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Additive Processes on Local Fields

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Abstract. The present paper aims at investigating some basic properties of additive processes on local fields which are rotation-invariant, i.e., invariant under multiplications by any elements of norm 1. We will give the transition probabilities of such processes explicitly, and prove that each process is a limit of processes expressed as integrals. The Lévy measures of their transition probabilities will be used to tell whether they are recurrent, and whether they are stable. For stable processes, we will give some detailed observations.

§0. Introduction

In §1, we will give explicitly all rotation-invariant additive processes on any local field K. §2 is devoted to show Lévy-Ito decomposition theorem for rotation-invariant additive processes on local fields, which represents them as limits of some integrals. In §3, we will give necessary and sufficient conditions for the processes to be recurrent or to hit every point with positive probability. §4 is close investigations of stable processes. In §5, we will see that under some conditions, we can define local times for rotation-invariant additive processes. As an application of this, we will determine the Hausdorff dimension of the set of times at which the process hits its starting point.

Albeverio and Karwowski constructed a family of rotation-invariant additive processes on p-adics $\mathbb{Q}_p$ and determined their generators ([1], [2]). In this paper, we will extend their theory to all local fields, and characterize and closely investigate rotation-invariant additive processes on local fields. As another work on measure theory on local fields, Satoh proved in [16] and [17] an extension theorem of measures on vector spaces over local fields.

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Throughout this paper, p will stand for a prime number. Let K be a finite algebraic extention field over p-adic field $\mathbb{Q}_p$. R, P, π, q will be respectively the unique maximal compact subring of K, the unique maximal ideal of R, a prime element of K, and the module of K ([18] Theorem 6 and Definition 3, §4 I). Let η_K denote the Haar measure on K normalized as $\eta_K(R) = 1$, and let $\int \cdot dx$, $\int \cdot dy$ etc. mean integrals with respect to η_K. Each non-zero element x in K is expressed in one and only one way in the form

$$x = \sum_{i=m}^{\infty} \alpha_{k_i} \pi^i, \qquad \alpha_{k_m} \neq 0, \tag{0.1}$$

where $\{\alpha_k\}_{1 \leq k \leq q}$ is a full set of representatives of the residual field R/P of K, and m is an integer ([18] Corollary 2, §4 I). Let e be the order of ramification of K over $\mathbb{Q}_p$ ([18] Definition 4, §4 I), and put $r = p^{1/e}$. The norm $|\cdot|$ on K is defined by

$$\left| \sum_{i=m}^{\infty} \alpha_{k_i} \pi^i \right| = r^{-m},$$

if $\alpha_{k_m} \neq 0$, and by $|0| = 0$. Let $D(x, r^M)$ denote the disc of radius r^M centered at x.

§1. Rotation-invariant additive processes on local fields

We denote by K^* the character group of K, that is, the group of all continuous homomorphisms from the additive group K into the circle group $\{c \in \mathbb{C} : |c| = 1\}$ in the complex plane. For each probability measure μ on K, its characteristic function $\hat{\mu}$ is a function on K^*, defined by

$$\hat{\mu}(\varphi) = \int_K \varphi(x) \mu(dx).$$

Note that the correspondence between a probability measure μ on K and its characteristic function is one to one ([14] Theorem 3.1, §3 IV). By [18] Proposition 12, §5 II, we can take $\varphi_1 \in K^*$ such that

$$\varphi_1(R) = \{1\},$$

and that

$$\varphi_1(\pi^{-1}R) \neq \{1\}.$$

Then any $\varphi \in K^*$ is expressed as $\varphi(\ \cdot\ ) = \varphi_1(y\ \cdot\ )$ for some $y \in K$ ([18] Corollary of Theorem 3, §5 II). Write

$$\varphi_y(\ \cdot\ ) := \varphi_1(y\ \cdot\ ),$$

for simplicity, then we have

$$\int_{|x|\le r^m} \varphi_y(x)dx = \begin{cases} q^m, & \text{if } |y| \le r^{-m}, \\ 0, & \text{if } |y| > r^{-m}, \end{cases} \tag{1.1}$$

([12] p275).

Let $\{\mu_t\}_{t\ge 0}$ be any convolution semigroup of distributions on K such that μ_t converges weakly to the distribution degenerate at the origin when $t \to 0$. Then, since K is totally disconnected, $\widehat{\mu_t}$ has the canonical representation

$$\widehat{\mu_t}(\varphi) = \exp\left[t\int_K \big(\varphi(x) - 1\big)F(dx)\right], \qquad \varphi \in K^*, \tag{1.2}$$

where F is a σ-finite measure on $K - \{0\}$ with finite mass outside every neighborhood of the origin. Conversely, for any measure F satisfying the conditions above, there exists a semigroup $\{\mu_t\}_{t\ge 0}$ whose characteristic function is given by (1.2) ([14] Theorem 10.1, §10 IV, and Remark 1, §7 IV). The measure F is uniquely determined by $\{\mu_t\}$ and we call it the Lévy measure of $\{\mu_t\}$.

Lemma 1.1. *Let Y_t be a spatially homogeneous additive process on K, and let $Q_t(x, dy)$ be its transition probability. Then the following three conditions are equivalent.*

(1) *For each $M \in \mathbb{Z}$ and $t \ge 0$,*

$$h_{M,t}(x) := Q_t\big(x, D(0, r^M)\big) = Q_t\big(0, D(-x, r^M)\big)$$

depends only on the norm $|x|$ of x.

(2) $|Y_t|$ *is a Markov process on* $|K| := \{|x|; x \in K\}$. *That is, for any function* g *on* K *such that the value* $g(x)$ *depends only on* $|x|$, *it follows that*

$$E_0 g(Y_t + x) = E_0 g(Y_t + x'),$$

whenever $|x| = |x'|$.

(3) *The Lévy measure* F *of* $\{Q_t = Q_t(0, \cdot)\}$ *satisfies*

$$F(A) = F(uA) \tag{1.3}$$

for any unit u *of* R *and any Borel set* A.

PROOF. (1) ⇒ (2) If g is such as in (2), then

$$g = \sum_{M \in \mathbb{Z}} b_M \left(\chi_{D(0,r^M)} - \chi_{D(0,r^{M-1})} \right) + b_{-\infty} \chi_{\{0\}},$$

for some b_M, $b_{-\infty} \in \mathbb{R}$, where χ_A is the characteristic function of a set A. Then

$$E_0 g(Y_t + x) = \sum_{M \in \mathbb{Z}} b_M \big(h_{M,t}(x) - h_{M-1,t}(x) \big) + b_{-\infty} \lim_{M \to -\infty} h_{M,t}(x),$$

which depends only on $|x|$.

