J. Math. Sci. Univ. Tokyo
3 (1996), 699–711.

Global Theta Liftings of General Linear Groups

By Takao WATANABE

Abstract. A global theta lifting of some irreducible type 2 dual reductive pair is studied. It is proved that the image of a global theta lifting of a given irreducible automorphic cuspidal representation is nonvanishing if and only if its standard L-function is nonzero at the point $1/2$, and then the image coincides with the initial automorphic cuspidal representation. As a corollary of this result, the global Howe correspondence is obtained.

Introduction

Let k be a global field and $\mathbb{A}$ the adele ring of k. The pair $(GL_n(k), GL_n(k))$ is a type 2 dual reductive pair in the symplectic group $Sp_{n^2}(k)$ of size $2n^2$ ([3]). If ω' denotes the Weil representation of the metaplectic group $Mp_{n^2}(\mathbb{A})$ of $Sp_{n^2}(\mathbb{A})$, then the restriction of ω' to $GL_n(\mathbb{A}) \times GL_n(\mathbb{A})$ is described as follows. Let $\mathcal{S}(M_n(\mathbb{A}))$ be the space of Schwartz - Bruhat functions on the set $M_n(\mathbb{A})$ of all $n \times n$ matrices with entries in $\mathbb{A}$. Then, for $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $h,\ g \in GL_n(\mathbb{A})$,

$$\omega'(h,g)f(x) = |\det h|_{\mathbb{A}}^{n/2} |\det g|_{\mathbb{A}}^{n/2} f({}^t hxg) .$$

Let ω be the representation of $GL_n(\mathbb{A}) \times GL_n(\mathbb{A})$ defined by $\omega(h,g) = \omega'({}^t h^{-1}, g)$. We use ω instead of ω' for convenience. The purpose of this paper is to study the theta lifting and the Howe correspondence of the irreducible automorphic cuspidal representations of $GL_n(\mathbb{A})$ with respect to ω.

In order to mention our results, we denote by $\mathcal{H}_n = \otimes_v' \mathcal{H}_{n,v}$ the global Hecke algebra of $GL_n(\mathbb{A})$ (cf. [1, Section 3]) and K_n the standard maximal

1991 *Mathematics Subject Classification.* Primary 11F70; Secondary 11F27, 22E50.

compact subgroup of $GL_n(\mathbb{A})$. Let $\mathcal{S}_0(M_n(\mathbb{A}))$ be the subspace of $\mathcal{S}(M_n(\mathbb{A}))$ consisting of all $K_n \times K_n$-finite functions. We consider ω as a representation of $\mathcal{H}_n \otimes \mathcal{H}_n$ acting on $\mathcal{S}_0(M_n(\mathbb{A}))$. Let π be an irreducible automorphic cuspidal representation of $GL_n(\mathbb{A})$. We always assume that the representation space H_π of π is contained in the space of cusp forms on $GL_n(\mathbb{A})$. Thus H_π is an $\mathcal{H}_n$-module, but not a $GL_n(\mathbb{A})$-module. For $\varphi \in H_\pi$, $f \in \mathcal{S}_0(M_n(\mathbb{A}))$ and a complex number $s \in \mathbf{C}$, the theta lifting φ_f^s of φ is defined to be

$$\varphi_f^s(h) = \int_{GL_n(k)\backslash GL_n(\mathbb{A})} \varphi(g)|\det g|_{\mathbb{A}}^s \sum_{\substack{x \in M_n(k) \\ x \neq 0}} \omega(h,g)f(x)dg \ .$$

This integral is absolutely convergent for $\mathrm{Re}(s) >> 0$ and analytically continued to the whole s-plane as an entire function (see Lemma 3). For a fixed $s \in \mathbf{C}$, we denote by $\Theta^s(\pi)$ the space spanned by functions φ_f^s, ($\varphi \in H_\pi$, $f \in \mathcal{S}_0(M_n(\mathbb{A}))$) on $GL_n(\mathbb{A})$. Then we prove the following theorem.

THEOREM 1. *Let π be an irreducible automorphic cuspidal representation of $GL_n(\mathbb{A})$ and $L(s,\pi)$ its standard automorphic L-function. Then, the space $\Theta^0(\pi)$ is nonzero if and only if $L(1/2,\pi)$ is nonzero. In this case, $\Theta^0(\pi)$ coincides with H_π.*

We write $\pi^\vee$ for the contragredient representation of π. Next theorem is obtained as a corollary of the proof of Theorem 1 and the strong multiplicity one theorem.

