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Complementary Note on Similitudes of Forms

By Nguyễn Quốc THẮNG

1. In [T1] we considered non-degenarate (skew-) hermitian forms Φ over division algebras over local or global fields and computed the group of multiplicators (i.e. similarity factors) $\mathrm{M}(\Phi)$ of Φ. In [T2] we complemented [T1] by considering the group of special (or proper) multiplicators of forms of type D. One of key points of the approach taken in [T1] is that we have the analogue of a lemma due to Dieudonne for J-hermitian forms over D, where D is either a separable quadratic extension of a field k of characteristic $\neq 2$ or a quaternion division algebra with center k and J is the standard involution of D. In the proof of Proposition 1 of [T1] we implicitly assumed that we can choose a *non-degenerate* plane P (see [T1] for notation used further) such that P and $g(P)$ are isometric, where g is a similitude of Φ. Since this fact is not obvious and there is also a request about the proof, we would like to present one here.

Denote by V the underlying right D-vector space for Φ. If V is isotropic then any hyperbolic plane can be taken as P. Otherwise we assume that V is anisotropic. We may assume that g is not a homothetia. It follows that there is $x \in V$, $x \neq 0$, such that x and $g(x)$ are not collinear. Denote by $(.,.)$ the sesquilinear form associated with Φ. If x and $g(x)$ are orthogonal, then the restriction of Φ to the plane P spanned by x and $g(x)$ is non-degenerate and we are done. Hence we may assume that x and $g(x)$ are not orthogonal.

We assume first that the multiplicator λ of g is a norm from D to k, say, $\lambda = aa^J$ for some $a \in D^*$. We prove by induction on the dimension n of V. If $n = 2$ there is nothing to prove. Assume that $n \geq 3$. We have

$$(g(x), g(x)) = \lambda(x, x) = (xa, xa).$$

Hence by Witt's Theorem, there is $u \in \mathrm{U}(\Phi, D)$ such that $u(g(x)) = xa$. It is clear that we may replace g by ug, hence we may assume that $g(x) = xa$. Then g maps the orthogonal complement $W := \langle x\rangle^\perp$ onto itself, so we can apply the induction hypothesis to W to finish the proof.

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Now we assume that λ is not a norm from D. With x, P above let A be the matrix of Φ restricted to P. Then one checks that P and $g(P)$ are isometric and a computation shows that

$$Nrd(A) = \{((x,x)(g(x),g(x)) - (x,g(x))(x,g(x))^J\}^2 = (\lambda b^2 - cc^J)^2,$$

where $b = (x,x) \in k^*$, $c = (x,g(x)) \in D^*$. Since λ is not a norm from D it follows that $\lambda \neq (c/b)(c/b)^J$ i.e. $Nrd(A) \neq 0$.

2. Recently there is an extensive study of the group of similitudes and their groups of multiplicators in connections with the theory of division algebras with involution, algebraic K-theory and algebraic geometry (see e.g. [MT], where one can find some problems that were considered also in [T1]). However, up to now, one considered only the similitudes of forms over a matrix algebra over a division algebra. Over an arbitrary associative algebra A of finite dimension over a field k of characteristic $\neq 2$ with a k-linear involution J, following [BL], one may define the algebraic k-group *norm-one-group* U_A of A with respect to J by

$$\mathrm{U}_A(L) = \{a \in A_L : aa^J = 1\},$$

for any field extension L of k. We now define in similar way the k-group of similitudes of A with respect to J by

$$\mathrm{GU}_A(L) = \{a \in A_L : aa^J \in L^*\}$$

for any field extension L of k. The image of $\mathrm{GU}_A(L)$ in L^* is denoted by $\mathrm{M}_A(L)$ which is called the *group of multiplicators of A with respect to J over L*. We may define similar notions while restricting to the connected components of the algebraic groups defined above to have the *special norm-one k-group* SU_A*, group of special similitudes* GU_A^+ *and the group* M_A^+ *of special multiplicators.* One may pose the problem of studying these groups in general, determine their structure, etc... especially for some classes of algebras which occur most often in the practice. In genereal this problem is a dificult one since even in the case when A is a central simple finite dimensional algebra the problem is non-trivial. In particular, we would like to state the following

Problem. Let A be a semisimple algebra of a finite dimension over a local or global field k of characteristic 0 with non-trivial k-involution J.

Determine the structure of the groups (defined by mean of J) GU_A, GU_A^+, $\mathrm{M}_A(k)$, and $\mathrm{M}_A^+(k)$.

In general, it is possible to determine the structure of the group $\mathrm{M}_A^+(k)$, since the group SU_A satisfies the cohomological Hasse principle in dimension 1, so we may apply the appproach adopted in [T2], by reducing the problem to the local field case.

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Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario
Canada L8S 4K1
E-mail: nguyen@math.icarus.mcmaster.ca