(2) ⇒ (1) Let $g = \chi_{D(0,r^M)}$. If $|x| = |x'|$, then

$$h_{M,t}(x) = E_0 g(Y_t + x) = E_0 g(Y_t + x') = h_{M,t}(x').$$

(1) ⇒ (3) Let u be a unit of R, then

$$\widehat{Q_t}(\varphi_{u^{-1}y}) = \int_K \varphi_1(yx) Q_t(udx) = \int_K \varphi_1(yx) Q_t(dx) = \widehat{Q_t}(\varphi_y).$$

Hence

$$\exp \left[t \int_K \big(\varphi_y(x) - 1 \big) F(dx) \right] = \exp \left[t \int_K \big(\varphi_y(x) - 1 \big) F(udx) \right],$$

and the uniqueness of the Lévy measure implies

$$F(A) = F(uA),$$

for any Borel set A.

(3) $\Rightarrow$ (1) Put $Q_t^{(u)}(A) := Q_t(uA)$ for each unit u of R, and let F be the Lévy measure of Q_t. Then for any $y \in K$,

$$\begin{aligned}\widehat{Q_t^{(u)}}(\varphi_y) &= \exp\left[t \int_K \big(\varphi_y(x) - 1\big) F(udx)\right] \\ &= \exp\left[t \int_K \big(\varphi_y(x) - 1\big) F(dx)\right] = \widehat{Q_t}(\varphi_y),\end{aligned}$$

which implies $Q_t^{(u)} = Q_t$. $\square$

Let (Y_t, Q_t) be any rotation-invariant additive process on K, then by Lemma 1.1, its Lévy measure F must satisfy (1.3). Set $a(M) = F\big(D(0, r^M)^c\big)$ for each integer M. Then $\{a(M)\}_{M \in \mathbb{Z}}$ satisfies

$$a(M + 1) \le a(M), \tag{1.4}$$

and

$$\lim_{M \to +\infty} a(M) = 0. \tag{1.5}$$

Conversely, for a given sequence $\{a(M)\}_{M \in \mathbb{Z}}$ with the properties (1.4) and (1.5), there exists a unique measure F on $K - \{0\}$ satisfying (1.3) and

$$F\big(D(0, r^M)^c\big) = a(M). \tag{1.6}$$

In other words, a sequence $\{a(M)\}_{M \in \mathbb{Z}}$ satisfying (1.4) and (1.5) corresponds in one-to-one way to a rotation-invariant additive process (Y_t, Q_t) whose Lévy measure is given by (1.6). We will show that when $\{a(M)\}_{M \in \mathbb{Z}}$ is given, we can give explicitly the transition probability Q_t of the corresponding process Y_t.

In [1] and [2], Albeverio and Karwowski constructed rotation-invariant additive processes on $\mathbb{Q}_p$. Modifying their method, we can construct similar additive processes on any local field K. Let $\{a(M)\}$, $M \in \mathbb{Z}$ be any sequence of real numbers satisfying (1.4) and (1.5). The expression (0.1) shows that for each $y \in K$, $M \in \mathbb{Z}$, and $m \geq 1$, there are just $(q-1)q^{m-1}$ disjoint discs $D(x, r^M)$ of radii r^M such that $|x-y| = r^{M+m}$. Therefore, by replacing p by q, r, or π suitably in the arguments in [2] §2, we can construct an additive process X_t on K. Its transition probability P_t is given by

$$(1.7) \quad \begin{aligned} &P_t\big(x, D(y, r^M)\big) \\ &\quad = q^{-1}(q-1)\sum_{i=0}^{\infty} q^{-i} \exp\left[-(q-1)^{-1}\big(qa(M+i) - a(M+i+1)\big)t\right] \\ &\quad =: P_M(t), \end{aligned}$$

if $|x-y| \leq r^M$, and

$$(1.8) \qquad P_t\big(x, D(y, r^M)\big) = (q-1)^{-1}q^{1-m}\big(P_{M+m}(t) - P_{M+m-1}(t)\big),$$

if $|x-y| = r^{M+m}$, $m \geq 1$. We can verify similarly as in [2] that $\{P_t\}$, $t \geq 0$ determines a Markovian semigroup. We will call (X_t, P_t) above the A-process on K given by $\{a(M)\}_{M \in \mathbb{Z}}$. We will show that for given $\{a(M)\}_{M \in \mathbb{Z}}$, our process (Y_t, Q_t) given by (1.6) coincides with the A-process (X_t, P_t) given by $\{a(M)\}_{M \in \mathbb{Z}}$. We will prepare two lemmas;

LEMMA 1.2. *Let (X_t, P_t) be the A-process on K given by $\{a(M)\}_{M \in \mathbb{Z}}$.*

(1) *If $\lim_{M \to -\infty} a(M) = \infty$, then for each $t > 0$ and $x \in K$, the probability measure $P_t(x, \cdot)$ is absolutely continuous with respect to η_K.*

(2) *If $\lim_{M \to -\infty} a(M) = W < \infty$, then for each $t > 0$ and $x \in K$, $P_t(x, \cdot)$ is absolutely continuous with respect to η_K, except at x.*

Furthermore the density function $p_t(x, \cdot)$ of $P_t(x, \cdot)$ is given by

$$p_t(x, y) = (q-1)^{-1}q^{1-m}\big(P_m(t) - P_{m-1}(t)\big), \qquad \text{if } |y-x| = r^m,$$

in the both cases (1) and (2).

PROOF. (1) Note that $\lim_{M \to -\infty} a(M) = \infty$ implies that $\lim_{M \to -\infty} P_M(t) = 0$ for any $t > 0$. Let $M \in \mathbb{Z}$. If $|z-x| \leq r^M$ then

$$\int_{D(z, r^M)} p_t(x, y)dy = \int_{D(x, r^M)} p_t(x, y)dy = P_M(t) = P_t\big(x, D(z, r^M)\big).$$

If $|z-x| = r^{M+m}$, $m \geq 1$, then $|y-z| \leq r^M$ implies that $|y-x| = r^{M+m}$. Therefore

$$\int_{D(z,r^M)} p_t(x,y)dy = (q-1)^{-1}q^{1-m}\big(P_{M+m}(t) - P_{M+m-1}(t)\big) = P_t\big(x, D(z,r^M)\big) \tag{1.9}$$

(2) Since $P_t(x,\{x\}) = \lim_{M\to-\infty} P_M(t) = \exp(-Wt) > 0$, P_t is not absolutely continuous at x. Whereas (1.9) holds also in this case and (2) follows. □

REMARK. If

$$\sum_{i=-\infty}^{0} q^{-i}\exp\left[-(q-1)^{-1}\big(qa(i) - a(i+1)\big)t\right] < \infty,$$

then we can take as the density a continuous one by putting

$$p_t(x,x) = q^{-1}(q-1)\sum_{i=-\infty}^{\infty} q^{-i}\exp\left[-(q-1)^{-1}\big(qa(i) - a(i+1)\big)t\right].$$

LEMMA 1.3. *Let (X_t, P_t) be the A-process on K given by $\{a(M)\}_{M\in\mathbb{Z}}$. Then the characteristic function $\widehat{P}_t$ of $P_t(\cdot) = P_t(0,\cdot)$ is given by*

$$\widehat{P}_t(\varphi_y) = \exp\left[-(q-1)^{-1}\big(qa(-n) - a(-n+1)\big)t\right],$$

where $n = \frac{\log|y|}{\log r}$.

