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Minimal Discrepancy for a Terminal cDV Singularity Is 1

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Abstract. An answer to a question raised by Shokurov on the minimal discrepancy of a terminal singularity of index 1 is given. It is proved that the minimal discrepancy is 1 (it is 2 for a non-singular point and 0 for all other canonical singularities of index 1). A rough classification of terminal singularities of index 1 based on finding certain low degree monomials in their equations, and the toric techniques of weighted blow ups are used. This result has been generalized to terminal singularities of index $r > 1$ by Y.Kawamata; his theorem states that the minimal discrepancy is $1/r$.

This note provides a proof for the following fact cited by Shokurov in [Sho], Remark (4.10.2), with a reference to my verbal communication.

Theorem 0.1. *Let (Y, P) be a three-dimensional isolated compound Du Val (cDV) singularity. For any resolution $\pi : (\tilde{Y}, P) \longrightarrow (Y, P)$, let $E = \bigcup_{i=1}^{i=m} E_i$ denote its exceptional locus, $(E = \pi^{-1}(P))$, $E_i (i = 1, \ldots, m)$ being its irreducible components. The discrepancy coefficients a_j are determined by the formula*

$$K_{\tilde{Y}} = \pi^* K_Y + \sum_{\operatorname{codim}_{\tilde{Y}} E_j = 1} a_j E_j \ ,$$

and when $\operatorname{codim}_{\tilde{Y}} E$ *is 1*

$$\operatorname{mdc}(\pi) = \min_{\operatorname{codim}_{\tilde{Y}} E_j = 1} a_j$$

denotes the minimal discrepancy coefficient of π. Then there exists a resolution π with at least one exceptional component of codimension 1, such that $\operatorname{mdc}(\pi) = 1$.

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A generalization of this theorem to terminal singularities of index $r > 1$ was obtained by Kawamata [Kaw]. It states that any resolution contains an exceptional divisor of discrepancy $1/r$.

1. Reminder on terminal singularities

DEFINITION 1.1. A cDV singularity is a germ of an algebraic variety (or of an analytic space) (Y, P) which is formally equivalent to the germ of a hypersurface singularity $(\{f = 0\}, 0)$ in the affine space $\mathbf{A}^4$, where

$$f(t, x, y, z) = f_{X_n}(t, x, y) + zg(t, x, y, z), \tag{1.1}$$

where X_n stands for A_n, D_n or E_n, and f_{X_n} is one of the following polynomials:

$$\begin{aligned} f_{A_n} &= t^2 + x^2 + y^{n+1} \;\; (n \geq 1) \\ f_{D_n} &= t^2 + x^2 y + y^{n-1} \;\; (n \geq 4) \\ f_{E_6} &= t^2 + x^3 + y^4 \\ f_{E_7} &= t^2 + x^3 + xy^3 \\ f_{E_8} &= t^2 + x^3 + y^5 \;. \end{aligned}$$

Let us order the symbols A_n, D_k, E_l by

$$\begin{gathered} A_n < D_k < E_l \;\; \forall\, n \geq 1 \,\forall\, k \geq 4 \,\forall\, l = 6, 7, 8 \\ X_n < X_m \;\; \forall\, n < m \,\forall\, X = A, D, E. \end{gathered}$$

The singularity (Y, P) is said to be cX_n if X_n is minimal in a representation of (Y, P) by equation (1.1).

According to Reid [Reid-1], the isolated cDV-points are exactly terminal singularities of index 1; this implies in particular that the minimal discrepancy coefficient is positive in any resolution having at least one exceptional divisor. Remark, that the singularities $f_{X_n} = 0$, where X_n runs over the symbols $A_n (n \geq 1), D_n (n \geq 4), E_6, E_7, E_8$, are exactly *canonical* singularities in dimension 2 up to analytic equivalence; 'canonical' means that all the discrepancies a_j are non-negative. Look [Reid-2] for further properties of these and related classes of singularities. We state here for future use a criterion for a hypersurface singularity to be canonical.

