}$$

(iii) *To each $n/(n+1) < p \le 1$ and each $\omega$ with $0 < \omega < \pi/2$, there corresponds a number $\eta' = \eta'(p, \omega, n) > 0$ with $\eta' \le \eta$ such that if*

$$\|\boldsymbol{w}\|_{(n,1)} + \|\nabla \boldsymbol{w}\|_{(n/2,1)} \le \eta',$$

*then we have*

$$\|(\lambda + L)^{-1}\boldsymbol{u}\|_{H^1} \le C\|\boldsymbol{u}\|_{H^p}/|\lambda|^{1-n(1/p-1)/2} \qquad (|\arg\lambda| \le \pi - \omega), \tag{5.8}$$

*and therefore*

$$\|e^{-tL}\boldsymbol{a}\|_{H^1} \le Ct^{-n(1/p-1)/2}\|\boldsymbol{a}\|_{H^p}. \tag{5.9}$$

Proof. (i) is proved in the same way as in Section 3, by using Lemma 5.1, the expansion

$$A^{j/2}(\lambda + L)^{-1} = A^{j/2}(\lambda + A)^{-1}\sum_{k=0}^{\infty}[-B(\lambda + A)^{-1}]^k,$$

and the estimate

$$\|A^{j/2}(\lambda + A)^{-1}\boldsymbol{u}\|_1 \le C\|\boldsymbol{u}\|_1/|\lambda|^{1-j/2} \qquad (|\arg\lambda| \le \pi - \omega, \quad j = 0, 1, 2).$$

This last estimate follows immediately from the fact that the semigroup $\{e^{-tA}\}_{t\ge0}$ is bounded analytic in the space $\boldsymbol{L}^1_\sigma$. Estimate (5.6) is obtained via (3.36) from (5.5). Since $\boldsymbol{C}^\infty_{0,\sigma}(\mathbb{R}^n)$ is dense in $\boldsymbol{L}^1_\sigma$ (see [17]), the semigroup $\{e^{-tL}\}_{t\ge0}$ is strongly continuous in $\boldsymbol{L}^1_\sigma$.

(ii) Firstly, from Lemma 5.1 and estimate (5.5) with $j = 2$, we get

$$C_1\|A\boldsymbol{u}\|_1 \le \|L\boldsymbol{u}\|_1 \le C_2\|A\boldsymbol{u}\|_1.$$

Since $A$ is injective in $\boldsymbol{L}^1_\sigma$, so is $L$. Secondly, we introduce the completion $\widehat{D}$ of $D(L)$ in the norm $\|A\boldsymbol{u}\|_1$, and see that

$$\widehat{D} = \{u \in \boldsymbol{L}^{n/(n-2)}_w(\mathbb{R}^n) \ : \ \nabla\boldsymbol{u} \in \boldsymbol{L}^{n/(n-1)}_w(\mathbb{R}^n), \ A\boldsymbol{u} \in \boldsymbol{L}^1_\sigma\}.$$

We can always solve in $\widehat{D}$ the equation

$$L\boldsymbol{u} = \boldsymbol{f} \in C^\infty_{0,\sigma}(\mathbb{R}^n)$$

assuming that $\|\boldsymbol{w}\|_{(n,1)} + \|\nabla \boldsymbol{w}\|_{(n/2,1)}$ is small enough. Since $\boldsymbol{C}_{0,\sigma}^{\infty}(\mathbb{R}^n)$ is dense in $\boldsymbol{L}_\sigma^1$ as was shown in [17], it follows that $R(L)$ is dense in $\boldsymbol{L}_\sigma^1$ ; and so (5.7) is valid.

(iii) Estimate (5.8) is obtained in the same way as in Section 3 by employing

$$\|B\boldsymbol{u}\|_{H^p} \le C(\|\boldsymbol{w}\|_{(n,1)}^p + \|\nabla \boldsymbol{w}\|_{(n/2,1)}^p)^{1/p} \|\nabla^2 \boldsymbol{u}\|_{H^p}$$

instead of (3.28). Estimate (5.9) is deduced from (5.8) via (3.36). The proof is complete. □

We can now prove our main result of this section.

Theorem 5.3. *There is a constant $\ell > 0$ such that if*

$$\|\boldsymbol{w}\|_{(n,1)} + \|\nabla \boldsymbol{w}\|_{(n/2,1)} \le \ell,$$

*then for each $\boldsymbol{a} \in \boldsymbol{L}_\sigma^2 \cap \boldsymbol{L}_\sigma^1$, problem* (3.6) *possesses a weak solution $\boldsymbol{u}$ such that*

$$\boldsymbol{u}(t) \in \boldsymbol{L}_\sigma^1 \qquad \textit{for all} \ \ t \ge 0, \tag{5.10}$$

*and*

$$\lim_{t\to\infty} \|\boldsymbol{u}(t)\|_1 = 0. \tag{5.11}$$

Proof. As in the proof of Theorem 4.1, we get a weak solution $\boldsymbol{u}$ satisfying the energy inequality (E). Since

$$|\langle \boldsymbol{u}\cdot\nabla \boldsymbol{w}, \boldsymbol{u}\rangle| = |\langle \boldsymbol{w}, \boldsymbol{u}\cdot\nabla \boldsymbol{u}\rangle| \le \|\boldsymbol{w}\|_n \|\boldsymbol{u}\|_{2n/(n-2)} \|\nabla \boldsymbol{u}\|_2 \le C_n \|\boldsymbol{w}\|_{(n,1)} \|\nabla \boldsymbol{u}\|_2^2,$$

we see that (4.5) holds provided $\|\boldsymbol{w}\|_{(n,1)}$ is small enough.

Now, an analogue of Corollary 3.8 gives

$$\|e^{-tL}\boldsymbol{a}\|_2 \le Ct^{-n/4}\|\boldsymbol{a}\|_1.$$

Furthermore, since

$$\begin{aligned} \|e^{-tL^*}E_\lambda \boldsymbol{v}\|_{\mathrm{BMO}} &\le Ct^{-1/2}\|E_\lambda \boldsymbol{v}\|_n \le Ct^{-1/2}\|A^{(n-2)/4}E_\lambda \boldsymbol{v}\|_2 \\ &\le Ct^{-1/2}\lambda^{(n-2)/4}\|\boldsymbol{v}\|_2, \end{aligned}$$

which follows by duality from $\|e^{-tL}\boldsymbol{a}\|_{n/(n-1)} \leq Ct^{-1/2}\|\boldsymbol{a}\|_1 \leq Ct^{-1/2}\|\boldsymbol{a}\|_{H^1}$, we can show in the same way as in Section 4 that

$$\|E_\lambda e^{-(t-\tau)L}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_2 \leq C(t-\tau)^{-1/2}\lambda^{(n-2)/4}\|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2.$$

Hence, as in [3] one can deduce the estimate

$$\|\boldsymbol{u}(t)\|_2 \leq C(1+t)^{-n/4}. \tag{5.12}$$

A detailed proof of (5.12) will be given also in Section 6. We next substitute in (4.1) the function $\varphi(\tau) = e^{-(t-\tau)L^*}P\psi$, $\psi \in \boldsymbol{C}_c^\infty(\mathbb{R}^n)$, $s \leq \tau \leq t$, to get

$$\langle \boldsymbol{u}(t),\psi\rangle = \langle e^{-(t-s)L}\boldsymbol{u}(s),\psi\rangle - \int_s^t \langle \boldsymbol{u}\cdot\nabla\boldsymbol{u}, e^{-(t-\tau)L^*}P\psi\rangle d\tau, \tag{5.13}$$

since $P\boldsymbol{u} = \boldsymbol{u}$ and $Pe^{-tL} = e^{-tL}$. We set $s = 0$ in (5.13) and estimate the right-hand side, to obtain (see *Remarks* below)

$$\begin{aligned} |\langle \boldsymbol{u}(t),\psi\rangle| &\leq \|e^{-tL}\boldsymbol{a}\|_1\|\psi\|_\infty + C\left(\int_0^t \|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{H^1}d\tau\right)\|P\psi\|_{\mathrm{BMO}} \\ &\leq \|e^{-tL}\boldsymbol{a}\|_1\|\psi\|_\infty + C\left(\int_0^t \|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2 d\tau\right)\|\psi\|_{\mathrm{BMO}} \\ &\leq \left(\|e^{-tL}\boldsymbol{a}\|_1 + C\int_0^t \|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2 d\tau\right)\|\psi\|_\infty. \end{aligned} \tag{5.14}$$