PROOF. If $|y| = r^n$ then by (1.1),

$$\begin{aligned}\widehat{P}_t(\varphi_y) &= \int_{D(0,r^{-n})} \varphi_y(x)P_t(dx) + \sum_{m\geq -n+1}\int_{|x|=r^m} \varphi_y(x)p_t(0,x)dx \\ &= (q-1)^{-1}\big(qP_{-n}(t) - P_{-n+1}(t)\big) \\ &= \exp\left[-(q-1)^{-1}\big(qa(-n) - a(-n+1)\big)t\right]. \ \square\end{aligned}$$

PROPOSITION 1.4. *Let* (X_t, P_t) *be the* A*-process on* K *given by* $\{a(M)\}_{M\in\mathbb{Z}}$*. Then the Lévy measure of* $\{P_t\}$ *coincides with the measure* F *given by* (1.3) and (1.6).

PROOF. By Lemma 1.3, it suffices to show

$$\int_K \big(\varphi_y(x)-1\big)F(dx) = -(q-1)^{-1}\big(qa(-n)-a(-n+1)\big),$$

for $|y| = r^n$. If $m \geq -n+1$, then

$$\begin{aligned}
&\int_{|x|=r^m} \big(\varphi_y(x)-1\big)F(dx)\\
&= \sum_{(\alpha_{k_{-m-n}},\cdots,\alpha_{k_{-1}})} \int_{x\in D(y^{-1}\sum_{i=-m-n}^{-1}\alpha_{k_i}\pi^i,\, r^{-n})} \\
&\qquad \left(\varphi_1\Big(\sum_{i=-m-n}^{-1}\alpha_{k_i}\pi^i\Big)-1\right)F(dx)\\
&= \left\{\sum_{(\alpha_{k_{-m-n}},\cdots,\alpha_{k_{-1}})} \varphi_1\Big(\sum_{i=-m-n}^{-1}\alpha_{k_i}\pi^i\Big) - (q-1)q^{m+n-1}\right\}\\
&\qquad \times (q-1)^{-1}q^{-m-n+1}\big(a(m-1)-a(m)\big),
\end{aligned}$$

where $\sum_{(\alpha_{k_{-m-n}},\cdots,\alpha_{k_{-1}})}$ is the sum over all the representatives α_{k_i} $(-m-n \leq i \leq -1)$ of R/P with $\alpha_{k_{-m-n}} \neq 0$. By (1.1) we have

$$\begin{aligned}
&\sum_{(\alpha_{k_{-m-n}},\cdots,\alpha_{k_{-1}})} \varphi_1\Big(\sum_{i=-m-n}^{-1}\alpha_{k_i}\pi^i\Big)\\
&= q^n \sum_{(\alpha_{k_{-m-n}},\cdots,\alpha_{k_{-1}})} \int_{x\in D(y^{-1}\sum_{i=-m-n}^{-1}\alpha_{k_i}\pi^i,\, r^{-n})} \varphi_y(x)dx\\
&= q^n \int_{|x|=r^m} \varphi_y(x)dx\\
&= \begin{cases} -1, & \text{if } m=-n+1,\\ 0, & \text{if } m \geq -n+2. \end{cases}
\end{aligned}$$

Therefore

$$\begin{aligned}\int_K \big(\varphi_y(x)-1\big)F(dx) &= -q(q-1)^{-1}\big(a(-n)-a(-n+1)\big)\\ &\quad - \sum_{m\geq -n+2}\big(a(m-1)-a(m)\big)\\ &= -(q-1)^{-1}\big(qa(-n)-a(-n+1)\big),\end{aligned}$$

as desired. □

Thus our process Y_t with the Lévy measure F in (1.3) and (1.6) is exactly the A-process X_t, whose transition probabilities P_t are given by (1.7) and (1.8).

§2. Representation theorem

It is well known that Lévy processes on $\mathbb{R}$ have Lévy-Ito decomposition. We will show a representation theorem for our A-processes on local fields, modifying Ito's theory. Since local fields are totally disconnected, our A-processes have no continuous part, and their representations are rather simple.

Lemma 2.1. *Let (X_t, P_t) be the A-process on K starting at 0 given by $\{a(M)\}_{M\in\mathbb{Z}}$, and let for each fixed integer m,*

$$\tau_1^{(m)} := \inf\{t>0 \ : \ |X_t| > r^m\},$$

$$\tau_k^{(m)} := \inf\{t>0 \ : \ |X_{\tau_{k-1}^{(m)}+t} - X_{\tau_{k-1}^{(m)}}| > r^m\}, \qquad k\geq 2.$$

Then $\sum_{k=1}^{n}\tau_k^{(m)}$ has Gamma distribution $\Gamma\big(n, a(m)^{-1}\big)$.

Proof. Let $K_0 := D(0, r^m)$, K_1, $K_2, \cdots$ be the sequence of all disjoint discs of radii r^m in K, and let M_t be the Markov chain on $\{0,1,2,\cdots\}$ defined by

$$M_t = i \Longleftrightarrow X_t \in K_i.$$

Let $P_{ij}(t)$, L be the transition probability and the generator of M_t respectively, then

$$
\begin{aligned}
L\chi_i(i) &= \lim_{t\to 0} t^{-1}\big(\sum_k \chi_i(k)P_{ik}(t) - \chi_i(i)\big) \\
&= \lim_{t\to 0} t^{-1}\Big(P_t\big(0, D(0,r^m)\big) - 1\Big) \\
&= -a(m),
\end{aligned} \tag{2.1}
$$

where $\chi_i(j) := \delta_{ij}$. Let $\gamma_j := P(X_{\tau_1^{(m)}} = j)$, then there exists $\lambda \in [0,+\infty]$ such that