THEOREM 2. *For any irreducible automorphic cuspidal representation π of $GL_n(\mathbb{A})$, one has $\mathrm{Hom}_{\mathcal{H}_n \otimes \mathcal{H}_n}(\omega, \pi \otimes \pi^\vee) \neq 0$. Furthermore, if σ is an irreducible automorphic cuspidal representation satisfying $\mathrm{Hom}_{\mathcal{H}_n \otimes \mathcal{H}_n}(\omega, \sigma \otimes \pi^\vee) \neq 0$, then σ is isomorphic to π.*

It should be noted that the explicit local theta correspondence of (GL_n, GL_n) was implicitly proved by Godement and Jacquet ([2], [12]). Thus one can formally prove the first assertion of Theorem 2 for any irreducible admissible representation of $GL_n(\mathbb{A})$ (see Remark after the proof of Theorem 2). However, it seems for the author that there is no article described these facts and the global theta liftings (cf. [10, Section 4.6.5]).

We will use the following notations. For an associative ring R with identity element, we denote by $M_n(R)$ the set of all $n \times n$ matrices with entries in R. For $A \in M_n(R)$, $\det A$ stands for its determinant. The identity matrix in $M_n(R)$ is denoted by 1_n. When a base field F is given, we set $G_n = GL_n(F)$. We denote by U_n the groups consisting of all upper triangular matrices with ones in the diagonals and by Z_n the center of G_n. If F is a local field, then $|\cdot|_F$ denotes the normalized absolute value on F and α_F denotes the character of G_n defined as $\alpha_F(g) = |\det g|_F$ for $g \in G_n$. If G is a locally compact abelian group, then $\mathcal{S}(G)$ denotes the space of Schwartz - Bruhat functions on G.

1. The local theta correspondence: Non-archimedean case

Let F be a local non-archimedean field and q the order of the residual field of F. We fix a non-trivial additive character ψ_F of F. The character $\psi_{n,F}$ of U_n is defined to be

$$\psi_{n,F}(u) = \psi_F(u_{12} + u_{23} + \cdots + u_{n-1\,n}), \qquad (u = (u_{ij}) \in U_n) . \tag{1.1}$$

Let $\mathbf{W}(\psi_{n,F})$ be the space of all locally constant functions W on G_n satisfying $W(ug) = \psi_{n,F}(u)W(g)$ for any $u \in U_n$ and $g \in G_n$. Then $g \in G_n$ acts on $\mathbf{W}(\psi_{n,F})$ by right translation; $\rho(g)W(g') = W(g'g)$. For an irreducible admissible generic representation π of G_n, we denote by $\mathbf{W}(\pi, \psi_{n,F})$ the Whittaker model of π in $\mathbf{W}(\psi_{n,F})$.

We define the smooth representation $(\omega_F, \mathcal{S}(M_n(F)))$ of $G_n \times G_n$ as follows: for $f \in \mathcal{S}(M_n(F))$ and $h,\ g \in G_n$,

$$\omega_F(h,g)f(x) = \alpha_F(h)^{-n/2}\alpha_F(g)^{n/2}f(h^{-1}xg) . \tag{1.2}$$

For $f \in \mathcal{S}(M_n(F))$, $W \in \mathbf{W}(\pi, \psi_{n,F})$ and a complex number $s \in \mathbf{C}$, we consider the integral

$$\begin{aligned} V^s_{(W,f)}(h) &= \int_{G_n} W(g)\omega_F(h,g)f(1_n)\alpha_F(g)^{s-1/2}dg , \qquad (h \in G_n) \\ &= \alpha_F(h)^{-n/2}\int_{G_n} W(g)f(h^{-1}g)\alpha_F(g)^{s+n/2-1/2}dg . \end{aligned} \tag{1.3}$$

This integral converges absolutely for $\mathrm{Re}(s)$ large and becomes a rational function of q^{-s} (cf. [7, (5.2)]). Let $I(\pi)$ be the linear span of rational functions $V^s_{(W,f)}(h)$, ($W \in \mathbf{W}(\pi, \psi_{n,F})$, $f \in \mathcal{S}(M_n(F))$, $h \in G_n$). It is known by [2, Theorem 3.3] and [7, (5.2)] that $I(\pi)$ equals a fractional ideal $L(s,\pi)\mathbf{C}[q^{-s}, q^s]$ of the polynomial ring $\mathbf{C}[q^{-s}, q^s]$, where $L(s,\pi)$ stands for the local factor of π defined by Godement and Jacquet. Therefore, as a function in s, $L(s,\pi)^{-1}V^s_{(W,f)}(h)$ is holomorphic on $\mathbf{C}$ and is denoted by $\widetilde{V}^s_{(W,f)}(h)$. On the other hand, as a function in $h \in G_n$, $V^s_{(W,f)}$ is contained in $\mathbf{W}(\psi_{n,F})$. We denote by $\mathbf{V}^s(\pi, \psi_{n,F})$ the linear span of functions $\widetilde{V}^s_{(W,f)}$, ($W \in \mathbf{W}(\pi, \psi_{n,F})$, $f \in \mathcal{S}(M_n(F))$). By the uniqueness of analytic continuation , we have