Here we have used the boundedness of $P$ in BMO and the fact that $L^\infty \subset$ BMO with continuous injection. This shows that $\boldsymbol{u}(t)$ is a finite Borel measure on $\mathbb{R}^n$ for all $t \geq 0$, so we find that $\boldsymbol{u}(t) \in \boldsymbol{L}^1_\sigma$ for all $t \geq 0$, and this shows (5.10). The same calculation gives

$$\|\boldsymbol{u}(t)\|_1 \leq \|e^{-(t-s)L}\boldsymbol{u}(s)\|_1 + C\int_s^t \|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2 d\tau,$$

for all $0 \leq s \leq t$; and the desired decay result (5.11) is now obtained in the same way as in the proof of Theorem 4.1, by applying (4.5), (5.7) and (5.12). This completes the proof. □

Remark. A comment on the duality argument in (5.13) and (5.14) will be in order. As we have shown in [17], any $\boldsymbol{a} \in \boldsymbol{L}^1_\sigma$ has mean value zero on $\mathbb{R}^n$. Thus, in (5.13) with $s = 0$, the first term on the right-hand side indicates the duality between the space $\boldsymbol{L}^1_0$ of $L^1$ vector-functions with mean value zero and the quotient space $\boldsymbol{L}^\infty/\mathbb{R}^n$, while the second term means the duality between $\boldsymbol{H}^1(\mathbb{R}^n)$ and BMO/$\mathbb{R}^n$. Now, the operator $P$ defines a continuous linear operator from BMO/$\mathbb{R}^n$ to itself, while the continuous inclusion $\boldsymbol{L}^\infty \subset$ BMO gives rise to the continuous inclusion $\boldsymbol{L}^\infty/\mathbb{R}^n \subset$ BMO/$\mathbb{R}^n$. So, estimate (5.14) implies that the map $\psi \mapsto \langle \boldsymbol{u}(t), \psi\rangle$ defines a continuous linear functional on $\boldsymbol{L}^\infty/\mathbb{R}^n$. But, obviously, the canonical projection from $\boldsymbol{L}^\infty$ onto $\boldsymbol{L}^\infty/\mathbb{R}^n$ is continuous. We thus conclude, by taking $\psi$ from $\boldsymbol{C}_c^\infty(\mathbb{R}^n)$, that the map $\psi \mapsto \langle \boldsymbol{u}(t), \psi\rangle$ defines a finite Borel measure on $\mathbb{R}^n$, and so the function $\boldsymbol{u}(t)$ is in $\boldsymbol{L}^1_\sigma$. This argument will be applied also in Section 6.

## 6. Decay Rates of Perturbations in $\boldsymbol{H}^1_\sigma$ and in $\boldsymbol{L}^1_\sigma$

When $\boldsymbol{w} = 0$, we proved in [17] that if

$$\|e^{-tA}\boldsymbol{a}\|_{H^1} = O(t^{-\beta/2}) \quad \text{or} \quad \|e^{-tA}\boldsymbol{a}\|_1 = O(t^{-\beta/2})$$

as $t \to \infty$ for some $\beta > 0$, then there is a weak solution $\boldsymbol{u}$ satisfying, respectively,

$$\|\boldsymbol{u}(t)\|_{H^1} = O(t^{-\gamma/2}) \quad \text{or} \quad \|\boldsymbol{u}(t)\|_1 = O(t^{-\gamma/2}), \qquad \text{with} \quad \gamma = \min(1, \beta).$$

This result is deduced with the aid of the result of Wiegner [29], which asserts that

$$\|\boldsymbol{u}(t)\|_2 = O(t^{-n/4-\gamma/2}).$$

As is shown in [24], the largest exponent $n/4+\gamma/2 = (n+2)/4$ in the above decay estimate is optimal. By carefully examining the idea of the proof, we readily see that this largest exponent comes from the exponent of the elementary estimate

$$\|e^{-tA}\boldsymbol{a}\|_{C^{0,1}} \le \|\nabla e^{-tA}\boldsymbol{a}\|_\infty \le Ct^{-(n+2)/4}\|\boldsymbol{a}\|_2, \tag{6.1}$$

where $\|\cdot\|_{C^{0,1}}$ is the norm of the homogeneous Lipschitz space $C^{0,1}(\mathbb{R}^n)$. In this section we deal with the decay problem in the case $\boldsymbol{w} \ne 0$ and prove the following

Theorem 6.1. (i) *Suppose that $\boldsymbol{w}$ satisfies* (1.5). *For each $0 < \beta < 1$ there is an $\eta > 0$ such that if*

$$\|\boldsymbol{w}\|_{(n,1)} + \|\nabla \boldsymbol{w}\|_{(n/2,1)} \le \eta,$$

*then we have the following : For each* $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{L}^1_\sigma$ *satisfying*

$$\|e^{-tL}\boldsymbol{a}\|_1 = O(t^{-\beta/2}) \qquad as \;\; t \to \infty, \tag{6.2}$$

*there is a weak solution* $\boldsymbol{u}$ *such that, as* $t \to \infty$,

$$\|\boldsymbol{u}(t)\|_2 = O(t^{-n/4-\beta/2}) \qquad and \qquad \|\boldsymbol{u}(t)\|_1 = O(t^{-\beta/2}). \tag{6.3}$$

(ii) *Suppose that* $\boldsymbol{w}$ *satisfies* (1.3). *For each* $0 < \beta < 1$ *there is an* $\eta' > 0$ *such that if*

$$\|\boldsymbol{w}\| + \|\nabla \boldsymbol{w}\| \le \eta',$$

*then we have the following : For each* $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{H}^1_\sigma$ *satisfying*

$$\|e^{-tL}\boldsymbol{a}\|_{H^1} = O(t^{-\beta/2}) \qquad as \;\; t \to \infty, \tag{6.2$'$}$$

*there is a weak solution* $\boldsymbol{u}$ *such that, as* $t \to \infty$,

$$\|\boldsymbol{u}(t)\|_2 = O(t^{-n/4-\beta/2}) \qquad and \qquad \|\boldsymbol{u}(t)\|_{H^1} = O(t^{-\beta/2}). \tag{6.3$'$}$$

Remark. When $\boldsymbol{w} = 0$ we have shown in [17] that the result holds also for $\beta = 1$. However, when $\boldsymbol{w} \neq 0$, our method cannot be applied, since we know nothing about the validity of the estimate

$$\|e^{-tL}\boldsymbol{a}\|_{H^1} \le Ct^{-1/2}\|\boldsymbol{a}\|_{H^{n/(n+1)}_w}, \tag{6.4}$$

where $\boldsymbol{H}^p_w$ denotes the *weak* Hardy space. If $\boldsymbol{w} = 0$, then (6.4) is valid (see [17]); and this fact was effectively applied in [17] to get the decay result including $\beta = 1$. To deduce (6.4) by means of the Neumann series expansion for the resolvent, we have first to show that

$$\|B\boldsymbol{u}\|_{H^{n/(n+1)}_w} \le C(\|\boldsymbol{w}\|^{\frac{n}{n+1}} + \|\nabla \boldsymbol{w}\|^{\frac{n}{n+1}})^{1+\frac{1}{n}} \|\nabla^2 \boldsymbol{u}\|_{H^{n/(n+1)}_w}$$

or

$$\|B\boldsymbol{u}\|_{H^{n/(n+1)}_w} \le C(\|\boldsymbol{w}\|^{\frac{n}{n+1}}_{(n,1)} + \|\nabla \boldsymbol{w}\|^{\frac{n}{n+1}}_{(n/2,1)})^{1+\frac{1}{n}} \|\nabla^2 \boldsymbol{u}\|_{H^{n/(n+1)}_w}.$$

However, the validity of these two estimates is an open problem. We here note only that these estimates are valid if we replace $\|\nabla^2 \boldsymbol{u}\|_{H^{n/(n+1)}_w}$ by $\|\nabla^2 \boldsymbol{u}\|_{H^{n/(n+1)}}$.

