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A Compact Imbedding of Semisimple Symmetric Spaces

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Abstract. A realization of a ε-family of semisimple symmetric spaces $\{G/H_\varepsilon\}$ in a compact real analytic manifold $\mathbb{X}$ is constructed. The realization $\mathbb{X}$ has the following properties: a) The action of G on $\mathbb{X}$ is real analytic; b) There exist open G-orbits that are isomorphic to G/H_ε for each signature of roots ε; c) The system $\mathcal{M}_\lambda$ of invariant differential equations on G/H_ε extends analytically on $\mathbb{X}$ and has regular singularities in the weak sense along the boundaries.

Introduction

Let $X = G/H$ be a semisimple symmetric space of split rank l. The purpose of this paper is to construct an imbedding of X into a compact real analytic manifold $\mathbb{X}$ without boundary. Our construction is similar to those in Kosters[K], Oshima[O1], [O2], Oshima and Sekiguchi[OS1], and Sekiguchi[Se]. The main idea of construction was first presented in [O1].

In [O1] and [O2] Oshima constructed an imbedding of X in a real analytic manifold $\mathbb{X}'$. The number of open G-orbits in $\mathbb{X}'$ is 2^l and all open orbits are isomorphic to X. For example, if $X = SL(2,\mathbb{R})/SO(2)$, then $\mathbb{X}'$ is $\mathbb{P}^1_{\mathbb{C}}$; there are two open orbits that are isomorphic to X and one compact orbit that is isomorphic to $G/P \simeq \{z \in \mathbb{C}\,;\,|z| = 1\}$, where P is the set of the lower triangular matrices in $G = SL(2,\mathbb{R})$. The idea of construction is as follows. By the Cartan decomposition $G = KAH$, we must compactify A. We choose a coordinate system on $A \simeq (0,\infty)^l$ so that the coefficients of vector fields that correspond to local one parameter groups of transformations of G/H continue real analytically to $\mathbb{R}^l$. In [O1] and [O2], Oshima

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used the coordinate system $(t_1, \cdots, t_l) = (a^{-\alpha_1}, \cdots, a^{-\alpha_l})\,(a \in A)$, where $\{\alpha_1, \ldots, \alpha_l\}$ is the set of simple restricted roots.

When $X = G/K$ is a Riemannian symmetric space, Oshima and Sekiguchi[OS1] used the coordinate system $(t_1, \cdots, t_l) = (a^{-2\alpha_1}, \cdots, a^{-2\alpha_l})\,(a \in A)$ and constructed a compact real analytic manifold $\mathbb{X}$. There exists a family of open orbits $\{G/K_\varepsilon\,;\, \varepsilon \in \{-1,1\}^l\}$, where G/K_ε are semisimple symmetric spaces. For example, if $X = SL(2,\mathbb{R})/SO(2)$, then there are three open orbits in $\mathbb{X}$, one of which is isomorphic to $SL(2,\mathbb{R})/SO(1,1)$ and the other two open orbits are isomorphic to X. The two orbits that are not open are isomorphic to G/P.

We shall generalize the construction in [OS1] for a semisimple symmetric space $X = G/H$ and construct a real analytic manifold $\mathbb{X}$. The main result is given in Theorem 2.6. There exists a family of open orbits $\{G/H_\varepsilon\,;\, \varepsilon \in \{-1,1\}^l\}$, where G/H_ε are semisimple symmetric spaces such that $(H_\varepsilon)_\mathbb{C} \simeq H_\mathbb{C}$ for all ε. If G/H_ε is a Riemannian symmetric space for some ε, $\mathbb{X}$ is identical with that was constructed by Oshima and Sekiguchi.

§1. Semisimple symmetric spaces

In this section we define a family of semisimple symmetric spaces and establish some results about it, to be used later.

1.1. Symmetric pairs

First we review some notation and results of Oshima and Sekiguchi[OS2] concerning symmetric pairs. Let $\mathfrak{g}$ be a noncompact real semisimple Lie algebra and let σ be an involution (i.e. an automorphism of order 2) of $\mathfrak{g}$. Denoting by $\mathfrak{h}$ (resp. $\mathfrak{q}$) the $+1$ (resp. -1) eigenspace of σ, we have a direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. We call $(\mathfrak{g}, \mathfrak{h})$ a *semisimple symmetric pair* or *symmetric pair* for brevity. We define that two symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}', \mathfrak{h}')$ are isomorphic if there exists a Lie algebra isomorphism ϕ of $\mathfrak{g}$ to $\mathfrak{g}'$ such that $\phi(\mathfrak{h}) = \mathfrak{h}'$.

There exists a Cartan involution θ of $\mathfrak{g}$ which commutes with σ. Hereafter we fix such θ. Denoting by $\mathfrak{k}$ (resp. $\mathfrak{p}$) the $+1$ (resp. -1) eigenspace of θ, we have a direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We call $(\mathfrak{g}, \mathfrak{k})$ a Riemannian symmetric pair. Since σ and θ commute, we have the direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q}.$$

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ and let $\mathfrak{a}^*$ be its dual space. For $\alpha \in \mathfrak{a}^*$, let $\mathfrak{g}^\alpha$ denote the linear subspace of $\mathfrak{g}$ given by

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{a}^* \,;\, [Y, X] = \alpha(Y)X \quad \text{for all } Y \in \mathfrak{a}\}.$$

Then the set $\Sigma = \{\alpha \in \mathfrak{a}^* \,;\, \mathfrak{g}^\alpha \neq \{0\}, \alpha \neq 0\}$ becomes a root system. We call Σ the restricted root system of the symmetric pair $(\mathfrak{g}, \mathfrak{h})$. Put

$$\Sigma_0 = \{\alpha \in \Sigma \,;\, \alpha/2 \notin \Sigma\}.$$

Let W denote the Weyl group of Σ. For $\alpha \in \Sigma$ let $s_\alpha \in W$ denote the reflection in the hyperplane $\alpha = 0$. Fix a linear order in $\mathfrak{a}^*$ and let Σ^+ be the set of positive elements in Σ. Let $\Psi = \{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots in Σ^+, where the number $l = \dim \mathfrak{a}$ is called the split rank of the symmetric pair $(\mathfrak{g}, \mathfrak{h})$. Let $\{H_1, \ldots, H_l\}$ be the basis of $\mathfrak{a}$ dual to $\{\alpha_1, \ldots, \alpha_l\}$.

Definition 1.1.

(i) A mapping $\varepsilon : \Sigma \to \{1, -1\}$ is called a *signature of roots* if it satisfies the following conditions:

$$\begin{cases} \varepsilon(-\alpha) = \varepsilon(\alpha) & \text{for any } \alpha \in \Sigma, \\ \varepsilon(\alpha + \beta) = \varepsilon(\alpha)\varepsilon(\beta) & \text{if } \alpha, \beta \quad \text{and} \quad \alpha + \beta \in \Sigma. \end{cases}$$

(ii) For a signature of roots ε of Σ, we define an involution σ_ε of $\mathfrak{g}$ by

$$\sigma_\varepsilon(X) = \begin{cases} \sigma(X) & \text{for } X \in Z_{\mathfrak{g}}(\mathfrak{a}) \\ \varepsilon(\alpha)\sigma(X) & \text{for } X \in \mathfrak{g}^\alpha, \alpha \in \Sigma \end{cases}$$

where $Z_{\mathfrak{g}}(\mathfrak{a}) = \{X \in \mathfrak{g} \,;\, [X, \mathfrak{a}] = 0\}$.

