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A Sheaf-Theoretic Approach to the Equivariant Serre Spectral Sequence

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Abstract. Let G be a finite group and $f : Y \to X$ a Hurewicz G-fibration. Using sheaf-theoretic methods, we show that, under suitable assumptions on the G-spaces X and Y, the equivariant cohomology of Y is the limit of a spectral sequence, whose E_2-term is given by the equivariant cohomology of X with coefficient system depending on the equivariant cohomology of the fibre of f.

Introduction

Let G be a finite group and $\mathcal{C} = \mathcal{C}_G = \mathrm{Or}(G)$ the orbit category of G; the objects of $\mathcal{C}$ are the G-sets G/H for all subgroups $H \leq G$ and the morphisms of $\mathcal{C}$ are all G-maps between the G-sets G/H.

In [5] we associated to a G-space X a Grothendieck topos $\Gamma(\widetilde{X})$ whose objects are certain families of sheaves on the fixed point subspaces X^K of X. The main result of [5] states that if X is paracompact, then for any contravariant G-coefficient system $m : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$, the equivariant cohomology groups $\bar{H}^n_G(X; m)$ of X are isomorphic to the cohomology groups of the topos $\Gamma(\widetilde{X})$ with coefficients in a family of constant sheaves determined by m. We also showed that, given a G-map $f : Y \to X$, there is a spectral sequence expressing a connection between the cohomology of $\Gamma(\widetilde{X})$ and $\Gamma(\widetilde{Y})$.

The purpose of the present paper is to obtain a more concrete form of this connection. In particular, we derive the following equivariant Serre spectral sequence:

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Theorem. *Let $f : Y \to X$ be a Hurewicz G-fibration. Assume that all subspaces of the G-spaces X and Y are paracompact, X is locally G-contractible, and all fixed point subspaces of X are non-empty and simply connected. Let also $m : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ be a contravariant G-coefficient system. Then there is a spectral sequence with*

$$E_2^{pq} = \bar{H}_G^p(X; \mathcal{H}^q(F)),$$

converging to $\bar{H}_G^{p+q}(Y; m)$. Here $\mathcal{H}^q(F)$ is a contravariant G-coefficient system associating to G/P the equivariant cohomology group $\bar{H}_P^q(F; m_P)$ of the fibre F of f, and m_P is the P-coefficient system with $m_P(P/H) = m(G/H)$ for $H \leq P$.

If the simply connectedness hypothesis on X is dropped, there still is a a spectral sequence, but in this case the E_2-term is an object which may be regarded as the equivariant cohomology of X with suitable local (i.e. locally constant) coefficients.

The spectral sequence of the Theorem, as well as that of [5], can be regarded as generalizations of the classical Leray spectral sequence in sheaf cohomology. They are both constructed by an application of Grothendieck's spectral sequence of composite functors, see Théorème 2.4.1 of [2].

We point out that our methods are sheaf theoretic. This is the reason for the strong hypotheses needed in the Theorem. Also for this reason, the equivariant cohomology theory we use is the equivariant Alexander-Spanier cohomology constructed in [4]. For paracompact G-spaces which are locally sufficiently nice, as X in the Theorem, this version of equivariant cohomology is, of course, isomorphic to, for example, the equivariant singular cohomology of Illman (see [6]).

A different construction of the equivariant Serre spectral sequence has been given by Moerdijk and Svensson in [7], using singular methods. Their approach avoids many of the extra assumptions needed in our sheaf-theoretic approach.

The organization of this paper is the following: In section 1 we recall the definition of $\Gamma(\widetilde{X})$ and also construct a new topos $\Sigma(X)$ associated to a G-space. Section 2 is devoted to a study of certain morphisms of topoi needed in the construction of the spectral sequence of the Theorem. In section 3 we consider the connection between the cohomology of X and

Y for a G-map $f : Y \to X$, and in section 4 we finally specialize to a G-fibration f.