To prove Theorem 6.1, we need a few preliminary lemmas. The next lemma gives us the key tool for deducing decay rates of $\|\boldsymbol{u}(t)\|_2$.

LEMMA 6.2. *Let $E_\lambda$, $(\lambda \geq 0)$, be the spectral measure associated with the positive self-adjoint operator $A$ on $\boldsymbol{L}^2_\sigma$. If $0 \leq \beta < 1$, $\beta = n(1/p - 1)$ and $1/q = 1/p - 1/2 = 1/2 + \beta/n$, then we have*

$$\|E_\lambda e^{-(t-\tau)L} P(\boldsymbol{u}\cdot\nabla\boldsymbol{v})\|_2 \leq C_\beta (t-\tau)^{-(\beta+1)/2}\lambda^{(n-2)/4}\|\boldsymbol{u}\|_q\|\nabla\boldsymbol{v}\|_2 \tag{6.5}$$

*provided that $\|\boldsymbol{w}\| + \|\nabla\boldsymbol{w}\|$ or $\|\boldsymbol{w}\|_{(n,1)} + \|\nabla\boldsymbol{w}\|_{(n/2,1)}$ is sufficiently small, depending on $\beta$.*

PROOF. Taking $\psi \in \boldsymbol{C}^\infty_{0,\sigma}$, we apply a duality argument to get, by (3.29),

$$\begin{aligned}
&|\langle E_\lambda e^{-(t-\tau)L} P(\boldsymbol{u}\cdot\nabla\boldsymbol{v}), \psi\rangle| \\
&\quad = |\langle e^{-(t-\tau)L/2} P(\boldsymbol{u}\cdot\nabla\boldsymbol{v}), e^{-(t-\tau)L^*/2}E_\lambda\psi\rangle| \\
&\quad \leq C(t-\tau)^{-\beta/2}\|\boldsymbol{u}\cdot\nabla\boldsymbol{v}\|_{H^p}\|e^{-(t-\tau)L^*/2}E_\lambda\psi\|_{\mathrm{BMO}} \\
&\quad \leq C\|\boldsymbol{u}\|_q\|\nabla\boldsymbol{v}\|_2(t-\tau)^{-\beta/2}\|e^{-(t-\tau)L^*/2}E_\lambda\psi\|_{\mathrm{BMO}}.
\end{aligned}$$

But,

$$\begin{aligned}
\|e^{-(t-\tau)L^*/2}E_\lambda\psi\|_{\mathrm{BMO}} &\leq C(t-\tau)^{-1/2}\|E_\lambda\psi\|_n \\
&\leq C(t-\tau)^{-1/2}\|A^{(n-2)/4}E_\lambda\psi\|_2 \\
&\leq C(t-\tau)^{-1/2}\lambda^{(n-2)/4}\|\psi\|_2,
\end{aligned}$$

and so the result is proved. □

Finally, Lemma 6.3 below enables us to apply a bootstrap argument with respect to the exponent $\beta$ for completing the proof of Theorem 6.1.

LEMMA 6.3. *Under the assumption of Theorem 6.1, suppose that $\boldsymbol{w}$ satisfies (1.5). Then for each $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{L}^1_\sigma$ with property (6.2), there is a weak solution $\boldsymbol{u}$ such that*

$$\int_0^\infty \|\nabla\boldsymbol{u}\|_2^2 dt < +\infty\,; \tag{6.6}$$

$$\|\boldsymbol{u}(t)\|_2 \leq C(1+t)^{-n/4-\beta/2} + Ct^{(1-n-\beta)/4}\left(\frac{1}{t}\int_0^t F d\tau\right)^{1/2} \tag{6.7}$$

*with*

$$(6.8) \qquad F(t) = \int_0^t (t-\tau)^{-(\beta+1)/2} \|\boldsymbol{u}(\tau)\|_q^2 d\tau, \qquad 1/q = 1/2 + \beta/n\,;$$

*and*

$$(6.9) \qquad \begin{aligned} \|\boldsymbol{u}(t)\|_1 \ &\leq \ C(1+t)^{-\beta/2} + Ct^{-\beta/2}\left(\int_0^{t/2} \|\boldsymbol{u}\|_q^2 d\tau\right)^{1/2} \\ &\quad + C\left(\int_{t/2}^t \|\boldsymbol{u}\|_2^2 d\tau\right)^{1/2}. \end{aligned}$$

*Let* $\boldsymbol{w}$ *satisfy* (1.3)*. Then, if* $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{H}^1_\sigma$ *satisfies* (6.2′)*, there hold* (6.7) *and*

$$(6.9') \qquad \begin{aligned} \|\boldsymbol{u}(t)\|_{H^1} &\leq C(1+t)^{-\beta/2} + Ct^{-\beta/2}\left(\int_0^{t/2} \|\boldsymbol{u}\|_q^2 d\tau\right)^{1/2} \\ &\quad + C\left(\int_{t/2}^t \|\boldsymbol{u}\|_2^2 d\tau\right)^{1/2}. \end{aligned}$$

Proof. We first prove (6.6) and (6.7). To do so, we assume that $\boldsymbol{u}$ is smooth so that the calculations below are all legitimate. The rigorous proof of (6.6) and (6.7) is then carried out first by applying the argument below to approximate solutions as given, *e.g.*, in [11, 22, 23] and then passing to the limit.

The argument below is essentially due to [22] (see also [3, 4, 11, 23, 29]). We start with

$$\frac{d}{dt}\|\boldsymbol{u}\|_2^2 + 2(\|\nabla \boldsymbol{u}\|_2^2 + \langle \boldsymbol{u}\cdot\nabla \boldsymbol{w}, \boldsymbol{u}\rangle) = 0.$$

Applying the estimate

$$\begin{aligned} |\langle \boldsymbol{u}\cdot\nabla\boldsymbol{w}, \boldsymbol{u}\rangle| \ &= \ |\langle \boldsymbol{w}, \boldsymbol{u}\cdot\nabla\boldsymbol{u}\rangle| \leq \|\boldsymbol{w}\|_n \|\boldsymbol{u}\|_{2n/(n-2)} \|\nabla\boldsymbol{u}\|_2 \\ &\leq \ C_n \|\boldsymbol{w}\|_{(n,1)} \|\nabla \boldsymbol{u}\|_2^2, \end{aligned}$$

we obtain

$$\|\nabla\boldsymbol{u}\|_2^2 + \langle \boldsymbol{u}\cdot\nabla\boldsymbol{w}, \boldsymbol{u}\rangle \geq (1 - C_n\|\boldsymbol{w}\|_{(n,1)})\|\nabla\boldsymbol{u}\|_2^2.$$

Thus, if $\boldsymbol{w}$ is small as assumed in Theorem 6.1, then

$$\frac{d}{dt}\|\boldsymbol{u}\|_2^2 + 2C_0\|\nabla\boldsymbol{u}\|_2^2 \le 0 \tag{6.10}$$

for some $C_0 > 0$. Integrating this gives (6.6). Furthermore, for an arbitrary $\varrho > 0$ we have

$$\begin{aligned}\|\nabla\boldsymbol{u}\|_2^2 &= \|A^{1/2}\boldsymbol{u}\|_2^2 = \int_0^\infty \lambda d\|E_\lambda\boldsymbol{u}\|_2^2 \ge \int_\varrho^\infty \lambda d\|E_\lambda\boldsymbol{u}\|_2^2 \\ &\ge \varrho\int_\varrho^\infty d\|E_\lambda\boldsymbol{u}\|_2^2 = \varrho(\|\boldsymbol{u}\|_2^2 - \|E_\varrho\boldsymbol{u}\|_2^2),\end{aligned}$$

and so (6.10) implies that

$$\frac{d}{dt}\|\boldsymbol{u}\|_2 + C_0\lambda\|\boldsymbol{u}\|_2 \le \lambda\|E_\lambda\boldsymbol{u}\|_2. \tag{6.11}$$

To estimate the right-hand side we invoke the integral equation

$$\boldsymbol{u}(t) = e^{-tL}\boldsymbol{a} - \int_0^t e^{-(t-\tau)L}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})(\tau)d\tau. \tag{6.12}$$

