J. Math. Sci. Univ. Tokyo
3 (1996), 399–443.

Two Transforms of Plane Curves and Their Fundamental Groups

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§1. Introduction

Let $C = \{(X;Y;Z) \in \mathbf{P}^2; F(X,Y,Z) = 0\}$ be a projective curve and let $C^a = \{f(x,y) = 0\} \subset \mathbf{C}^2$ be the corresponding affine plane curve with respect to the affine coordinate space $\mathbf{C}^2 = \mathbf{P}^2 - \{Z = 0\}$, $x = X/Z$, $y = Y/Z$ and $f(x,y) = F(x,y,1)$. In this paper, we study two basic operations. First we consider an n-fold cyclic covering $\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2$, $\varphi_n(x,y) = (x,(y-\beta)^n + \beta)$, branched along a line $D = \{y = \beta\}$ for an arbitrary positive integer $n \geq 2$. Let $\mathcal{C}_n(C;D)$ be the projective closure of the pull back $\varphi_n^{-1}(C^a)$ of C^a. The behavior of φ_n at infinity gives an interesting effect on the fundamental group. In our previous paper [O6], we have studied the double covering φ_2 to construct some interesting plane curves, such as a Zariski's three cuspidal quartic and a conical six cuspidal sextic.

Secondly we consider the following Jung transform of degree n, J_n : $\mathbf{C}^2 \to \mathbf{C}^2$, $J_n(x,y) = (x + y^n, y)$ and let $\mathcal{J}_n(C;L_\infty)$ be the projective compactification of $J_n^{-1}(C^a)$. Though J_n is an automorphism of $\mathbf{C}^2$, the behavior of J_n or $\mathcal{J}_n(C)$ at infinity is quite interesting.

Both of φ_n and J_n can be extended canonically to rational mapping from $\mathbf{P}^2$ to $\mathbf{P}^2$ and they are not defined only at $[1;0;0]$ and constant along the line at infinity $L_\infty = \{Z = 0\}$. They have also the following similarity. For a generic φ_n and a generic J_n, there exist surjective homomorphisms

$$\Phi_n : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) \to \pi_1(\mathbf{P}^2 - C),$$
$$\Psi_n : \pi_1(\mathbf{P}^2 - \mathcal{J}_n(C)) \to \pi_1(\mathbf{P}^2 - C)$$

1991 *Mathematics Subject Classification.* Primary 14H30; Secondary 32S20.
The author is partially supported by Inamori foundation.

and both kernels $\mathrm{Ker}\,\Phi_n$ and $\mathrm{Ker}\,\Psi_n$ are cyclic group of order n which are subgroups of the respective centers of $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C))$ (Theorem (3.7) and Theorem (4.7)).

Both operations are useful to construct examples of interesting plane curves, starting from a simple plane curve. Applying this operation to a Zariski's three cuspidal quartic Z_4, we obtain new examples of plane curves $\mathcal{C}_n(Z_4)$ and $\mathcal{J}_n(Z_4)$ of degree $4n$ whose complement in $\mathbf{P}^2$ has a non-commutative finite fundamental group of order $12n$ (§5). We will construct a new example of Zariski pair $\{\mathcal{C}_3(Z_4), C_2\}$ of curves of degree 12 (§5).

In §6, we study non-atypical curves and their Jung transforms. We use a non-generic Jung transform to construct a rational curve $\widetilde{C}$ of degree pq for any p, q with $\gcd(p, q) = 1$ such that $\widetilde{C}$ has two irreducible singularities and the fundamental group $\pi_1(\mathbf{P}^2 - \widetilde{C})$ is isomorphic to the free product $\mathbf{Z}/p\mathbf{Z} * \mathbf{Z}/q\mathbf{Z}$ (Corollary (6.7.1)). This paper is composed as follows.

§2. Basic properties of $\pi_1(\mathbf{P}^2 - C)$ and Zariski's pencil method.
§3. Cyclic transforms of plane curves.
§4. Jung transforms of plane curves.
§5. Zariski's quartic and Zariski pairs.
§6. Non-atypical curves and some examples.