THEOREM 1.2. *A necessary condition for a hypersurface $\{f = 0\} \subset k^n$, $f = \sum a_m x^m$, to have a canonical singularity at zero is that the point*

$(1,\ldots,1)$ *lies above the Newton diagram* $\Delta(f)$ *of the function* f. *The condition is also sufficient provided* f *is a non-degenerate series in the sense of Khovanskiĭ, that is for any face* $\Delta \prec \Delta(f)$, *the polynomial* $f_\Delta = \sum_{m\in\Delta} a_m x^m$ *defines a non-singular (maybe empty) hypersurface in* $(k^*)^n$.

Proof. See [Mar-2], Theorem 3, and also [Reid-2] for the "necessary" part. In fact, the sufficiency follows immediately from the structure of the Khovanskiĭ embedded toric resolution of a non-degenerate singularity [Kho]: in any coordinate patch of this resolution the exceptional locus Γ is either empty, or its irreducible components Γ satisfy the hypotheses of Proposition 2.3 below, and $d_\Gamma = 1$ since the intersection $\Gamma \cap (k^*)^n$ is non-singular by the non-degeneracy assumption. So the non-negativity of the discrepancy a_Γ implies $a_\alpha \geq 0$ (in the notation of Proposition 2.3), which is equivalent to saying that the point $(1,\ldots,1)$ lies above the face Δ. □

Proposition 1.3. *Let* (Y,P) *be an isolated cDV singularity. Then it is formally equivalent to a hypersurface singularity* $(\{f=0\},0)$, *where* f *is one of the following polynomials:*

(i) $f = t^2+x^2+y^2+z^n$ $(n\geq 2)$ *if* (Y,P) *is* cA_1;

(ii) $f = t^2+x^2+g(y,z)$, *where* $j_2 g = 0$, *if* (Y,P) *is* cA_n $(n \geq 2)$;

(iii) $f = t^2+g(x,y,z)$, *where* $j_2 g = 0$ *and* $g_3(x,y,z)$ *is not divisible by a square of a linear form, if* (Y,P) *is* cD_4;

(iv) $f = t^2+x^2y+g(x,y,z)$, *where* $j_3 g = 0$, *if* (Y,P) *is* cD_n $(n\geq 5)$;

(v) $f = t^2+x^3+g(x,y,z)$, *where* $j_3 g = 0$ *and* $j_5 g = g_4+g_5$ *contains at least one of the monomials*

$$z^4, yz^3, y^2z^2, z^5, yz^4, y^2z^3, xz^3, xyz^2 \tag{1.2}$$

with a non-zero coefficient, if (Y,P) *is* cE_n $(n = 6,7,8)$.

(We denote by $j_k g$ *the* k*-th jet of* g, *and by* g_k *the homogeneous component of degree* k *of* g*).*

Proof. (i), (ii), (iii) and (iv) are easy consequencies of the Morse Lemma and Definition 1.1. (v) follows from the following Proposition. □

Proposition 1.4. *Assume that the equation* $f=0$, *where*

$$f = t^2+x^3+g(x,y,z) \quad (j_3 g = 0) \tag{1.3}$$

defines an isolated singularity at $0 \in A^4$*. Then it is a* cE_n *point, if and only if* g *contains, possibly after a permutation of* y, z*, one of the monomials (1.2).*

PROOF. For reader's convenience, I reproduce the proof given in [Mar-1]; see also Corollary 3 in [Mar-2]. □

Sufficiency. By a change of variables $y \to y + az$, one can reduce the problem to the case when g contains one of the monomials z^4, xz^3, z^5. If the coefficient of z^4 is non-zero, then after a homothety, we have

$$t^2 + x^3 + g(x, 0, z) = t^2 + x^3 + z^4 + \eta(t, x, z), \tag{1.4}$$

where the exponents of all the monomials of η lie above the Newton diagram of $f_{E_6}(t, x, z) = t^2 + x^3 + z^4$. By Lemma in Sect. 2 of [Mar-2], the function (1.4) is formally equivalent to f_{E_6}, hence (1.3) defines a cDV singularity whose hyperplane section $y = 0$ is E_6, hence it is of type $\leq cE_6$. As it is neither cA_n, nor cD_n, it is cE_6. The cases when g contains the sum $c_1 z^4 + c_2 xz^3 + c_3 z^5$ with $c_1 = 0, c_2 \neq 0$ or $c_1 = c_2 = 0, c_3 \neq 0$ are considered in a similar way.