By Lemma 6.2, the boundedness of the semigroup $\{e^{-tL}\}_{t\ge 0}$ in $\boldsymbol{L}^2_\sigma$, and the fact that

$$\|e^{-tL}\boldsymbol{a}\|_2 = \|e^{-tL/2}e^{-tL/2}\boldsymbol{a}\|_2 \le Ct^{-n/4}\|e^{-tL/2}\boldsymbol{a}\|_1 \le Ct^{-n/4-\beta/2}$$

for large $t > 0$, we obtain

$$\begin{aligned}\|E_\lambda\boldsymbol{u}\|_2 &\le \|e^{-tL}\boldsymbol{a}\|_2 + \int_0^t \|E_\lambda e^{-(t-\tau)L}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_2 d\tau \\ &\le \|e^{-tL}\boldsymbol{a}\|_2 + C\lambda^{(n-2)/4}\int_0^t (t-\tau)^{-(\beta+1)/2}\|\boldsymbol{u}\|_q\|\nabla\boldsymbol{u}\|_2 d\tau \\ &\le C(1+t)^{-\beta/2-n/4} + C\lambda^{(n-2)/4}F(t)^{1/2}G(t)^{1/2},\end{aligned}$$

where

$$G(t) = \int_0^t (t-\tau)^{-(\beta+1)/2}\|\nabla\boldsymbol{u}\|_2^2 d\tau.$$

We substitute this in (6.11), take $\lambda = m/(C_0 t)$ with $m = \frac{n+2}{4}$, multiply both sides by $t^m$, and integrate in $t$, to get

$$\begin{aligned}\|\boldsymbol{u}(t)\|_2 &\le C(1+t)^{-\beta/2-n/4} \\ &\quad + Ct^{(2-n)/4}\left(\frac{1}{t}\int_0^t F d\tau\right)^{1/2}\left(\frac{1}{t}\int_0^t G d\tau\right)^{1/2}.\end{aligned} \tag{6.13}$$

But, direct calculation using (6.6) gives

$$\frac{1}{t}\int_0^t G d\tau \le C t^{-(\beta+1)/2}$$

and so we get (6.7) from (6.13).

To prove (6.9) we invoke

$$\langle \boldsymbol{u}(t), \psi\rangle = \langle e^{-tL/2}\boldsymbol{u}(t/2), \psi\rangle - \int_{t/2}^t \langle \boldsymbol{u}\cdot\nabla\boldsymbol{u}, e^{-(t-\tau)L^*}P\psi\rangle d\tau$$

with $\psi \in \boldsymbol{C}_c^\infty(\mathbb{R}^n)$, to get

$$\begin{aligned}
&|\langle \boldsymbol{u}(t), \psi\rangle| \\
&\quad \le \|e^{-tL/2}\boldsymbol{u}(t/2)\|_1\|\psi\|_\infty + C\int_{t/2}^t \|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2\|P\psi\|_{\mathrm{BMO}} d\tau \\
&\quad \le C\left(\|e^{-tL/2}\boldsymbol{u}(t/2)\|_1 + \int_{t/2}^t \|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2 d\tau\right)\|\psi\|_\infty \\
&\quad \le C\left[\|e^{-tL/2}\boldsymbol{u}(t/2)\|_1 + \left(\int_{t/2}^t \|\boldsymbol{u}\|_2^2 d\tau\right)^{1/2}\left(\int_{t/2}^t \|\nabla\boldsymbol{u}\|_2^2 d\tau\right)^{1/2}\right]\|\psi\|_\infty,
\end{aligned}$$

so that, by (6.6) and Remark at the end of Section 5,

$$\|\boldsymbol{u}(t)\|_1 \le \|e^{-tL/2}\boldsymbol{u}(t/2)\|_1 + C\left(\int_{t/2}^t \|\boldsymbol{u}\|_2^2 d\tau\right)^{1/2}. \tag{6.14}$$

On the other hand, from (6.12) we have

$$\begin{aligned}
\langle e^{-tL/2}\boldsymbol{u}(t/2), \psi\rangle &= \langle \boldsymbol{u}(t/2), e^{-tL^*/2}P\psi\rangle \\
&= \langle e^{-tL/2}\boldsymbol{a}, e^{-tL^*/2}P\psi\rangle \\
&\quad - \int_0^{t/2}\langle \boldsymbol{u}\cdot\nabla\boldsymbol{u}, e^{-(t/2-\tau)L^*}e^{-tL^*/2}P\psi\rangle d\tau \\
&= \langle e^{-tL}\boldsymbol{a}, \psi\rangle - \int_0^{t/2}\langle e^{-(t-\tau)L}P(\boldsymbol{u}\cdot\nabla\boldsymbol{u}), \psi\rangle d\tau
\end{aligned}$$

so that, by (3.29) and (6.6),

$$|\langle e^{-tL/2}\boldsymbol{u}(t/2), \psi\rangle| \le \|e^{-tL}\boldsymbol{a}\|_1\|\psi\|_\infty$$

$$
\begin{aligned}
&+ C\int_0^{t/2}(t-\tau)^{-\beta/2}\|\boldsymbol{u}\|_q\|\nabla\boldsymbol{u}\|_2\|\psi\|_{\mathrm{BMO}}d\tau\\
\le\ & \left[\|e^{-tL}\boldsymbol{a}\|_1 + Ct^{-\beta/2}\int_0^{t/2}\|\boldsymbol{u}\|_q\|\nabla\boldsymbol{u}\|_2 d\tau\right]\|\psi\|_\infty\\
\le\ & \left[\|e^{-tL}\boldsymbol{a}\|_1 + Ct^{-\beta/2}\left(\int_0^{t/2}\|\boldsymbol{u}\|_q^2 d\tau\right)^{1/2}\right]\|\psi\|_\infty.
\end{aligned}
$$

Therefore,

$$
\|e^{-tL/2}\boldsymbol{u}(t/2)\|_1 \le C(1+t)^{-\beta/2} + Ct^{-\beta/2}\left(\int_0^{t/2}\|\boldsymbol{u}\|_q^2 d\tau\right)^{1/2}.
$$

Combining this with (6.14) gives (6.9). Estimate (6.9′) is deduced similarly. This proves Lemma 6.3. □

PROOF OF THEOREM 6.1. We give a detailed proof of assertion (i); assertion (ii) is proved similarly by employing (6.9′) instead of (6.9).

(I) Since $\|e^{-tL}\boldsymbol{a}\|_2 \le C(1+t)^{-n/4}$, we already know that

$$
\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-n/4} \qquad \text{and} \qquad \lim_{t\to\infty}\|\boldsymbol{u}(t)\|_1 = 0.
$$

Hence $\|\boldsymbol{u}(t)\|_q \le \|\boldsymbol{u}(t)\|_1^{2\beta/n}\|\boldsymbol{u}(t)\|_2^{1-2\beta/n} \le C(1+t)^{\beta/2-n/4}$. This implies that

$$
\frac{1}{t}\int_0^t F d\tau \le \begin{cases} Ct^{\beta/2-1} & (\beta > n/2-1,\ n=3)\\ Ct^{-(\beta+1)/2} & (0<\beta<n/2-1)\\ C_\delta t^{-3/4+\delta} & (\beta = 1/2,\ n=3)\end{cases}
$$

for any small $\delta > 0$. From (6.7) we see that, for any small $\delta > 0$,

$$
\begin{aligned}
(6.15)\quad \|\boldsymbol{u}(t)\|_2 \ \le\ & C(1+t)^{-\beta/2-n/4}\\
&+\begin{cases} C(1+t)^{-1} & (\beta > n/2-1,\ n=3)\\ C(1+t)^{-\beta/2-n/4} & (0<\beta<n/2-1)\\ C_\delta(1+t)^{-1+\delta} & (\beta=1/2,\ n=3).\end{cases}
\end{aligned}
$$

(II) Suppose $0<\beta<n/2-1$ so that $\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-\beta/2-n/4}$ by (6.15). Then (6.9) gives

$$
\begin{aligned}
\|\boldsymbol{u}(t)\|_1 \ \le\ & C(1+t)^{-\beta/2} + C(1+t)^{1/2-\beta/2-n/4}\\
&+ Ct^{-\beta/2}\left(\int_0^{t/2}\|\boldsymbol{u}\|_q^2 d\tau\right)^{1/2}.
\end{aligned}
$$

But, (6.15) and the boundedness of $\|\boldsymbol{u}(t)\|_1$ together imply that

$$\|\boldsymbol{u}(t)\|_q^2 \leq C(1+t)^{-(1-2\beta/n)(\beta+n/2)} = C(1+t)^{-n/2+2\beta^2/n},$$

and so

$$\begin{aligned}\|\boldsymbol{u}(t)\|_1 \quad &\leq \quad C(1+t)^{-\beta/2} + C(1+t)^{1/2-\beta/2-n/4} \\ &\qquad + C(1+t)^{\beta^2/n+1/2-n/4-\beta/2}.\end{aligned}$$

Since $0 < \beta < n/2 - 1$, this proves the desired result for $\|\boldsymbol{u}(t)\|_1$.