Denoting by $\mathfrak{h}_\varepsilon$ (resp. $\mathfrak{q}_\varepsilon$) the $+1$ (resp. -1) eigenspace of σ_ε, we have a direct sum decomposition $\mathfrak{g} = \mathfrak{h}_\varepsilon \oplus \mathfrak{q}_\varepsilon$. By definition, σ_ε commutes with θ and σ, and $\mathfrak{a}$ is also a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}_\varepsilon$. This implies that Σ is also the restricted root system of the symmetric pair $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$. For a real Lie algebra $\mathfrak{u}$ let $\mathfrak{u}_{\mathbb{C}}$ denote its complexification. The following lemma can be proved easily in the same way as the proof of Lemma 1.3 in [OS1].

LEMMA 1.2. *The automorphism*

$$f_\varepsilon = \mathrm{Ad}\left(\exp\left(\sum_{j=1}^{l} \frac{\pi\sqrt{-1}}{4}(1-\varepsilon(\alpha_j))H_j\right)\right)$$

of $\mathfrak{g}_\mathbb{C}$ *maps* $\mathfrak{h}_\mathbb{C}$ *onto* $(\mathfrak{h}_\varepsilon)_\mathbb{C}$. *Hence the complexifications of* $\mathfrak{h}$ *and* $\mathfrak{h}_\varepsilon$ *are isomorphic in* $\mathfrak{g}_\mathbb{C}$.

For a symmetric pair $(\mathfrak{g}, \mathfrak{h})$, let $F((\mathfrak{g}, \mathfrak{h}))$ denote the totality of symmetric pairs $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$ for all signatures ε of roots and we call it an *ε-family* of symmetric pairs (obtained from $(\mathfrak{g}, \mathfrak{h})$).

For each $\alpha \in \Sigma$, $\theta\sigma$ leaves $\mathfrak{g}^\alpha$ invariant. Denoting by $\mathfrak{g}^\alpha_+$ (resp. $\mathfrak{g}^\alpha_-$) the $+1$ (resp. -1) eigenspace of $\theta\sigma$ in $\mathfrak{g}^\alpha$, we have a direct sum decomposition $\mathfrak{g}^\alpha = \mathfrak{g}^\alpha_+ \oplus \mathfrak{g}^\alpha_-$. The number $m(\alpha) = \dim \mathfrak{g}^\alpha$ is called the *multiplicity* of α and the pair $(m^+(\alpha), m^-(\alpha)) = (\dim \mathfrak{g}^\alpha_+, \dim \mathfrak{g}^\alpha_-)$ is called the signature of α. If we denote by $(m^+(\alpha,\varepsilon), m^-(\alpha,\varepsilon))$ the *signature* of α as a restricted root of $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$, then

$$((m^+(\alpha,\varepsilon), m^-(\alpha,\varepsilon)) = \begin{cases} (m^+(\alpha), m^-(\alpha)) & \text{if } \varepsilon(\alpha) = 1 \\ (m^-(\alpha), m^+(\alpha)) & \text{if } \varepsilon(\alpha) = -1. \end{cases} \tag{1.1}$$

DEFINITION 1.3. A symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is called *basic* if

$$m^+(\alpha) \geq m^-(\alpha) \quad \text{for any} \quad \alpha \in \Sigma_0.$$

PROPOSITION 1.4. ([OS2, Proposition 6.5])*Let F be an ε-family of symmetric pairs. Then there exists a basic symmetric pair in F that is unique up to isomorphism.*

Example 1.5.

(i) Riemannian symmetric pairs are basic. If an ε-family F contains a Riemannian symmetric pair, then the mutually non-isomorphic symmetric pairs contained in F are determined in [OS1, Appendix]. For a Riemannian symmetric pair $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(2))$, the ε-family is up to isomorphism given by

$$F((\mathfrak{g}, \mathfrak{k})) = \{(\mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(2)), (\mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(1,1))\}.$$

(ii) For a real semisimple Lie algebra $\mathfrak{g}'$ let $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'$ and $\mathfrak{h} = \{(X, X)\,;\, X \in \mathfrak{g}'\} \simeq \mathfrak{g}'$. In this case $m^+(\alpha) = m^-(\alpha)$ for any $\alpha \in \Sigma$ and hence the pair $(\mathfrak{g}, \mathfrak{h})$ is basic.

(iii) The ε-families obtained from irreducible symmetric pairs such that they are neither of type (i) nor (ii) are determined in [OS2, Table V]. For example, the symmetric pair $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(3,6), \mathfrak{so}(3,1) + \mathfrak{so}(5))$ is basic and the ε-family is up to isomorphism given by

$$F = \{(\mathfrak{so}(3,6), \mathfrak{so}(3-k, 1+k) + \mathfrak{so}(k, 5-k))\,;\, 0 \leq k \leq 2\}.$$

1.2. Definition of symmetric spaces G/H_ε

For an ε-family of symmetric pairs, we will define a family of symmetric spaces. Hereafter we assume that $(\mathfrak{g}, \mathfrak{h})$ is a basic symmetric pair and consider the ε-family obtained from $(\mathfrak{g}, \mathfrak{h})$.

For a Lie group L with Lie algebra $\mathfrak{l}$ and a subalgebra $\mathfrak{t}$ of $\mathfrak{l}$, let $Z_L(\mathfrak{t})$ and $Z_{\mathfrak{l}}(\mathfrak{t})$ denote the centralizer of $\mathfrak{t}$ in L and that of $\mathfrak{t}$ in $\mathfrak{l}$ respectively and let L_0 denote the connected component of the identity element in L.

Let $G_{\mathbb{C}}$ be a connected complex Lie group whose Lie algebra is $\mathfrak{g}_{\mathbb{C}}$ and let G be the analytic subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}$. We extend σ and θ to $\mathfrak{g}_{\mathbb{C}}$ as $\mathbb{C}$-linear involutions.

We assume that the involution σ is lifted to G (i.e. there exists an analytic automorphism $\tilde{\sigma}$ of G such that $\tilde{\sigma}(\exp X) = \exp \sigma(X)$ for any $X \in \mathfrak{g}$) and denote the lifting by the same letter. If $G_{\mathbb{C}}$ is simply connected or is the adjoint group of $\mathfrak{g}_{\mathbb{C}}$, then any involution of $\mathfrak{g}$ is lifted to G (c.f. [OS2, Lemma 1.5]).

LEMMA 1.6. *Under the above assumption, the involution σ_ε of $\mathfrak{g}$ is lifted to G for each signature of roots ε.*

PROOF. We fix a signature of roots ε. Let $\widetilde{G}_{\mathbb{C}}$ denote the universal covering group of $G_{\mathbb{C}}$ and let $\widetilde{G}$ be the analytic subgroup of $\widetilde{G}_{\mathbb{C}}$ corresponding to $\mathfrak{g}$ and let π denote the covering map $\pi : \widetilde{G} \to G$. The involutions σ and σ_ε are lifted to $\widetilde{G}_{\mathbb{C}}$.

Let U be the analytic subgroup of $\widetilde{G}_{\mathbb{C}}$ corresponding to $\mathfrak{u} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$. Then the center $\widetilde{Z}$ of $\widetilde{G}_{\mathbb{C}}$ is contained in $Z_U(\sqrt{-1}\mathfrak{a})$. It follows from [H, Chapter VII, Corollary 2.8] that $Z_U(\sqrt{-1}\mathfrak{a})$ is connected. By definition, σ

and σ_ε coincide on $Z_{\mathfrak{u}}(\sqrt{-1}\mathfrak{a})$, hence their liftings to $\widetilde{G}_{\mathbb{C}}$ coincide on the connected Lie group $Z_U(\sqrt{-1}\mathfrak{a})$. Since σ is lifted to G, $\ker\pi \subset Z_U(\sqrt{-1}\mathfrak{a})$ is σ-stable, hence it is σ_ε-stable. It follows from [H, Chapter VII, Lemma 1.3] that σ_ε is lifted to G. □

We define $G^\sigma = \{g \in G\,;\, \sigma(g) = g\}$ and let H be a closed subgroup of G between G^σ and its identity component $(G^\sigma)_0$. The homogeneous space G/H is called a *semisimple symmetric space* associated with the symmetric pair $(\mathfrak{g}, \mathfrak{h})$. Hereafter we fix a symmetric space G/H associated with $(\mathfrak{g}, \mathfrak{h})$.

Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$. The Weyl group W of the restricted root system Σ can be identified with $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, where $N_K(\mathfrak{a})$ is the normalizer of $\mathfrak{a}$ in K. For a signature of roots ε, we put $H_\varepsilon = (G^{\sigma_\varepsilon})_0 Z_{K\cap H}(\mathfrak{a})$.

LEMMA 1.7. *H_ε is a closed subgroup of G that is contained in G^{σ_ε}.*

PROOF. It follows from the proof of Lemma 1.6 that σ and σ_ε coincide on $Z_{K\cap H}(\mathfrak{a})$, hence $H_\varepsilon \subset G^{\sigma_\varepsilon}$.

For any $z \in Z_{K\cap H}(\mathfrak{a})$ we have $\sigma_\varepsilon \circ \mathrm{Ad}(z) = \mathrm{Ad}(\sigma_\varepsilon z) \circ \sigma_\varepsilon = \mathrm{Ad}(z) \circ \sigma_\varepsilon$, hence $\mathrm{Ad}(z)(\mathfrak{h}_\varepsilon) = \mathfrak{h}_\varepsilon$. It shows that H_ε is a group with Lie algebra $\mathfrak{h}_\varepsilon$. Since $(G^{\sigma_\varepsilon})_0$ is a closed subgroup of G and H_ε has finitely many connected components, H_ε is a closed subgroup of G. □

The above lemma shows that G/H_ε is a semisimple symmetric space associated with the symmetric pair $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$. We give an important lemma that will be used later;

LEMMA 1.8. *For each signature of roots ε,*

(i) $Z_{K\cap(G^{\sigma_\varepsilon})_0}(\mathfrak{a}) \subset Z_{K\cap(G^\sigma)_0}(\mathfrak{a})$
(ii) $Z_{K\cap H}(\mathfrak{a}) = Z_{K\cap H_\varepsilon}(\mathfrak{a})$

PROOF. (i) Let ε be a signature of roots. We put $\mathfrak{h}_\varepsilon^a = \mathfrak{k}\cap\mathfrak{h}_\varepsilon + \mathfrak{p}\cap\mathfrak{q}_\varepsilon$ and let $(H_\varepsilon^a)_0$ be the analytic subgroups of G corresponding to $\mathfrak{h}_\varepsilon^a$. If $\varepsilon = (1, \cdots, 1)$, then we drop ε in our notation and write $\mathfrak{h}^a$, H_0^a etc. Then $(\mathfrak{h}_\varepsilon^a, \mathfrak{h}_\varepsilon^a \cap \mathfrak{k})$ is a Riemannian symmetric pair and $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{h}_\varepsilon^a \cap \mathfrak{p}$. The groups $K \cap (G^{\sigma_\varepsilon})_0$ and $K \cap (H_\varepsilon^a)_0$ are maximal compact subgroups of $(G^{\sigma_\varepsilon})_0$ and $(H_\varepsilon^a)_0$ respectively, thus $K \cap (G^{\sigma_\varepsilon})_0$ and

$K\cap(H^a_\varepsilon)_0$ are connected. Moreover $K\cap(G^{\sigma_\varepsilon})_0$ and $K\cap(H^a_\varepsilon)_0$ have same Lie algebra $\mathfrak{k}\cap\mathfrak{h}_\varepsilon$. Therefore they coincide. It follows from [W, Lemma 1.1.3.8] and its proof that

$$Z_{K\cap(G^{\sigma_\varepsilon})_0}(\mathfrak{a})=Z_{K\cap(H^a_\varepsilon)_0}(\mathfrak{a})=(Z_{K\cap(H^a_\varepsilon)_0}(\mathfrak{a}))_0(K\cap(H^a_\varepsilon)_0\cap\exp\sqrt{-1}\mathfrak{a})$$

Since $(Z_{K\cap(H^a_\varepsilon)_0}(\mathfrak{a}))_0=(Z_{K\cap H^a_0}(\mathfrak{a}))_0$ for each ε, it suffices to prove

$$K\cap(H^a_\varepsilon)_0\cap\exp\sqrt{-1}\mathfrak{a}\subset K\cap H^a_0\cap\exp\sqrt{-1}\mathfrak{a}\,, \tag{1.2}$$

for each signature of roots ε.

Let $(\widetilde{H}^a_\varepsilon)_{\mathbb{C}}$ be the simply connected connected Lie group with Lie algebra $(\mathfrak{h}^a_\varepsilon)_{\mathbb{C}}$. Let $\widetilde{H}^a_\varepsilon$ and $K(\widetilde{H}^a_\varepsilon)$ be the analytic subgroups of $(\widetilde{H}^a_\varepsilon)_{\mathbb{C}}$ corresponding to $\mathfrak{h}^a_\varepsilon$ and $\mathfrak{k}\cap\mathfrak{h}_\varepsilon$ respectively. By [H, Chapter VII, Theorem 8.5], the lattice

$$\mathfrak{a}_{K(\widetilde{H}^a_\varepsilon)}=\{X\in\mathfrak{a}\,;\,\exp\sqrt{-1}X\in K(\widetilde{H}^a_\varepsilon)\}$$

in $\mathfrak{a}$ is spanned by

$$\frac{2\pi\sqrt{-1}}{\langle\alpha,\alpha\rangle}A_\alpha\quad(\alpha\in\Sigma(\mathfrak{h}^a_\varepsilon,\mathfrak{a})),$$

where $A_\alpha\in\mathfrak{a}$ is determined by $\alpha(X)=B(A_\alpha,X)$ for all $X\in\mathfrak{a}$. Here B denotes the Killing form of $\mathfrak{h}^a_\varepsilon$ and $\Sigma(\mathfrak{h}^a_\varepsilon,\mathfrak{a})$ is the restricted root system for the symmetric pair $(\mathfrak{h}^a_\varepsilon,\mathfrak{k}\cap\mathfrak{h}_\varepsilon)$. Notice that $m^+(\alpha,\varepsilon)$ is the multiplicity of $\alpha\in\Sigma$ considered as an element of $\Sigma(\mathfrak{h}^a_\varepsilon,\mathfrak{a})$. By (1.1) and Definition 1.3, $m^+(\alpha)\geq m^+(\alpha,\varepsilon)$ for any $\alpha\in\Sigma_0$ and $\varepsilon(\alpha)=\varepsilon(\alpha/2)^2=1$ for $\alpha\in\Sigma\setminus\Sigma_0$. Therefore we have $\Sigma(\mathfrak{h}^a_\varepsilon,\mathfrak{a})\subset\Sigma(\mathfrak{h}^a,\mathfrak{a})$, hence $\mathfrak{a}_{K(\widetilde{H}^a_\varepsilon)}\subset\mathfrak{a}_{K(\widetilde{H}^a)}$. By Lemma 1.2, the center of $(\widetilde{H}^a_\varepsilon)_{\mathbb{C}}$ coincides with that of $(\widetilde{H}^a)_{\mathbb{C}}$ and σ_ε coincides with σ on it, hence (1.2) follows.