Necessity. Suppose that all the monomials (1.2) and those obtained by the permutation $y \leftrightarrow z$ have zero coefficients in g. Then f has the following form:

$$f = t^2 + x^3 + \sum_{k=4}^{5} \sum_{\substack{a+b+c=k \\ a \geq 6-k}} A_{abc} x^a y^b z^c + f_{>5}(x, y, z) \tag{1.5}$$

We should verify that the generic section of the hypersurface $f = 0$ by a plane $u = 0$, where $u = \alpha_1 t + \alpha_2 x + \alpha_3 y + \alpha_4 z$ is a linear form, is a non-canonical singularity. Apply the coordinate change $t \to t, x \to x, y \to \frac{1}{\alpha_3} u, z \to z$ in (1.5). In new coordinates,

$$f = t^2 + x^3 + \sum_{k \geq 4} \sum_{\substack{a+b+c+d=k \\ a \geq \max\{0, 6-k\}}} A_{abcd} x^a y^b z^c t^d \tag{1.6}$$

The hyperplane section $u = 0$ becomes $y = 0$ in new coordinates, and substituting $y = 0$ into (1.6), we obtain the surface singularity $\phi(t, x, z) = 0$, where

$$\phi = t^2 + x^3 + \sum_{k\geq 4} \sum_{\substack{a+c+d=k \\ a\geq \max\{0,6-k\}}} A_{a0cd}x^a z^c t^d. \tag{1.7}$$

Hence, there exists a face Δ of the Newton diagram of f spanned by the exponents of three monomials t^2, x^3 and $x^a z^c t^d$ such that $A_{a0cd} \neq 0$. Let $w = (w_1, w_2, w_3)$ be the normal of Δ normalized so that $< w, m >= 1$ for $m \in \Delta$. Then we have $w_1 = 1/2, w_2 = 1/3, w_3 = \frac{1}{c}(1 - \frac{a}{3} - \frac{d}{2})$. As w_3 should be positive, we have very few possibilities for the values of a, d. In the case when $a = d = 0$, we have $k = a + c + d \geq 6$, hence $c = k \geq 6$, and $|w| = w_1 + w_2 + w_3 \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$. This is equivalent to say that the point $(1, \ldots 1)$ lies on or under Δ, hence, by Theorem 1.2, the singularity is non-canonical. If $d = 1, a = 0$, then $k \geq 6, c = k - 1 \geq 5$, and $w_3 \leq \frac{1}{10}$. If $a = 1, d = 0$, we have $k \geq 5$, and $w_3 \leq \frac{2}{3c} \leq \frac{1}{6}$. If $a = 1, d = 1$, we have $k \geq 5, c = k - 2 \geq 3$, and $w_3 \leq \frac{1}{6c} \leq \frac{1}{18}$. If $a = 2, d = 0$, then $k \geq 4, c \geq 2$, and $w_3 \leq \frac{1}{3c} \leq \frac{1}{6}$. In all the cases, $|w| \leq 1$, hence the singularity is non-canonical.

2. Weighted blow ups

We fix the lattice $N = \mathbf{Z}^n \subset V = \mathbf{R}^n$ and the coordinate octant $\tau = \mathbf{R}^n_+ = \{(y_1, \ldots, y_n) \in \mathbf{R}^n | y_i \geq 0 \ \forall\ i\}$. Then the affine space $\mathbf{A}^n$ can be thought of as the toric variety

$$X_\tau = X_{V,N,\tau} := \mathrm{Spec}k[\tau^* \cap N^*],$$

where τ^*, N^* denote the dual objects in the dual $\mathbf{R}$-vector space $W = V^* \simeq \mathbf{R}^n$:

$$M = N^* = \{w \in W | w(N) \subset \mathbf{Z}\}$$
$$\tau^* = \{w \in W | w_{|\tau} \geq 0\}.$$

See, e.g. [Da] for more details on toric varieties.