$$
P(\tau_1^{(m)} > t) = \exp(-\lambda t), \quad t \geq 0, \tag{2.2}
$$

and that

$$
Lf(0) = \lambda \sum_j \big(f(j) - f(0)\big)\gamma_j, \tag{2.3}
$$

for $f \in \mathcal{D}(L)$. Putting $f = \chi_0$ in (2.3) gives

$$
L\chi_0(0) = -\lambda \sum_{j\neq 0} \gamma_j = -\lambda.
$$

This combined with (2.1) and (2.2) shows

$$
P(\tau_1^{(m)} > t) = \exp\big(-a(m)t\big).
$$

Since $\tau_k^{(m)}$, $k \geq 1$ are i.i.d., we have for each $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
E\exp\left(i\alpha \sum_{k=1}^n \tau_k^{(m)}\right) &= \left(\int_0^\infty \exp(i\alpha t)\Big(1 - \exp\big(-a(m)t\big)\Big)' dt\right)^n \\
&= \big(1 - ia(m)^{-1}\alpha\big)^{-n},
\end{aligned}
$$

which proves our assertion. □

Lemma 2.2. *Let (X_t, P_t) be the A-process on K given by $\{a(M)\}_{M\in\mathbb{Z}}$. Write $X_{t-} := \lim_{s\uparrow t} X_s$ and let $J_t := X_t - X_{t-}$. Then for each integer m,*

$$W_t^{(m)} := \sharp\{s \le t \ : \ |J_s| > r^m\}$$

has Poisson distribution of mean $a(m)t$.

Proof. By Lemma 2.1 we have

$$\begin{aligned} P(W_t^{(m)} \le n) &= P\left(\sum_{i=1}^{n+1} \tau_i^{(m)} > t\right) \\ &= 1 - \int_0^t \Gamma(n+1)^{-1} a(m)^{n+1} \beta^n \exp\bigl(-a(m)\beta\bigr) d\beta \\ &= 1 - \int_0^{a(m)t} \frac{\exp(-\beta)\beta^n}{n!} d\beta. \end{aligned}$$

That is, if we write

$$G_n(\alpha) := 1 - \int_0^\alpha \frac{\exp(-\beta)\beta^n}{n!} d\beta, \qquad \alpha > 0,$$

then

$$P(W_t^{(m)} \le n) = G_n\bigl(a(m)t\bigr).$$

On the other hand, if P_α is the Poisson distribution of mean α then

$$H_n(\alpha) := P_\alpha\bigl(\{0, 1, \cdots, n\}\bigr) = \sum_{i=0}^n \frac{\exp(-\alpha)\alpha^i}{i!}.$$

Hence

$$\frac{d}{d\alpha} H_n(\alpha) = -\frac{\exp(-\alpha)\alpha^n}{n!} = \frac{d}{d\alpha} G_n(\alpha).$$

Since $G_n(0) = H_n(0) = 1$, we have $G_n(\alpha) = H_n(\alpha)$ for any $\alpha > 0$. Thus the proof is completed. □

Lemma 2.3 (Ottaviani's inequality). *Let Y_1, Y_2, $\cdots$, Y_m be independent random variables on a normed space with norm $|\cdot|$. If*

$$P\{|Y_{k+1} + Y_{k+2} + \cdots + Y_m| \le a\} \ge \beta, \quad k = 0, 1, \cdots, m-1,$$

for some positive a and β then

$$P\{\max_{k=1}^{m}|Y_1+Y_2+\cdots+Y_k|>2a\}\leq\frac{1}{\beta}P\{|Y_1+Y_2+\cdots+Y_m|>a\}.$$

PROOF. Let

$$S_k:=Y_1+Y_2+\cdots+Y_k,\qquad T_k:=\max_{i=1}^{k}|S_i|,$$

$$\mathcal{B}_k:=\sigma[Y_k],$$

and let

$$A_k:=\{T_{k-1}\leq 2a,\ |S_k|>2a\},\qquad B_k:=\{|S_m-S_k|\leq a\},$$

for $1\leq k\leq m$. Then A_1, A_2, $\cdots$, A_m are disjoint and

$$\{T_m>2a\}=A_1+A_2+\cdots+A_m.$$

If $\omega\in A_k\cap B_k$ then

$$|S_m(\omega)|\geq|S_k(\omega)|-|S_m(\omega)-S_k(\omega)|>a,$$

and hence we have

$$\sum_{k=1}^{n}A_k\cap B_k\subset\{|S_m|>a\}.$$

Since A_k and B_k are independent, we obtain

$$P\{|S_m|>a\}\geq\sum_{k=1}^{m}P(A_k\cap B_k)\geq\beta\sum_{k=1}^{m}P(A_k)=\beta P\{T_m>2a\}.\ \square$$

Let us prove a representation theorem for A-processes;

THEOREM 2.4. *Let X_t be an A-process on K. Let*

$$N\big((s,t]\times E,\omega\big):=\sharp\{s<u\leq t\ :\ J_u(\omega)\in E\},$$

for $t > s \geq 0$ *and* E *a Borel set in* $K - \{0\}$. *Then* $N_t(E, \omega) := N\big((0,t] \times E, \omega\big)$ *is a Poisson random measure with mean measure* $dt \cdot F(dx)$. *Set*

$$X_t^{(n)}(\omega) := \int_{r^{-n} < |x| \leq r^n} x N_t(dx, \omega),$$

for $n \in \mathbb{N}$. *Then* $X_t^{(n)}(\omega)$ *converges almost surely to* $X_t(\omega)$ *and the convergence is uniform in* t *in any compact sets.*

Proof. We will prove first that

$$\lim_{n\to\infty} P\{\sup_{s\leq t} |X_s(\omega) - X_s^{(n)}(\omega)| > \varepsilon\} = 0, \tag{2.4}$$

for any $\varepsilon > 0$. Note that $X_s(\omega) - X_s^{(n)}(\omega)$ is the sum of jumps $J_u(\omega)$ of $X_u(\omega)$, $u \leq s$, such that $0 < |J_u(\omega)| \leq r^{-n}$ or $|J_u(\omega)| > r^n$. If n is so large that $r^{-n} < \varepsilon < r^n$, then Lemma 2.2 yields

$$\begin{aligned}
&P\{\sup_{s\leq t} |X_s(\omega) - X_s^{(n)}(\omega)| > \varepsilon\} \\
&\quad = P\{X_s(\omega) - X_s^{(n)}(\omega) \text{ makes a jump larger than } r^n \text{ at some } s \in [0,t]\} \\
&\quad = P\{X_s(\omega) \text{ makes a jump larger than } r^n \text{ at some } s \in [0,t]\} \\
&\quad = P\{W_t^{(n)} \geq 1\} \\
&\quad = 1 - e^{-a(n)t} \\
&\quad \to 0,
\end{aligned}$$

as $n \to \infty$. Thus (2.4) holds. In other words, let $\mathcal{D}_K\big([0,t]\big)$ be the space of all K-valued right-continuous functions on $[0,t]$ with left limits, normed by

$$\|\xi\| := \sup_{0\leq s\leq t} |\xi(s)|, \quad \xi \in \mathcal{D}_K\big([0,t]\big).$$

