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On Volumes and Chern-Simons Invariants of Geometric 3-Manifolds

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Abstract. Let $M_n(K)$ be the hyperbolic 3-manifold obtained by n-cyclic covering of S^3 branched over a hyperbolic knot K. A method to compute the volume and the Chern-Simons invariant of $M_n(K)$ is given. The value of the volume of M_n is n times the value of the volume of the corresponding hyperbolic orbifold. This volume can be obtained by appying the Schläffli Formula for the volume to the cone-manifold family, (K, α), with singularity K. The same approch is followed for the Chern-Simons invariant, after proving a "Schläffli Formula" for a generalized Chern-Simons function on the family of cone-manifold structures (K, α).

0. Introduction

After the Mostow Rigidity Theorem [Mos], each geometrical invariant of a hyperbolic 3-manifold is a topological invariant. Among the most important geometric invariants we list are the volume and the Chern-Simons invariant. In this paper we give a method to compute the volume and the Chern-Simons invariant of the hyperbolic 3-manifold $M_n(K)$, obtained as an n-cyclic covering of S^3 branched over a hyperbolic knot K.

The volume of $M_n(K)$ is n times the volume of the geometric orbifold $S^3(K, 2\pi/n)$, whose underlying space is S^3 and the singularity is the knot K with cyclic isotropy group of rank n. These orbifolds $S^3(K, 2\pi/n)$, belong to the continuous family of cone manifolds $S^3(K, \alpha)$, whose underlying space is S^3 and whose singular set is the knot K with angle α, $0 < \alpha < \alpha_0$. The Schläffli Formula for the volume, (see [Vi], [C], [M]), applies to this family

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of cone manifolds (see [H]) and therefore we can obtain the volume of each orbifold $S^3(K, 2\pi/n)$, and then the volume of the manifold $M_n(K)$.

To follow the same program to compute the Chern-Simons invariant of the geometric 3-manifold $M_n(K)$, we associate a real number $I(M, \Sigma_\alpha)$ to each geometric cone manifold (M, Σ_α),whose underlying space is M and whose singular set is a nullhomologous knot Σ, with angle α, and with constant curvature geometry (> 0, $= 0$, < 0). This number depends on some choices and therefore is not an invariant of the cone manifold, but it is equivalent (mod 1) to the Chern-Simons invariant if the cone-manifold is a manifold, and is additive in appropriate branched coverings. If we consider a one parameter family of cone manifolds, $(S^3, \Sigma_{\alpha(t)})$, we have a function $I(S^3, \Sigma_\alpha)(t)$. We prove a "Schläffli Formula" for this function. One formula of this type for a family of flat $SU(2)$-connections in the exterior of a knot in a 3-manifold M was obtained by Kirk and Klassen [K-K1], and after generalized for flat $SL(2, \mathbb{C})$-connections in [K-K2], but our context is different. We consider in each *compact* cone-manifold of the family the Riemannian connection.

In Sec. 1 we present some relevant concepts of cone-manifolds, namely the jump and twist of a singular curve, which are related to the torsion of a curve in a Riemannian manifold.

In Sec. 2 the number $I(M, \Sigma_\alpha)$ is defined, and some of its properties are studied.

In Sec. 3 a "Schläffli Formula" for the function $I(S^3, K_\alpha)(t)$, where K is a hyperbolic knot in S^3 is stated and proved. Although the proof of the Schläffli Formula for spherical cone-manifolds, that we give here, can easily be adapted to the hyperbolic cone-manifolds case, we have decided to include a different proof, for the latter case, to emphasize its relationship with the work in [Y] and [N-Z]. Finally we prove Theorem 3.9 which states the formulas for Chern-Simons invariant and volume of the hyperbolic manifold $M_n(K)$. Detailed computations of these invariants for cyclic branched coverings of rational knots appear in [HLM$_2$].

1. Geometric cone manifolds

We are interested in 3-dimensional cone-manifolds with a fixed geometric structure (geometric cone-manifolds). A cone-manifold is a PL manifold together with a, possibly empty, codimension two locally flat, submanifold called the singular set. (In a 3-dimensional cone-manifold the singular set will consist of curves, but not graphs.) In this paper, a geometric cone-manifold will be modeled on some space of constant curvature k (compare [T]). Points off the singular set have neighborhoods homeomorphic to neighborhoods in the model. Points on the singular set have neighborhoods homeomorphic to neighborhoods constructed as follows: take an angle α wedge in the model. (A wedge is the intersection or union of two half spaces that intersect; the angle α is the dihedral angle where $0 < \alpha < 2\pi$). Then identify the two boundaries of the wedge, using the natural rotation by α, to form a topological space W_α. Points on the singular set have neighborhoods homeomorphic to neighborhoods in this topological space. The homeomorphism carries the singular set to the axis of rotation in the topological space. Transition functions are isometries. A singular curve Σ whose points have only neighborhoods homeomorphic to neighborhoods in W_α will be called an α-curve. Note that if we let α be equal to 2π, then a 2π-curve is actually a regular geodesic. But, of course, in general Σ can be thought of as a singular geodesic.

The differences between our definition of a cone manifold and the common definition of orbifold are:

1. In dimension three the singular set in a cone manifold is a curve, not, as is sometimes the case in an orbifold, a graph.
2. The "cone angle" is any angle α, $0 < \alpha \leqq 2\pi$. In an orbifold this angle is always $2\pi/n$.

Let $(\overrightarrow{M}^3, \overrightarrow{\Sigma}_\alpha)$ be an oriented 3-dimensional cone-manifold $\overrightarrow{M}^3$ of constant curvature k, where the singular set $\overrightarrow{\Sigma}_\alpha$ consists of an oriented α-curve ($0 < \alpha \leq 2\pi$) which is a nullhomologous knot in M^3. Consider an oriented meridian disc $\overrightarrow{D}$ of the neighborhood $U = \{p \in \overrightarrow{M}^3; d(p, \overrightarrow{\Sigma}_\alpha) \leq \epsilon\}$ of $\overrightarrow{\Sigma}_\alpha$. The orientation of $\overrightarrow{D}$ is chosen such that the orientation of $\overrightarrow{D}$ followed by the orientation of $\overrightarrow{\Sigma}_\alpha$ coincides with the orientation of $\overrightarrow{M}$. Let $\overrightarrow{m} = \partial \overrightarrow{D}$. Call $\overrightarrow{l_c}$ the canonical longitude of $\overrightarrow{\Sigma}_\alpha$, i. e. $l_c = \partial U \cup S$ where S is an oriented surface in M^3 bounded by $\overrightarrow{\Sigma}_\alpha$, and $\overrightarrow{l_c}$, $\overrightarrow{\Sigma}_\alpha$ are parallel.

Next, we shall define two invariants of the cone manifold associated to the singular set, the twist and jump.