1. Topoi Associated to a G-Space

Let G be a finite group and $\mathcal{C} = \mathcal{C}_G = \mathrm{Or}(G)$ the orbit category of G (see the Introduction). As in [5], we let $\mathcal{D} = \mathcal{D}_G$ be the category whose objects are the morphisms $u : G/H \to G/K$ of $\mathcal{C}$ and whose morphisms $u \to u'$ are the pairs (α, β) of morphisms of $\mathcal{C}$ making the square

(1.1)
$$\begin{array}{ccc} G/H & \xleftarrow{\beta} & G/H' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ G/K & \xrightarrow{\alpha} & G/K' \end{array}$$

commutative. In the terminology of [1] we have $\mathcal{D} = F\mathcal{C}$, the category of factorizations in $\mathcal{C}$.

Let X be a (Hausdorff) G-space. To X we can associate the functor $X : \mathcal{C}^{\mathrm{op}} \to \mathbf{Top}$ given by $G/K \mapsto X^K \cong \mathrm{Map}_G(G/K, X)$ on objects. Using the methods of [8, ch. I] we obtain a $\mathcal{C}$-topos $\overline{X} \to \mathcal{C}$ whose fibre over G/K is $\mathrm{Sh}(X^K)$, the topos of sheaves on X^K; a morphism $\alpha : G/K \to G/K'$ of $\mathcal{C}$ induces the morphism

$$(X(\alpha)_*, X(\alpha)^*) : \mathrm{Sh}(X^{K'}) \longrightarrow \mathrm{Sh}(X^K)$$

of topoi between the fibres of $\overline{X}$. Let $\widetilde{X} = \mathcal{D} \times_{\mathcal{C}} \overline{X}$ be the fibre product of $\overline{X} \to \mathcal{C}$ and the "target" functor $T : \mathcal{D} \to \mathcal{C}$ mapping $u \mapsto G/K$ ($u \in \mathrm{Ob}(\mathcal{D})$ as above). Then $\widetilde{X} \to \mathcal{D}$ is a $\mathcal{D}$-topos whose fibre over the object $u : G/H \to G/K$ of $\mathcal{D}$ is $\mathrm{Sh}(X^K)$.

The topos $\Gamma(\widetilde{X})$ of sections of $\widetilde{X} \to \mathcal{D}$ was the main object of study in [5]. We recall that the objects of $\Gamma(\widetilde{X})$ are the families $\mathcal{F} = (\mathcal{F}(u))_{u \in \mathrm{Ob}(\mathcal{D})}$, where for each $u : G/H \to G/K$, $\mathcal{F}(u) \in \mathrm{Sh}(X^K)$, such that each morphism (α, β) of $\mathcal{D}$, as in 1.1, induces functorially a morphism

$$\mathcal{F}(\alpha, \beta) : \mathcal{F}(u) \longrightarrow X(\alpha)_* \mathcal{F}(u').$$

We also recall that the above construction is natural with respect to G-maps. Namely, if $f : Y \to X$ is a G-map between G-spaces, inducing the map $f^K : Y^K \to X^K$ between fixed point subspaces for $K \leq G$, we have an obvious natural transformation between the functors $Y, X : \mathcal{C}^{\mathrm{op}} \to \mathbf{Top}$, and this gives rise to a morphism

$$(f_*, f^*) : \Gamma(\widetilde{Y}) \longrightarrow \Gamma(\widetilde{X})$$

of topoi. Explicitly, if $\mathcal{F} = (\mathcal{F}(u))$ is an object of $\Gamma(\widetilde{Y})$, then $(f_*\mathcal{F})(u) = f_*^K \mathcal{F}(u)$ for $u : G/H \to G/K$.

Now we associate to the G-space X a new category $\Sigma(X)$. The objects of $\Sigma(X)$ are the families $\mathcal{E} = (\mathcal{E}_H)_{H \leq G}$, where $\mathcal{E}_H \in \mathrm{Sh}(X^H)$ for each $H \leq G$, such that every morphism $\beta : G/H' \to G/H$ of $\mathcal{C}$ induces functorially a morphism of sheaves

$$\mathcal{E}(\beta) : \mathcal{E}_H \longrightarrow X(\beta)^* \mathcal{E}_{H'} .$$

A morphism $\mathcal{E} \to \mathcal{E}'$ in $\Sigma(X)$ is a family of morphisms $\mathcal{E}_H \to \mathcal{E}'_H$ of $\mathrm{Sh}(X^H)$ for $H \leq G$ such that the square

$$\begin{array}{ccc}
\mathcal{E}_H & \longrightarrow & \mathcal{E}'_H \\
\big\downarrow & & \big\downarrow \\
X(\beta)^*\mathcal{E}_{H'} & \longrightarrow & X(\beta)^*\mathcal{E}'_{H'}
\end{array}$$

commutes for every $\beta : G/H' \to G/H$.