Then the $\mathcal{D}_K\big([0,t]\big)$-valued random variable $\mathbb{X}_t^{(n)}(\omega) := \big(X_s^{(n)}(\omega),\ 0 \leq s \leq t\big)$ converges in probability to $\mathbb{X}_t(\omega) := \big(X_s(\omega),\ 0 \leq s \leq t\big)$ with respect to $\|\cdot\|$. Hence for large n, m, we have

$$P\{\|\mathbb{X}_t^{(m)} - \mathbb{X}_t^{(n)}\| > \varepsilon\} < \frac{1}{2}.$$

By Lemma 2.3, we have

$$P\{\max_{k=1}^{m}\|\mathbb{X}_t^{(n+k)}-\mathbb{X}_t^{(n)}\|>2\varepsilon\}\le 2P\{\|\mathbb{X}_t^{(n+m)}-\mathbb{X}_t^{(n)}\|>\varepsilon\}.$$

Therefore

$$P\{\max_{1\le k,l\le m}\|\mathbb{X}_t^{(n+k)}-\mathbb{X}_t^{(n+l)}\|>4\varepsilon\}\le 4P\{\|\mathbb{X}_t^{(n+m)}-\mathbb{X}_t^{(n)}\|>\varepsilon\}.$$

Since the ω-set in the left hand side increases to $\{\sup_{k,l}\|\mathbb{X}_t^{(n+k)}-\mathbb{X}_t^{(n+l)}\|>4\varepsilon\}$ as $m\to\infty$,

$$P\{\sup_{k,l}\|\mathbb{X}_t^{(n+k)}-\mathbb{X}_t^{(n+l)}\|>4\varepsilon\}\le 4\sup_m P\{\|\mathbb{X}_t^{(n+m)}-\mathbb{X}_t^{(n)}\|>\varepsilon\}.$$

As $n\to\infty$, the ω-set in the left hand side decreases and the right hand side tends to 0. Therefore

$$P\{\lim_{n\to\infty}\sup_{k,l}\|\mathbb{X}_t^{(n+k)}-\mathbb{X}_t^{(n+l)}\|>4\varepsilon\}=0.$$

Letting $\varepsilon\to 0$ shows

$$\lim_{n\to\infty}\sup_{k,l}\|\mathbb{X}_t^{(n+k)}-\mathbb{X}_t^{(n+l)}\|=0,\quad \text{a.s.}. \ \square$$

§3. Recurrence

In the sequel additive processes on K are always assumed to be right continuous and to have left limits for all t with probability one. The existence of such a version is proved in [13] T3, XIII.

In this section we will give criteria for A-processes to be recurrent or to hit a single point with positive probability.

Theorem 3.1. *Let (X_t, P_t) be the A-process on K given by $\{a(M)\}_{M\in\mathbb{Z}}$. Then X_t is recurrent if and only if*

$$\sum_{i=1}^{\infty}\frac{q^{-i}}{a(i)}=\infty.$$

Proof. For $M \in \mathbb{Z}$,

$$\int_0^\infty P_t\big(x, D(x, r^M)\big)dt = q^{-1}(q-1)^2 q^M \sum_{i=M}^\infty q^{-i}\big(qa(i) - a(i+1)\big)^{-1},$$

which diverges for any M if and only if

$$\sum_{i=1}^\infty q^{-i}\big(qa(i) - a(i+1)\big)^{-1} = \infty.$$

Since $(q-1)a(i) \leq qa(i) - a(i+1) \leq qa(i)$, our assertion follows. □

Now we will investigate the hitting probabilities of single points for A-processes (X_t, P_t). Define

$$V_x := \inf\{t > 0 \ : \ X_t = x\},$$

for $x \in K$. Let for $\lambda > 0$, $C^\lambda := C^\lambda(\{x\})$ be the λ-capacity of a one-point set.

Lemma 3.2. *Let (X_t, P_t) be a spatially homogeneous standard Markov process on K.*

(1) *Assume that for any $t > 0$, $P_t(x, dy)$ has a density $p_t(x, y)$ with respect to η_K. Then $C^\lambda > 0$ if and only if*

$$g^\lambda(x) := \int_0^\infty \exp(-\lambda t) p_t(0, x) dt$$

is bounded.

(2) *If for some $\lambda > 0$, $C^\lambda = 0$ then $C^\lambda = 0$ for all $\lambda > 0$ and $P_0(V_x < \infty) = 0$ for a.e.x.*

(3) *If P_t has a density and if $g^\lambda(x)$ is bounded and continuous, then*

$$E_0\left[\exp(-\lambda V_x) \ : \ V_x < \infty\right] = \frac{g^\lambda(x)}{g^\lambda(0)},$$

for any $x \in K$.

PROOF. See [15] §6 and 7. □

Applying the previous lemma, we shall give a criterion on λ-capacities of A-processes;

PROPOSITION 3.3. *Let* (X_t, P_t) *be the* A*-process on* K *given by* $\{a(M)\}_{M\in\mathbb{Z}}$ *such that* $\lim_{M\to-\infty} a(M) = \infty$. *Then for any* $\lambda > 0$, $C^\lambda > 0$ *if and only if*

$$\sum_{i=-\infty}^{0} \frac{q^{-i}}{1+a(i)} < \infty. \tag{3.1}$$

PROOF. By Lemma 3.2 (1), $C^\lambda > 0$ if and only if $g^\lambda(x)$ is bounded. If $|x| = r^m$, we have

$$\begin{aligned} g^\lambda(x) = (1-q^{-1}) \sum_{i=m}^{\infty} &\frac{q^{-i}}{\lambda + (q-1)^{-1}\big(qa(i) - a(i+1)\big)} \\ &- \frac{q^{-m}}{\lambda + (q-1)^{-1}\big(qa(m-1) - a(m)\big)}. \end{aligned} \tag{3.2}$$

Since the right-hand side tends to zero as $m \to +\infty$, $g^\lambda(x)$ is bounded for large $|x|$. Hence in order that g^λ is bounded, it is necessary and sufficient that the right-hand side of (3.2) converges as $m \to -\infty$. The inequality $(q-1)a(i) \le qa(i) - a(i+1) \le qa(i)$ establishes the proposition. □

Here we prove a theorem concerning with the hitting probabilities of single points;

THEOREM 3.4. *Let* (X_t, P_t) *be the* A*-process on* K *given by* $\{a(M)\}$.