Definition 2.1. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in N \cap \mathrm{Int}\ (\tau)$ be a primitive lattice vector in the interior of τ. The weighted blow up $\sigma_\alpha : \mathbf{A}^n_\alpha \longrightarrow \mathbf{A}^n$ is the toric morphism defined by the subdivision of the standard coordinate octant τ into a minimal fan having the ray $\mathbf{R}_+ \cdot \alpha$ as one of its edges. The n-dimensional cones of this fan are

$$\Sigma_n = \{< \alpha, e_2, \ldots, e_n >, < e_1, \alpha, \ldots, e_n >, \ldots, < e_1, e_2, \ldots, \alpha >\},$$

and the fan itself is the union of Σ_n and the set of all the faces of the cones from Σ_n.

The discrete valuation $v_\alpha = \mathrm{ord}_{E_\alpha}$ of the function field $k(\mathbf{A}^n) = k(y_1, \ldots y_n)$ associated to the prime exceptional divisor E_α of σ_α is given by the formula

$$v_\alpha(y^m) =< \alpha, m >,$$

where $m \in M$, $y^m = y_1^{m_1} \cdots y_n^{m_n}$, and $< \, , \, >$ denotes the natural coupling between M and N. For a function $f = \sum_{m \in M} a_m y^m$ we have

$$v_\alpha(f) = \min_{a_m \neq 0} v_\alpha(x^m) = \min_{a_m \neq 0} < \alpha, m > . \tag{2.1}$$

Let $Y = \{f = 0\}$ be a hypersurface in $\mathbf{A}^n$, and $Y_\alpha \subset \mathbf{A}^n_\alpha$ its proper transform in $\mathbf{A}^n_\alpha$. Let Γ be any component of $Y_\alpha \cap E_\alpha$ of dimension $n-2$ such that Y_α is normal at the generic point of Γ. Then E_α is Cartier at the generic point of Γ, and the multiplicity $d = d_\Gamma$ in $E_\alpha|_{Y_\alpha} = d\Gamma$ is well defined. Let $\tilde{v}_\Gamma$ be the valuation on $k(Y_\alpha)$ induced by v_α:

$$\tilde{v}_\Gamma(h) = \min_{\tilde{h}|_{Y_\alpha} = h, \tilde{h} \in k(\mathbf{A}^n_\alpha)} v_\alpha(\tilde{h}), \; h \in k(Y_\alpha).$$

Then we have

LEMMA 2.2. $\tilde{v}_\Gamma(h) = \left[\frac{1}{d_\Gamma} v_\Gamma(h)\right]$.

PROOF. Let t be a local parameter of $\mathcal{O}_{Y_\alpha, \Gamma}$, and z that of $\mathcal{O}_{\mathbf{A}_\alpha, E_\alpha}$. One can choose z in such a way that $vz = t^{d_\Gamma}$ with v invertible in $\mathcal{O}_{\mathbf{A}_\alpha, \Gamma}$. For any $h \in k(Y_\alpha)$ we can write $h = ut^k$ with u invertible in $\mathcal{O}_{\mathbf{A}_\alpha, \Gamma}$, then $k = v_\Gamma(h)$, and we are done. □

Now, let

$$\omega_0 = \mathrm{res}_Y \left(\frac{dy_1 \wedge \ldots \wedge dy_n}{f} \right)$$

be a base of $\Gamma(Y, \omega_Y)$. The valuation v_α, and hence $\tilde{v}_\Gamma$, extends in an obvious way to the canonical differentials. We have

PROPOSITION 2.3. *If Γ is not a toric subvariety of $\mathbf{A}^n_\alpha$, then the following formula holds:*

$$v_\Gamma(\sigma^*_\alpha \omega_0) = a_\alpha d_\Gamma,$$

where $\sigma^*_\alpha \omega_0$ is the lift of ω_0 to the weighted blow up, $a_\alpha = -v_\alpha(f) + |\alpha| - 1$, and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