$$\rho(g_1)\widetilde{V}^s_{(\pi(g_2)W,f)} = \alpha_F(g_2)^{-s+1/2}\widetilde{V}^s_{(W,\omega(g_1,g_2^{-1})f)}\ , \qquad (g_1,\ g_2 \in G_n)$$

for all $s \in \mathbf{C}$. Therefore, $\mathbf{V}^s(\pi, \psi_{n,F})$ is a nonzero G_n-submodule of $\mathbf{W}(\psi_{n,F})$ for each s.

LEMMA 1. *The space $\mathbf{V}^s(\pi, \psi_{n,F})$ coincides with the space $\alpha_F^{s-1/2} \otimes \mathbf{W}(\pi, \psi_{n,F})$ for all $s \in \mathbf{C}$.*

PROOF. We first assume $\mathrm{Re}(s) >> 0$. By changing g to hg in the integral (1.3), we obtain

$$V^s_{(W,f)}(h) = \alpha_F(h)^{s-1/2}\int_{G_n} W(hg)f(g)\alpha_F(g)^{s+n/2-1/2}dg\ .$$

We take an open compact subgroup Ω in G_n such that $f(kg) = f(g)$ for any $k \in \Omega$. Let dk be the Haar measure on Ω normalized so that the volume of Ω equals 1. Then, we have

$$\begin{aligned}\int_{G_n} & W(hg)f(g)\alpha_F(g)^{s+n/2-1/2}dg \\ &= \int_\Omega\int_{G_n} W(hkg)f(kg)\alpha_F(kg)^{s+n/2-1/2}dgdk \\ &= \int_{G_n}\left(\int_\Omega W(hkg)dk\right) f(g)\alpha_F(g)^{s+n/2-1/2}dg\ .\end{aligned}$$

Let $\mathbf{W}(\pi, \psi_{n,F})^\Omega$ be the subspace of $\mathbf{W}(\pi, \psi_{n,F})$ consisting of all elements fixed by Ω. The admissibility of π implies that $\mathbf{W}(\pi, \psi_{n,F})^\Omega$ is of finite dimension. Thus we can take a basis $\{W_1, \cdots, W_m\}$ of $\mathbf{W}(\pi, \psi_{n,F})^{\Omega}$. Then, by the same argument as in [7, page 434], there exist matrix coefficients $\phi_1, \cdots, \phi_m$ of π such that

$$\int_{\Omega} W(hkg)dk = \sum_{j=1}^{m} W_j(h)\phi_j(g) \ .$$

Therefore, if we set

$$Z(f, s+n/2-1/2; \phi_j) = \int_{G_n} \phi_j(g) f(g) \alpha_F(g)^{s+n/2-1/2} dg \ , \tag{1.4}$$

then we have

$$\widetilde{V}^s_{(W,f)}(h) = \alpha_F(h)^{s-1/2} \sum_{j=1}^{m} \frac{Z(f, s+n/2-1/2; \phi_j)}{L(s,\pi)} W_j(h) \ . \tag{1.5}$$

Since the right-hand side is holomorphic on $\mathbf{C}$ by [2, Theorem 3.3], (1.5) holds for all $s \in \mathbf{C}$, and hence $\widetilde{V}^s_{(W,f)}$ is contained in $\alpha_F^{s-1/2} \otimes \mathbf{W}(\pi, \psi_{n,F})$. The irreducibility of π concludes that $\mathbf{V}^s(\pi, \psi_{n,F}) = \alpha_F^{s-1/2} \otimes \mathbf{W}(\pi, \psi_{n,F})$. □

As we mentioned in Introduction, Godement and Jacquet ([2]) essentially proved that $\mathrm{Hom}_{G_n \times G_n}(\omega_F, \pi \otimes \pi^{\vee}) \neq 0$ for any irreducible admissible representation π of G_n. Furthermore, Weil ([12]) noted that $\dim \mathrm{Hom}_{G_n \times G_n}(\omega_F, \pi \otimes \pi^{\vee}) = 1$ if π is a supercuspidal representation.