Proposition 1.2. *$\Sigma(X)$ is a Grothendieck topos.*

Proof. It is enough to note that $\Sigma(X)$ satisfies the conditions of Giraud's theorem (see [3, Théorème 1.2]). The first three conditions follow from the fact that $X(\beta)^*$ commutes with all inductive limits (being a left adjoint functor) and with finite projective limits (being exact). Thus it remains to show that $\Sigma(X)$ has a set of generators.

Let $K \leq G$ and $x \in X^K$. Assume that for every $\gamma : G/H \to G/K$ we are given an open neighbourhood V_γ of $X(\gamma)(x)$ in X^H. Then we can find an arbitrarily small open G_x-invariant neighbourhood U of x in X with the following properties:

$$\begin{aligned}
gx \neq x \quad &\Longrightarrow \quad U \cap gU = \emptyset \\
gx \notin X^H \quad &\Longrightarrow \quad (gU) \cap X^H = \emptyset \\
X(\gamma)(x) = gx \quad &\Longrightarrow \quad (gU) \cap X^H \subset V_\gamma \quad (\gamma : G/H \to G/K) .
\end{aligned}$$

Define
$$\mathcal{E}_H = \coprod_{g\in G/G_x} (gU) \cap X^H \in \mathrm{Sh}(X^H)\,,$$
if $\mathrm{Map}_G(G/H, G/K) \neq \emptyset$ and $\mathcal{E}_H = \emptyset$ otherwise. In this way we obtain an object $\mathcal{E} = (\mathcal{E}_H)_{H\leq G}$ of $\Sigma(X)$, and all such objects form a set of generators for $\Sigma(X)$. □

To end this section, we point out that $\Gamma(\widetilde{X})$ and $\Sigma(X)$ can in an evident way be regarded as ringed topoi, the ring being in both cases the family consisting of constant sheaves $\mathbb{Z}$.

2. Some Morphisms

In the notation of the previous section, we have an obvious functor $\Lambda^* : \Sigma(X) \to \Gamma(\widetilde{X})$, mapping the object $(\mathcal{E}_H)_{H\leq G}$ of $\Sigma(X)$ to the object $(\mathcal{F}(u))_{u\in\mathrm{Ob}(\mathcal{D})}$ of $\Gamma(\widetilde{X})$ defined by
$$\mathcal{F}(u) = X(u)^*\mathcal{E}_H\,, \qquad u : G/H \to G/K\,.$$

LEMMA 2.1. *The functor Λ^* is the inverse image part of a morphism of topoi*
$$\Lambda = (\Lambda_*, \Lambda^*) : \Gamma(\widetilde{X}) \longrightarrow \Sigma(X)\,.$$

PROOF. Since Λ^* is clearly exact, it is enough to construct a right adjoint Λ_* to Λ^*.

Let $\mathcal{F} = (\mathcal{F}(u))$ be an object of $\Gamma(\widetilde{X})$ and $P \leq G$ a subgroup. We consider the comma category $\mathcal{C}/(G/P)$ whose objects are all G-maps $G/H \to G/P$ for $H \leq G$ and morphisms are all commutative triangles

(2.2)
$$\begin{tikzcd} G/H \arrow[rr, "u"] \arrow[dr, "\varphi"'] & & G/K \arrow[dl, "\psi"] \\ & G/P & \end{tikzcd}$$

To the object object $\mathcal{F}$ of $\Gamma(\widetilde{X})$ we associate the functor $F(\mathcal{C}/(G/P)) \to \mathrm{Sh}(X^P)$ which maps the object 2.2 of $F(\mathcal{C}/(G/P))$ to the sheaf