PROOF. It is well-known that the form of the canonical differential

$$\nu = \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_n}{y_n}$$

is invariant up to a multiplicative constant under toric changes of variables. This implies that $\mathrm{ord}_D \nu = -1$ for any toric divisor D, in particular, for $D = E_\alpha$ we have $v_\alpha(\nu) = -1$. Hence

$$v_\alpha\left(\frac{dy_1 \wedge \ldots \wedge dy_n}{f}\right) = -v_\alpha(f) + v_\alpha(y_1 \cdots y_n) + v_\alpha(\nu) = a_\alpha.$$

Now, let $X_\sigma \simeq (\mathbf{A}^1 \setminus \{0\})^{n-1} \times \mathbf{A}^1 \subset X_\Sigma$ be the open subset corresponding to the one-dimensional cone $\sigma = \mathbf{R}_+ \cdot \alpha \in \Sigma$. The exceptional divisor $E_\alpha \cap X_\sigma = (\mathbf{A}^1 \setminus \{0\})^{n-1}$ is given by $z_n = 0$. We can choose any coordinate system $z_1 = x^{m^{(1)}}, \ldots, z_n = x^{m^{(n)}}$ associated to a basis of M of the following form: $m^{(1)}, \ldots, m^{(n-1)}$ is a basis of $M \cap \alpha^\perp$, and $m^{(n)} \in \mathrm{Int}\sigma^* \cap M$ completes it to a basis of M. Then

$$f = z_n^N f_0(z_1, \ldots, z_n), \quad N = v_\alpha(f),$$
$$f_0(z_1, \ldots, z_n) = g_0(z_1, \ldots, z_{n-1}) + z_n g_1(z_1, \ldots, z_{n-1}) + \ldots$$

so, Y_α is defined by the equation $f_0 = 0$. As $X_\sigma \cap \{z_n = 0\}$ is an open subset of E_α whose complement in E_α is a union of toric subvarieties, we see that, by our hypotheses, the intersection

$$E_\alpha \cap Y_\alpha \cap X_\sigma = \{z_n = g_0(z_1, \ldots, z_{n-1}) = 0\} \subset (\mathbf{A}^1 \setminus \{0\})^{n-1}$$

is non-empty and contains a component Γ of multiplicity d_Γ. We have

$$\frac{dy_1 \wedge \ldots \wedge dy_n}{f} = u \frac{z_n^{-N+|\alpha|-1}}{f_0} dz_1 \wedge \ldots \wedge dz_n$$

with u invertible on X_σ, which implies the result. $\square$

Remark 2.4. If the hyperplane $H = \{w \in W| < \alpha, m >= v_\alpha(f)\}$ contains a $(n-1)$-dimensional face of the Newton diagram of f, then all the components of $E_\alpha \cap Y_\alpha$ are non-toric.

3. Proof of Theorem 0.1

Let $(Y, P) = (\{f = 0\}, 0)$ be an isolated cDV singularity defined by one of the equations (i)–(v) of Proposition 1.3. We will use the notations of Section 2 in the case $n = 4$ with coordinates $(y_1, y_2, y_3, y_4) = (t, x, y, z)$.

Definition 3.1. A vector $\alpha \in N \cap \operatorname{Int} \tau$ is called an admissible weight for the equation f, if $E_\alpha \cap Y_\alpha$ contains at least one simple non-toric component Γ and $a_\alpha = -v_\alpha(f) + |\alpha| - 1 = 1$.

If α is admissible, then Y is normal at the generic point of Γ, $d_\Gamma = 1$, and by Proposition 2.3, we have $v_\Gamma(\sigma_\alpha^* \omega_0) = 1$. But the orders of $\sigma_\alpha^* \omega_0$ on prime exceptional divisors are exactly the discrepancy coefficients, so for the partial resolution $\sigma_\alpha : Y_\alpha \longrightarrow Y$ we have an exceptional divisor Γ with discrepancy $a_\Gamma = 1$. Then any resolution of Y which dominates σ_α has an exceptional divisor of discrepancy 1.