2. The local theta correspondence: Archimedean case

In this section, we denote by F a local archimedean field. Let $\mathcal{G}_n$ be the Lie algebra of G_n as a real Lie group and K the standard maximal compact subgroup of G_n. We define a non-trivial additive character ψ_F of F as

$$\psi_F(a) = \begin{cases} \exp(2\pi\sqrt{-1}a\lambda) & \text{if } F = \mathbf{R} \\ \exp(2\pi\sqrt{-1}(a\lambda + \overline{a\lambda})) & \text{if } F = \mathbf{C} \end{cases},$$

where $\lambda \in F$ is a nonzero constant. The character $\psi_{n,F}$ of U_n is defined similarly as (1.1).

Let (π, H^∞) be an irreducible admissible representation of G_n realized as a smooth Fréchet representation of moderate growth (cf. [9, Section 2]). We denote by H the space of K-finite vectors in the Fréchet space H^∞. We assume that π is generic, i.e. there exists a nonzero continuous linear functional λ on H^∞ satisfying

$$\lambda(\pi(u)v) = \psi_{n,F}(u)\lambda(v)$$

for all $u \in U_n$ and $v \in H^\infty$. Such a λ is unique up to constant ([11, Theorem 3.1]). Then we denote by $\mathbf{W}^\infty(\pi, \psi_{n,F})$ the space of functions W_v on G_n of the form

$$W_v(g) = \lambda(\pi(g)v), \qquad (v \in H^\infty).$$

We also denote by $\mathbf{W}(\pi, \psi_{n,F})$ the subspace of $\mathbf{W}^\infty(\pi, \psi_{n,F})$ consisting of W_v with $v \in H$, so that $\mathbf{W}(\pi, \psi_{n,F})$ is an underlying irreducible $(\mathcal{G}_n, K)$-module of π. We write $L(s, \pi)$ for the local factor of π defined by Godement and Jacquet ([2, Theorem 8.7]).

We define the smooth representation $(\omega_F, \mathcal{S}(M_n(F)))$ of $G_n \times G_n$ by the same way as (1.2). Let $\mathcal{S}_0(M_n(F))$ be the subspace of $\mathcal{S}(M_n(F))$ consisting of all $K \times K$-finite functions. Then $(\omega, \mathcal{S}_0(M_n(F)))$ is a $(\mathcal{G}_n \oplus \mathcal{G}_n, K \times K)$-module. For $W \in \mathbf{W}^\infty(\pi, \psi_{n,F})$, $f \in \mathcal{S}(M_n(F))$ and $s \in \mathbf{C}$, we set

$$V^s_{(W,f)}(h) = \int_{G_n} W(g)\omega_F(h, g)f(1_n)\alpha_F(g)^{s-1/2}dg\ .$$

By [9, Section 6] (or [6, Section 9]), this integral is absolutely convergent for $\mathrm{Re}(s) >> 0$ and extends to a meromorphic function on the whole $\mathbf{C}$. Furthermore, if we set $\widetilde{V}^s_{(W,f)}(h) = L(s,\pi)^{-1}V^s_{(W,f)}(h)$, it becomes an entire function in s. By the uniqueness of analytic continuation, $\widetilde{V}^s_{(W,f)}$ satisfies the following for all $s \in \mathbf{C}$:

$$\widetilde{V}^s_{(W,f)}(uh) = \psi_{n,F}(u)\widetilde{V}^s_{(W,f)}(h), \qquad (u \in U_n),$$
$$\widetilde{V}^s_{(\pi(g_2)W,f)}(hg_1) = \alpha_F(g_2)^{-s+1/2}\widetilde{V}^s_{(W,\omega(g_1,g_2^{-1})f)}(h), \qquad (g_1,\ g_2 \in G_n).$$

Let $\mathbf{V}^s(\pi, \psi_{n,F})$ denote the linear span of $\widetilde{V}^s_{(W,f)}$, $(W \in \mathbf{W}(\pi, \psi_{n,F}),\ f \in \mathcal{S}_0(M_n(F)))$. Since the linear span of functions $\widetilde{V}^s_{(W,f)}(1_n)$, $(W \in$

$\mathbf{W}(\pi, \psi_{n,F})$, $f \in \mathcal{S}_0(M_n(F))$) in s contains the set $\{P(s)|\lambda|_F^{-ns/2} \colon P(s) \in \mathbf{C}[s]\}$ (cf. [2, Theorem 8.7]), the space $\mathbf{V}^s(\pi, \psi_{n,F})$ is nonzero for all $s \in \mathbf{C}$.