Since we have $Z_{K\cap H_\varepsilon}(\mathfrak{a})=Z_{K\cap(G^{\sigma_\varepsilon})_0}(\mathfrak{a})Z_{K\cap H}(\mathfrak{a})$ by the definition of H_ε, (ii) follows from (i). □

§2. Construction of compact imbedding

2.1. Parabolic subgroups

We assume that $(\mathfrak{g},\mathfrak{h})$ is a basic symmetric pair. We define a standard parabolic subalgebra $\mathfrak{p}_\sigma$ of $\mathfrak{g}$ by $\mathfrak{p}_\sigma=Z_{\mathfrak{k}}(\mathfrak{a})+\mathfrak{n}_\sigma$, where $\mathfrak{n}_\sigma=\sum_{\alpha\in\Sigma^+}\mathfrak{g}^\alpha$. Let $\mathfrak{p}_\sigma=\mathfrak{m}_\sigma+\mathfrak{a}_\sigma+\mathfrak{n}_\sigma$ be a Langlands decomposition of $\mathfrak{p}_\sigma$ (c.f. [OS2, Section 8]). Let P_σ denote the parabolic subgroup of G with Lie algebra $\mathfrak{p}_\sigma$ and let $P_\sigma = M_\sigma A_\sigma N_\sigma$ be the Langlands decomposition corresponding to $\mathfrak{p}_\sigma = \mathfrak{m}_\sigma + \mathfrak{a}_\sigma + \mathfrak{n}_\sigma$. Let N_σ^- be the analytic subgroup of G corresponding to $\mathfrak{n}_\sigma^- = \theta(\mathfrak{n}_\sigma)$. If $(\mathfrak{g}, \mathfrak{h})$ is a Riemannian symmetric pair, then $\mathfrak{p}_\sigma$ is a minimal parabolic subalgebra of $\mathfrak{g}$.

DEFINITION 2.1. A mapping $\varepsilon : \Sigma \to \{-1, 0, 1\}$ is called an *extended signature of roots* when it satisfies the condition:

$$\varepsilon(\alpha) = \prod_{i=1}^{l} \varepsilon(\alpha_i)^{|m_i|} \quad \text{for } \alpha = \sum_{i=1}^{l} m_i \alpha_i \in \Sigma. \tag{2.1}$$

Note that any mapping of $\Psi = \{\alpha_1, \cdots, \alpha_l\}$ to $\{-1, 0, 1\}$ is uniquely extended to a mapping of Σ to $\{-1, 0, 1\}$ which satisfies (2.1). Therefore we can identify the set of all extended signatures of roots with $\{-1, 0, 1\}^l$ by $\varepsilon \mapsto (\varepsilon(\alpha_1), \ldots, \varepsilon(\alpha_l))$. For an extended signature of roots ε, we define a signature of roots $\tilde{\varepsilon}$ by

$$\tilde{\varepsilon}(\alpha_j) = \begin{cases} \varepsilon(\alpha_j) & \text{if } \varepsilon(\alpha_j) \neq 0 \\ 1 & \text{if } \varepsilon(\alpha_j) = 0. \end{cases} \tag{2.2}$$

For an extended signature of roots we define $\Theta_\varepsilon = \{\alpha \in \Psi\,;\, \varepsilon(\alpha) \neq 0\}$, $\langle \Theta_\varepsilon \rangle = \Sigma \cap \sum_{\alpha \in \Theta_\varepsilon} \mathbb{R}\alpha$ and $\langle \Theta \rangle^+ = \Sigma^+ \cap \langle \Theta \rangle$. Let W_{Θ_ε} be the subgroup of W generated by the reflections with respect to the elements of $\langle \Theta_\varepsilon \rangle$. Notice that $\langle \Theta_\varepsilon \rangle$ become a root system and W_{Θ_ε} is its Weyl group.

We define a parabolic subalgebra $\mathfrak{p}_\varepsilon$ by

$$\mathfrak{p}_\varepsilon = \mathfrak{m}_\sigma + \mathfrak{a}_\sigma + \sum_{\alpha \in \langle \Theta_\varepsilon \rangle} \mathfrak{g}^\alpha + \sum_{\alpha \in \Sigma^+ \setminus \langle \Theta_\varepsilon \rangle} \mathfrak{g}^\alpha$$

and let $\mathfrak{p}_\varepsilon = \mathfrak{m}_\varepsilon + \mathfrak{a}_\varepsilon + \mathfrak{n}_\varepsilon$ be the Langlands decomposition of $\mathfrak{p}_\varepsilon$ such that $\mathfrak{a}_\varepsilon \subset \mathfrak{a}_\sigma$. Let P_ε be the parabolic subgroup of G with Lie algebra $\mathfrak{p}_\varepsilon$ and let $P_\varepsilon = M_\varepsilon A_\varepsilon N_\varepsilon$ be the Langlands decomposition of P_ε corresponding to $\mathfrak{p}_\varepsilon = \mathfrak{m}_\varepsilon + \mathfrak{a}_\varepsilon + \mathfrak{n}_\varepsilon$. We define subalgebras $\mathfrak{a}^\varepsilon$, $\mathfrak{m}(\varepsilon)$ and $\mathfrak{p}(\varepsilon)$ of $\mathfrak{g}$ by $\mathfrak{a}^\varepsilon = \sum_{\alpha_j \in \Theta_\varepsilon} \mathbb{R}H_j$, $\mathfrak{m}(\varepsilon) = \mathfrak{m}_\varepsilon \cap \mathfrak{h}_{\tilde{\varepsilon}} = Z_{\mathfrak{h}_{\tilde{\varepsilon}}}(\mathfrak{a}_\varepsilon)$ and $\mathfrak{p}(\varepsilon) = \mathfrak{m}(\varepsilon) + \mathfrak{a}_\varepsilon + \mathfrak{n}_\varepsilon$. We have a direct sum decomposition $\mathfrak{a}_\sigma = \mathfrak{a}^\varepsilon + \mathfrak{a}_\varepsilon$.

Let A, A^ε and $M(\varepsilon)_0$ be analytic subgroup of G corresponding to $\mathfrak{a}$, $\mathfrak{a}^\varepsilon$ and $\mathfrak{m}(\varepsilon)$ respectively. We define $M(\varepsilon) = M(\varepsilon)_0 Z_{K\cap H}(\mathfrak{a})$ and $P(\varepsilon) = M(\varepsilon)A_\varepsilon N_\varepsilon$. If ε is a signature of roots, $\Theta_\varepsilon = \Psi$, $W_{\Theta_\varepsilon} = W$ and $P(\varepsilon) = H_\varepsilon$. On the other hand, if $\varepsilon = (0, \dots, 0)$, $\Theta_\varepsilon = \varnothing$, $W_{\Theta_\varepsilon} = \{e\}$ and $P_\varepsilon = P_\sigma$.

Lemma 2.2. *$M(\varepsilon)$ and $P(\varepsilon)$ are closed subgroups of G.*

Proof. Since $\mathrm{Ad}(z)\sigma_{\tilde\varepsilon}(X) = \sigma_{\tilde\varepsilon}(\mathrm{Ad}(z)X)$ for all $z \in Z_{K\cap H}(\mathfrak{a}) = Z_{K\cap H_{\tilde\varepsilon}}(\mathfrak{a})$ and $X \in \mathfrak{g}$, we have $\mathrm{Ad}(z)(\mathfrak{m}(\varepsilon)) = \mathfrak{m}(\varepsilon)$ for all $z \in Z_{K\cap H}(\mathfrak{a})$. Therefore $M(\varepsilon)$ is a group. It is closed, because $M(\varepsilon)_0$ is a connected component of $H_{\tilde\varepsilon} \cap M_\varepsilon$ and $Z_{K\cap H}(\mathfrak{a})$ is compact.