(1) *If* $\lim_{M\to-\infty} a(M) = W < \infty$, *then*

$$\begin{aligned} &P_0(V_x < \infty) = 0, \qquad x \neq 0, \\ &P_0(V_0 < \infty) = 1. \end{aligned}$$

(2) *If* $\lim_{M\to-\infty} a(M) = \infty$ *and if* $\sum_{i=-\infty}^{0} \frac{q^{-i}}{1+a(i)} = \infty$*, then*

$$P_0(V_x < \infty) = 0,$$

for any $x \in K$*.*

(3) *If* $\sum_{i=-\infty}^{0} \frac{q^{-i}}{1+a(i)} < \infty$*, then*

$$P_0(V_x < \infty) = \lim_{\lambda\downarrow 0} \frac{g^\lambda(x)}{g^\lambda(0)},$$

for any $x \in K$*.*

PROOF. (1) We have proved in Lemma 2.2 that $W_t^{(m)}(\omega) = \sharp\{s \le t \ : |J(s)| > r^m\}$ has Poisson distribution of mean $a(m)t$. If we let $\tau := \inf\{t > 0 : X_t \neq 0\}$, then

$$\begin{aligned} P_0(\tau \ge t) &= P_0(\text{no jumps occur during } [0,t]) \\ &= \lim_{m\to-\infty} P_0(W_t^{(m)} = 0) \\ &= \exp(-Wt). \end{aligned}$$

Hence

$$P_0(\tau > 0) = \lim_{t\to 0} \exp(-Wt) = 1,$$

which implies $P_0(V_0 < \infty) = 1$. Now let $x \neq 0$. For each $t > 0$, $|J_s| > 0$ for only finitely many $s \in (0,t]$, since

$$E\big[\sharp\{s \le t \ : \ |J_s| > 0\}\big] = \lim_{M\to-\infty} a(M)t < \infty.$$

Then we shall define $\xi_k = \xi_k(\omega)$ the time of k'th jump, and we can write

$$X_t = \sum_{s\le t} J_s. \tag{3.3}$$

Therefore

$$P_0(V_x < \infty) \le \sum_{k=1}^{\infty} P\big(J(\xi_1) + \cdots + J(\xi_k) = x\big). \tag{3.4}$$

On the other hand,

$$P\big(J(\xi_i) \in A\big) = \frac{F(A - \{0\})}{F(K - \{0\})},$$

and $F|_{K-\{0\}}$ has no atoms, so are the distributions of $J(\xi_i)$. Since $J(\xi_i)$ are independent,

$$P\big(J(\xi_1) + \cdots + J(\xi_k) = x\big) = 0,$$

for each k. This combined with (3.4) shows $h(x) = 0$.

(2) By Lemma 3.2 (2) and Proposition 3.3, we have

$$P_0(V_x < \infty) = 0, \tag{3.5}$$

for a.e.x. Whereas for each $m \in \mathbb{Z}$, $A_m := D(0, r^m) - D(0, r^{m-1})$ has positive Haar measure, and $P_0(V_x < \infty)$ are equal for all $x \in A_m$ since X_t is rotation-invariant. Hence (3.5) must hold for all $x \neq 0$. We have proved in Lemma 1.2 that $P_t(x, dy)$ is absolutely continuous, and we can see that (3.5) also holds for $x = 0$.

(3) g^λ is bounded by the assumption, and we can easily see that it is continuous. Therefore Lemma 3.2 (3) yields

$$E_0\left[\exp(-\lambda V_x) \ : \ V_x < \infty\right] = \frac{g^\lambda(x)}{g^\lambda(0)}.$$

Letting $\lambda \to 0$ gives our assertion. □

§4. Stable case

In this section and the next, we will give some rather close investigations of stable A-processes.

Proposition 4.1. *Let (X_t, P_t) be the A-process on K given by $a(M) = c^M a(0)$, $M \in \mathbb{Z}$, where $0 < c < 1$ and $a(0) > 0$. Then the law of X_{ct} is equivalent to that of πX_t.*

Proof. Let $\widetilde{P}_t$ be the transition probability of πX_t. It suffices to show

$$P_{ct}\big(0, D(x, r^M)\big) = \widetilde{P}_t\big(0, D(x, r^M)\big),$$

for any $x \in K$ and $M \in \mathbb{Z}$. If $|x| \le r^M$,

$$\begin{aligned}\widetilde{P}_t\big(0, D(x, r^M)\big) &= P_t\big(0, D(\pi^{-1}x, r^{M+1})\big)\\ &= q^{-1}(q-1)\sum_{i=0}^{\infty} q^{-i}\exp\left[-(q-1)^{-1}(q-c)c^{M+i+1}a(0)t\right]\\ &= P_{ct}\big(0, D(x, r^M)\big),\end{aligned}$$

since $|\pi^{-1}x| = r|x| \le r^{M+1}$. If $|x| = r^{M+m}$, $m \ge 1$,

$$\begin{aligned}\widetilde{P}_t\big(0, D(x, r^M)\big) &= (q-1)^{-1}q^{1-m}\big(P_{M+m+1}(t) - P_{M+m}(t)\big)\\ &= (q-1)^{-1}q^{1-m}\big(P_{M+m}(ct) - P_{M+m-1}(ct)\big)\\ &= P_{ct}\big(0, D(x, r^M)\big),\end{aligned}$$

since $|\pi^{-1}x| = r|x| = r^{M+m+1}$. □

Let us see when a stable A-process is recurrent and when it hits a point with positive probability.

Proposition 4.2. *Let (X_t, P_t) be the A-process on K given by $a(M) = c^M a(0)$. Then X_t is recurrent if and only if $c \le q^{-1}$.*

Proof. Immediate from Theorem 3.1 and the equality:

$$\sum_{i=1}^{\infty} \frac{q^{-i}}{a(i)} = a(0)^{-1}\sum_{i=1}^{\infty}(qc)^{-i}. \ \square$$

Proposition 4.3. *Let (X_t, P_t) be the A-process on K given by $a(M) = c^M a(0)$. Then*

$$P_0(V_x < \infty)\begin{cases} = 1, & \text{if } c < q^{-1},\\ = 0, & \text{if } c \ge q^{-1},\end{cases}$$

for any $x \in K$.

Proof. We have $\lim_{M\to-\infty} a(M) = \lim_{M\to-\infty} c^{-M}a(0) = \infty$, and

$$\sum_{i=-\infty}^{0} \frac{q^{-i}}{1+a(i)} = \sum_{i=-\infty}^{0} \frac{q^{-i}}{1+c^i a(0)} \begin{cases} <\infty, & \text{if } c < q^{-1}, \\ =\infty, & \text{if } c \geq q^{-1}. \end{cases}$$

Hence by Theorem 3.4, $P_0(V_x < \infty) = 0$ if $c \geq q^{-1}$. Let $c < q^{-1}$, $|x| = r^m$, and

$$f_\lambda(i) := q^{-i}\Big(\big(\lambda + (q-1)^{-1}(q-c)c^i a(0)\big)^{-1} - \big(\lambda + (q-1)^{-1}(q-c)c^{i-1}a(0)\big)^{-1}\Big).$$

Then by Theorem 3.4 (3) we obtain

$$P_0(V_x < \infty) = \lim_{\lambda \downarrow 0} \left(\frac{\sum_{i=-\infty}^{m-1} f_\lambda(i)}{\sum_{i=m}^{\infty} f_\lambda(i)} + 1 \right)^{-1} = 1. \ \square$$

Now we investigate the density functions in stable case.