(III) Consider next the case $\beta \geq 1/2$ and $n = 3$. From (6.15) we have

$$\|\boldsymbol{u}(t)\|_q \leq \begin{cases} C(1+t)^{-1+2\beta/3} & (\beta > 1/2) \\ C_\delta(1+t)^{-2(1-\delta)/3} & (\beta = 1/2) \end{cases}$$

for any small $\delta > 0$. Then (6.9) gives

$$\|\boldsymbol{u}(t)\|_1 \leq C(1+t)^{-\beta/2} + \begin{cases} C(1+t)^{-\beta/2} & (\beta < 3/4) \\ C_\delta(1+t)^{-3/8+\delta} & (\beta = 3/4) \\ C(1+t)^{-1/2+\beta/6} & (\beta > 3/4) \end{cases} \tag{6.16}$$

for any small $\delta > 0$. Thus, for $1/2 \leq \beta < 3/4$, estimate (6.16) gives the desired bound for the norm $\|\boldsymbol{u}(t)\|_1$. In this case we have

$$\|\boldsymbol{u}(t)\|_q^2 \leq C_\delta(1+t)^{-2\beta^2/3+2(-1+\delta)(1-2\beta/3)}$$

for any small $\delta > 0$. Since $2\beta/3 < 1/2$, we can take $\delta > 0$ so that the last term is integrable in $t \in (0, \infty)$. We thus have

$$\left(\frac{1}{t}\int_0^t F d\tau\right)^{1/2} \leq Ct^{-(\beta+1)/4}$$

and so (6.7) gives the desired result for $\|\boldsymbol{u}(t)\|_2$ with $1/2 \leq \beta < 3/4$ and $n = 3$.

(IV) If $3/4 \leq \beta < 1$ and $n = 3$, then (6.16) and (6.15) together yield

$$\|\boldsymbol{u}(t)\|_q^2 \leq \begin{cases} C(1+t)^{-2+2\beta/3+2\beta^2/9} & (\beta > 3/4) \\ C_\delta(1+t)^{-11/8+\delta} & (\beta = 3/4) \end{cases}$$

for any small $\delta > 0$. Thus, $\|\boldsymbol{u}\|_q^2$ is integrable in $t \in (0, \infty)$, and so

$$\left(\frac{1}{t}\int_0^t F d\tau\right)^{1/2} \le Ct^{-(\beta+1)/4}.$$

Substituting this in (6.7) gives

$$\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-\beta/2-n/4}$$

and this shows the desired result for $\|\boldsymbol{u}(t)\|_2$. Since $\|\boldsymbol{u}\|_q^2$ is integrable in $t \in (0, \infty)$, the desired bound for $\|\boldsymbol{u}(t)\|_1$ in the case $3/4 \le \beta < 1$ is now obtained from (6.9). This completes the proof of Theorem 6.1. □

As is seen from the above argument, estimates (3.34) and (5.9) play a crucial role for deducing the exponent $\beta$. Thus, to improve our decay result, it would be desirable to improve (3.34) and (5.9) to the form which covers some exponent $p \le n/(n+1)$. However, even if this is the case, it seems impossible to get an improved decay rate for solutions to the nonlinear problem. Indeed, we have the followng

PROPOSITION 6.4. *Let* $\boldsymbol{u} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{L}^q_\sigma$ *and* $\nabla\boldsymbol{u} \in \boldsymbol{L}^2 \cap \boldsymbol{L}^r$ *for some* $1 < q < \infty$ *and* $1 < r < \infty$ *with*

$$1/q + 1/r = 1 + 1/n.$$

*If* $\boldsymbol{u}\cdot\nabla\boldsymbol{u} \in \boldsymbol{H}^{n/(n+1)}$, *then* $\boldsymbol{u} \equiv 0$.

Because of Proposition 6.4, we cannot apply our method in order to deduce an improved decay rate from the nonlinear term $\int_0^t \langle \boldsymbol{u}\cdot\nabla\boldsymbol{u}, e^{-(t-\tau)L^*}P\psi\rangle d\tau$. In this sense, our decay result seems to be optimal in the case $\boldsymbol{w} \ne 0$, insofar as (1.3) or (1.5) is assumed.

PROOF OF PROPOSITION 6.4. The assumption implies $\boldsymbol{u} \in \boldsymbol{H}^{n/(n+1)} \cap \boldsymbol{L}^1$. We thus see that (see [26, p. 128]) the function $|x|\cdot|\boldsymbol{u}\cdot\nabla\boldsymbol{u}|$ is integrable on $\mathbb{R}^n$ and

$$\int x_j \boldsymbol{u}\cdot\nabla u_\ell dx = \sum_{k=1}^n \int x_j\partial_k(u_k u_\ell)dx = 0 \qquad \text{for } j, \ell = 1, \cdots, n. \tag{6.17}$$

Here we take $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\psi(x) = 1$ for $|x| \le 1$, $\psi(x) = 0$ for $|x| \ge 2$, and set $\psi_N(x) = \psi(x/N)$ with $N > 0$. An integration by parts then gives

$$
\begin{aligned}
&\sum_{k=1}^n \int x_j \partial_k (u_k u_\ell) dx \\
&\qquad = \lim_{N\to\infty} \sum_{k=1}^n \int x_j \psi_N \partial_k (u_k u_\ell) dx \\
&\qquad = -\lim_{N\to\infty} \sum_{k=1}^n \left[ \int \delta_{jk} \psi_N u_k u_\ell dx + \int_{N\le|x|\le 2N} x_j (\partial_k \psi_N) u_k u_\ell dx \right] \\
&\qquad \equiv -\lim_{N\to\infty} (I_{1,N} + I_{2,N}).
\end{aligned}
$$

Since $\boldsymbol{u}$ is in $\boldsymbol{L}^2$, an elementary calculation shows

$$
\lim_{N\to\infty} I_{1,N} = \lim_{N\to\infty} \sum_{k=1}^n \int \delta_{jk} \psi_N u_k u_\ell dx = \lim_{N\to\infty} \int \psi_N u_j u_\ell dx = \int u_j u_\ell dx,
$$

and, with $M = \sup |\nabla\psi|$,

$$
\begin{aligned}
|I_{2,N}| &\le 2nMN^{-1} \int_{N\le|x|\le 2N} N |\boldsymbol{u}|^2 dx \\
&= 2nM \int_{N\le|x|\le 2N} |\boldsymbol{u}|^2 dx \to 0 \qquad \text{as} \quad N \to \infty.
\end{aligned}
$$