LEMMA 2. *The space* $\mathbf{V}^s(\pi, \psi_{n,F})$ *coincides with the space* $\alpha_F^{s-1/2} \otimes \mathbf{W}(\pi, \psi_{n,F})$ *for all* $s \in \mathbf{C}$.

PROOF. For $f \in \mathcal{S}_0(M_n(F))$, there exists an elementary idempotent ξ in the Hecke algebra of G_n ([2, Section 8]) such that

$$\int_K f(k^{-1}x)\xi(k)dk = f(x) \ .$$

The admissibility of π implies that the image $\pi(\xi)\mathbf{W}(\pi, \psi_{n,F})$ of $\pi(\xi)$ is of finite dimension. Let $\{W_1, \cdots, W_m\}$ be a basis of $\pi(\xi)\mathbf{W}(\pi, \psi_{n,F})$. From the similar argument as in the proof of [6, Proposition 9.2], it follows that, for each $W \in \mathbf{W}(\pi, \psi_{n,F})$, there exist bi-$K$-finite matrix coefficients $\phi_1, \cdots, \phi_m$ of π such that

$$\pi(\xi)(\pi(g)W)(h) = \int_K W(hkg)\xi(k)dk = \sum_{j=1}^{m} W_j(h)\phi_j(g) \ .$$

Then, by the analogous calculation as in the proof of Lemma 1, we have

$$\widetilde{V}^s_{(W,f)}(h) = \alpha_F(h)^{s-1/2} \sum_{j=1}^{m} \frac{Z(f, s+n/2-1/2, \phi_j)}{L(s,\pi)} W_j(h) \ , \tag{2.1}$$

if $\mathrm{Re}(s) >> 0$. Here $Z(f, s+n/2-1/2, \phi_j)$ is defined similarly as (1.4). It is known by [2, Theorem 8.7] or [5, Proof of Proposition 4.5] that $L(s,\pi)^{-1}Z(f, s+n/2-1/2, \phi_j)$ extends to an entire function of s. Thus the assertion follows from the same argument as in the proof of Lemma 1. □

3. The global theta correspondence

In the rest of this paper, we denote by k a global field and by $\mathbb{A} = \prod'_v k_v$ the adele ring of k. For a k-subgroup G of $G_n = GL_n(k)$, $G(\mathbb{A}) = \prod'_v G(k_v)$ denotes the corresponding adele group. We fix a non-trivial additive character ψ of $k\backslash\mathbb{A}$ and define the character ψ_n of $U_n(\mathbb{A})$ similarly as (1.1). The restriction of ψ_n to $U_n(k_v)$ is denoted by $\psi_{n,v}$. We define the character α of $G_n(\mathbb{A})$ by $\alpha(g) = |\det g|_{\mathbb{A}}$. Let $\pi = \otimes_v' \pi_v$ be an irreducible automorphic cuspidal representation of $G_n(\mathbb{A})$. There exists a unique real number t so that $\alpha^{-t}\varphi$ is square integrable on $Z(\mathbb{A})G_n\backslash G_n(\mathbb{A})$ for any $\varphi \in H_\pi$. Thus $\alpha^{-t} \otimes \pi$ becomes a unitary cuspidal representation. For each $\varphi \in H_\pi$, we set

$$W_\varphi(g) = \int_{U_n\backslash U_n(\mathbb{A})} \psi_n(u)^{-1}\varphi(ug)du \, .$$

The space $\mathbf{W}(\pi, \psi_n)$ of all W_φ ($\varphi \in H$) is decomposed into the restricted tensor product of local Whittaker models $\mathbf{W}(\pi_v, \psi_{n,v})$, i.e.

$$\mathbf{W}(\pi, \psi_n) = \otimes_v' \mathbf{W}(\pi_v, \psi_{n,v}) \, .$$

Let μ_π be the central character of π. For $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $s \in \mathbf{C}$, we define a modified theta series $\theta(s, \mu_\pi, f)$ as

$$\theta(s, \mu_\pi, f) = \int_{Z_n\backslash Z_n(\mathbb{A})} \mu_\pi(z)\alpha(z)^{s+n/2} \sum_{\substack{x\in M_n(k) \\ x\neq 0}} f(zx)dz \, .$$