Proposition 4.4. *Let (X_t, P_t) be the A-process on K given by $a(M) = c^M a(0)$ with $0 < c < 1$. Then*

$$p_t(0,x) \sim \frac{q(1-c)}{c(q-1)} a(0) |x|^{\frac{\log(q^{-1}c)}{\log r}} t \qquad (x \to \infty).$$

Proof. For $|x| = r^m$, we have proved in Lemma 1.2 that

$$\begin{aligned} p_t(0,x) &= (q-1)^{-1} q^{1-m} \big(P_m(t) - P_{m-1}(t)\big) \\ &= q^{-m} \Bigg(\sum_{i=0}^{\infty} q^{-i} \Big(\exp[-(q-1)^{-1}(q-c)c^{m+i}a(0)t] \\ &\qquad - \exp[-(q-1)^{-1}(q-c)c^{m-1+i}a(0)t] \Big) \Bigg). \end{aligned}$$

Since

$$q^{-m} = q^{-\frac{\log|x|}{\log r}} = |x|^{-\frac{\log q}{\log r}},$$

and

$$c^m = c^{\frac{\log|x|}{\log r}} = |x|^{\frac{\log c}{\log r}},$$

we shall write for $x \neq 0$,

$$p_t(0,x) = |x|^{-\frac{\log q}{\log r}} f\left(h|x|^{\frac{\log c}{\log r}} t\right),$$

where $h := (q-1)^{-1}(q-c)a(0)$, and $f(y) := \sum_{i=0}^{\infty} q^{-i}\exp(-c^i y) - \sum_{i=0}^{\infty} q^{-i}\exp(-c^{i-1}y)$. Since $\frac{\log c}{\log r} < 0$, we have

$$\begin{aligned} p_t(0,x) &\sim |x|^{-\frac{\log q}{\log r}} f'(0)h|x|^{\frac{\log c}{\log r}} t \\ &= |x|^{-\frac{\log q}{\log r}} \sum_{i=0}^{\infty} \left(\frac{c}{q}\right)^i (c^{-1}-1)h|x|^{\frac{\log c}{\log r}} t \\ &= \frac{q(1-c)}{c(q-1)} a(0)|x|^{\frac{\log(q^{-1}c)}{\log r}} t, \end{aligned}$$

as $x \to \infty$. □

Proposition 4.5. *Let (X_t, P_t) be the A-process on K given by $a(M) = c^M a(0)$ with $0 < c < 1$. Then there exists a continuous periodic function ψ of period $\log c^{-1}$ such that*

$$p_t(0,0) = t^{\frac{\log q}{\log c}} \exp\left[\psi(\log t)\right].$$

Proof. We have

$$\begin{aligned} p_{ct}(0,0) &= (q-1)\sum_{i=-\infty}^{\infty} q^{-i}\exp\left[-(q-1)^{-1}(q-c)c^i a(0)t\right] \\ &= qp_t(0,0). \end{aligned}$$

Hence if we write $g(u) := p_{e^u}(0,0)$, then

$$\begin{aligned} g(u + \log c^{-1}) &= p_{c^{-1}e^u}(0,0) \\ &= q^{-1} g(u). \end{aligned}$$

Let $\psi(u) := \log g(u) + \frac{\log q}{-\log c} u$. Then we have $\psi(u + \log c^{-1}) = \psi(u)$, and

$$\begin{aligned} p_t(0,0) &= g(\log t) \\ &= \exp\left[\psi(\log t) - \frac{\log q}{-\log c}\log t\right] \\ &= t^{\frac{\log q}{\log c}} \exp\left[\psi(\log t)\right]. \ \square \end{aligned}$$

§5. Local times

In this section we will see that if X_t is a stable A-process with positive probabilities of hitting single points, we can determine the Hausdorff dimensions of the sets $\{t > 0 \ : \ X_t = X_0\}$.

We first exhibit some known results;

LEMMA 5.1. *Let (X_t, P_t) be an additive process on K.*

(1) *If the transformations P_t, $t \geq 0$ leave invariant the Banach space $\mathcal{C}(K)$ of functions which are continuous on K and vanish at infinity, and if P_t converges strongly on $\mathcal{C}(K)$ to the identity transformation as $t \to 0$, then $P_t(x, dy)$ satisfies the Hunt's hypotheses (A) in [6].*

(2) *Let $P_t(x, dy)$ have a density $p_t(x, y)$ with respect to η_K and satisfy Hunt's hypotheses (F) ([8]). Assume that for some $\lambda > 0$ and for each $x_0 \in K$, the function*

$$x \mapsto U^\lambda(x_0, x) := \int_0^\infty \exp(-\lambda t) p_t(x_0, x) dt$$

is bounded and continuous at $x = x_0$. Then each $x_0 \in K$ is regular for $\{x_0\}$.

(3) *Let P_t satisfy Hunt's hypotheses (A) and the assumptions in* (2) *above. Put for each $x_0 \in K$,*

$$\beta := \inf\{\alpha \ : \ \lambda^\alpha U^\lambda(x_0, x_0) \to \infty \ as \ \lambda \to \infty\},$$

$$\sigma := \sup\{\alpha \ : \ \lambda^\alpha U^\lambda(x_0,x_0) \to 0 \ \textit{as} \ \lambda \to \infty\},$$

and let $Z_{x_0} := \{t \ : \ X_t = x_0\}$. *Then*

$$\sigma \le \dim Z_{x_0} \le \beta \qquad P_{x_0}\text{-}a.e..$$

Proof. See [6] pp.50-51, [3] Theorems 3.1 and 4.1. □

Lemma 5.2. *Let* (X_t, P_t) *be an A-process on* K. *Then* P_t *satisfies Hunt's hypotheses (A) in [6].*