Hence, (6.17) gives

$$
0 = \int x_j \boldsymbol{u} \cdot \nabla \boldsymbol{u} dx = -\int u_j u_\ell dx \qquad \text{for} \quad j, \ell = 1, \cdots, n,
$$

and therefore $\boldsymbol{u} \equiv 0$. This completes the proof. □

## 7. More on Decay Rates in $\boldsymbol{L}^1_\sigma$

Up to the previous section we have treated stationary flows $\boldsymbol{w}$ satisfying (1.3) or (1.5). However, the flows satisfying (1.2) or (1.6) contain more informations on the spatial decay. For example, property (1.6) implies $\boldsymbol{w} \in \boldsymbol{L}^r$ for $n/(n-1) < r \le \infty$; and when $n \ge 5$ and $\ell = n$, property (1.2) implies $\boldsymbol{w} \in \boldsymbol{L}^r$ for $2 \leq r \leq \infty$. With these examples in mind, we shall consider in this section the flows $\boldsymbol{w}$ satisfying

$$\|\boldsymbol{w}\| < \frac{n-2}{2}, \qquad \boldsymbol{w} \in \boldsymbol{L}^2(\mathbb{R}^n), \tag{7.1}$$

and discuss the time-decay of perturbations in $\boldsymbol{L}^2_\sigma$ and in $\boldsymbol{L}^1_\sigma$. As will be seen from the argument below, one can replace (7.1) by

$$\boldsymbol{w} \in \boldsymbol{L}^2(\mathbb{R}^n) \cap \boldsymbol{L}^n(\mathbb{R}^n), \qquad \text{and} \quad \|\boldsymbol{w}\|_n \quad \text{is sufficiently small.} \tag{7.1$'$}$$

Assuming (7.1) or (7.1′), we shall show that there are lots of weak solutions of the perturbation problem which decay in $\boldsymbol{L}^1_\sigma$ like $t^{-1/2}$. Furthermore, it should be noticed that we will need in this section neither conditions on the derivative $\nabla \boldsymbol{w}$, nor detailed analysis on the linearized operator $L = A + B$. Indeed, the result will be directly obtained from the integral equation :

$$\begin{aligned} \boldsymbol{u}(t) &= e^{-tA}\boldsymbol{a} - \int_0^t e^{-(t-\tau)A} B\boldsymbol{u}(\tau)d\tau - \int_0^t e^{-(t-\tau)A} P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})(\tau)d\tau \\ &= e^{-tA}\boldsymbol{a} - \int_0^t P\nabla e^{-(t-\tau)A}(\boldsymbol{w}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{w} + \boldsymbol{u}\otimes\boldsymbol{u})(\tau)d\tau. \end{aligned} \tag{7.2}$$

The result is stated as follows.

THEOREM 7.1. (i) *Under the assumption* (7.1) *or* (7.1′), *there exists for each* $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma$ *a weak solution* $\boldsymbol{u}$ *such that*

$$\lim_{t\to\infty} \|\boldsymbol{u}(t)\|_2 = 0. \tag{7.3}$$

*Furthermore, if* $\|e^{-tA}\boldsymbol{a}\|_2 \leq C(1+t)^{-\alpha}$ *for some* $\alpha > 0$, *then*

$$\|\boldsymbol{u}(t)\|_2 \leq C(1+t)^{-\beta} \qquad \text{with} \quad \beta = \min(\alpha, (n+2)/4). \tag{7.4}$$

(ii) *For any* $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{L}^1_\sigma$, *the weak solution* $\boldsymbol{u}$ *treated in* (i) *lies in* $\boldsymbol{L}^1_\sigma$ *for all* $t \geq 0$ *and satisfies*

$$\lim_{t\to\infty} \|\boldsymbol{u}(t)\|_1 = 0. \tag{7.5}$$

*Furthermore, if* $\|e^{-tA}\boldsymbol{a}\|_1 \leq C(1+t)^{-\alpha}$ *for some* $\alpha > 0$, *then*

$$\|\boldsymbol{u}(t)\|_1 \leq C(1+t)^{-\gamma} \qquad \text{with} \quad \gamma = \min(\alpha, 1/2). \tag{7.6}$$

REMARKS. (i) Part (i) of Theorem 7.1 is due to Grunau [9]. He treated the case where

$$n=3, \qquad \|\boldsymbol{w}\|<(n-2)/2, \qquad \boldsymbol{w}\in \boldsymbol{L}^r(\mathbb{R}^n) \quad (2\le r<n)$$

and proved that if $\|e^{-tA}\boldsymbol{a}\|_2\le C(1+t)^{-\alpha}$, there is a weak solution $\boldsymbol{u}$ such that

$$\|\boldsymbol{u}(t)\|_2\le C(1+t)^{-\beta} \qquad \text{with} \ \ \beta=\min\left(\alpha,(n+2)/4,1/2+n/2r\right). \tag{7.7}$$

Our proof of Theorem 7.1 given below can be adapted to deducing (7.7) in the case of general space dimensions $n\ge 3$.

(ii) Part (ii) is intended merely to show the existence of a weak solution $\boldsymbol{u}$ which decays in $\boldsymbol{L}^1_\sigma$ like $t^{-1/2}$ under the assumption (7.1) or (7.1′), and we aim at no generality on the conditions to be satisfied by the stationary flows $\boldsymbol{w}$.

(iii) Examples of initial velocities $\boldsymbol{a}\in \boldsymbol{L}^1_\sigma\cap \boldsymbol{L}^2_\sigma$ satisfying $\|e^{-tA}\boldsymbol{a}\|_1\le C(1+t)^{-\alpha}$ are furnished by (higher-order) derivatives of vector fields in $\boldsymbol{C}^\infty_{0,\sigma}(\mathbb{R}^n)$. Indeed, such vector fields satisfy the moment condition

$$\int x^\gamma \boldsymbol{a}(x)dx=0 \qquad (|\gamma|\le m)$$

for some integer $m\ge 0$, and this implies the estimate

$$\|e^{-tA}\boldsymbol{a}\|_1\le C(1+t)^{-(m+1)/2}.$$

The case $m=0$ is discussed in [17].

PROOF OF THEOREM 7.1. We consider only the case where $\boldsymbol{w}$ satisfies (7.1). The other case that $\boldsymbol{w}$ satisfies (7.1′) is treated similarly.

(i) Assuming (7.1), we obtain as in Section 6

$$\frac{d}{dt}\|\boldsymbol{u}\|_2^2+2C_0\|\nabla \boldsymbol{u}\|_2^2\le 0,$$

and so

$$\|\boldsymbol{u}(t)\|_2^2\le \|\boldsymbol{a}\|_2^2\,; \qquad \int_0^\infty \|\nabla \boldsymbol{u}\|_2^2 d\tau\le C\|\boldsymbol{a}\|_2^2. \tag{7.8}$$

In the same way as in Sections 4, 5 and 6, we introduce the spectral measure $E_\lambda$ associated with $A_2$ in $\boldsymbol{L}^2_\sigma$, to deduce

$$\frac{d}{dt}\|\boldsymbol{u}\|_2 + C_0\lambda\|\boldsymbol{u}\|_2 \le \lambda\|E_\lambda \boldsymbol{u}\|_2. \tag{7.9}$$

But, since $E_\lambda e^{-tA} = e^{-tA}E_\lambda$, from (7.2) we get

$$\|E_\lambda \boldsymbol{u}\|_2 \le \|e^{-tA}\boldsymbol{a}\|_2 + \int_0^t \|E_\lambda B\boldsymbol{u}\|_2 d\tau + \int_0^t \|E_\lambda P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_2 d\tau.$$

Here we invoke the following

LEMMA 7.2. *Under the assumption* (7.1), *we have*

$$\begin{aligned} &\|E_\lambda B\boldsymbol{u}\|_2 \le C\lambda^{(n+2)/4}\|\boldsymbol{w}\|_2\|\boldsymbol{u}\|_2\,; \\ &\|E_\lambda P(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_2 \le C\lambda^{(n+2)/4}\|\boldsymbol{u}\|_2^2. \end{aligned} \tag{7.10}$$

PROOF. For $\varphi \in \boldsymbol{L}^2_\sigma$, we have $E_\lambda\varphi \in C^\infty(A) \subset \bigcap_{m=1}^{\infty} \boldsymbol{W}^{m,2}(\mathbb{R}^n)$, and so

$$\begin{aligned} |\langle E_\lambda B\boldsymbol{u}, \varphi\rangle| &= |\langle \boldsymbol{w}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{w}, \nabla E_\lambda\varphi\rangle| \le 2\|\boldsymbol{w}\|_2\|\boldsymbol{u}\|_2\|\nabla E_\lambda\varphi\|_\infty\,; \\ |\langle E_\lambda P(\boldsymbol{u}\cdot\nabla\boldsymbol{u}), \varphi\rangle| &= |\langle \boldsymbol{u}\otimes\boldsymbol{u}, \nabla E_\lambda\varphi\rangle| \le \|\boldsymbol{u}\|_2^2\|\nabla E_\lambda\varphi\|_\infty. \end{aligned}$$