From [2, Lemmas 11.5 and 11.6], it follows that the integral of the right-hand side is absolutely convergent for $\mathrm{Re}(s) > n/2 - t$ and the function $(h, g) \mapsto \theta(s, \mu, \omega(h, g)f)$ is slowly increasing on $G_n\backslash G_n(\mathbb{A}) \times G_n\backslash G_n(\mathbb{A})$. By using $\theta(s, \mu_\pi, f)$, the theta lifting φ_f^s of $\varphi \in H_\pi$ is written as

$$\varphi_f^s(h) = \int_{Z_n(\mathbb{A})G_n\backslash G_n(\mathbb{A})} \varphi(g)\alpha(g)^s\theta(s, \mu_\pi, \omega(h, g)f)dg \, .$$

Since $\varphi(h)$ is rapidly decreasing on $Z_n(\mathbb{A})G_n\backslash G_n(\mathbb{A})$, this integral is absolutely convergent for $\mathrm{Re}(s) > n/2 - t$.

LEMMA 3. *$\varphi_f^s(h)$ is analytically continued to an entire function of s.*

PROOF. If $\mathrm{Re}(s) > n/2 - t$, we have

$$\varphi_f^s(h) = \sum_{j=1}^{n} \int_{G_n\backslash G_n(\mathbb{A})} \varphi(g)\alpha(g)^s \sum_{\substack{x\in M_n(k) \\ \mathrm{rank}(x)=j}} \omega(h, g)f(x)dg \, .$$

It follows from [2, Lemma 12.13] that the sum over $1 \le j \le n-1$ is equal to zero. Thus, $\varphi_f^s(h)$ equals

$$\int_{G_n\backslash G_n(\mathbb{A})} \varphi(g)\alpha(g)^s \sum_{x\in G_n} \omega(h,g)f(x)dg$$
$$= \alpha(h)^{-n/2} \int_{G_n(\mathbb{A})} \varphi(g)\alpha(g)^{s+n/2} f(h^{-1}g)dg\ .$$

By [2, Theorem 12.4], the last integral can be continued analytically to the whole s-plane as an entire function. □

By the above expression of φ_f^s, it is known that the space $\Theta^s(\pi)$ is contained in the space of cusp forms on $G_n(\mathbb{A})$ if $\mathrm{Re}(s) > n/2 - t$.

Proof of Theorem 1. First we assume $\mathrm{Re}(s) >> 0$. For $\varphi_f^{s-1/2} \in \Theta^{s-1/2}(\pi)$, we set

$$V^s_{(\varphi,f)}(h) = \int_{U_n\backslash U_n(\mathbb{A})} \psi_n(u)^{-1}\varphi_f^{s-1/2}(uh)du$$
$$= \alpha(h)^{s-1/2} \int_{G_n(\mathbb{A})} W_\varphi(hg)\alpha(g)^{s+n/2-1/2} f(g)dg\ .$$

We may assume that W_φ and f are decomposable, i.e. they are of the forms

$$W_\varphi(g) = \prod_v W_v(g_v), \qquad f(g) = \prod_v f_v(g_v)\ .$$

Then we set

$$V^s_{(W_v,f_v)}(h_v) = |\det h_v|_v^{s-1/2} \int_{G_n(k_v)} W_v(h_v g_v)|\det g_v|_v^{s+n/2-1/2} f_v(g_v)dg_v\ .$$

Let $S(\varphi,f)$ be the finite set of places of k such that W_v is a class one Whittaker function and f_v the characteristic function of the set $M_n(\mathcal{O}_v)$ consisting of integral matrices if $v \notin S(\varphi,f)$. It follows from [2, Lemma 6.10], (1.5) and (2.1) that if $v \notin S(\varphi,f)$, then

$$V^s_{(W_v,f_v)}(h_v) = L(s,\pi_v)|\det h_v|_v^{s-1/2} W_v(h_v),$$

and if $v \in S(\varphi, f)$, then $V^s_{(W_v, f_v)}$ is of the form

$$V^s_{(W_v,f_v)}(h_v) = L(s, \pi_v)|\det h_v|_v^{s-1/2} \sum_j \Xi_{v,j}(s) W_{v,j}(h_v) ,$$

where $\Xi_{v,j}(s)$ are entire functions of s and $W_{v,j}$ are elements in $\mathbf{W}(\pi_v, \psi_{n,v})$. We have

$$\begin{aligned} V^s_{(\varphi,f)}(h) &= L(s,\pi)\alpha(h)^{s-1/2} \\ &\quad \times \prod_{v \in S(\varphi,f)} \left\{ \sum_j \Xi_{v,j}(s) W_{v,j}(h_v) \right\} \prod_{v \notin S(\varphi,f)} W_v(h_v) . \end{aligned}$$