Proof. Let $f \in \mathcal{C}(K)$. Since P_t is shift-invariant, it can be easily seen that $P_t f$ is continuous. Let us prove that $P_t f$ vanishes at infinity. We shall suppose that $f \not\equiv 0$. For any $\varepsilon > 0$, there exists $N \in \mathbb{Z}$ such that $|f(x)| < \frac{\varepsilon}{2}$ provided that $|x| > r^N$. Since $\lim_{n\to\infty} P_n(t) = 1$, there exists $N' \ge 1$ such that

$$P_n(t) - P_{n-1}(t) < \frac{\varepsilon}{2}(\max|f|)^{-1}(q-1)q^{-1},$$

for $n \ge N'$. Let $N_0 = \max(N, N')$ and $|x| = r^{N_0+m}$, $m \ge 1$, then

$$\begin{aligned} |P_t f(x)| &\le \left|\int_{|y|>r^{N_0}} P_t(x,dy)f(y)\right| + \left|\int_{|y|\le r^{N_0}} P_t(x,dy)f(y)\right| \\ &< \frac{\varepsilon}{2} + \max|f|\cdot P_t\big(x, D(0,r^{N_0})\big) \\ &< \varepsilon. \end{aligned}$$

Finally, it is easily seen that

$$\| P_t f - f \| \to 0,$$

as $t \to 0$ for $f \in \mathcal{C}(K)$, using a similar argument as in [2]. By Lemma 5.1 (1), our assertion is proved. □

Lemma 5.3. *Let* (X_t, P_t) *be an A-process on* K. *Then* $P_t(x,dy)$ *satisfies Hunt's hypotheses (F) in [8].*

Proof. We have

$$
\begin{aligned}
\int_K & P_t\big(x, D(y, r^N)\big)\eta_K(dx) \\
&= \int_{|x-y|\le r^N} P_N(t)\eta_K(dx) \\
&\quad + \sum_{m\ge 1}\int_{|x-y|=r^{N+m}} (q-1)^{-1}q^{1-m}\big(P_{N+m}(t)-P_{N+m-1}(t)\big)\eta_K(dx) \\
&= q^N\Big(P_N(t)+\sum_{m\ge 1}\big(P_{N+m}(t)-P_{N+m-1}(t)\big)\Big) \\
&= q^N \\
&= \eta_K\big(D(y, r^N)\big).
\end{aligned}
$$

Especially η_K is excessive relative to $P_t(x, dy)$. For every positive continuous function γ on $(0, \infty)$ with compact support, let

$$h(\gamma, x, y) := \int_0^\infty p_t(x, y)\gamma(t)dt.$$

Then for any Borel set B in K, we have

$$\int_B h(\gamma, x, y)\eta_K(dy) = \int_0^\infty \gamma(t)P_t(x, B)dt.$$

Since $p_t(x, y)$ belongs to $\mathcal{C}(K)$ as a function of x or of y, we can easily verify that so is $h(\gamma, x, y)$. Now let f be any continuous function on K with compact support. Since $p_t(x, y)$ is bouded, so is $h(\gamma, x, y)$, and therefore it is easily seen that $\int \eta_K(dx)f(x)h(\gamma, x, y)$ belongs to $\mathcal{C}(K)$ as a function of y. □

Lemma 5.4. *Let (X_t, P_t) be the A-process on K given by $a(M) = c^M a(0)$ with $0 < c < q^{-1}$. Then any point $x_0 \in K$ is regular for $\{x_0\}$.*

Proof. By Lemma 5.1 (2) and Lemma 5.3, it suffices to show that for some $\lambda > 0$ the function $x \mapsto U^\lambda(x_0, x)$ is bounded. Whereas since $U^\lambda(x_0, x) = g^\lambda(x - x_0)$, Proposition 3.3 implies our assertion. □

Theorem 5.5. *Let* (X_t, P_t) *be the A-process on* K *given by* $a(M) = c^M a(0)$ *with* $0 < c < q^{-1}$*. Then*

$$P_{x_0}\left(\dim Z_{x_0} = 1 + \frac{\log q}{\log c}\right) = 1,$$

for any $x_0 \in K$*.*

Proof. By Proposition 4.5, there exist $a > b > 0$ such that

$$bt^{\frac{\log q}{\log c}} < p_t(0,0) < at^{\frac{\log q}{\log c}},$$

for any $t > 0$. Then we have

$$b\lambda^{-1-\frac{\log q}{\log c}} \int_0^\infty e^{-t} t^{\frac{\log q}{\log c}} dt < U^\lambda(x_0, x_0) < a\lambda^{-1-\frac{\log q}{\log c}} \int_0^\infty e^{-t} t^{\frac{\log q}{\log c}} dt.$$

Applying Lemma 5.1 (3) completes the proof. □

References

[1] Albeverio, S. and W. Karwowski, Diffusion on p-adic numbers, Gaussian Random Fields, Proc. Nagoya Conf. (1991), edited by K. Ito and T. Hida, World Scientific.

[2] Albeverio, S. and W. Karwowski, A random walk on p-adics–the generator and its spectrum, Stochastic Processes and Their Applications **53** (1994).

[3] Blumenthal, R. M. and R. K. Getoor, Markov processes and potential theory, Academic Press, 1968.

[4] Bosch, S., Guntzer, U. and R. Remmert, Non-archimedean analysis, Springer-Verlag, 1984.

[5] Chung, K. L., Markov chains with stationary transition probabilities, Springer-Verlag, 1960.

[6] Hunt, G. A., Markov processes and potentials I, Illinois J. Math. **1** (1957), 44–93.

[7] Hunt, G. A., Markov processes and potentials II, Illinois J. Math. **1** (1957), 316–369.

[8] Hunt, G. A., Markov processes and potentials III, Illinois J. Math. **2** (1958), 151–213.

[9] Kesten, H., Hitting probabilities of single points for processes with stationary independent increments, Memories of the American Mathematical Society **93** (1969), American Mathematical Society.
[10] Koblitz, N., p-adic numbers, p-adic analysis, and zeta-functions, 2-nd edition, Graduate Texts in Mathematics 58, Springer-Verlag, 1974.
[11] Kuo, H. H., Gaussian measures in Banach spaces, Lecture Notes in Mathematics 463, Springer-Verlag, 1975.
[12] Lang, S., Algebraic number theory, Addison-Wesley, 1970.
[13] Meyer, P. A., Processus de Markov, Lecture Notes in Mathematics 26, Springer-Verlag, 1967.
[14] Parthasarathy, K. R., Probability measures on metric spaces, Academic Press, 1967.
[15] Port, S. C. and C. J. Stone, Infinitely divisible processes and their potential theory, first part, Ann. Inst. Fourier, Grenoble **21**(2) (1971), 157–275.
[16] Satoh, T., Wiener measures on certain normed spaces over non-archimedean local fields, Suriken-Kokyuroku **792** (1992), 39–63.
[17] Satoh, T., Wiener measures on certain Banach spaces over non-archimedean local fields, Compositio Math. **93** (1994), 81–108.
[18] Weil, A., Basic number theory, Springer-Verlag, 1974.

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