Case 1. $k = -1$: Let $h : \pi_1(U \setminus \Sigma, o) \longrightarrow PSL(2, \mathbb{C})$ be the holonomy of $U \setminus \Sigma_\alpha$. Then (see [GM]) h admits two liftings to $SL(2, \mathbb{C})$. The image of $\overrightarrow{l_c}$ in $SL(2, \mathbb{C})$ under these two liftings is the same since $\overrightarrow{l_c}$ is nullhomologous outside $\overrightarrow{\Sigma}_\alpha$. So up to conjugation in $SL(2,\mathbb{C})$,

$$(1) \qquad h(\overrightarrow{m}) = \pm \begin{bmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{bmatrix}, \qquad h(\overrightarrow{l}_c) = \begin{bmatrix} e^{\frac{v}{2}} & 0 \\ 0 & e^{-\frac{v}{2}} \end{bmatrix}$$

were $v = \delta + i\beta$, δ is the length of $\overrightarrow{\Sigma}_\alpha$, and β, $-2\pi \leq \beta < 2\pi$, is the angle of the lifted holonomy of $\overrightarrow{\Sigma}_\alpha$ (notice that $h(\overrightarrow{l}_c)$ is well defined regardless of the lifting).

REMARK. There exists a $2:1$ map $\lambda : SU(2) \times SU(2) \longrightarrow SO(4)$ defined as follows.

$\mathbb{R}^4$ is represented by matrices $x = \begin{bmatrix} u+iv & w+it \\ -w+it & u-iv \end{bmatrix} = u\overrightarrow{1} + v\overrightarrow{i} + w\overrightarrow{j} + t\overrightarrow{k}$, where $\overrightarrow{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\overrightarrow{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\overrightarrow{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\overrightarrow{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. $S^3 \subset \mathbb{R}^4$ are the matrices with determinant $+1$: $u^2+v^2+w^2+t^2 = 1$, i.e. $S^3 = SU(2)$. By stereographic projection S^3 is $\mathbb{R}^3 + \infty$ with the standard orientation given by the vectors $\{\overrightarrow{i} - \overrightarrow{1}, \overrightarrow{j} - \overrightarrow{1}, \overrightarrow{k} - \overrightarrow{1}\}$. Define $\lambda : SU(2) \times SU(2) \longrightarrow SO(4)$ as follows. $\lambda(A,B)$ is the homomorphism of $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by

$$x \to A^t x \overline{B}$$

then $\lambda(\begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}, \begin{bmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{bmatrix})$ sends 1 to $\cos(\alpha-\beta) + \sin(\alpha-\beta)i$, and j to $\cos(\alpha+\beta)j + \sin(\alpha+\beta)k$. The map λ is $2:1$ since $\lambda(A,B) = \lambda(-A,-B)$. The restriction of λ to the diagonal $\hat{\lambda}(A) = \lambda(A,A)$ defines a map $\hat{\lambda} : S^3 \longrightarrow SO(3)$ by $\hat{\lambda}(A) = \{x \to A^t x \overline{A}\}$, where $trace(x) = 0$ (this implies that $x \in \{0\} \times \mathbb{R}^3$). Using [GM] it is easy to see that any homomorphism from $\pi_1(U \setminus \Sigma, o) \longrightarrow SO(4)$ lifts to two homomorphisms into $SU(2) \times SU(2)$.

Case 2. $k = 0$: Let $h : \pi_1(U \setminus \Sigma, o) \longrightarrow Iso^+(E^3)$ be the holonomy of $U \setminus \Sigma_\alpha$. $Iso^+(E^3)$ is $\mathbb{R}^3 \ltimes SO(3)$. Then h has two lifts to $\mathbb{R}^3 \ltimes SU(2)$ and as in Case 1 the image of $\overrightarrow{l}_c$ in these two lifts coincide. Up to conjugation we have

$$(1') \qquad h(\overrightarrow{m}) = \left(\overrightarrow{0}, \begin{bmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{bmatrix}\right), \quad h(\overrightarrow{l}_c) = \left(\delta\overrightarrow{k}, \begin{bmatrix} e^{i\frac{\beta}{2}} & 0 \\ 0 & e^{-i\frac{\beta}{2}} \end{bmatrix}\right)$$

were δ is the length of $\overrightarrow{\Sigma}_\alpha$, and β, $-2\pi \leq \beta < 2\pi$, is the angle of the lifted holonomy of $\overrightarrow{\Sigma}_\alpha$.

Case 3. $k = 1$: Let $h : \pi_1(U \setminus \Sigma, o) \longrightarrow SO(4)$ be the holonomy of $U \setminus \Sigma_\alpha$. As before we have two lifts of h into $SU(2) \times SU(2)$. We assume up to conjugation in $SU(2) \times SU(2)$ that

$$\begin{aligned} h(\overrightarrow{m}) &= \left(\pm \begin{bmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{bmatrix}, \pm \begin{bmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{bmatrix} \right), \\ h(\overrightarrow{l}_c) &= \left(\begin{bmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{bmatrix}, \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \right) \end{aligned} \tag{2}$$

In this case $\delta = \gamma - \phi$ is length of $\overrightarrow{\Sigma}_\alpha$, and $\beta = \gamma + \phi$, $-2\pi \leq \beta < 2\pi$, is the angle of the lifted holonomy of $\overrightarrow{\Sigma}_\alpha$

For any $k \neq 0$ we can normalize by multiplying the metric by a constant such that the new cone manifold, $\overrightarrow{M}_n$, belongs to case 1 or 3. This process does not change angles.

1.1 Definition. Let $(\overrightarrow{M}^3, \overrightarrow{\Sigma}_\alpha)$ be an oriented 3-dimensional cone-manifold of constant curvature k, where $\overrightarrow{\Sigma}_\alpha$ is a nullhomologous knot in $\overrightarrow{M}$. The *jump of* $\overrightarrow{\Sigma}_\alpha$ is the equivalence class $\overline{\beta}$ in $\mathbb{R}/4\pi\mathbb{Z}$ represented by the angle β, $-2\pi \leq \beta < 2\pi$ of the holonomy of $\overrightarrow{\Sigma}_\alpha$ in the associated cone manifold $\overrightarrow{M}_n$, of constant curvature 1, 0, -1. The *twist of* $\overrightarrow{\Sigma}_\alpha$, $tw(\overrightarrow{\Sigma}_\alpha)$, is the real number $\beta\frac{\alpha}{2\pi}$, and $\overline{tw}(\overrightarrow{\Sigma}_\alpha) = \overline{\beta}\frac{\alpha}{2\pi}$ is an equivalence class in $\mathbb{R}/2\alpha\mathbb{Z}$.