Consequently, we can take a finite number of entire functions $\Xi_j(s)$ and cusp forms $\varphi_j \in H_\pi$ such that

$$V^s_{(\varphi,f)}(h) = L(s,\pi)\alpha(h)^{s-1/2} \sum_j \Xi_j(s) W_{\varphi_j}(h) . \tag{3.1}$$

It is known by [11, Theorem 5.9] that

$$\begin{aligned} \varphi_f^{s-1/2}(h) &= \sum_{\gamma \in U_{n-1} \backslash G_{n-1}} V^s_{(\varphi,f)}(\gamma h) \\ \varphi_j(h) &= \sum_{\gamma \in U_{n-1} \backslash G_{n-1}} W_{\varphi_j}(\gamma h) . \end{aligned}$$

Here we regard G_{n-1} as a subgroup of G_n by the embedding

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} .$$

Therefore, when $\mathrm{Re}(s)$ is sufficiently large, we obtain

$$\varphi_f^s(h) = L(s+1/2,\pi)\alpha(h)^s \sum_j \Xi_j(s+1/2)\varphi_j(h) .$$

Since the right-hand side is an entire function of s, this expression holds for all $s \in \mathbf{C}$. This implies the first assertion of Theorem. Furthermore, by the irreducibility of π, we have $\Theta^0(\pi) = H_\pi$ if $\Theta^0(\pi) \neq 0$. □

Proof of Theorem 2. For $\varphi \in H_\pi$, $f \in \mathcal{S}_0(M_n(\mathbb{A}))$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) >> 0$, we set

$$\widetilde{V}^s_{(\varphi,f)}(h) = L(s,\pi)^{-1} \int_{G_n(\mathbb{A})} W_\varphi(g)\alpha(g)^{s-1/2}\omega(h,g)f(1_n)dg \ .$$

By (3.1), $\widetilde{V}^s_{(\varphi,f)}(h)$ extends to an entire function of s, and as a function in h, $\widetilde{V}^s_{(\varphi,f)}$ is contained in $\alpha^{s-1/2} \otimes W(\pi,\psi_n)$. Thus we can consider

$$\theta(\varphi,f)(h) = \sum_{\gamma \in U_{n-1}\backslash G_{n-1}} \widetilde{V}^{1/2}_{(\varphi,f)}(\gamma h) \ .$$

The space spanned by $\theta(\varphi,f)$, $(\varphi \in H_\pi,\ f \in \mathcal{S}_0(M_n(\mathbb{A})))$ equals H_π. We identify the dual space $H_\pi^\vee$ of H_π with the space of functions $\alpha^{-2t}\overline{\varphi}$, $(\varphi \in H_\pi)$ by the pairing

$$< \alpha^{-2t}\overline{\varphi}_1, \varphi_2 >= \int_{Z_n(\mathbb{A})G_n\backslash G_n(\mathbb{A})} \overline{\varphi}_1(g)\varphi_2(g)\alpha(g)^{-2t}dg \ .$$

Then the pairing $(\alpha^{-2t}\overline{\varphi}_1 \otimes \varphi_2) \otimes f \mapsto < \alpha^{-2t}\overline{\varphi}_1, \theta(\varphi_2,f) >$ on $(H_\pi^\vee \otimes H_\pi) \otimes \mathcal{S}_0(M_n(\mathbb{A}))$ gives rise to a nonzero $\mathcal{H}_n \otimes \mathcal{H}_n$-morphism from ω to the contragredient representation $(\pi^\vee \otimes \pi)^\vee$ of $\pi^\vee \otimes \pi$. Next, let $\sigma = \otimes'_v \sigma_v$ be an irreducible autormophic cuspidal representation satisfying $\mathrm{Hom}_{\mathcal{H}_n \otimes \mathcal{H}_n}(\omega, \sigma \otimes \pi^\vee) \neq 0$. Then we have

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{H}_{n,v} \otimes \mathcal{H}_{n,v}}(\omega_v, \sigma_v \otimes \pi_v^\vee) \neq 0 \quad \text{and} \\ &\mathrm{Hom}_{\mathcal{H}_{n,v} \otimes \mathcal{H}_{n,v}}(\omega_v, \pi_v \otimes \pi_v^\vee) \neq 0 \end{aligned} \tag{3.2}$$

for each place v. We denote by $S(\pi)$ the finite set of finite places v where π_v is not a spherical representation. Since the local Howe duality conjecture is true for the case of real reductive dual pairs ([4, Theorem 1]) and the case of spherical representations of unramified reductive dual pairs ([3, Theorem 7.1]), we have $\sigma_v \cong \pi_v$ for all $v \notin S(\pi)$ by (3.2). Then the strong multiplicity one theorem ([8, Corollary 4.10]) implies $\sigma \cong \pi$. □