Owing to the Langlands decomposition, $P(\varepsilon)$ is closed because $M(\varepsilon)$ is closed in M_ε. It is easy to see that $M(\varepsilon)$ and A_ε normalize N_ε. Thus $P(\varepsilon)$ is a group. □

2.2. Root systems and Weyl groups

Let

$$\Psi' = \{\alpha \in \Psi\,;\, 2\alpha \notin \Sigma \text{ and } m^-(\alpha) = 0\} \tag{2.3}$$

and $\Sigma' = \Sigma \cap \sum_{\alpha\in\Psi'} \mathbb{R}\alpha$. For an extended signature of roots ε, we define $\Sigma'_\varepsilon = \{\alpha \in \Sigma'\,;\, \varepsilon(\alpha) = 1\}$ and $\Sigma_\varepsilon = \{\alpha \in \langle\Theta_\varepsilon\rangle\,;\, \varepsilon(\alpha) = 1 \text{ or } m^-(\alpha) > 0\}$. By [B, Chapter IV, Proposition 23], Σ_ε and Σ'_ε are root systems. Let W', W_ε, W'_ε and W'_{Θ_ε} denote the subgroups of W generated by the reflections with respect to the roots in Σ', Σ_ε, Σ'_ε and $\Sigma' \cap \langle\Theta_\varepsilon\rangle$ respectively. We put

$$W(\varepsilon) = \{w \in W_{\Theta_\varepsilon}\,;\, \Sigma_\varepsilon \cap w\Sigma^+ = \Sigma_\varepsilon \cap \Sigma^+\}.$$

Lemma 2.3.

(i) $W(\varepsilon) = \{w \in W_{\Theta_\varepsilon}\,;\, \Sigma_\varepsilon \cap \Phi_w = \varnothing\}$. *Here* $\Phi_w = \{\alpha \in \Sigma^+\,;\, w^{-1}\alpha \in -\Sigma^+\}$.

(ii) $W(\varepsilon) = \{w \in W'_{\Theta_\varepsilon}\,;\, \Sigma'_\varepsilon \cap w\Sigma^+ = \Sigma'_\varepsilon \cap \Sigma^+\}$.

(iii) *Let the pair $(W^*_{\Theta_\varepsilon}, W^*_\varepsilon)$ be equal to $(W_{\Theta_\varepsilon}, W_\varepsilon)$ or $(W'_{\Theta_\varepsilon}, W'_\varepsilon)$. Then every element $w \in W^*_{\Theta_\varepsilon}$ can be written in a unique way in the form*

$$w = w_\varepsilon w(\varepsilon) \quad (w_\varepsilon \in W^*_\varepsilon,\ w(\varepsilon) \in W(\varepsilon)).$$

Proof. The proof is almost the same as that of [OS1, Lemma 2.5]. So we omit it. □

Let ε be a signature of roots. Let $W(\mathfrak{a}; H_\varepsilon)$ be the set of all elements w in W such that the representative $\bar{w}$ of w can be taken from $N_{K\cap H_\varepsilon}(\mathfrak{a})$. We have $W(\mathfrak{a}; H_\varepsilon) \simeq N_{K\cap H_\varepsilon}(\mathfrak{a})/Z_{K\cap H_\varepsilon}(\mathfrak{a})$. We put $W(\mathfrak{a}; (H_\varepsilon)_0) = N_{K\cap(H_\varepsilon)_0}(\mathfrak{a})/Z_{K\cap(H_\varepsilon)_0}(\mathfrak{a})$. For $\alpha \in \Sigma_0$, let $\mathfrak{g}(\alpha)$ denote the Lie subalgebra of $\mathfrak{g}$ that is generated by $\mathfrak{g}^\alpha$ and $\theta\mathfrak{g}^\alpha$.

Proposition 2.4. *Let ε be a signature of roots.*

(i) *Let $\alpha \in \Sigma_0$. Then $\mathfrak{h}_\varepsilon^a \cap \mathfrak{g}(\alpha) \neq \{0\}$ if and only if $s_\alpha \in W(\mathfrak{a}; (H_\varepsilon)_0)$.*
(ii) *$W(\mathfrak{a}; H_\varepsilon) = W_\varepsilon$.*

Proof. We use the method of rank one reduction. Let $\alpha \in \Sigma_0$. If $\mathfrak{h}_\varepsilon^a \cap \mathfrak{g}^\alpha \neq \{0\}$, then α can be considered as an element of the restricted root system $\Sigma(\mathfrak{h}_\varepsilon^a, \mathfrak{a})$ of the symmetric pair $(\mathfrak{h}_\varepsilon^a, \mathfrak{k} \cap \mathfrak{h}_\varepsilon^a)$. Thus there exists $X_\alpha \in \mathfrak{g}^\alpha \cap \mathfrak{h}_\varepsilon^a$ such that $\exp(X_\alpha + \theta X_\alpha) = \bar{s}_\alpha$ (c.f. [H, Chapter VII]).

If $\mathfrak{h}_\varepsilon^a \cap \mathfrak{g}^\alpha = \{0\}$, then by [OS2, Remark 7.4], $(\mathfrak{g}(\alpha), \mathfrak{g}(\alpha) \cap \mathfrak{h}) = (\mathfrak{so}(n+1,1), \mathfrak{so}(n,1))$ for some n. Thus $s_\alpha \notin W(\mathfrak{a}; (H_\varepsilon)_0)$.

Since $W(\mathfrak{a}; (H_\varepsilon)_0)$ is generated by the reflections $s_\alpha\,(\alpha \in \Sigma)$ such that $s_\alpha \in W(\mathfrak{a}; (H_\varepsilon)_0)$, $W(\mathfrak{a}; (H_\varepsilon)_0)$ is the Weyl group of the root system

$$\Sigma_\varepsilon = \{\alpha \in \Sigma\,;\, (\mathfrak{g}(\alpha), \mathfrak{g}(\alpha) \cap \mathfrak{h}) \neq (\mathfrak{so}(n+1,1), \mathfrak{so}(n,1)) \text{ for any } n\}.$$

Thus $W(\mathfrak{a}; (H_\varepsilon)_0) = W_\varepsilon$. Since $H_\varepsilon = (H_\varepsilon)_0 Z_{K\cap H}(\mathfrak{a})$, we have $W(\mathfrak{a}; H_\varepsilon) = W_\varepsilon$. □

By Proposition 2.4, we have $W(\mathfrak{a}; H) = W$. Hereafter we fix representatives $\bar{w} \in N_{K\cap H}(\mathfrak{a})$ for all w in W.

2.3. Construction of compact imbedding

Let $\tilde{\mathbb{X}}$ denote the product manifold $G \times \mathbb{R}^l \times W'$. For $s \in \mathbb{R}$ define sgn s to be 1 if $s > 0$, 0 if $s = 0$ and -1 if $s < 0$. For $x = (g, t, w) \in \tilde{\mathbb{X}}$ we define an extended signature of roots ε_x by $\varepsilon_x(\alpha_j) = \text{sgn } t_j$ $(j = 1, \ldots, l)$. We have

$A_{\varepsilon_x},\ W_{\varepsilon_x},\ \Theta_{\varepsilon_x},\ P_{\varepsilon_x},\ P(\varepsilon_x)$ etc., which we write $A_x,\ W_x,\ \Theta_x,\ P_x,\ P(x)$ etc. for short. For $(x,t,w)\in\tilde{\mathbb{X}}$ we define $a(x)\in A^x$ by

$$a(x)=\exp(-\tfrac{1}{2}\textstyle\sum_{t_j\neq 0}\log|t_j|H_j). \tag{2.3}$$

DEFINITION 2.5. We say that two elements $x=(g,t,w)$ and $x'=(g',t',w')$ of $\tilde{\mathbb{X}}$ are equivalent if and only if the following conditions hold.

(i) $\varepsilon_x(w^{-1}\alpha)=\varepsilon_{x'}(w'^{-1}\alpha)$ for any $\alpha\in\Sigma$.
(ii) $w^{-1}w'\in W(x)$.
(iii) $ga(x)P(x)\bar{w}^{-1}=g'a(x')P(x')\bar{w}'^{-1}$.