The following theorem gives a list of admissible weights for all the cDV singularities.

Theorem 3.2. *The following weights are admissible for the singularity (Y, P) defined by one of the equations (i)–(v) of Proposition 1.3, after an eventual linear change of coordinates (y_2, y_3, y_4):*

(1) $\alpha = (1, 1, 1, 1)$ in the case cA_n $(n \geq 1)$;

(2) $\alpha = (2, 1, 1, 1)$ in the case cD_4;

(3) $\alpha = (2, 1, 2, 1)$ in the case cD_n $(n \geq 5)$;

(4) $\alpha = (3, 2, 1, 2)$ in the case cE_n, if f does not contain any one of the monomials y_3^4, y_3^5;

(5) $\alpha = (2, 2, 1, 1)$ in the case cE_n, if $g_4(0, y_3, y_4) \neq 0$;

(6) $\alpha = (3, 2, 1, \epsilon)$ with ϵ =1 or 2 in the case cE_n, if $g_4(0, y_3, y_4) = 0$ and g_5 contains y_3^5.

PROOF. (1) $f = t^2 + x^2 + y^2 + z^n$ $(n \geq 1)$ or $f = t^2 + x^2 + g(y, z)$ with $j_2 g = 0$; $\alpha = (1, 1, 1, 1)$. Make an ordinary blow up $\sigma = \sigma_{(1,1,1,1)} : \tilde{\mathbf{A}}^4 \longrightarrow \mathbf{A}^4$:

$$y_1 = z_4 z_1, y_2 = z_4 z_2, y_3 = z_4 z_3, y_4 = z_4 .$$

We have:

$$\begin{gathered}\sigma^* f = z_4^N f_0,\ N = v_\alpha(f) = 2,\ |\alpha| = 4,\ a_\alpha = -v_\alpha(f) + |\alpha| - 1 = 1, \\ f_0(z_1, z_2, z_3, z_4) = z_1^2 + z_2^2 + z_3^2 + z_4^{n-2} \text{ or } \ z_1^2 + z_2^2 + z_4 \tilde{g}(z_3, z_4), \\ E_\alpha \cap Y_\alpha = \{z_1^2 + z_2^2 + z_3^2 = z_4 = 0\} \text{ or } \ \{z_1^2 + z_2^2 = z_4 = 0\}\end{gathered}$$

In the first case the last intersection is a simple irreducible non-toric divisor, and in the second it is the union of two simple irreducible non-toric divisors $\Gamma_1 \cup \Gamma_2$.

(2) $f = y_1^2 + g(y_2, y_3, y_4)$, $g = g_3 + g_4 + \ldots$, g_3 is not divisible by the square of a linear form; $\alpha = (2, 1, 1, 1)$. Look at the open subset $X_\sigma \subset \mathbf{A}^4_\alpha$ defined in the proof of Proposition 2.3 and choose coordinates on X_σ as indicated there, for example,

$$z_1 = y_1 y_2^{-2}, z_2 = y_2 y_3^{-1}, z_3 = y_3 y_4^{-1}, z_4 = y_2 .$$

We have:

$$\begin{gathered}\sigma_\alpha^* f = z_4^N f_0,\ N = v_\alpha(f) = 3,\ |\alpha| = 5,\ a_\alpha = -v_\alpha(f) + |\alpha| - 1 = 1, \\ f_0(z_1, z_2, z_3, z_4) = g_3(1, z_2^{-1}, z_2^{-1} z_3^{-1}) + z_4(z_1^2 + g_4(1, z_2^{-1}, z_2^{-1} z_3^{-1})) + \ldots, \\ X_\sigma \cap E_\alpha \cap Y_\alpha = \{g_3(1, z_2^{-1}, z_2^{-1} z_3^{-1}) = z_4 = 0\}.\end{gathered}$$

The intersection is empty iff $g_3(y_2, y_3, y_4) = y_2 y_3 y_4$. In this case all the components of $E_\alpha \cap Y_\alpha$ are toric, and we should apply a linear change of coordinates, say $y_2 \to y_2, y_3 \to y_3, y_4 \to y_3 + y_4$, and repeat the same construction. Then the above intersection will contain a simple component $\Gamma = \{1 + z_3^{-1} = z_4 = 0\}$.