§2. Basic properties of $\pi_1(\mathbf{P}^2 - \mathbf{C})$ and Zariski's pencil method

Let C be a reduced projective curve of degree d and let $C_1, \dots, C_r$ be the irreducible components of C and let d_i be the degree of C_i. So $d = d_1 + \cdots + d_r$. First we recall that the first homology of the complement is given by the Lefschetz duality and by the exact sequence of the pair $(\mathbf{P}^2, C)$ as follows.

$$(2.1) \qquad H_1(\mathbf{P}^2 - C) \cong \mathbf{Z}^r/(d_1, \dots, d_r) \cong \mathbf{Z}^{r-1} \oplus \mathbf{Z}/d_0\mathbf{Z}$$

where $d_0 = \gcd(d_1, \dots, d_r)$ and $\mathbf{Z}^r = \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ (r factors). In particular, if C is irreducible ($r = 1$), we have $H_1(\mathbf{P}^2 - C) \cong \mathbf{Z}/d\mathbf{Z}$ and $H_1(\mathbf{C}^2 - C^a) \cong \mathbf{Z}$ where $\mathbf{C}^2 := \mathbf{P}^2 - L_\infty$ and $C^a := C \cap L_\infty$.

(A) van Kampen-Zariski's pencil method

We fix a point $B_0 \in \mathbf{P}^2$ and we consider the pencil of lines $\{L_\eta,\ \eta \in \mathbf{P}^1\}$ through B_0. Taking a linear change of coordinates if necessary, we may assume that L_η is defined by $L_\eta = \{X - \eta Z = 0\}$ and $B_0 = [0;1;0]$ in homogeneous coordinates. Take $L_\infty = \{Z = 0\}$ as the line at infinity and we write $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$. Note that $L_\infty = \lim_{\eta\to\infty} L_\eta$. We assume that $L_\infty \not\subset C$. We consider the affine coordinates $(x, y) = (X/Z, Y/Z)$ on $\mathbf{C}^2$ and let $F(X, Y, Z)$ be the defining homogeneous polynomial of C and let $f(x, y) := F(x, y, 1)$ be the affine equation of C. In this affine coordinates, the pencil line L_η is simply defined by $\{x = \eta\}$. As we consider two fundamental groups $\pi_1(\mathbf{P}^2 - C)$ and $\pi_1(\mathbf{P}^2 - C \cup L_\infty)$ simultaneously, we use the notations : $C^a = C \cap \mathbf{C}^2$ and $L_\eta^a = L_\eta \cap \mathbf{C}^2 \cong \mathbf{C}$. We identify hereafter L_η and L_η^a with $\mathbf{P}^1$ and $\mathbf{C}$ respectively by $y : L_\eta \cong \mathbf{P}^1$ for $\eta \neq \infty$. Note that the base point of the pencil B_0 corresponds to $\infty \in \mathbf{P}^1$.

We say that the pencil $\{L_\eta = \{x = \eta\},\ \eta \in \mathbf{C}\}$, is *admissible* if there exists an integer $d' \le d$ which is independent of $\eta \in \mathbf{C}$ such that $C^a \cap L_\eta^a$ consists of d' points counting the multiplicity. This is equivalent to : $f(x, y)$ *has degree d' in y and the coefficient of $y^{d'}$ is a non-zero constant.* Note that if $B_0 \notin C$, L_η is admissible and $d' = d$. If $d' < d$, $B_0 \in C$ and the intersection multiplicity $I(C, L_\infty; B_0) = d - d'$.

Hereafter we assume that the pencil $\{L_\eta\}$ is admissible. A line L is called *generic* with respect to C if $C \cap L$ consists of d distinct points. A pencil line L_η is called *non-generic* with respect to C if L_η passes through a singular point of C^a or L_η is tangent to C^a. Otherwise L_η is called *generic.* Here we note that a generic pencil line L_{η_0} may not be generic as a line in $\mathbf{P}^2$ if $B_0 \in C$ and $d - d' \ge 2$ but L_{η_0} intersects transversely with C^a at d' points.