$X(\psi)^*\mathcal{F}(u)$ on X^P. We define $\Lambda_*(\mathcal{F})_P$ to be the projective limit of this functor $F(\mathcal{C}/(G/P)) \to \mathrm{Sh}(X^P)$, i.e.

$$\Lambda_*(\mathcal{F})_P = \varprojlim X(\psi)^*\mathcal{F}(u) \in \mathrm{Sh}(X^P)\,,$$

the limit taken over all objects 2.2 of $F(\mathcal{C}/(G/P))$.

We let $\Lambda_*(\mathcal{F}) = (\Lambda_*(\mathcal{F})_P)_{P\leq G}$. In this way we obtain a functor $\Lambda_* : \Gamma(\widetilde{X}) \to \Sigma(X)$, and we leave it to the reader to verify that Λ_* is right adjoint to Λ^*. $\square$

Let $\mathcal{F} = (\mathcal{F}(u))_{u\in \mathrm{Ob}(\mathcal{D})}$ be an object of $\Gamma(\widetilde{X})$ and $P \leq G$. We give a more concrete description of the sheaf $\Lambda_*(\mathcal{F})_P$ constructed in the proof of Lemma 2.2. Let us denote $\mathcal{C}_P = \mathrm{Or}(P)$ and $\mathcal{D}_P = F(\mathcal{C}_P)$. We have the functor $\mathcal{C}_P \to \mathcal{C}$ mapping a P-orbit P/H to the G-orbit $G/H \cong G\times_P (P/H)$ and a P-map $v : P/H \to P/K$ to the G-map $G\times_P v : G/H \to G/K$; this functor induces a functor $\iota_P : \mathcal{D}_P \to \mathcal{D}$.

Let X_P be the G-space X regarded as a P-space. Then we have the topos $\Gamma(\widetilde{X}_P)$ and the functor $\iota_P^* : \Gamma(\widetilde{X}) \to \Gamma(\widetilde{X}_P)$ induced by ι_P. Further, restriction to the subspace $X^P \subset X^K$ for $K \leq P$ defines an obvious functor

$$\rho_P : \Gamma(\widetilde{X}_P) \longrightarrow \mathbf{Hom}(\mathcal{D}_P, \mathrm{Sh}(X^P))\,.$$

We claim that the projective limit of $(\rho_P \circ \iota_P^*)(\mathcal{F})$ is isomorphic to the sheaf $\Lambda_*(\mathcal{F})_P$ on X^P, that is

$$\Lambda_*(\mathcal{F})_P \cong \varprojlim_{v\in\mathcal{D}_P} \mathcal{F}(G\times_P v) \mid X^P\,. \tag{2.3}$$

This claim is, actually, a consequence of the simple observation that the functor $\mathcal{C}_P \to \mathcal{C}/(G/P)$ mapping P/H to the canonical surjection $G/H \cong G\times_P (P/H) \to G\times_P (P/P) \cong G/P$, is an equivalence of categories, and it induces an equivalence $\mathcal{D}_P \to F(\mathcal{C}/(G/P))$.

The following fact about the functor $\iota_P^* : \Gamma(\widetilde{X}) \to \Gamma(\widetilde{X}_P)$ will be useful in section 4:

LEMMA 2.4. *The functor ι_P^* is the direct image part of a morphism of topoi $\Gamma(\widetilde{X}) \to \Gamma(\widetilde{X}_P)$.*

PROOF. By Proposition (1.2.9) of [8], ι_P^* has a left adjoint $\iota_{P!}$: $\Gamma(\widetilde{X}_P) \to \Gamma(\widetilde{X})$. To prove the Lemma, we must show that $\iota_{P!}$ is exact.