(3) $f = y_1^2 + y_2^2 y_3 + g(y_2, y_3, y_4)$, $g = g_4 + g_5 + \ldots$; $\alpha = (2, 1, 2, 1)$. We choose

$$z_1 = y_1 y_2^{-2}, z_2 = y_1 y_3^{-1}, z_3 = y_2 y_4^{-1}, z_4 = y_2 .$$

We have:

$$\begin{gathered}\sigma_\alpha^* f = z_4^N f_0,\ N = v_\alpha(f) = 4,\ |\alpha| = 6,\ a_\alpha = -v_\alpha(f) + |\alpha| - 1 = 1, \\ f_0(z_1, z_2, z_3, z_4) = z_1^2 + z_1 z_2^{-1} + g_4(1, 0, z_3^{-1}) + z_4 \tilde{g}(z_1, z_2, z_3), \\ X_\sigma \cap E_\alpha \cap Y_\alpha = \{z_1^2 + z_1 z_2^{-1} + g_4(1, 0, z_3^{-1}) = z_4 = 0\}.\end{gathered}$$

This intersection is non-empty and reduced irreducible independently of the vanishing or non-vanishing of $g_4(1,0,z_3^{-1})$. If $g_4(1,0,z_3^{-1})=0$, then the invertible factor z_1 cancels out and we have $X_\sigma \cap E_\alpha \cap Y_\alpha = \{z_1 + z_2^{-1} = z_4 = 0\}$.

(4) $f = y_1^2 + y_2^3 + g(y_2,y_3,y_4)$, $j_3 g = 0$, and g does not contain the monomials y_3^4, y_3^5; $\alpha = (3,2,1,2)$. We choose

$$z_1 = y_1 y_3^{-3}, z_2 = y_2 y_3^{-2}, z_3 = y_2 y_4^{-1}, z_4 = y_3\,.$$

We have:

$$\begin{gathered}\sigma_\alpha^* f = z_4^N f_0,\ N = v_\alpha(f) = 6,\ |\alpha| = 8,\ a_\alpha = -v_\alpha(f) + |\alpha| - 1 = 1,\\ f_0(z_1,z_2,z_3,z_4) = z_1^2 + z_2^3 + c_1 z_2^2 + c_2 z_2 + c_3 z_2^2 z_3^{-1}\\ +c_4 z_2 z_3^{-1} + c_5 z_2^2 z_3^{-2} + c_6,\end{gathered}$$

where

$$c_1 y_2^2 y_3^2 + c_2 y_2 y_3^4 + c_3 y_2 y_3^2 y_4 + c_4 y_3^4 y_4 + c_5 y_3^2 y_4^2 + c_6 y_3^6 = g_{N,\alpha}(y_2,y_3,y_4)$$

is the α-principal part of g, and

$$X_\sigma \cap E_\alpha \cap Y_\alpha = \{z_1^2 + z_2(z_2^2 + (c_1 + \frac{c_3}{z_3} + \frac{c_5}{z_3^2})z_2 + c_2 + \frac{c_4}{z_3}) + c_6 = z_4 = 0\}.$$

This intersection is non-empty and reduced irreducible because all its slices $\{z_3 = 0\}$ are. Indeed, the equation $z_1^2 + z_2(z_2^2 + Az_2 + B) + C = 0$ is irreducible for any $A, B, C \in k$.