1.2 Remarks.

1) If $\alpha = 2\pi$, $\overrightarrow{\Sigma}_{2\pi}$ is a regular geodesic in $\overrightarrow{M}^3$. Then l_c can be used to give an orthonormal framing along $\overrightarrow{\Sigma}$ as follows. The e_1-vector is tangent to $\overrightarrow{\Sigma}$; the e_2-vector defines a geodesic intersecting ∂U in a point of l_c; and e_3 is determined by e_1, e_2 and the orientation of $\overrightarrow{M}$. This framing s can be extended to a neighborhood of Σ and is used to define the real number

$$\tau(\overrightarrow{\Sigma}, l_c) = \tau(\overrightarrow{\Sigma}, s) = -\int_{s(\Sigma)} \theta_{23}$$

where (θ_{ij}) is the Riemannian connection form on the $SO(3)$ oriented frame bundle $F(M)$ of M. In this case

$$tw(\overrightarrow{\Sigma}) = \tau(\overrightarrow{\Sigma}, l_c) \equiv \tau(\Sigma)\,(mod\, 2\pi)$$

where $\tau(\Sigma)$ is the torsion of Σ. (See [Y] and [M-R].)

2) Let $p : \tilde{M} \longrightarrow M$ be a n-cyclic covering branched over the α-curve $\overrightarrow{\Sigma}_\alpha$ in the cone-manifold $(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$, where Σ is a nullhomologous knot in M. If the branching index n of p satisfies $n\alpha \leq 2\pi$, then $\tilde{M}$ inherits a natural cone-manifold structure from M with the $n\alpha$-curve $\tilde{\Sigma}_{n\alpha} = p^{-1}(\Sigma_\alpha)$ as singular set. We say that

$$p : (\overrightarrow{\tilde{M}}, \overrightarrow{\tilde{\Sigma}}_{n\alpha}) \longrightarrow (\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$$

is a *covering between cone-manifolds.* Then

$$tw(\overrightarrow{\tilde{\Sigma}}_{n\alpha}) = n\, tw(\overrightarrow{\Sigma}_\alpha) \quad \text{and} \quad \overline{tw}(\overrightarrow{\tilde{\Sigma}}_{n\alpha}) \equiv n\, \overline{tw}(\overrightarrow{\Sigma}_\alpha) \qquad (\text{mod } 2n\alpha)$$

In fact, $\text{jump}(\overrightarrow{\tilde{\Sigma}}_{n\alpha}) = \text{jump}(\overrightarrow{\Sigma}_\alpha) = \beta$ implies $tw(\overrightarrow{\tilde{\Sigma}}_{n\alpha}) = \beta\frac{n\alpha}{2\pi} = n(\beta\frac{\alpha}{2\pi}) = n\, tw(\overrightarrow{\Sigma}_\alpha)$. Therefore $\overline{tw}(\overrightarrow{\tilde{\Sigma}}_{n\alpha}) \equiv n\, \overline{tw}(\overrightarrow{\Sigma}_\alpha)$ (mod $2n\alpha$).

2. The number $I(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$

Next we define, $I(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$, associated to an oriented cone-manifold $(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$, where the singular set Σ_α is an α-curve which is a nullhomologous knot in M. This number (mod 1) is equal to the Chern-Simons invariant of the Riemannian manifold M when $\alpha = 2\pi$; and, if $p : (\overrightarrow{\tilde{M}}, \overrightarrow{\tilde{\Sigma}}_{n\alpha}) \longrightarrow (\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$ is a n-cyclic covering between cone-manifolds, then $I(\overrightarrow{\tilde{M}}, \overrightarrow{\tilde{\Sigma}}_{n\alpha}) \equiv nI(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$.

Let $(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$ be an oriented cone-manifold where the singular set Σ is an α-curve which is a nullhomologous knot in M. Let $\overrightarrow{m}$ be a meridian of $\overrightarrow{\Sigma}_\alpha$. It follows from [Me, Th.4.3] that there exists a frame field

$$s : \overrightarrow{M} \setminus (\Sigma \cup m) \longrightarrow F(\overrightarrow{M} \setminus (\Sigma \cup m))$$

having special singularities at $\Sigma \cup m$.

Recall ([Y;Def. 1.3], and compare [Me]) that a frame field on a 3-manifold N is *special* on a link J if it is an orthonormal frame field on

$N \setminus J$ which has the following behavior near each component K of J. Let U_K be an open neighborhood of radius ϵ of K. For each point x of $U_K \setminus K$

i) $e_3(x)$ is tangent to $\gamma(x, y)$ and it has direction opposite to y, where $\gamma(x, y)$ is the unique geodesic in U_K such that $d(x, y) = d(x, K) = length(\gamma(x, y)) = \delta$,

ii) $e_2(x)$ is tangent to $\overrightarrow{S}_\delta(y)$, where $\overrightarrow{S}_\delta(y) = \{z \in U_K | d(z, K) = d(z, y) = \delta\}$

It follows that in the limit $e_1(x)$ is tangent to $\overrightarrow{K}$.

Following the notation of [Y] let Q be the Chern-Simons form defined on the positively-oriented orthonormal frame bundle, $F(\overrightarrow{M} \setminus \Sigma)$

$$Q = \frac{1}{4\pi^2}(\theta_{12} \wedge \theta_{13} \wedge \theta_{23} + \theta_{12} \wedge \Omega_{12} + \theta_{13} \wedge \Omega_{13} + \theta_{23} \wedge \Omega_{23})$$

where (θ_{ij}) is the connection 1-form and (Ω_{ij}) is the curvature 2-form of the Riemannian connection on the 3-manifold $\overrightarrow{M} \setminus \Sigma$.

2.1 Proposition. *Then*

$$\frac{1}{2}\int_{s(M-\Sigma-m)} Q \qquad (\text{mod } 1)$$

is an invariant of $(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$.

Proof. The value of $\frac{1}{2}\int_{s(M-\Sigma-m)} Q$ is independent of the frame field s because if $\bar{s}$ is another frame field on $\overrightarrow{M} \setminus (\Sigma \cup m)$ having special singularities at $\Sigma \cup m$, then $s = \bar{s}$ on a neighborhood of $\Sigma \cup m$. Then the proof follows as in [CS] for the closed case. □

Let $s' = (f_1, f_2, f_3)$ be an orthonormal framing defined on a subset of $\overrightarrow{M} \setminus \Sigma$ containing m such that $f_1(y)$ is the tangent vector to m at each $y \in m$ having the same direction as the e_1-vectors of s near y, and $f_2(y)$ is tangent to the meridian disc of Σ bounded by m.