Let $\mathcal{F}$ be an object of $\Gamma(\widetilde{X}_P)$ and $w : G/L \to G/M$ an object of $\mathcal{D}$, i.e. a G-map. Let $\mathcal{D}_P/w$ be the category whose objects are the pairs $(v,(\alpha,\beta))$, where $v : P/H \to P/K$ is an object of $\mathcal{D}_P$ and $(\alpha,\beta) : G \times_P v \to w$ is a morphism of $\mathcal{D}$, so the square in the diagram

$$\begin{tikzcd} G/H & G \times_P (P/H) \arrow[l, "\sim"'] \arrow[d, "G\times_P v"'] & G/L \arrow[l, "\beta"'] \arrow[d, "w"] \\ G/K & G \times_P (P/K) \arrow[l, "\sim"'] \arrow[r, "\alpha"] & G/M \end{tikzcd}$$

must commute; a morphism of $\mathcal{D}_P/w$ is a morphism of $\mathcal{D}_P$ making evident triangles commutative. Now, by the construction of Proposition (1.2.9) of [8],

$$\iota_{P!}(\mathcal{F}) : w \mapsto \varinjlim_{\mathcal{D}_P/w} X(\alpha)^*\mathcal{F}(v) \in \mathrm{Sh}(X^M)\,.$$

Consider the normalizer $N(L,P) = \{b \in G \mid b^{-1}Lb \le P\}$. The group P acts on the right on $N(L,P)$. We choose one representative from each P-orbit of $N(L,P)$, and for each such a representative b we define the G-map

$$\beta : G/L \to G/L_\beta\,, \qquad gL \mapsto gbL_\beta\,; \qquad L_\beta = b^{-1}Lb\,.$$

Let B be the set of these G-maps β. We write G/M as a disjoint union of P-orbits:

$$G/M \cong \coprod_j P/P_j\,.$$

For every $\beta \in B$, let

$$\begin{aligned} w \circ \beta^{-1} \in & \mathrm{Map}_G(G/L_\beta, G/M) \cong \mathrm{Map}_P(P/L_\beta, G/M) \\ & \cong \coprod_j \mathrm{Map}_P(P/L_\beta, P/P_j) \end{aligned}$$

correspond to the P-map $v_\beta : P/L_\beta \to P/P_{j_\beta}$; let also $\alpha_\beta : G/P_{j_\beta} \to G/M$ be the G-map induced by $P/P_{j_\beta} \hookrightarrow G/M$. Then we see that the diagrams

$$\begin{array}{ccccc} G/L_\beta & \xleftarrow{\sim} & G\times_P (P/L_\beta) & \xleftarrow{\beta} & G/L \\ & & {\scriptstyle G\times_P v_\beta}\downarrow & & \downarrow{\scriptstyle w} \\ G/P_{j_\beta} & \xleftarrow{\sim} & G\times_P (P/P_{j_\beta}) & \xrightarrow{\alpha_\beta} & G/M \end{array} \qquad (\beta \in B)$$

represent the objects of a cofinal discrete subcategory of $\mathcal{D}_P/w$. Hence

$$\iota_{P!}(\mathcal{F})(w) \cong \coprod_{\beta\in B} X(\alpha_\beta)^*\mathcal{F}(v_\beta),$$

and the exactness of $\iota_{P!}$ is clear. □

3. Cohomology of a G-Map

Let $f : Y \to X$ be a G-map. In this and the next section we have to assume that all subspaces of X and Y are paracompact.

We consider modules in the ringed topoi $\Gamma(\widetilde{Y})$, $\Gamma(\widetilde{X})$ and $\Sigma(X)$; the corresponding abelian categories are denoted $\operatorname{Mod}\Gamma(\widetilde{Y})$, $\operatorname{Mod}\Gamma(\widetilde{X})$ and $\operatorname{Mod}\Sigma(X)$. The global section functors are denoted $\Gamma_Y : \operatorname{Mod}\Gamma(\widetilde{Y}) \to \mathbf{Ab}$, $\Gamma_X : \operatorname{Mod}\Gamma(\widetilde{X}) \to \mathbf{Ab}$ and $\Gamma_{\Sigma(X)} : \operatorname{Mod}\Sigma(X) \to \mathbf{Ab}$. Their derived functors are the cohomology functors of the corresponding topoi: $R^n\Gamma_X(\cdot) = H^n(\Gamma(\widetilde{X});\cdot)$ etc.