Since the Gagliardo–Nirenberg inequality ([8]) gives

$$\begin{aligned} \|\nabla E_\lambda\varphi\|_\infty &\le C\|\nabla E_\lambda\varphi\|_{2n}^{1/2}\|\nabla^2 E_\lambda\varphi\|_{2n}^{1/2} \le C\|A^{1/2}E_\lambda\varphi\|_{2n}^{1/2}\|AE_\lambda\varphi\|_{2n}^{1/2} \\ &\le C\|A^{(n+1)/4}E_\lambda\varphi\|_2^{1/2}\|A^{(n+3)/4}E_\lambda\varphi\|_2^{1/2} \le C\lambda^{(n+2)/4}\|\varphi\|_2, \end{aligned}$$

we obtain (7.10) via the duality $(\boldsymbol{L}^2_\sigma)^* = \boldsymbol{L}^2_\sigma$. This proves the lemma. □

Now we combine (7.10) with (7.9) to get

$$\begin{aligned} &\frac{d}{dt}\|\boldsymbol{u}\|_2 + C_0\lambda\|\boldsymbol{u}\|_2 \\ &\qquad \le C\lambda\left(\|e^{-tA}\boldsymbol{a}\|_2 + \lambda^{(n+2)/4}\int_0^t (\|\boldsymbol{u}\|_2 + \|\boldsymbol{u}\|_2^2) d\tau\right). \end{aligned} \tag{7.11}$$

We set $\lambda = m[C_0(1+t)]^{-1}$ with $m > 1 + \frac{n+2}{4}$ in (7.11), multiply both sides by $(1+t)^m$, and then integrate the resulting inequality with respect to $t$, to get

$$(7.12) \qquad \begin{aligned} \|\boldsymbol{u}(t)\|_2 \; &\leq \; \frac{C}{t}\int_0^t \|e^{-\tau A}\boldsymbol{a}\|_2 d\tau \\ &\quad + C(1+t)^{-(n+2)/4}\int_0^t (\|\boldsymbol{u}\|_2 + \|\boldsymbol{u}\|_2^2)d\tau. \end{aligned}$$

Since $\|e^{-tA}\boldsymbol{a}\|_2 \to 0$ as $t \to \infty$ for all $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma$, and since $\|\boldsymbol{u}\|_2 \in L^\infty(\mathbb{R}_+)$ by (7.8), it follows from (7.12) that

$$\|\boldsymbol{u}(t)\|_2 \leq \frac{C}{t}\int_0^t \|e^{-\tau A}\boldsymbol{a}\|_2 d\tau + C(1+t)^{1/2-n/4} \to 0 \qquad \text{as} \;\; t \to \infty.$$

This proves (7.3).

Suppose now that $\|e^{-tA}\boldsymbol{a}\|_2 \leq C(1+t)^{-\alpha}$ for some $\alpha > 0$. Since $\|\boldsymbol{u}\|_2 \in L^\infty(\mathbb{R}_+)$ by (7.8), we get from (7.12)

$$\|\boldsymbol{u}(t)\|_2 \leq C[(1+t)^{-\alpha} + (1+t)^{1/2-n/4}].$$

Hence, $\|\boldsymbol{u}(t)\|_2 = O(t^{-\alpha})$ if $\alpha < 1/4$. In the opposite case we have $\|\boldsymbol{u}(t)\|_2 \leq C(1+t)^{-1/4}$; so

$$\int_0^t \|\boldsymbol{u}\|_2 d\tau \leq C(1+t)^{3/4}, \qquad \int_0^t \|\boldsymbol{u}\|_2^2 d\tau \leq C(1+t)^{1/2},$$

and therefore by (7.12)

$$\|\boldsymbol{u}(t)\|_2 \leq C[(1+t)^{-\alpha} + (1+t)^{1/4-n/4}].$$

Thus, $\|\boldsymbol{u}(t)\|_2 \leq C(1+t)^{-\alpha}$ if $\alpha < 1/4 - n/4$. If $\alpha \geq 1/4 - n/4$, then $\|\boldsymbol{u}(t)\|_2 \leq C(1+t)^{1/4-n/4}$; so

$$\int_0^t \|\boldsymbol{u}\|_2 d\tau \leq \begin{cases} C(1+t)^{1/2} & (n=3) \\ C(1+t)^{1/4} & (n=4) \\ C\log(1+t) & (n=5) \\ C & (n \geq 6), \end{cases}$$

and

$$\int_0^t \|\boldsymbol{u}\|_2^2 d\tau \leq \begin{cases} C\log(1+t) & (n=3) \\ C & (n \geq 4). \end{cases}$$

By (7.12) we see that

$$(7.13)\quad \|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-\alpha} + \begin{cases} C(1+t)^{-n/4} & (n=3) \\ C(1+t)^{-(1+n)/4} & (n=4) \\ C(1+t)^{-(n+2)/4}\log(1+t) & (n=5) \\ C(1+t)^{-(n+2)/4} & (n \ge 6). \end{cases}$$

Hence, if $n \ge 6$, we conclude that

$$(7.14)\qquad \|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-\beta} \quad \text{with} \;\; \beta = \min(\alpha, (n+2)/4),$$

and (7.4) is deduced in all cases of $\alpha$. When $n = 4$ or $n = 5$, we see from (7.13) that

$$\int_0^t \|\boldsymbol{u}\|_2 d\tau \le C, \qquad \int_0^t \|\boldsymbol{u}\|_2^2 d\tau \le C.$$

Hence we get (7.14) via (7.12) and so (7.4) is completely proved also for $n = 4, 5$. When $n = 3$ and $\alpha \ge n/4$, we have $\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-n/4} = C(1+t)^{-3/4}$, and so

$$\int_0^t \|\boldsymbol{u}\|_2 d\tau \le C(1+t)^{1/4}, \qquad \int_0^t \|\boldsymbol{u}\|_2^2 d\tau \le C.$$

Thus, (7.12) gives

$$\|\boldsymbol{u}(t)\|_2 \le C[(1+t)^{-\alpha} + (1+t)^{-1}].$$

This shows $\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-\alpha}$ if $\alpha < 1$. In the opposite case we have $\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-1}$, so that

$$\int_0^t \|\boldsymbol{u}\|_2 d\tau \le C\log(1+t), \qquad \int_0^t \|\boldsymbol{u}\|_2^2 d\tau \le C.$$

Hence (7.12) yields

$$\|\boldsymbol{u}(t)\|_2 \le C[(1+t)^{-\alpha} + (1+t)^{-5/4}\log(1+t)].$$

This gives $\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-\alpha}$ if $\alpha < 5/4$. When $\alpha \ge 5/4$, we have $\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{\varepsilon-5/4}$ for all $\varepsilon > 0$, and so

$$\int_0^t \|\boldsymbol{u}\|_2 d\tau \le C, \qquad \int_0^t \|\boldsymbol{u}\|_2^2 d\tau \le C.$$

We thus get (7.14) via (7.12), and so (7.4) is completely proved also for $n = 3$.