2.2 Definition. We define

$$I(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha) = \frac{1}{2}\int_{s(M-\Sigma-m)} Q - \frac{1}{4\pi}\tau(m, s') - \frac{1}{4\pi}tw(\Sigma)$$

$$I_1(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha) \equiv I(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha) \ (\text{mod } 1)$$

$$I_{\frac{\alpha}{2\pi}}(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha) \equiv I(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha) \ (\text{mod } \frac{\alpha}{2\pi}) \tag{3}$$

The real number $I(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$ depends only on the frame field s, since the value of β is chosen so that $-2\pi \leq \beta < 2\pi$. Is follows from Proposition 2.1 that the class $I_1(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$ is independent of the frame field s. The class $I_{\frac{\alpha}{2\pi}}(\overrightarrow{M}, \overrightarrow{\Sigma}_\alpha)$ depends on the frame field s, but is independent on the representative in the equivalence class $\overline{\beta}$. In the case of $\alpha = \frac{2\pi}{n}$ the cone-manifold is an orbifold. Then $\frac{\alpha}{2\pi} = \frac{1}{n}$ and

$$I_{\frac{\alpha}{2\pi}}(\overrightarrow{M}, \overrightarrow{\Sigma}_{\frac{2\pi}{n}}) = I_{\frac{1}{n}}(\overrightarrow{M}, \overrightarrow{\Sigma}_{\frac{2\pi}{n}}) \equiv I(\overrightarrow{M}, \overrightarrow{\Sigma}_{\frac{2\pi}{n}}) \pmod{\frac{1}{n}}$$

is both independent of the frame field s and of the representative of the equivalence class $\overline{\beta}$. Then $I_{\frac{1}{n}}(\overrightarrow{M}, \overrightarrow{\Sigma}_{\frac{2\pi}{n}}) \pmod{\frac{1}{n}}$ is **an invariant of the orbifold** $(\overrightarrow{M}, \overrightarrow{\Sigma}, n)$, which will naturally be called *the Chern-Simons invariant of that orbifold*, denoted $CS(\overrightarrow{M}, \overrightarrow{\Sigma}, n)$, i.e.

$$CS(\overrightarrow{M}, \overrightarrow{\Sigma}, n) := I_{\frac{1}{n}}(\overrightarrow{M}, \overrightarrow{\Sigma}_{\frac{2\pi}{n}}) \pmod{\frac{1}{n}}.$$

The following remarks justify the above definition.

2.3 REMARKS.

1) Suppose $\alpha = 2\pi$. Then the cone-manifold $(\overrightarrow{M}, \Sigma_{2\pi})$ is a geometric manifold $\overrightarrow{M}$. The Chern-Simons invariant $CS(\overrightarrow{M})$ is well defined mod 1. Because $\frac{\alpha}{2\pi} = 1$ the two classes I_1 and $I_{\frac{\alpha}{2\pi}}$ are equal and then

$$I_1(\overrightarrow{M}, \Sigma_\alpha) \equiv CS(\overrightarrow{M}) \quad (mod\, 1)$$

This is a consequence of the extended torsion formula of Meyerhoff-Ruberman [M-R] because we can consider a surface S in M such that $\partial S = \Sigma \cup m$ which has the shape depicted in Figure 1 in a neighborhood of m. This surface exists because Σ is null-homologous in M. Observe that $\tau(m, s') = \tau(m, S) \pm 2\pi$, where $\tau(m, S)$ is the torsion of m with respect to the framing induced by S, and $tw(\Sigma) = \tau(\overrightarrow{\Sigma}, l_c) = \tau(\overrightarrow{\Sigma}, S) \pm 2\pi$.

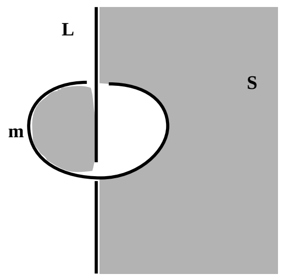

Figure 1

2) Let $p : (\overrightarrow{\tilde{M}}, \overrightarrow{\tilde{L}}_{n\alpha}) \longrightarrow (\overrightarrow{M}, \overrightarrow{L}_{\alpha})$ be a n-cyclic covering between cone-manifolds branched over $\overrightarrow{L}$. Then

$$\begin{aligned} I(\overrightarrow{\tilde{M}}, \overrightarrow{\tilde{L}}_{n\alpha}) &= n\, I(\overrightarrow{M}, \overrightarrow{L}_{\alpha}) \\ I_1(\overrightarrow{\tilde{M}}, \overrightarrow{\tilde{L}}_{n\alpha}) &\equiv n\, I_1(\overrightarrow{M}, \overrightarrow{L}_{\alpha}) \pmod 1 \\ I_{\frac{n\alpha}{2\pi}}(\overrightarrow{\tilde{M}}, \overrightarrow{\tilde{L}}_{n\alpha}) &\equiv n\, I_{\frac{\alpha}{2\pi}}(\overrightarrow{M}, \overrightarrow{L}_{\alpha}) \pmod{\frac{n\alpha}{2\pi}} \end{aligned}$$

To prove this, note that the frame field s lifts to a frame field $\tilde{s}$ in $\tilde{M}\setminus(\tilde{\Sigma}\cap\tilde{m})$ having special singularities at $\tilde{\Sigma}\cap\tilde{m}$, then $\int_{\tilde{s}(\tilde{M}-\tilde{\Sigma}-\tilde{m})} Q = n\int_{s(M-\Sigma-m)} Q$. On the other hand, the frame field s' lifts to a frame field $\tilde{s}'$ in a subset of $\tilde{M}\setminus\tilde{\Sigma}$ containing $\tilde{m}$ such that $\tau(\tilde{m},\tilde{s}') = n\tau(m,s')$. Finally, recall that $tw(\overrightarrow{\tilde{\Sigma}}) = n\, tw(\overrightarrow{\Sigma})$.

Note that if $\alpha = \frac{2\pi}{n}$ then the n-cyclic covering of M branched over Σ is a Riemannian manifold. This gives a procedure to compute the Chern-Simons invariant of some manifolds.

3) Suppose $\alpha = 0$ and the manifold is hyperbolic. Then $I_1(\overrightarrow{M}, \overrightarrow{L}_0)$ is equal to the extended definition of the Chern-Simons invariant for manifolds with cusps, made by Meyerhoff in [Me].

3. A "Schläffli" Formula for $I(S^3, K_\alpha)$.

In this section we deduce a "Schläffli" formula for the invariants $I(S^3, K_\alpha)$, $I_1(S^3, K_\alpha)$ and $I_{\frac{\alpha}{2\pi}}(S^3, K_\alpha)$ of a one parameter family of cone-manifolds $(S^3, K_{\alpha(t)})$ where the angle $\alpha(t)$ and the function $\beta(t)$ (one lift of

$\overline{\beta}$) are differentiable functions of t, and $\overrightarrow{K}$ is a hyperbolic knot in S^3. We suppose in the proof that $\beta(t) \in [-2\pi, 2\pi]$ (otherwise the same result follows for $I_1(S^3, K_\alpha)$ and $I_{\frac{\alpha}{2\pi}}(S^3, K_\alpha)$ using a different lift $I'(S^3, K_{\alpha_1})$ associated to $\beta(t)$).

First observe that $dI(S^3, K_{\alpha(t)}) = dI_1(S^3, K_{\alpha(t)}) = dI_{\frac{\alpha}{2\pi}}(S^3, K_{\alpha(t)})$. We distinguish two cases according as the curvature is negative or positive. As a matter of fact, there is a proof of Case 1 analogous to the proff of Case 2. However, we have decided to include here a different proof, (more in the line of [Y] and [N-Z]) to relate our result to the interesting work of these mathematicians.