Let $m : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ be a contravariant coefficient system and $m/Y \in \operatorname{Mod}\Gamma(\widetilde{Y})$ the corresponding object consisting of constant sheaves (see section 1 of [5]). By the main result of [5],

$$H^n(\Gamma(\widetilde{Y}); m/Y) \cong \bar{H}^n_G(Y; m),$$

the equivariant Alexander-Spanier cohomology of Y with coefficients m. We want to establish a connection between $\bar{H}^n_G(Y;m)$ and the cohomology of X; a start in this direction was made in section 7 of [5].

First of all, we have a factorization

$$\Gamma_Y : \operatorname{Mod}\Gamma(\widetilde{Y}) \xrightarrow{f_*} \operatorname{Mod}\Gamma(\widetilde{X}) \xrightarrow{\Gamma_X} \mathbf{Ab},$$

where f_* and Γ_X are the direct image parts of two morphisms of topoi. Thus there is the following identity for the derived functors between the appropriate derived categories of bounded below complexes:

$$R\Gamma_Y = R\Gamma_X \circ Rf_* \,. \tag{3.1}$$

The spectral sequence arising from this identity was studied in section 7 of [5].

Secondly, to compute $Rf_*(m/Y)$, consider the Alexander-Spanier resolution $\mathcal{C}^{\cdot}(m/Y)$ of m/Y (cf. section 2 of [5]). Explicitly, for $u : G/H \to G/K$ we have $\mathcal{C}^n(m/Y)(u) = \mathcal{C}^n(Y^K; m(G/H))$, and the sections of this sheaf on an open set $U \subset Y^K$ are

$$\Gamma(U, \mathcal{C}^n(Y^K; m(G/H))) = \bar{C}^n(U; m(G/H)) \,,$$

because, under the present hypothesis, U is paracompact. Here $\bar{C}^n$ refers to the group of Alexander-Spanier cochains, with the locally zero cochains factored out. In particular, the sheaves $\mathcal{C}^n(Y^K; m(G/H))$ are flabby, whence

$$Rf_*(m/Y) = f_*\mathcal{C}^{\cdot}(m/Y) \,. \tag{3.2}$$

Thirdly, we note

Lemma 3.3. *There is a factorization*

$$\Gamma_X : \mathrm{Mod}\,\Gamma(\widetilde{X}) \xrightarrow{\Lambda_*} \mathrm{Mod}\,\Sigma(X) \xrightarrow{\Gamma_{\Sigma(X)}} \mathbf{Ab} \,.$$

Proof. The functor which to an abelian group A associates the family of constant sheaves A, is left adjoint to both Γ_X and $\Gamma_{\Sigma(X)} \circ \Lambda_*$. □

It follows that $R\Gamma_X = R\Gamma_{\Sigma(X)} \circ R\Lambda_*$. Combining this with 3.1 and 3.2 we get

$$R\Gamma_Y(m/Y) = R\Gamma_{\Sigma(X)}(R\Lambda_*(f_*\mathcal{C}^{\cdot}(m/Y))) \,.$$

This leads to the following result:

PROPOSITION 3.4. *In the above situation, there is a spectral sequence with*

$$E_2^{pq} = H^p(\Sigma(X); R^q\Lambda_*(f_*\mathcal{C}^{\cdot}(m/Y)))\,,$$

converging to $\bar{H}_G^{p+q}(Y;m)$.

In the next section we give a more concrete description of the E_2-term in some special cases.

4. Cohomology of a G-Fibration

In this section we assume, in addition to the hypotheses of section 3, that $f : Y \to X$ is a Hurewicz G-fibration and X is locally G-contractible. This last condition, which is satisfied if, for example, X is a $G-CW$-complex, means that every orbit $Gx \subset X$ has arbitrarily small open G-neighbourhoods V such that Gx is a G-deformation retract of V. Our objective is to study the E_2-term of the spectral sequence of Proposition 3.4. We prove the following fact about the coefficient system $R^q\Lambda_*(f_*\mathcal{C}^{\cdot}(m/Y)) = (R^q\Lambda_*(f_*\mathcal{C}^{\cdot}(m/Y))_P)_{P\leq G} \in \operatorname{Mod}\Sigma(X)$:

PROPOSITION 4.1. *For* $P \leq G$ *, the sheaf* $R^q\Lambda_*(f_*\mathcal{C}^{\cdot}(m/Y))_P$ *on* X^P *is locally constant, with stalk over* $x \in X^P$ *isomorphic to* $\bar{H}_P^q(f^{-1}(x); m_P)$.