The condition (i) implies $w\Theta_x=w'\Theta_{x'},\ w\Sigma'_x=w'\Sigma'_{x'}$, and $wW'_{\Theta_x}w^{-1}=w'W'_{\Theta_{x'}}w'^{-1}$. Therefore, under the condition (i), the condition (ii) is equivalent to

$$w^{-1}w'\in W'_{\Theta_x}=W'_{\Theta_{x'}}\quad\text{and}\quad w(\Sigma'_x\cap\Sigma^+)=w'(\Sigma'_{x'}\cap\Sigma^+).$$

Therefore this is in fact an equivalent relation, which we write $x\sim x'$.

Assume that $x,\ x'\in\tilde{\mathbb{X}}$ satisfy the conditions (i) and (ii). The Lie algebra $\mathfrak{p}(x)=\mathfrak{p}(\varepsilon_x)$ equals

$$Z_{\mathfrak{h}}(\mathfrak{a})+\sum_{\alpha_j\in\Psi\setminus\Theta_x}\mathbb{R}H_j+\sum_{\alpha\in\Sigma}\{X+\varepsilon_x(\alpha)\sigma(X)\,;\,X\in\mathfrak{g}^\alpha\},$$

where $Z_{\mathfrak{h}}(\mathfrak{a})$ is a centralizer of $\mathfrak{a}$ in $\mathfrak{h}$. Since $\bar{w}'^{-1}\bar{w}\in H$, it is easy to see that $\mathrm{Ad}(\bar{w}'^{-1}\bar{w})\mathfrak{p}(x)=\mathfrak{p}(x')$. Moreover since $\bar{w}'^{-1}\bar{w}Z_{K\cap H}(\mathfrak{a})\bar{w}^{-1}\bar{w}'=Z_{K\cap H}(\mathfrak{a})$, we have $\bar{w}P(x)\bar{w}^{-1}=\bar{w}'P(x')\bar{w}'^{-1}$. Therefore the condition (iii) is equivalent to

$$ga(x)P(x)=g'a(x')\bar{w}'^{-1}\bar{w}P(x)\quad\text{in } G/P(x).$$

Therefore the equivalent relation is compatible with an action of G on $\tilde{\mathbb{X}}$ given by $g'(g,t,w)=(g'g,t,w)\ (g'\in G)$.

Let $\mathbb{X}$ denote the topological space $\tilde{\mathbb{X}}/\sim$ and let $\pi:\tilde{\mathbb{X}}\to\mathbb{X}$ be the projection. The space $\mathbb{X}$ inherits from $\tilde{\mathbb{X}}$ a continuous action of G, given by $g\pi(x)=\pi(gx)$.

We state the main theorem of this paper:

THEOREM 2.6.

(i) $\mathbb{X}$ *is a compact connected real analytic manifold without boundary.*

(ii) *The action of* G *on* $\mathbb{X}$ *is analytic and the* G*-orbit structure is normal crossing type in the sense of* [O1, Remark 6].

(iii) *For a point* x *in* $\tilde{\mathbb{X}}$, *the orbit* $G\pi(x)$ *is isomorphic to* $G/P(x)$ *and* $\mathbb{X}$ *has the orbital decomposition*

$$\mathbb{X} = \bigsqcup_{\substack{\varepsilon\in\{-1,0,1\}^l \\ w\in W'_\varepsilon}} G\pi(e,\varepsilon,w).$$

(iv) *There are* $|W'|$ *orbits which are isomorphic to* G/H (*also to* $G/P((e,0,1))$). *For a signature of roots* ε *and* $w \in W'_\varepsilon$, *the number of compact orbits in* $\mathbb{X}$ *that is contained in the closure of the open orbit* $G\pi(e,\varepsilon,w) \simeq G/H_\varepsilon$ *equals* $|W(\varepsilon)|$.

REMARK 2.7.

(i) If $(\mathfrak{g},\mathfrak{h})$ is a Riemannian symmetric pair, then the space $\mathbb{X}$ was constructed in [OS1, Section 2] and the above theorem was proved there ([OS1, Theorem 2.6]).

(ii) In [O2, Section 1] Oshima studies a realization of semisimple symmetric spaces. Let X be a semisimple symmetric space and let $\mathbb{X}'$ denote the compact real analytic manifold that is constructed in [O2]. All open orbits in $\mathbb{X}'$ are isomorphic to X. The construction of $\mathbb{X}$ is similar to that of $\mathbb{X}'$. The difference is that $a(x)$ is defined by $\exp\left(-\sum_t \log|t_j| H_j\right)$ in [O2] in place of (2.3).

Example 2.8. For the $\mathbb{R}$-, $\mathbb{C}$- and $\mathbb{H}$-hyperbolic spaces, the space $\mathbb{X}$ is constructed by Sekiguchi [Se, Section 3]. For example, consider the case of the real hyperbolic space. Let $G = SO_0(p,q)$ and $H = SO_0(p,q-1)$ $(p \geq q \geq 1)$. We take $K = SO(p) \times SO(q)$ and $\mathfrak{a} = \mathbb{R}Y$ where $Y = E_{1,p+q} + E_{p+q,1}$, then $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{p} \cap \mathfrak{q}$. We have $\Sigma = \{\pm\alpha\}$ where $\alpha(Y) = 1$ with signature $(m^+(\alpha), m^-(\alpha)) = (p-1, q-1)$. Therefore the rank one symmetric space $X = G/H$ is basic. The space $\mathbb{X}$ has the orbital decomposition $\mathbb{X} = X^+ \cup X^0 \cup X^-$, where $X^+ \simeq X$ and $X^- \simeq SO_0(p,q)/SO_0(p-1,q)$.

§3. Proof of Theorem 2.6

In this section we prove Theorem 2.6. The proof goes in a similar way as the proof of [OS1, Theorem 2.7]. We will give an outline of the proof here.

Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$. Let $\Sigma(\mathfrak{a}_\mathfrak{p})$ be the restricted root system of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$. Let $\mathfrak{g}(\sigma)$ be the reductive Lie algebra generated by

$$\{\mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \lambda)\,;\, \lambda \in \Sigma(\mathfrak{a}_\mathfrak{p}) \text{ with } \lambda|\mathfrak{a} = 0\},$$

where $\mathfrak{g}(\mathfrak{a}_\mathfrak{p}; \lambda)$ denotes the root space for $\lambda \in \Sigma(\mathfrak{a}_\mathfrak{p})$. Put

$$\mathfrak{m}(\sigma) = \{X \in \mathfrak{m}_\sigma\,;\, [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}(\sigma)\}.$$

Let $G(\sigma)$ and $M(\sigma)_0$ denote the analytic subgroups of G corresponding to $\mathfrak{g}(\sigma)$ and $\mathfrak{m}(\sigma)$ respectively and put

$$M(\sigma) = M(\sigma)_0(K \cap \exp \sqrt{-1}\mathfrak{a}_\mathfrak{p}).$$

By [O2, Lemma 1.4] we may assume that the representative $\bar{w}$ of $w \in W$ in $N_K(\mathfrak{a})$ normalize $G(\sigma)$ and $M(\sigma)$ for all $w \in W$.