Remark. We prove $\mathrm{Hom}_{\mathcal{H}_n \otimes \mathcal{H}_n}(\omega, \pi \otimes \pi^\vee) \neq 0$ for any irreducible admissible representation π of $G_n(\mathbb{A})$. Let S be the finite set of places containing all archimedean places and all finite places v where π_v is not a spherical representation. We take nonzero spherical vectors $e_v \in H_{\pi_v}$ and $e_v^\vee \in H_{\pi_v}^\vee$ for each $v \notin S$. Then H_π and $H_\pi^\vee$ are decomposed into restricted tensor products of the H_{π_v} and the $H_{\pi_v}^\vee$ with respect to $\{e_v\}_{v\notin S}$ and $\{e_v^\vee\}_{v\notin S}$, respectively. It is known by [2, Theorems 3.3 and 8.7] that $\mathrm{Hom}_{\mathcal{H}_{n,v}\otimes\mathcal{H}_{n,v}}(\omega_v, \pi_v \otimes \pi_v^\vee) \neq 0$ for each v. If $v \notin S$, we can take a nonzero $T_v \in \mathrm{Hom}_{\mathcal{H}_{n,v}\otimes\mathcal{H}_{n,v}}(\omega_v, \pi_v \otimes \pi_v^\vee)$ normalized so that $T_v(f_v) = e_v \otimes e_v^\vee$ for the characteristic function f_v of $M_n(\mathcal{O}_v)$ (cf. [3, Theorem 10.2]). If $v \in S$, we take an arbitrary nonzero $T_v \in \mathrm{Hom}_{\mathcal{H}_{n,v}\otimes\mathcal{H}_{n,v}}(\omega_v, \pi_v \otimes \pi_v^\vee)$. Then $T = \otimes_v T_v$ gives a nonzero element in $\mathrm{Hom}_{\mathcal{H}_n\otimes\mathcal{H}_n}(\omega, \pi \otimes \pi^\vee)$.

References

[1] Borel, A. and H. Jacquet, Automorphic forms and automorphic representations, Automorphic Forms, Representations, and L-functions, Proc. of Symp. in Pure Math. Vol. 33 part 1, Amer. Math. Soc., 1979, pp. 189–202.

[2] Godement, R. and H. Jacquet, Zeta Functions of Simple Algebras, Springer Verlag, Lecture Notes in Math. 260, 1972.

[3] Howe, R., θ-series and invariant theory, Automorphic Forms, Representations, and L-functions, Proc. of Symp. in Pure Math. Vol. 33 part 1, Amer. Math. Soc., 1979, pp. 275–285.

[4] Howe, R., Transcending classical invariant theory, J. Amer. Math. Soc. **2** (1989), 535–552.

[5] Jacquet, H., Principal L-functions of the linear group, Automorphic Forms, Representations, and L-functions, Proc. of Symp. in Pure Math. Vol. 33 part 2, Amer. Math. Soc., 1979, pp. 63–86.

[6] Jacquet, H., Piatetski-Shapiro, I. I. and J. Shalika, Automorphic forms on $GL(3)$ II, Ann. of Math. **109** (1979), 213–258.

[7] Jacquet, H., Piatetski-Shapiro, I. I. and J. Shalika, Rankin - Selberg convolutions, Amer. J. Math. **105** (1983), 367–464.

[8] Jacquet, H. and J. Shalika, On Euler products and the classification of automorphic representations I, Amer. J. Math. **103** (1981), 499–558.

[9] Jacquet, H. and J. Shalika, Rankin-Selberg convolutions: Archimedean Theory, Festschrift in honor of I.I. Piatetski-Shapiro, Israel Math. Conf. Proc. Vol. 2, 1990, pp. 125–207.

[10] Prasad, D., Weil representation, Howe duality and the theta correspondence, Theta functions: from the classical to the modern, CRM Proceedings and Lecture Notes, 1, Amer. Math. Soc., 1993, pp. 105–127.

[11] Shalika, J., The multiplicity one theorem for GL_n, Ann. of Math. **100** (1974), 171–193.

[12] Weil, A., Commentaire de "Fonction zêta et distributions", Collected Papers, Vol. III (1964–1978), Springer Verlag, 1980, pp. 448–449.

(Received September 1, 1995)

Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560
Japan
E-mail: watanabe@math.wani.osaka-u.ac.jp