Case 1. The curvature of the cone manifold $(S^3, K_{\alpha(t)})$ is -1 for $t \in (t_0, t_1)$.

We recall some notation and results of [Y] and [N-Z], in particular the important Theorem 3.1 of [Y] and the formula (46) of [N-Z], which can be thought of as a "Schläffli" formula.

Let $\overrightarrow{K}$ be a hyperbolic knot in S^3. Let N be the oriented complete hyperbolic structure on $S^3 \setminus K$. Let U be the deformation space of N. For $u \in U$, the corresponding hyperbolic manifold is denoted by N_u.

The holonomy of the hyperbolic structure N_u is $\rho_u : \pi_1(S^3 \setminus K) \longrightarrow PSL(2, \mathbb{C})$. Then, up to conjugation in $PSL(2, \mathbb{C})$,

$$\rho_u(m) = \begin{bmatrix} e^{\frac{u}{2}} & 0 \\ 0 & e^{-\frac{u}{2}} \end{bmatrix} \qquad \rho_u(l_c) = \begin{bmatrix} e^{\frac{v}{2}} & 0 \\ 0 & e^{-\frac{v}{2}} \end{bmatrix}$$

Let s be an orthonormal framing on $N \setminus m$ obtained as in the following proposition.

3.1 PROPOSITION. *Let $\overrightarrow{K}$ be a hyperbolic knot in S^3. Consider its complete hyperbolic structure N. Then there exists an orthonormal framing on $S^3 \setminus (K \cup m)$, where $\overrightarrow{m}$ is a meridian of $\overrightarrow{K}$, having a special singularity at $K \cup m$.*

PROOF. The proof follows the idea of the construction made in [Y], for the Figure-Eight knot. Consider a tubular neighborhood E of K. Let m be the boundary of a disc D in the interior of E such that $D \cap K$=one point. First, define a framing s_0 on the exterior of the Hopf link, $H = S_1 \cup S_2$, with special singularity at H ([Y;pg 503]). Second, define an orientation-preserving map $\phi : S^3 \to S^3$ such that

i) $\phi(K) = S_1$ and $\phi(m) = S_2$

ii) ϕ maps the neighborhood E of $(K \cup m)$ diffeomorphically onto a similar neighborhood E_0 of H, and $\phi(S^3 \setminus E) \subset (S^3 \setminus E_0)$ ([Y;pg 501]). (E_0 is a tubular neighborhood of S_1 containing S_2.)

Then the map ϕ induces a framing s on $S^3 \setminus (K \cup m)$ having special singularity at $K \cup m$. $\square$

By the Schmidt orthonormalization process applied to s with respect to the hyperbolic structure of N_u, one obtains an orthonormal framing s_u on $N_u \setminus m$. By deforming it in a neighborhood of m if necessary, we may assume that s_u has a special singularity at m.

Given any Riemannian manifold R, the differential form C is defined on the positively-oriented orthonormal frame bundle by the following expression

$$\begin{aligned} C =& \frac{1}{4\pi^2}(4\theta_1 \wedge \theta_2 \wedge \theta_3 - d(\theta_1 \wedge \theta_{23} + \theta_2 \wedge \theta_{31} + \theta_3 \wedge \theta_{12}) \\ &+ \frac{1}{4i\pi^2}(\theta_{12} \wedge \theta_{13} \wedge \theta_{23} + \theta_{12} \wedge \Omega_{12} + \theta_{13} \wedge \Omega_{13} + \theta_{23} \wedge \Omega_{23}) \end{aligned}$$

where (θ_i), (θ_{ij}) and (Ω_{ij}) are, respectively, the fundamental 1-form, the connection 1-form and the curvature 2-form of the Riemannian connection on R.

Let $s' = (f_1, f_2, f_3)$ be an orthonormal framing defined on a subset of $\overrightarrow{M} \setminus \Sigma$ containing m such that $f_1(y)$ is the tangent vector to m at each $y \in m$ having the same direction as the e_1-vectors of s near y, and $f_2(y)$ is tangent to the meridian disc of Σ bounded by m. For $u \in U$ let s'_u be the result of orthonormalizing s' with respect to the hyperbolic metric of N_u.

3.2 Theorem of Yoshida. *[Y; Theorem 3.1]. The function*

$$f(u) = \int_{s_u(N_u \setminus m)} C - \frac{1}{2\pi} \int_{s'_u(m)} (\theta_1 - i\theta_{23}) \tag{4}$$

is a holomorphic function on a neighborhood of u^0 in U, where u^0 represents the original complete hyperbolic structure N.

3.3 Theorem. *For $u \in U$*

$$Re(f(u)) = \frac{1}{\pi^2} Vol(N_u) + \frac{1}{4\pi^2} Im(u\overline{v}) \tag{5}$$

Proof. Following the first part of the proof of Theorem 5.1 of [Y], for each $u \in U$ consider X_ϵ, the closure of $s_u(N_u \setminus (E_\epsilon(e) \cup m))$ in $F(N_u)$, where $E_\epsilon(e)$ is the ϵ-neighborhood of the end e of N_u. Then $\partial X_\epsilon = s(\partial E_\epsilon(e) \cup R)$, where R is mapped onto m by the bundle projection $F(N_u) \longrightarrow N_u$.

Then

$$\begin{aligned} Re(f(u)) &= \int_{s(N_u \setminus m)} Re(C) - \frac{1}{2\pi} \int_{s'(m)} \theta_1 \\ &= \frac{1}{\pi^2} Vol(N_u) - lim_{\epsilon \to 0} \int_{s(\partial E_\epsilon(e))} \frac{1}{4\pi^2} \Theta \end{aligned}$$

where $\Theta = \theta_1 \wedge \theta_{23} + \theta_2 \wedge \theta_{31} + \theta_3 \wedge \theta_{12}$.

To compute $-\int_{s(\partial E_\epsilon(e))} \frac{1}{4\pi^2}\Theta$, consider $p : \widetilde{N}_u \to N_u$, the universal cover of N_u and let $d_u : \widetilde{N}_u \to H^3$ be a developing map of N_u. We may assume that the image by d_u of a connected component of $p^{-1}E_\epsilon(e)$ lies into the hyperbolic cylinder around the t-axis

$$E_\delta = \{(r, \gamma.\phi) \in H^3 | log\, cot(\frac{\pi}{4} - \frac{\gamma}{2}) \leq \delta\},$$

Using these polar coordinates on $T_\epsilon = \partial E_\epsilon$ and putting $z = log\, r + i\phi$ on $\widetilde{T}_\epsilon$, the universal cover of T_ϵ, then T_ϵ is the quotient of $\widetilde{T}_\epsilon$ by the $\mathbb{Z} \times \mathbb{Z}$-action generated by the translations by $\{u, v\}$. Let I be the parallelogram spanned by them. We have

$$-\int_{s(\partial E_\epsilon(e))} \frac{1}{4\pi^2}\Theta = \frac{1}{4\pi^2} \frac{1 + sin^2\gamma}{cos^2\gamma} \int_I d\, log\, r \wedge d\,\phi = \frac{1}{4\pi^2} \frac{1 + sin^2\gamma}{cos^2\gamma} Area(I)$$