We recall that the contravariant P-coefficient system m_P is determined by the G-coefficient system m through $m_P(P/H) = m(G\times_P(P/H))$.

By 2.3 we have $R^q\Lambda_*(f_*\mathcal{C}^{\cdot}(m/Y))_P \cong R^q\Lambda_P(f_*\mathcal{C}^{\cdot}(m/Y))$, where Λ_P is the composite

$$\Lambda_P : \Gamma(\widetilde{X}) \xrightarrow{\iota_P^*} \Gamma(\widetilde{X}_P) \xrightarrow{\rho_P} \mathbf{Hom}(\mathcal{D}_P, \operatorname{Sh}(X^P)) \xrightarrow{\varprojlim} \operatorname{Sh}(X^P)\,.$$

Here ι_P^* and $\varprojlim$ are the direct image parts of two morphisms of topoi, and ι_P^* and ρ_P are exact. Furthermore, the derived functors of $\varprojlim \circ \rho_P$ can be calculated with aid of the acyclic objects described in (1.3.10) of [8], and ρ_P maps these objects to $\varprojlim$-acyclic objects of $\mathbf{Hom}(\mathcal{D}_P, \operatorname{Sh}(X^P))$. It follows that

$$R\Lambda_P \cong (R\varprojlim) \circ \rho_P \circ \iota_P^*\,,$$

and in particular

$$R^q\Lambda_P(f_*\mathcal{C}^\cdot(m/Y)) \cong \varprojlim{}^q(\rho_P\iota_P^* f_*\mathcal{C}^\cdot(m/Y))\,.$$

Let $f_P : f^{-1}(X^P) \to X^P$ be the restriction of f. Then f_P is a P-map, with P acting trivially on X^P.

LEMMA 4.2. *The complex $\rho_P\iota_P^* f_*\mathcal{C}^\cdot(m/Y)$ is quasi-isomorphic to the complex $(f_P)_*\mathcal{C}^\cdot(m_P/f^{-1}(X^P))$.*

REMARK 4.3. The sections of the complex $(f_P)_*\mathcal{C}^\cdot(m_P/f^{-1}(X^P))$ over an open set $V \subset X^P$ form, in the notation of section 2 of [5], the complex $\bar{A}^\cdot(f^{-1}(V); m_P)$, which to an object $v : P/H \to P/K$ of $\mathcal{D}_P$ associates the complex $\bar{C}^\cdot(f^{-1}(V)^K; m(G/H))$ of abelian groups. On the other hand, $\rho_P\iota_P^* f_*\mathcal{C}^\cdot(m/Y)$ maps $v : P/H \to P/K$ to the restriction to X^P of the complex of sheaves on X^K whose sections over an open set $U \subset X^K$ are $\bar{C}^\cdot(f^{-1}(U)^K; m(G/H))$.

PROOF OF 4.2. Restriction of cochains defines a natural morphism $\rho_P\iota_P^* f_*\mathcal{C}^\cdot(m/Y) \to (f_P)_*\mathcal{C}^\cdot(m_P/f^{-1}(X^P))$. We take the values of these complexes on an object $v : P/H \to P/K$ of $\mathcal{D}_P$ and look at stalks over a point $x \in X^P$. Our hypotheses guarantee that x has arbitrarily small open neighbourhoods V in X^P and U in X^K, contractible to $\{x\}$, and there are K-fibre homotopy equivalences $f^{-1}(V) \simeq V \times f^{-1}(x)$ and $f^{-1}(U) \simeq U \times f^{-1}(x)$, inducing $f^{-1}(U)^K \simeq U \times f^{-1}(x)^K$ and $f^{-1}(V)^K \simeq V \times f^{-1}(x)^K$. Using 4.3 we see that our morphism induces a quasi-isomorphism on the stalks over x, the cohomology of both stalks being isomorphic to $\bar{H}^\cdot(f^{-1}(x)^K; m(G/H))$. $\square$