Let $\mathbf{C}_B$ be the line of the parameters of the pencil ($\mathbf{C}_B \cong \mathbf{C}$) and $\Sigma := \{\eta_1, \dots, \eta_\ell\}$ be parameters in $\mathbf{C}_B$ which corresponds to non-generic pencil lines. We fix a generic pencil line L_{η_0} and put $L_{\eta_0}^a \cap C^a = \{Q_1, \dots, Q_{d'}\}$. The complement $L_{\eta_0}^a - L_{\eta_0}^a \cap C^a$ is topologically $\mathbf{C}$ minus d'-points . We take a base point $b_0 \in L_{\eta_0}^a$ on the imaginary axis which is sufficiently near to B_0 and $b_0 \neq B_0$. We take a large disk Δ_{η_0} in the generic pencil line $L_{\eta_0}^a$ such that $\Delta_{\eta_0} \supset C \cap L_{\eta_0}^a$ and $b_0 \notin \Delta_{\eta_0}$. We orient the boundary of Δ_{η_0} counter-clockwise and let $\Omega = \partial\Delta_{\eta_0}$. We join Ω to the base point by a path L connecting b_0 and Ω along the imaginary axis. Let ω be the class of this loop $L \circ \Omega \circ L^{-1}$ in $\pi_1(L_{\eta_0}^a - L_{\eta_0}^a \cap C; b_0)$. We take free generators $g_1, \dots, g_{d'}$ of $\pi_1(L_{\eta_0}^a - L_{\eta_0}^a \cap C; b_0)$ so that g_i goes around Q_i counter-clockwise along a small circle and

$$\omega = g_{d'} \cdots g_1 \tag{2.2}$$

Put $G = \pi_1(L^a_{\eta_0} - L^a_{\eta_0} \cap C^a; b_0)$. Note that G is a free group of rank d' with generators $g_1, \dots, g_{d'}$. The fundamental group $\pi_1(\mathbf{C}_B - \Sigma; \eta_0)$ acts on G which we refer by *the monodromy action* of $\pi_1(\mathbf{C}_B - \Sigma; \eta_0)$. We recall this action quickly.

Take a large disk $\Delta \subset \mathbf{C}_B$ on the base space so that $\Delta \supset \Sigma$ and $\eta_0 \in \Delta$. So we have $\pi_1(\mathbf{C}_B - \Sigma; \eta_0) \cong \pi_1(\Delta - \Sigma; \eta_0)$. We take a system of free generators $\sigma_1, \dots, \sigma_\ell$ of $\pi_1(\Delta - \Sigma; \eta_0)$ which are represented by smooth loops in Δ, so that the product $\sigma_\ell \cdots \sigma_1$ is homotopic to the counter-clockwise oriented boundary of Δ. We take a large disk of radius R, $B(R) := \{y \in \mathbf{C}; |y| \le R\}$ so that $B(R) \supset \bigcup_{\eta \in \Delta} C^a \cap L_\eta$ under the identification $y : L^a_\eta \cong \mathbf{C}$. We may assume that $b_0 \in L_{\eta_0} - B(2R)$. Take $g \in \pi_1(L^a_{\eta_0} - C^a \cap L^a_{\eta_0}; b_0)$ and $\sigma \in \pi_1(\mathbf{C}_B - \Sigma; \eta_0)$. Represent them by smooth loops $\alpha : (I, \partial I) \to (L^a_{\eta_0} - L^a_{\eta_0} \cap C; b_0)$ and $\tau : (I, \partial I) \to (\Delta - \Sigma; \eta_0)$ and construct a one-parameter family of deffeomorphisms $h_\theta : (L_{\eta_0}, C \cap L_{\eta_0}) \to (L_{\tau(\theta)}, C \cap L_{\sigma(\theta)})$, $0 \le \theta \le 1$ such that the composition

$$\mathbf{C} \xrightarrow{y^{-1}} L^a_{\eta_0} \xrightarrow{h_\theta} L^a_{\tau(\theta)} \xrightarrow{y} \mathbf{C}$$

is identity on $\mathbf{C} - B(2R)$. The action of $\sigma \in \pi_1(\mathbf{C}_B - \Sigma; \eta_0)$ on $g \in G$ is defined by $(g, \sigma) \mapsto [h_{2\pi} \circ \alpha]$. We denote this class by g^σ. Note that $\omega^g = \omega$ for any $g \in \pi_1(\mathbf{C}_B - \Sigma; \eta_0)$. The normal subgroups of G which is normally generated by $\{g^{-1} g^\sigma \ ; \ g \in G, \sigma \in \pi_1(\mathbf{C}_B - \Sigma; \eta_0)\}$ is called *the group of the monodromy relations* and we denote it by $\mathcal{M}$. Let $\mathcal{M}(\sigma_i) = \{g_j^{\sigma_i} g_j^{-1}; j = 1, \dots, d\}$. Then the group of the monodromy relations $\mathcal{M}$ is the minimal normal subgroup of G generated by $\bigcup_{i=1}^{\ell} \mathcal{M}(\sigma_i)$. By the definition, we have the relation $R(\sigma_i) : \ g_j = g_j^{\sigma_i}$ in the quotient group $G/\mathcal{M}$. We call $R(\sigma_i)$ *the monodromy relation for* σ_i. The following is a reformulation of a theorem of van-Kampen ([K]) to an affine situation with an admissible pencil. Let $j : L^a_{\eta_0} - L_{\eta_0} \cap C^a \to \mathbf{C}^2 - C^a$ and $\iota : \mathbf{C}^2 - C^a \to \mathbf{P}^2 - C$ be the respective inclusions.