As $\epsilon \to 0$, $\gamma \to 0$ and $Area(I) \to Im(u\overline{v})$, which proves the theorem. $\square$

3.4 Theorem [N-Z]. *The real part of $df(u)$ is equal to the real part of $g(u)$, where g is the holomorphic differential*

$$g(u) = \frac{i}{4\pi^2} (vdu - udv) \tag{6}$$

PROOF. The generalized Schläffli formula (46) of [N-Z] with our choices of orientation (change of sign on v) is

$$dVol(N_u) = -\frac{\pi}{2}d(\frac{1}{2\pi}Im(u\overline{v})) - \frac{1}{4}\,Im(vdu - udv)$$

Then

$$\begin{aligned} dRe(f(u)) &= \frac{1}{\pi^2}dVol(N_u) + \frac{1}{4\pi^2}dIm(u\overline{v}) \\ &= -\frac{1}{4\pi^2}\,Im(vdu - udv) = Re(g(u)).\ \square \end{aligned}$$

The above theorem shows that g(u) is an exact holomorphic differential. Thus the integral

$$h(u) = \int_0^u g(u) = \frac{i}{4\pi}\int_0^u (vdu - udv)$$

defines a holomorphic function of u, such that $h(u) - f(u)$ =constant. In particular the imaginary part of $df(u)$ is equal to the imaginary part of $g(u)$.

3.5 THEOREM. *Let $(S^3, K_{\alpha(t)})$ be a family of hyperbolic cone-manifold structures in S^3, where $\alpha(t)$ and $\beta(t)$ are differentiable functions of t. Then the following equation ("the Schläffli formula") between differential forms holds*

$$dI(S^3, K_\alpha) = -\frac{1}{4\pi^2}\beta d\alpha \tag{7}$$

PROOF. Observe that

$$Im(f(u)) = \int_{s_u(N_u\backslash m)} Q + \frac{1}{2\pi}\int_{s'_u(m)} \theta_{23} = \int_{s_u(N_u\backslash m)} Q - \frac{1}{2\pi}\tau(m, s'_u)$$

and by (3) (Definition 2.2)

$$I(S^3, K_\alpha) = \frac{1}{2}\int_{s_u(S^3-K-m)} Q \; - \; \frac{1}{4\pi}\, tw(K) \; - \; \frac{1}{4\pi}\,\tau(m, s'_u)$$

Then

$$I(S^3, K_\alpha) \equiv \frac{1}{2} Im(f(u)) \; - \; \frac{1}{4\pi}\, tw(K)$$

This implies that

$$dI(S^3, K_\alpha) = \frac{1}{2} Im(g(u)) \; - \; \frac{1}{4\pi}\, d(\frac{1}{2\pi}\alpha\beta)$$

Now, because $u = i\alpha$, $v = \delta + i\beta$, we have that $Im(g(u)) = \frac{1}{4\pi^2}(\alpha d\beta - \beta d\alpha)$. One obtains

$$dI(S^3, K_\alpha) = \frac{1}{8\pi^2}(\alpha d\beta - \beta d\alpha) \; - \; \frac{1}{8\pi^2}\,(\alpha d\beta + \beta d\alpha) = -\frac{1}{4\pi^2}\beta d\alpha. \; \square$$

Case 2. The curvature of the cone manifold $(S^3, K_{\alpha(t)})$ is 1 for $t \in [t_0, t_1]$.

By a straightforward adaptation of an argument of Neuwirth [N, Chapter III], we can assume that for each $t \in [t_0, t_1]$ there exists a polyhedron (we do not require it to be either convex or geodesic, only of the same combinatorial type for each $t \in [t_0, t_1]$) $D_t \subset S^3$ such that $(S^3, K_{\alpha(t)})$ is the result of identifying the faces of D_t by isometries $\{g_1, ..., g_l\}$ of S^3. The set of edges of D_t that project onto K is $\{A_1, ..., A_h\}$; the set of edges of D_t that project onto the meridian m of K is $\{B_1, ..., B_k\}$. Denote $D_t - \{A_1, ..., A_h, B_1, ..., B_k\}$ by D_t^0. Since K is a singular geodesic we can suppose that for each $t \in [t_0, t_1]$, A_1 is part of a fixed geodesic γ, that a point P in $A_1 \subset \gamma$ projects onto the same point $\pi(P) \in K$ and that an orthogonal direction to K in P is also the same for each t. The identifications of pairs of faces in D_t are realized by the isometries $\{g_1(t), ..., g_l(t)\}$. Let D be the combinatorial polyhedron, then the family D_t defines a differential map

$$\eta : M^4 = D \times [t_0, t_1] \longrightarrow S^3$$

such that $\eta(\hat{P}, t) = P$, for each t, $\eta(A_1 \times [t_0, t_1]) \subset \gamma)$ and $d\eta(\hat{v}, t) = r(t)v$, where $\hat{v} \in T_{\hat{P}}D$, $v \perp \gamma$.

There exists an orthonormal framing s_t on $X_t = D_t^0/\{g_1, ..., g_l\}$ having special singularities at $K \cup m$ obtained by orthonormalization with respect to each Riemannian structure, of the framing s in $S^3 - (K \cup m)$. Let

$$\mu : M_0^4 = D^0 \times [t_0, t_1] \longrightarrow F(S^3) = SO(4)$$

be the map defined by $\mu(x,t) = \widetilde{s}_t(\eta(x,t))$ where $\widetilde{s}_t(\eta(x,t))$ is the framing induced on D_t^0 by the framing s_t on $X_t = D_t^0/\{g_1, ..., g_l\}$

By the Stokes formula

$$\int_{M_0} \mu^*(dQ) \;=\; \int_{\mu(M_0)} dQ \;=\; 0 \;=\; \int_{\partial\overline{\mu(M_0)}} Q \tag{8}$$

Observe that $\partial\overline{\mu(M_0)} \;=\; \mu(\partial D^0 \times [t_0, t_1]) \cup \mu(-D \times t_0) \cup \mu(D \times t_1) \cup C_1 \cup C_2$, where C_1 and C_2 are diffeomorphic to the image by μ of ((the boundary of a small neighborhood of $\{A_1, ..., A_h\}$ in $D) \times [t_0, t_1]$) and the image by μ of ((the boundary of a small neighborhood of $\{B_1, ..., B_k\}$ in $D) \times [t_0, t_1]$) respectively.