We fix a basis $\{X_1, \cdots, X_L\}$ so that $X_i \in \mathfrak{g}^{\alpha(i)}$ for some $\alpha(i) \in \Sigma^+$, where $L = \dim \mathfrak{n}_\sigma$. We fix an basis $\{Z_1, \cdots, Z_{L'}\}$ of $\mathfrak{m}_\sigma$ so that $\{Z_1, \cdots, Z_{L''}\}$ is a basis of $\mathfrak{m}(\sigma)$ and $\{Z_{L''+1}, \cdots, Z_{L'}\}$ is a basis of $\mathfrak{g}(\sigma)$, where $L' = \dim \mathfrak{m}_\sigma$ and $L'' = \dim \mathfrak{m}(\sigma)$. Moreover we put $l'' = \dim \mathfrak{a}_\sigma$ and choose $H_{l+1}, \cdots, H_{l''} \in \mathfrak{a}_\sigma \cap \mathfrak{h}$ so that $\{H_1, \cdots, H_l, H_{l+1}, \cdots, H_{l''}\}$ is a basis of $\mathfrak{a}_\sigma$. We put $X_{-i} = \sigma(X_i)$. Then $\{X_{-1}, \cdots, X_{-L}\}$ is a basis of $\mathfrak{n}_\sigma^-$ and

$$\{X_1, \cdots, X_L, X_{-1}, \cdots, X_{-L}, Z_1, \cdots, Z_{L'}, H_1, \cdots, H_{l''}\}$$

forms a basis of $\mathfrak{g}$.

Lemma 3.1. *Fix an element g of G and consider the map*

$$\tilde{\pi}_g : N_\sigma^- \times M(\sigma) \times A^\varepsilon \to G/P(\varepsilon)$$

defined by $\tilde{\pi}_g(n, m, a) = gnmaP(\varepsilon)$.

(i) *The map $\tilde{\pi}_g$ induces an analytic diffeomorphism of $N_\sigma^- \times M(\sigma)/(M(\sigma) \cap H) \times A^\varepsilon$ onto an open subset of $G/P(\varepsilon)$.*

(ii) *For an element Y in $\mathfrak{g}$ let Y_ε be the vector field on $G/P(\varepsilon)$ corresponding to the* 1*-parameter group which is defined by the action* $\exp(tY)$ $(t \in \mathbb{R})$ *on $G/P(\varepsilon)$. For $p = (n, m, a) \in N_\sigma^- \times M(\sigma) \times A^\varepsilon$, we have*

$$(Y_\varepsilon)_{\tilde{\pi}(p)} = d\tilde{\pi}_p \left(\left(\sum_{i=1}^{L} (\varepsilon(\alpha_i) c_i^+(nm) a^{-2\alpha_i} + c_i^-(nm)) \mathrm{Ad}(m) X_{-i} + \sum_{j=1}^{L''} c_j^0(nm) Z_j + \sum_{k=1}^{l} c_k(nm) H_k \right)_p \right).$$

Here X_{-i}, Z_j and H_k are identified with left invariant vector fields on N_σ^-, $M(\sigma)$ and A^ε respectively. Moreover the analytic functions c_i^+, c_i^-, c_j^0 and c_k on G are defined by

$$\mathrm{Ad}(g)^{-1} Y = \sum_{i=1}^{L} (c_i^+(g) X_i + c_i^-(g) X_{-i}) + \sum_{j=1}^{L''} c_j^0(g) Z_j + \sum_{k=1}^{l} c_k(g) H_k$$

for $g \in G$.

PROOF. Notice that $\sigma = \sigma_\varepsilon$ on $M(\sigma)$. We have

$$M(\sigma) \cap H \subset Z_{K \cap H}(\mathfrak{a}) = Z_{K \cap H_\varepsilon}(\mathfrak{a}) \subset H_\varepsilon.$$

Thus $M(\sigma) \cap H \subset M(\sigma) \cap H_\varepsilon$. The inclusion $M(\sigma) \cap H_\varepsilon \subset M(\sigma) \cap H$ can be proved in the same way. Therefore we have $M(\sigma) \cap H = M(\sigma) \cap H_\varepsilon$. Now (i) follows from [O2, Lemma 1.6].

The proof of (ii) can be done in the same way as that of [O2, Lemma 1.6 (ii)], where the statement is proved when ε does not take the value -1. So we omit it. □

For $g \in G$ and $w \in W'$, we define the set U_g^w by

$$U_g^w = \pi((g N_\sigma^- \times M(\sigma)) \times \mathbb{R}^l \times \{w\}).$$

Then Lemma 3.1 shows that the map

$$\phi_g^w : N_\sigma^- \times M(\sigma)/(M(\sigma) \cap H) \times \mathbb{R}^l \to U_g^w \subset \mathbb{X}$$

defined by $(n, m, t) \mapsto \pi((gn\bar{m}, t, w))$ is bijective. We put $U = N_\sigma^- \times M(\sigma)/(M(\sigma) \cap H) \times \mathbb{R}^l$.

LEMMA 3.2. *Fix $g, g' \in G$ and $w, w' \in W'$.*

(i) *For an element Y of $\mathfrak{g}$ the local one parameter group of transformation $(\phi_g^w)^{-1} \circ \exp(tY) \circ \phi_g^w$ $(t \in \mathbb{R})$ defines an analytic vector field on U.*
(ii) *The map $(\phi_{g'}^{w'})^{-1} \circ \phi_g^w$ of $(\phi_g^w)^{-1}(U_g^w \cap U_{g'}^{w'})$ onto $(\phi_{g'}^{w'})^{-1}(U_g^w \cap U_{g'}^{w'})$ defines an analytic diffeomorphism between these open subsets of $\mathbb{R}^l$.*
(iii) *ϕ_g^w is a homeomorphism onto an open subset U_g^w of $\mathbb{X}$.*

PROOF. To prove (i), we may assume that $w = e$. By Lemma 3.1, $Y \in \mathfrak{g}$ determines an analytic vector field on $N_\sigma^- \times M(\sigma)/(M(\sigma) \cap H) \times \mathbb{R}_\varepsilon^l$, because H_k determines the vector field $-2t_k \frac{\partial}{\partial t_k}$ on $\mathbb{R}_\varepsilon^l$ by the correspondence $t \mapsto a(t)$. Here $\mathbb{R}_\varepsilon^l$ denotes the set $\{t \in \mathbb{R}^l\,;\, t_j = 0 \quad \text{if } \varepsilon(\alpha_j) = 0\}$. They piece together and define an analytic vector field on U.

We can prove (ii) and (iii) in the same way as the proof of [O2, Lemma 1.9] and [OS1, Lemma 2.8]. So we omit it. □

We put $V = \{t \in \mathbb{R}^l\,;\, t^\alpha < 1 \text{ for all } \alpha \in \Sigma^+\}$. Since $(gkm, t, w) \sim (gk, t, w)$ for any $g \in G,\ k \in K,\ m \in Z_{K\cap H}(\mathfrak{a}),\ t \in \mathbb{R}^l$ and $w \in W'$, we can define the map

$$\psi_g^w : K/Z_{K\cap H}(\mathfrak{a}) \times V \to \mathbb{X}$$

by $(kZ_{K\cap H}(\mathfrak{a}), t) \mapsto \pi((gk, t, w))$.

LEMMA 3.3. *For any $g,\ g' \in G$ and $w \in W'$, the map*

$$(\phi_{g'}^{w'})^{-1} \circ \psi_g^w : (\psi_g^w)^{-1}(\operatorname{Im}\psi_g^w \cap U_{g'}^{w'}) \mapsto (\phi_{g'}^{w'})^{-1}(\operatorname{Im}\psi_g^w \cap U_{g'}^{w'})$$

is an analytic diffeomorphism between the open subsets of $K/Z_{K\cap H}(\mathfrak{a}) \times V$ and U.