(5) $f = y_1^2 + y_2^3 + g(y_2,y_3,y_4)$, $j_3 g = 0$, $g_4(0,y_3,y_4) \neq 0$; take $\alpha = (2,2,1,1)$. Choose coordinates

$$z_1 = y_1 y_2^{-1}, z_2 = y_1 y_3^{-2}, z_3 = y_3 y_4^{-1}, z_4 = y_3\,.$$

We have:

$$\begin{gathered}\sigma_\alpha^* f = z_4^N f_0,\ N = v_\alpha(f) = 4,\ |\alpha| = 6,\ a_\alpha = -v_\alpha(f) + |\alpha| - 1 = 1,\\ f_0(z_1,z_2,z_3,z_4) = z_2^2 + z_1^{-3} z_2^3 z_4^2 + g_4(0,1,z_3^{-1}) + z_4 \tilde{g}(z_1,z_2,z_3,z_4),\\ X_\sigma \cap E_\alpha \cap Y_\alpha = \{z_2^2 + g_4(0,1,z_3^{-1}) = z_4 = 0\}\,.\end{gathered}$$

The last intersection has one or two irreducible components of multiplicity 1.

(6) $f = y_1^2 + y_2^3 + g(y_2, y_3, y_4),\ j_3 g = 0,\ g_4(0, y_3, y_4) = 0,\ g_5$ contains y_3^5; take $\alpha = (3, 2, 1, 1)$. Choose coordinates

$$z_1 = y_1 y_3^{-3}, z_2 = y_2 y_3^{-2}, z_3 = y_3 y_4^{-1}, z_4 = y_3 .$$

Remind, that in the case cE_n we should suppose that g contains one of the monomials (1.2). So,

$$g_5(0, y_3, y_4) = \sum_{i=0}^{5} c_i y_3^{5-i} y_4^i,\ c_0 \neq 0,$$

and at least one of the coefficients c_3, c_4, c_5 is different from 0. We have:

$$\begin{gathered}\sigma_\alpha^* f = z_4^N f_0,\ N = v_\alpha(f) = 5,\ |\alpha| = 7,\ a_\alpha = -v_\alpha(f) + |\alpha| - 1 = 1,\\ f_0(z_1, z_2, z_3, z_4) = z_1^2 z_4 + z_2^3 z_4 + \sum_{i=0}^{5} c_i z_3^{-i} + z_2 \sum_{i=0}^{3} c_i' z_3^{-i}\\ +z_4 \tilde{g}(z_1, z_2, z_3, z_4),\\ X_\sigma \cap E_\alpha \cap Y_\alpha = \{\sum_{i=0}^{5} c_i z_3^{-i} + z_2 \sum_{i=0}^{3} c_i' z_3^{-i} = z_4 = 0\} .\end{gathered}$$

The above conditions on c_i imply that the intersection is always non-empty. But it may be multiple. There are no components of multiplicity 1 only if $c_i' = 0$ $(i = 0, 1, 2, 3)$ and:

$$\begin{aligned} & g_5(0, y_3, y_4) = y_3^k (y_3 - \gamma_1 y_4)^{5-k},\ \gamma_1 \neq 0\ (k = 0, 1, 2, 3),\\ \text{or}\quad & g_5(0, y_3, y_4) = y_3 (y_3 - \gamma_1 y_4)^2 (y_3 - \gamma_2 y_4)^2,\ \gamma_1 \neq 0, \gamma_2 \neq 0, \gamma_1 \neq \gamma_2,\\ \text{or}\quad & g_5(0, y_3, y_4) = (y_3 - \gamma_1 y_4)^3 (y_3 - \gamma_2 y_4)^2,\ \gamma_1 \neq 0, \gamma_2 \neq 0, \gamma_1 \neq \gamma_2. \end{aligned}$$

In all the cases the change of variables $y_3 \to y_3, y_4 \to y_3 - \gamma_1 y_4$ brings us to the case (4), in which the existence of a simple non-toric component has been verified for the weight $\alpha = (3, 2, 1, 2)$.

Thus, we can suppose that the polynomial defining Y_α in $X_\sigma \cap E_\alpha$ has a simple factor of the form $1 - \gamma_1 z_3^{-1}$, giving rise to the wanted component of multiplicity 1. $\square$

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