The faces of D can be paired as $(S_1, g_1 S_1), ..., (S_l, g_l S_l)$, then $\mu(g_i(x), t) = dg_i(\mu(x,t))$ for all $x \in S_i$, $i = 1, ...l$ and for all $t \in [t_0, t_1]$. This implies that

$$\int_{\mu(\partial D^0 \times [t_0,t_1])} Q = \sum_{i=1}^{l} \left(\int_{\mu(S_i \times [t_0,t_1])} Q - \int_{\mu(g_i S_i \times [t_0,t_1])} Q \right) \;=\; 0. \tag{9}$$

On the other hand

$$\begin{aligned} \int_{\mu(D_{t_1})} Q \;&=\; \int_{s_1(S^3 - K - m)} Q \\ \int_{\mu(D_{t_0})} Q \;&=\; \int_{s_0(S^3 - K - m)} Q \end{aligned} \tag{10}$$

Now we compute $\int_{C_1} Q$ and $\int_{C_2} Q$.

Observe that the image of the quotient map

$$q_t : \overline{\mu(D^0 \times \{t\})} \longrightarrow \overline{\mu(D^0 \times \{t\})}/\{dg_1(t), ..., dg_l(t)\}$$

is a 3-manifold $\widehat{X}_t$ with boundary which is diffeomorphic to the compactification of X_t with two tori: $(K \cup m) \times S^1$. The "quotient" map

$$q : \overline{\mu(D^0 \times [t_0, t_1])} \longrightarrow \cup_{t\in[t_0,t_1]} \widehat{X}_t := X$$

given by $q(\mu(x,t)) = q_t(x)$, sends C_1 and C_2 to $\cup_{t\in[t_0,t_1]} \partial\widehat{X}_t = \widehat{C}_1 \cup \widehat{C}_2$.

3.6 Lemma.

$$\int_{C_1} Q = \frac{1}{4\pi^2} \int_{[t_0,t_1]} (\beta\alpha' - \beta'\alpha)dt \tag{11}$$

where $\alpha(t)$ is the angle around K and $\beta(t)$ is the jump of K.

Proof. Consider the following parametrization of $\widehat{C}_1$:

$$\rho : K \times S^1 \times [t_0, t_1] \longrightarrow \widehat{C}_1$$
$$(\phi, \psi, t) \to \left(\delta(t)e_1(\phi), \left(\frac{\beta(t)}{2\pi}\phi + \frac{\alpha(t)}{2\pi}\psi\right) E_{23}, \frac{\partial}{\partial t}\right)$$

where $\phi \in [0, 2\pi]$ is a parametrization of K, e_1 is the first vector of s_t, E_{23} is the unit twist around e_1, $\alpha(t)$ is the angle around K and $\beta(t)$ is the jump of K. Observe that ρ is well defined because the factor S^1 projects onto a point in (S^3, K_α), and is described as a twist (E_{23}) around a vector e_1 tangent to K. The coefficient is $\left(\frac{\beta(t)}{2\pi}\phi + \frac{\alpha(t)}{2\pi}\psi\right)$, because for ϕ_0, $\psi \in [0, 2\pi]$ the angle is α, and for ψ_0, $\phi \in [0, 2\pi]$ the angle is β.

To compute ρ^*Q, note that $\rho^*\theta_{12} = \rho^*\theta_{13} = 0$ and $\rho^*\Omega_{23} = d\rho^*\theta_{23}$, so that $\rho^*Q = \frac{1}{4\pi^2}\rho^*\theta_{23} \wedge d\rho^*\theta_{23}$. On the other hand $\rho^*\theta_{23}\left(\frac{\partial}{\partial\phi}\right) = \theta_{23}\left(d\rho\left(\frac{\partial}{\partial\phi}\right)\right) = \frac{\beta(t)}{2\pi}$, and $\rho^*\theta_{23}\left(\frac{\partial}{\partial\psi}\right) = \theta_{23}\left(d\rho\left(\frac{\partial}{\partial\psi}\right)\right) = \frac{\alpha(t)}{2\pi}$. Therefore $\rho^*\theta_{23} = \frac{\alpha(t)}{2\pi}d\psi + \frac{\beta(t)}{2\pi}d\phi$, and $d\rho^*\theta_{23} = -\frac{\alpha'(t)}{2\pi}d\psi \wedge dt - \frac{\beta'(t)}{2\pi}d\phi \wedge dt$. Thus $\rho^*Q = -\frac{1}{4\pi^2}\frac{1}{4\pi^2}(\beta\alpha' - \beta'\alpha)d\phi \wedge d\psi \wedge dt$.

$$\begin{aligned}
\int_{C_1} Q &= -\int_{K\times S^1\times[t_0,t_1]} \rho^*Q \\
&= \frac{1}{4\pi^2}\int_{K\times S^1\times[t_0,t_1]} \frac{1}{4\pi^2}(\beta\alpha' - \beta'\alpha)d\phi \wedge d\psi \wedge dt \\
&= \frac{1}{4\pi^2}\int_{[t_0,t_1]} (\beta\alpha' - \beta'\alpha)dt. \ \square
\end{aligned}$$

3.7 Lemma.

$$\int_{C_2} Q = \frac{1}{2\pi}\left(\tau_0(m) - \tau_1(m)\right) \tag{12}$$

where $\tau_0(m)$ and $\tau_1(m)$ are the torsion of the meridian m of K in the cone-manifold structures t_0 and t_1, respectively, with respect to the same longitude.

Proof. The computation for C_2 is analogous to the computation for C_1, but now the angle is constant (2π) and the jump is the torsion τ_t. Then

$$\begin{aligned}\int_{C_2} Q &= -\int_{m\times S^1\times[t_0,t_1]} \rho^* Q \\ &= -\frac{1}{4\pi^2}\int_{m\times S^1\times[t_0,t_1]} \frac{1}{2\pi}\tau_t' d\phi\wedge d\psi\wedge dt \\ &= -\frac{1}{2\pi}\int_{[t_0,t_1]} \tau_t' dt = \frac{1}{2\pi}\left(\tau_0(m)-\tau_1(m)\right). \ \square\end{aligned}$$

3.8 Theorem. *Let $(S^3, K_{\alpha(t)})$ be a family of spherical cone-manifold structures in S^3, where $\alpha(t)$ and $\beta(t)$ are differentiable functions of t. Then the following equation ("the Schläffli formula") between differential forms holds*

$$dI(S^3, K_\alpha) = -\frac{1}{4\pi^2}\beta d\alpha \tag{13}$$

Proof. By (3), (8), (9), (10), (11) and (12), we have

$$\begin{aligned}I(S^3,K_{\alpha_1}) - I(S^3,K_{\alpha_0}) &= \frac{1}{2}\int_{s_1(S^3-K-m)} Q - \frac{1}{4\pi}\tau_1(m) - \frac{1}{4\pi}tw(K_{\alpha(t_1)}) \\ &\quad - \frac{1}{2}\int_{s_0(S^3-K-m)} Q + \frac{1}{4\pi}\tau_0(m) + \frac{1}{4\pi}tw(K_{\alpha(t_0)}).\end{aligned}$$