Proof. We fix an arbitrary point x in $(\psi_g^w)^{-1}(\mathrm{Im}\,\psi_g^w \cap U_{g'}^{w'})$. We can prove in the same way as the proof of [OS1, Lemma 2.9] that the differential of the map $(\phi_{g'}^{w'})^{-1} \circ \psi_g^w$ at x is bijective, hence the map $(\phi_{g'}^{w'})^{-1} \circ \psi_g^w$ is an analytic local isomorphism between open subsets. The injectivity of the map also can be proved in the same way as the proof of [OS1, Lemma 2.9] by using the Cartan decomposition [Sc, Proposition 7.1.3]. So we do not give the proof in detail here. □

Proof of Theorem 2.6. It remains to prove that $\mathbb{X}$ is connected, compact and Hausdorff. The proof can be done in the same way as the proof of [OS1, Theorem 2.7] by using Lemma 2.3, Lemma 3.2, Lemma 3.3 and the Cartan decomposition [Sc, Proposition 7.1.3]. So we omit it. □

The following are easy consequences of Theorem 2.6 and Lemma 3.3.

Corollary 3.4. *For a signature ε of roots and an element w of W', we put $\mathbb{X}_\varepsilon^w = \pi(G \times \{\varepsilon(\alpha_1), \cdots, \varepsilon(\alpha_l)\} \times \{w\})$ and $B_w = \pi(G \times \{0\} \times \{w\})$. Then we have natural identifications $G/H_\varepsilon \simeq \mathbb{X}_\varepsilon^w$ and $G/P_\sigma \simeq B_w$. Moreover B_w is contained in the closure of $\mathbb{X}_\varepsilon^1$ if and only if $w \in W(\varepsilon)$.*

Corollary 3.5. *The map*

$$\psi_g^w : K/Z_{K\cap H}(\mathfrak{a}) \times V \ni (kZ_{K\cap H}(\mathfrak{a}), t) \mapsto \pi((gk, t, w)) \in \mathbb{X}$$

is an analytic diffeomorphism and $\bigcup_{g\in G,\, w\in W'} \mathrm{Im}\,\psi_g^w$ *is an open covering of* $\mathbb{X}$.

§4. Invariant differential operators

In this section we shall show that the system of invariant differential equations on G/H_ε extends analytically on $\mathbb{X}$ and has regular singularities in the weak sense along the boundaries. For the notion of the systems of differential equations with regular singularities we refer [KO], [OS1] and [Sc]. First we recall after [O2] and [Sc] on the structure of the algebra of invariant differential operators on G/H_ε.

For a real or complex Lie subalgebra $\mathfrak{u}$ of $\mathfrak{g}_\mathbb{C}$ let $U(\mathfrak{u})$ denote the universal enveloping algebra of $\mathfrak{u}'$, where $\mathfrak{u}'$ is the complex subalgebra of $\mathfrak{g}_\mathbb{C}$ generated by $\mathfrak{u}$.

Retain the notation of Section 1. Let $\mathfrak{j}$ be a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}$. Then by the definition of σ_ε, $\mathfrak{j}$ is also a maximal abelian subspace of $\mathfrak{q}_\varepsilon$. Let $\Sigma(\mathfrak{j})$ denote the root system for the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$. Let $\Sigma(\mathfrak{j})^+$ denote the set of positive roots with respect to a compatible orders for $\Sigma(\mathfrak{j})$ and Σ. Put $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma(\mathfrak{j})^+}\alpha$. Let $\mathfrak{n}_\mathbb{C}$ be the nilpotent subalgebra of $\mathfrak{g}_\mathbb{C}$ corresponding to $\Sigma(\mathfrak{j}_\mathbb{C})^+$ and put $\mathfrak{n}_\mathbb{C}^- = \sigma(\mathfrak{n}_\mathbb{C})$.

From the Iwasawa decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{n}_\mathbb{C}^- \oplus \mathfrak{j}_\mathbb{C} \oplus (\mathfrak{h}_\varepsilon)_\mathbb{C}$ and the Poincaré-Birkoff-Witt theorem it follows that

$$U(\mathfrak{g}) = (\mathfrak{n}_\mathbb{C}^- U(\mathfrak{g}) + U(\mathfrak{g})(\mathfrak{h}_\varepsilon)_\mathbb{C}) \oplus U(\mathfrak{j}).$$

Let δ_ε be the projection of $U(\mathfrak{g})$ to $U(\mathfrak{j})$ with respect to this decomposition. Let η be the algebra automorphism of $U(\mathfrak{j})$ generated by $\eta(Y) = Y - \rho(Y)$ for $Y \in \mathfrak{j}$ and put $\tilde{\gamma}_\varepsilon = \eta \circ \delta_\varepsilon$. Then the map $\tilde{\gamma}_\varepsilon$ induces an isomorphism:

$$\gamma_\varepsilon : U(\mathfrak{g})^{\mathfrak{h}_\varepsilon}/(U(\mathfrak{g})^{\mathfrak{h}_\varepsilon} \cap U(\mathfrak{g})(\mathfrak{h}_\varepsilon)_\mathbb{C}) \xrightarrow{\sim} U(\mathfrak{j})^{W(\mathfrak{j})},$$

where $U(\mathfrak{g})^{\mathfrak{h}_\varepsilon}$ is the set of $\mathfrak{h}_\varepsilon$-invariant elements in $U(\mathfrak{h}_\varepsilon)$ and $U(\mathfrak{j})^{W(\mathfrak{j})}$ is the set of the elements in $U(\mathfrak{j})$ that are invariant under the Weyl group $W(\mathfrak{j})$ of $\Sigma(\mathfrak{j})$.

Let $\mathbb{D}(G/H_\varepsilon)$ denote the algebra of invariant differential operators on G/H_ε. Since $\mathbb{D}(G/H_\varepsilon) \simeq U(\mathfrak{g})^{\mathfrak{h}_\varepsilon}/(U(\mathfrak{g})^{\mathfrak{h}_\varepsilon} \cap U(\mathfrak{g})(\mathfrak{h}_\varepsilon)_\mathbb{C})$ (c.f. [O2, P 618]), we have the algebra isomorphism:

$$\gamma_\varepsilon : \mathbb{D}(G/H_\varepsilon) \xrightarrow{\sim} U(\mathfrak{j})^{W(\mathfrak{j})} \tag{4.1}$$

Let w be an element in W' and ε be a signature of roots. Put $\mathbb{X}_\varepsilon^w = G\pi(e, \varepsilon, w)$ and let

$$\iota_\varepsilon^w : G/H_\varepsilon \xrightarrow{\sim} \mathbb{X}_\varepsilon^w$$

be the natural isomorphism. Let $\mathbb{D}(\mathbb{X})$ denote the algebra of G-invariant differential operators on $\mathbb{X}$ whose coefficients are analytic.

Proposition 4.1.

(i) *There exists a surjective algebra isomorphism*

$$\gamma : \mathbb{D}(\mathbb{X}) \to U(\mathfrak{j})^{W(\mathfrak{j})}$$

that is given by $\gamma(D) = \gamma_\varepsilon \circ (\iota_\varepsilon^w)^{-1}(D|\mathbb{X}_\varepsilon^w)$, *which does not depend on the choice of* $w \in W'$ *and* $\varepsilon \in \{\pm 1\}^l$.

(ii) *The system of invariant differential equations*

$$\mathcal{M}_\lambda : (D - \gamma(D)(\lambda))u = 0 \quad \text{for all } D \in \mathbb{D}(\mathbb{X})$$

has regular singularities in the weak sense along the set of walls $\{\pi(G\{(e,t,w)\,;\,t_j = 0\}\,;\,j = 1,\ldots,l\}$ *with the edge* $\pi(G(e,0,w))$ *for each* $w \in W'$. *The set of characteristic exponents of* $\mathcal{M}_\lambda$ *is* $\{s_{w\lambda} = (s_{w\lambda,i})_{1\le i\le l}\}$, *where* $s_{w\lambda,i} = \frac{1}{2}(\rho - \lambda)(H_i)$.

PROOF. The proof can be done in a similar way with the proof of Proposition 2.26 and Lemma 2.28 in [OS1] (c.f. [O2]). So we omit it. □

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