PROPOSITION (2.3). (1) *The canonical homomorphism* $j_\sharp : \pi_1(L^a_{\eta_0} - L^a_{\eta_0} \cap C^a; b_0) \to \pi_1(\mathbf{C}^2 - C^a; b_0)$ *is surjective and the kernel* $\operatorname{Ker} j_\sharp$ *is equal*

to $\mathcal{M}$ and therefore $\pi_1(\mathbf{C}^2 - C^a; b_0)$ is isomorphic to the quotient group $G/\mathcal{M}$.
(2) *The canonical homomorphism $\iota_\sharp : \pi_1(\mathbf{C}^2 - C^a; b_0) \to \pi_1(\mathbf{P}^2 - C; b_0)$ is surjective. If $B_0 \notin C$ (so $d' = d$), the kernel* $\operatorname{Ker} \iota_\sharp$ *is normally generated by $\omega = g_d \cdots g_1$.*
Assume further that $B_0 \notin C$ and L_∞ is generic. Then
(3) ([O3]) *ω is in the center of $\pi_1(\mathbf{C}^2 - C^a)$. Therefore* $\operatorname{Ker}(\iota_\sharp) = \langle \omega \rangle \cong \mathbf{Z}$.
(4) *$\iota_\sharp$ induces an isomorphism of the commutator groups: $\iota_{\sharp\mathcal{D}} : \mathcal{D}(\pi_1(\mathbf{C}^2 - C^a)) \overset{\cong}{\longrightarrow} \mathcal{D}(\pi_1(\mathbf{P}^2 - C))$ and an exact sequence of first homologies:* $0 \to \langle \omega \rangle \cong \mathbf{Z} \to H_1(\mathbf{C}^2 - C) \to H_1(\mathbf{P}^2 - C) \to 0$.

Proof. The assertions are well-known except (4). So we only need to show the assertion (4). First $\iota_{\sharp\mathcal{D}}$ is surjective. As the homology class $[\omega]$ of ω is given by $[(0, d_1, \dots, d_r)]$ under the identification $H_1(\mathbf{C}^2 - C^a) \cong \mathbf{Z}^{r+1}/(1, d_1, \dots, d_r)$, $[\omega]$ generates an infinite cyclic group. Thus the injectivity of $\iota_{\sharp\mathcal{D}}$ follows from $\mathcal{D}(\pi_1(\mathbf{C}^2 - C)) \cap \operatorname{Ker} \iota_\sharp = \{e\}$. The exact sequence follows from the first isomorphism and the property: $\langle \omega \rangle \cap \mathcal{D}(\pi_1(\mathbf{C}^2 - C^a)) = \{e\}$. $\square$

We usually denote $G/\mathcal{M}$ as $\pi_1(\mathbf{C}^2 - C^a; b_0) = \langle g_1, \dots, g_d; R(\sigma_1), \dots, R(\sigma_\ell) \rangle$. We call $\pi_1(\mathbf{C}^2 - C^a)$ *the fundamental group of a generic affine complement of* C if L_∞ is generic. Note that if L_∞ is generic, $\pi_1(\mathbf{C}^2 - C^a)$ does not depend on the choice of a line at infinity L_∞.