Using the fact that $tw(K_{\alpha(t)}) = \frac{\alpha(t)\beta(t)}{2\pi}$, we obtain

$$
\begin{aligned}
& I(S^3, K_{\alpha_1}) - I(S^3, K_{\alpha_0}) \frac{1}{8\pi^2}\alpha(t_0)\beta(t_0) \\
& \mu\mu - \frac{1}{8\pi^2}\alpha(t_1)\beta(t_1) - \frac{1}{8\pi^2}\int_{[t_0,t_1]}(\beta\alpha' - \beta'\alpha)dt \\
& \qquad = -\frac{1}{8\pi^2}\int_{[t_0,t_1]}((\beta\alpha' - \beta'\alpha) + (\beta\alpha)')dt = -\frac{1}{8\pi^2}\int_{[t_0,t_1]} 2\beta\alpha' dt \\
& \qquad = -\frac{1}{4\pi^2}\int_{[t_0,t_1]} \beta\alpha' dt
\end{aligned}
$$

Therefore

$$dI(S^3, K_\alpha) = -\frac{1}{4\pi^2}\beta d\alpha. \ \square$$

Suppose we compute directly the volume, and also the Chern-Simons invariant of the complete hyperbolic structure of the exterior of a hyperbolic knot K in S^3, using the framing defined by the canonical longitude of K. These are $V(S^3, K_0)$ and $I_1(S^3, K_0)$, respectively. Then the following theorem applies.

3.9 THEOREM. *Let $\overrightarrow{M}_n(K)$ be the n-cyclic cover of S^3 branched over the hyperbolic knot K, and let $(\overrightarrow{S}^3, \overrightarrow{K}_{\alpha(t)})$ be a family of cone manifold structures such that the angle $\alpha(t) \in (0, \alpha_0]$, the jump $\overline{\beta(t)} \in \mathbb{R}/(4\pi\mathbb{Z})$ and the length δ of K are differentiable functions of t. The Chern-Simons invariant of the geometric manifold $\overrightarrow{M}_n(K)$, can be obtained by the following formulas.*

$$CS(\overrightarrow{M}_n(K)) \equiv nI_1(\overrightarrow{S}^3, \overrightarrow{K}_0) - \frac{n}{4\pi^2}\int_0^{\frac{2\pi}{n}} \beta d\alpha \qquad (\mathrm{mod}\ 1) \tag{14}$$

where β is any differentiable lift of the jump to the universal covering $\mathbb{R}$ of $\mathbb{R}/(4\pi\mathbb{Z})$. The volume of the hyperbolic manifold $\overrightarrow{M}_n(p/q)$, $n > n_0$, is

$$V(\overrightarrow{M}_n(K)) = nV(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}}) = nV(\overrightarrow{S}^3, \overrightarrow{K}_0) - \frac{n}{2}\int_0^{\frac{2\pi}{n}} \delta d\alpha. \tag{15}$$

PROOF. Lets prove (14). Let β be a differentiable lift to $\mathbb{R}$ of $\overline{\beta(t)}$, then

$$-\frac{1}{4\pi^2}\int_0^{\frac{2\pi}{n}} \beta d\alpha = \tilde{I}(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}}) - \tilde{I}(\overrightarrow{S}^3, \overrightarrow{K}_0) \tag{16}$$

where $\tilde{I}(\overrightarrow{S}^3, \overrightarrow{K}_\alpha)(t)$ is a differentiable lift to $\mathbb{R}$ of $I_{\frac{\alpha}{2\pi}}(\overrightarrow{S}^3, \overrightarrow{K}_\alpha)(t)$. As $\tilde{I}(\overrightarrow{S}^3, \overrightarrow{K}_\alpha)(t)$ and $I(\overrightarrow{S}^3, \overrightarrow{K}_\alpha)(t)$ are two lifts to the universal covering $\mathbb{R}$ of $\mathbb{R}/(\frac{\alpha}{2\pi})$ for the same function $I_{\frac{\alpha}{2\pi}}(\overrightarrow{M}, \overrightarrow{K}_\alpha)(t)$, they differ by a multiple of $\frac{\alpha}{2\pi}$. In particular for $\alpha = \frac{2\pi}{n}$, they differ by $\frac{k}{n}$

$$\tilde{I}(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}}) = I(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}}) + \frac{k}{n}.$$

Then

$$\begin{aligned}
\tilde{I}_1(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}}) &\equiv I_1(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}}) + \frac{k}{n} \pmod 1\\
\tilde{I}_1(\overrightarrow{S}^3, \overrightarrow{K}_0) &\equiv I_1(\overrightarrow{S}^3, \overrightarrow{K}_0) \pmod 1.
\end{aligned}$$

From (16) we have

$$\begin{aligned}
\tilde{I}(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}}) &= \tilde{I}(\overrightarrow{S}^3, \overrightarrow{K}_0) - \frac{1}{4\pi^2}\int_0^{\frac{2\pi}{n}} \beta d\alpha\,,\\
\tilde{I}_1(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}}) &\equiv I_1(\overrightarrow{M}, \overrightarrow{K}_{\frac{2\pi}{n}}) + \frac{k}{n}\\
&\equiv I_1(\overrightarrow{S}^3, \overrightarrow{K}_0) - \frac{1}{4\pi^2}\int_0^{\frac{2\pi}{n}} \beta d\alpha \pmod 1
\end{aligned}$$

And therefore

$$\begin{aligned}
nI_1(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}}) &\equiv nI_1(\overrightarrow{S}^3, \overrightarrow{K}_0) - \frac{n}{4\pi^2}\int_0^{\frac{2\pi}{n}} \beta d\alpha \pmod 1\\
CS(\overrightarrow{M}_n(K)) &\equiv nI_1(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}})\\
&\equiv nI_1(\overrightarrow{S}^3, \overrightarrow{K}_0) - \frac{n}{4\pi^2}\int_0^{\frac{2\pi}{n}} \beta d\alpha \pmod 1.
\end{aligned}$$

To prove (15) note that $V(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}})$ is easly computed, (see [H]), by using the Formula of Schläffli for the volume (see for instance [M]). In a one parameter family of polytopes in the hyperbolic space, $dV = -(1/2)\sum \ell_i d\alpha_i$, where V is volume, the sum is taken over all the edges, ℓ_i is the length of the ith edge and α_i is its dihedral angle. The volume of a cone manifold is the volume of the polyhedron from which it is constructed before identifications are made. If several edges of a polytope are identified and the resulting identified edge is <u>not</u> part of the singular set then the sum of the corresponding dihedral angles is 2π and, since the differential of the constant 2π equals zero, these edges make no contribution to dV. Hence, in our case (see [HLM$_1$]),

$$dV(S^3, (p/q)_\alpha) = -(1/2)\delta d\alpha.$$

Therefore

$$V(\overrightarrow{S}^3, \overrightarrow{K}_{\frac{2\pi}{n}}) = V(\overrightarrow{S}^3, \overrightarrow{K}_0) - \frac{1}{2}\int_0^{\frac{2\pi}{n}} \delta d\alpha.$$

and this implies (15). □

See [HLM$_2$] for a detailed computation of the Chern-Simons invariants and volumes of the cyclic coverings of S^3 branched over a rational knot p/q.

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