Let $V \subset X^P$ be an open set. By Remark 4.3 above and Propositions 2.9 and 2.10 of [5] we have

$$\begin{aligned}
\Gamma(V, \varprojlim_{\mathcal{D}_P}{}^q (f_P)_*\mathcal{C}^\cdot(m_P/f^{-1}(X^P))) &\cong \varprojlim_{\mathcal{D}_P}{}^q \Gamma(V, (f_P)_*\mathcal{C}^\cdot(m_P/f^{-1}(X^P))) \\
&\cong \varprojlim_{\mathcal{D}_P}{}^q \bar{A}^\cdot(f^{-1}(V); m_P) \\
&\cong \bar{H}^q_P(f^{-1}(V); m_P)\,.
\end{aligned}$$

If V is contractible to $x \in V$, there is a P-fibre homotopy equivalence $f^{-1}(V) \simeq V \times f^{-1}(x)$ and so $\bar{H}^q_P(f^{-1}(V); m_P) \cong \bar{H}^q_P(f^{-1}(x); m_P)$ in this case. This completes the proof of Proposition 4.1.

Let us now add the assumption that all fixed point subspaces X^P of X are non-empty and simply connected. Fix a point $x_0 \in X^G$ and denote $F = f^{-1}(x_0)$. Then, for any $P \leq G$, the locally constant sheaf $R^q\Lambda_*(f_*\mathcal{C}^{\cdot}(m/Y))_P$ on X^P is isomorphic to the constant sheaf $\bar{H}^q_P(F; m_P)$, and the object $R^q\Lambda_*(f_*\mathcal{C}^{\cdot}(m/Y)) \in \operatorname{Mod}\Sigma(X)$ is essentially a contravariant coefficient system $\mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$. To finish the proof of the Theorem, we therefore only have to prove the following

Proposition 4.4. *Under the above assumptions, if $m : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ is a contravariant coefficient system, then*

$$H^p(\Sigma(X); m) \cong \bar{H}^p_G(X; m).$$

Proof. Proposition 4.1 applied to the G-fibration $\mathrm{id} : X \to X$ gives $R\Lambda_*(m/X) \cong m$, so

$$R\Gamma_{\Sigma(X)}(m) \cong (R\Gamma_{\Sigma(X)} \circ R\Lambda_*)(m/X) \cong R\Gamma_X(m/X).$$

The claim follows from the main result of [5], which says that $R^p\Gamma(m/X) \cong \bar{H}^p_G(X; m)$. □

References

[1] Baues, H.-J. and G. Wirsching, Cohomology of small categories, J. Pure Appl. Algebra **38** (1985) 187–211.

[2] Grothendieck, A., Sur quelques points d'algebre homologique, Tôhoku Math. J. **9** (1957), 119–221.

[3] Grothendieck, A. and J. L. Verdier, Topos, Lecture Notes in Mathematics 269 (SGA 4, Tome 1), Springer-Verlag, Berlin-Heidelberg-New York, 1972, pp. 299–519.

[4] Honkasalo, H., Equivariant Alexander-Spanier cohomology, Math. Scand. **63** (1988), 179–195.

[5] Honkasalo, H., Sheaves on fixed point sets and equivariant cohomology, Math. Scand. **78** (1996), 37–55.

[6] Illman, S., Equivariant singular homology and cohomology, Mem. Amer. Math. Soc. **156** (1975).
[7] Moerdijk, I. and J.-A. Svensson, The equivariant Serre spectral sequence, Proc. Amer. Math. Soc. **118** (1993), 263–278.
[8] Saint-Donat, B., Techniques de descente cohomologique, Lecture Notes in Mathematics 270 (SGA 4, Tome 2), Springer-Verlag, Berlin-Heidelberg-New York, 1972, pp. 83–162.

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