(ii) To deduce $L^1$-decay rates, we start with the second equality of (7.2):

$$\text{(7.15)} \quad \boldsymbol{u}(t) = e^{-tA}\boldsymbol{a} - \int_0^t P\nabla e^{-(t-\tau)A}(\boldsymbol{w}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\boldsymbol{w} + \boldsymbol{u}\otimes\boldsymbol{u})d\tau.$$

The integral on the right-hand side is estimated in $\boldsymbol{L}^1_\sigma$ with the aid of the following

LEMMA 7.3. *The kernel function $K_t$ of the operator $\nabla e^{-tA}$ belongs to the Hardy space $\boldsymbol{H}^1(\mathbb{R}^n)$, with norm*

$$\|K_t\|_{H^1} \le Ct^{-1/2},$$

*where $C > 0$ is independent of $t > 0$.*

Admitting Lemma 7.3 for a moment, we continue our discussion. From (7.1), Lemma 7.3 and (7.15) it follows that

$$\text{(7.16)} \quad \begin{aligned} \|\boldsymbol{u}(t)\|_1 &\le \|e^{-tA}\boldsymbol{a}\|_1 + C\int_0^t (t-\tau)^{-1/2}(\|\boldsymbol{w}\otimes\boldsymbol{u}\|_1 + \|\boldsymbol{u}\otimes\boldsymbol{u}\|_1)d\tau \\ &\le \|e^{-tA}\boldsymbol{a}\|_1 + C\int_0^t (t-\tau)^{-1/2}(\|\boldsymbol{u}\|_2 + \|\boldsymbol{u}\|_2^2)d\tau. \end{aligned}$$

Since $\|e^{-tA}\boldsymbol{a}\|_2 \le C(1+t)^{-n/4}$ for $\boldsymbol{a} \in \boldsymbol{L}^2_\sigma \cap \boldsymbol{L}^1_\sigma$, part (i) shows that $\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-n/4}$. Thus, (7.16) implies

$$\begin{aligned} \|\boldsymbol{u}(t)\|_1 &\le \|e^{-tA}\boldsymbol{a}\|_1 + C\int_0^t (t-\tau)^{-1/2}[(1+\tau)^{-n/4} + (1+\tau)^{-n/2}]d\tau \\ &\le \|e^{-tA}\boldsymbol{a}\|_1 + \begin{cases} C(1+t)^{-1/4} & (n=3) \\ C(1+t)^{-1/2}\log(1+t) & (n=4) \\ C(1+t)^{-1/2} & (n\ge 5). \end{cases} \end{aligned}$$

Since $\lim_{t\to\infty}\|e^{-tA}\boldsymbol{a}\|_1 = 0$ because $\int \boldsymbol{a}(x)dx = 0$ for all $\boldsymbol{a} \in \boldsymbol{L}^1_\sigma$ (see [17]), we conclude that

$$\lim_{t\to\infty}\|\boldsymbol{u}(t)\|_1 = 0,$$

and so (7.5) is proved.

Suppose next that $\|e^{-tA}\boldsymbol{a}\|_1 \le C(1+t)^{-\alpha}$ for some $\alpha > 0$. Then $\|e^{-tA}\boldsymbol{a}\|_2 \le C(1+t)^{-\alpha-n/4}$, and so part (i) gives $\|\boldsymbol{u}(t)\|_2 \le C(1+t)^{-\beta-n/4}$ with $\beta = \min(\alpha, 1/2)$. Suppose first $n \ge 4$. Then $\beta + n/4 > 1$, and so (7.16) gives

$$\|\boldsymbol{u}(t)\|_1 \le C[(1+t)^{-\alpha} + (1+t)^{-1/2}].$$

This proves (7.6) for $n \ge 4$. Suppose next $n = 3$. If $\alpha < 1/4$, then (7.16) gives

$$\begin{aligned}\|\boldsymbol{u}(t)\|_1 &\le C(1+t)^{-\alpha} + C\int_0^t (t-\tau)^{-1/2}[(1+\tau)^{-\alpha-3/4} \\ &\quad + (1+\tau)^{-2\alpha-3/2}]d\tau \\ &\le C[(1+t)^{-\alpha} + (1+t)^{-\alpha-1/4}] \le C(1+t)^{-\alpha}.\end{aligned}$$

If $\alpha = 1/4$, then from (7.16) we have

$$\|\boldsymbol{u}(t)\|_1 \le C[(1+t)^{-1/4} + (1+t)^{-1/2}\log(1+t)] \le C(1+t)^{-1/4}.$$

If $\alpha > 1/4$, then $\beta + n/4 > 1$; so (7.16) implies

$$\|\boldsymbol{u}(t)\|_1 \le C[(1+t)^{-\alpha} + (1+t)^{-1/2}].$$

We have thus proved (7.6) for all $\alpha$ in the case $n = 3$. The proof of Theorem 7.1 is complete. □

PROOF OF LEMMA 7.3. Consider the function $g(x) = \pi^{-n/2}\nabla e^{-|x|^2} = -2\pi^{-n/2}xe^{-|x|^2}$. Then

$$K_t(x) = (4\pi t)^{-n/2}\nabla e^{-|x|^2/4t} = (4t)^{-(n+1)/2}g(x/\sqrt{4t}) \equiv (4t)^{-(n+1)/2}g_t(x).$$

Thus, if $g \in \boldsymbol{H}^1(\mathbb{R}^n)$, then $K_t \in \boldsymbol{H}^1(\mathbb{R}^n)$ and

$$\|K_t\|_{H^1} = (4t)^{-(n+1)/2}\|g_t\|_{H^1} = (4t)^{-(n+1)/2+n/2}\|g\|_{H^1} = Ct^{-1/2},$$

and the proof will be complete.

It thus remains to show that $g \in \boldsymbol{H}^1(\mathbb{R}^n)$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be supported by the unit ball centered at the origin, satisfying $\int \varphi dx = 1$; and set $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$. Observe first that Young's inequality gives $|\varphi_t * g| \le \|\varphi\|_1\|g\|_\infty$; so we get

$$\sup_{t>0}|\varphi_t * g| \in L^\infty(\mathbb{R}^n) \subset L^1(B_1), \tag{7.17}$$

where $B_1$ is the unit ball centered at the origin. To find an appropriate estimate for $|x| > 1$, we write $g = \nabla f$ with $f = \pi^{-n/2} e^{-|x|^2}$, to get

$$\begin{aligned}(\varphi_t * g)(x) &= \frac{1}{t^n} \int_{B_t(x)} \varphi\left(\frac{x-y}{t}\right) g(y)dy \\ &= \frac{1}{t^{n+1}} \int_{B_t(x)} (\nabla\varphi)\left(\frac{x-y}{t}\right) f(y)dy.\end{aligned}$$

For any $x \in \mathbb{R}^n$ with $|x| > 1$ and any $t \geq (1+|x|)/2$, we get

$$|(\varphi_t * g)(x)| \leq \frac{A}{t^{n+1}} \int_{B_t(x)} |f(y)|dy \leq A(1+|x|)^{-n-1} \|f\|_1,$$

with $A > 0$ depending only on $n$ and $\varphi$. If $0 < t \leq (1+|x|)/2$, then we get

$$\begin{aligned}|(\varphi_t * g)(x)| &\leq \frac{C}{|B_t(x)|} \int_{B_t(x)} |g(y)|dy \leq C \sup_{|y-x|<(1+|x|)/2} |g(y)| \\ &\leq C \sup_{|y|>(|x|-1)/2} |g(y)|,\end{aligned}$$

with $C > 0$ depending only on $n$ and $\varphi$. Since $g$ is in $\mathscr{S}$, for $0 < t \leq (1+|x|)/2$ we have

$$|(\varphi_t * g)(x)| \leq C \sup_{|y|>(|x|-1)/2} (1+|y|)^{-n-1} \leq C(1+|x|)^{-n-1},$$

where $C > 0$ depends only on $n$, $\varphi$ and $g$. We thus conclude that

$$\sup_{t>0} |(\varphi_t * g)(x)| \leq A(1+|x|)^{-n-1} \qquad \text{whenever} \quad |x| > 1,$$

with $A > 0$ depending only on $n$, $\varphi$, $f$ and $g$. Combining this with (7.17) gives

$$\sup_{t>0} |\varphi_t * g| \in L^1(\mathbb{R}^n)\,; \qquad \text{and so} \quad g \in \boldsymbol{H}^1(\mathbb{R}^n).$$

This proves Lemma 7.3. □

Remark. The above proof actually shows that if $g = \nabla f$ and $f \in \mathscr{S}$, then $g \in \boldsymbol{H}^p(\mathbb{R}^n)$ for all $p$ with $n/(n+1) < p \leq 1$, and even more, $g \in \boldsymbol{H}_w^{n/(n+1)}(\mathbb{R}^n)$. The same method applies to showing that if $g = \nabla^k f$ for some integer $k > 1$ and $f \in \mathscr{S}$, then $g \in \boldsymbol{H}^p(\mathbb{R}^n)$ for all $p$ with $n/(n+k) < p \leq 1$, and even more, $g \in \boldsymbol{H}_w^{n/(n+k)}(\mathbb{R}^n)$.

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