(B) Bracelets and lassos

An element $\rho \in \pi_1(\mathbf{P}^2 - C; b_0)$ is called *a lasso* for C_i if it is represented by a loop $\mathcal{L} \circ \tau \circ \mathcal{L}^{-1}$ where τ is a counter-clockwise oriented boundary of a small normal disk $D_i(P)$ of C_i at a regular point $P \in C_i$ such that $D_i(P) \cap (C \cup L_\infty) = \{P\}$ and $\mathcal{L}$ is a path connecting b_0 and τ. We call τ *a bracelet* for C_i. It is easy to see that any two bracelets τ and τ' for the same irreducible component, say C_i, are free homotopic. Therefore *the homotopy class of a lasso for C_i (or L_∞) is unique up to a conjugation.* We say that the line at infinity L_∞ is *central* for C if there is a lasso ω for L_∞ which is in the center of $\pi_1(\mathbf{C}^2 - C^a) = \pi_1(\mathbf{P}^2 - C \cup L_\infty)$. If L_∞ is generic for C, L_∞ is central by Proposition (2.3) but the converse is not always true (see Corollary (3.3.1) and Theorem (4.3)).

Assume that L_∞ is central for C and take an admissible pencil $\{L_\eta, \eta \in \mathbf{C}\}$ with the base point $B_0 \notin C$. Then $d' = d$ and ω defined by (2.2) is in the center of $\pi_1(\mathbf{C}^2 - C^a; b_0)$ as ω^{-1} is a lasso for L_∞. Thus we can replace the homotopy deformation of ω by free homotopy deformation of Ω. This viewpoint is quite useful in the later sections.

REMARK (2.4). Suppose that $B_0 \notin C$ and L_∞ is *not generic*. Take $\Delta = \{\eta \in \mathbf{C}_B; |\eta| \le R\} \subset \mathbf{C}_B$ as before and we may assume that $\eta_0 \in \partial\Delta$ and let $\sigma_\infty := \partial\Delta$. The monodromy relation $g_i^{-1} g_i^{\sigma_\infty}$ is contained in the group of monodromy relations $\mathcal{M}$. We can also consider the monodromy relation around $\eta = \infty$. For this purpose, we identify $L_\eta \cong \mathbf{P}^1$ through another rational function $\varphi := Y/X$ for $|\eta| \ge R$. For $\eta \ne 0$, $\varphi : L_\eta \to \mathbf{C}$ is written as $\varphi(\eta, y) = y/\eta$. Let $j_\theta : L_{\eta_0} \to L_{\eta_0 \exp(\theta i)}$, $0 \le \theta \le 2\pi$ be a family of homeomorphisms which is identity outside of a big disk under this identification $\varphi : L_\eta \to \mathbf{C}$. Then the base point b_0 stays constant under the identification by φ but under the first identification of $y : L_\eta \to \mathbf{P}^1$, the base point is rotated by $\theta \mapsto b_0 \exp(\theta i)$. Putting $h' = j_{2\pi}$, this implies that the monodromy relation around L_∞ is given by

$$h'_\sharp(g) = \omega g^{-\sigma_\infty} \omega^{-1}, \quad g \in G \tag{2.4.1}$$

This gives the following corollary.

COROLLARY (2.5). *Take another generic line $L_{\eta_0'}$ for C with $\eta_0' \ne \eta_0$. Let $R_1, \ldots, R_\ell$ be the monodromy relation along σ_i as before. Then the fundamental group of a generic affine complement $\pi_1(\mathbf{P}^2 - C \cup L_{\eta_0'}; b_0)$ is isomorphic to the quotient group of $\pi_1(\mathbf{C}^2 - C^a; b_0)$ by the relation $\omega g_i = g_i \omega$, $i = 1, \ldots, d$. In particular, if ω is in the center of $\pi_1(\mathbf{C}^2 - C^a; b_0)$, $\pi_1(\mathbf{C}^2 - C^a; b_0)$ is isomorphic to the fundamental group of a generic affine complement $\pi_1(\mathbf{P}^2 - C \cup L_{\eta_0'}; b_0)$.*

PROOF. Changing coordinates if necessary, we may assume that $\eta_0' = 0$. Using the second identification $Y/X : L_\eta \cong \mathbf{P}^1$ for $\eta \ne 0$, we can write the monodromy relation $R(\infty)$ at $\eta = \infty$ as

$$g_j = h'_\sharp(g_j), \quad \text{for} \quad j = 1, \ldots, d \tag{$R(\infty)$}$$

and the other monodromy relations $R_i, i = 1, \ldots, \ell$ are the same with those which are obtained from the first identification. Therefore we have $\pi_1(\mathbf{P}^2 -$

$C \cup L_{\eta'_0}; b_0) \cong \langle g_1, \dots, g_d; R_1, \dots, R_\ell, R(\infty) \rangle$. On the other hand, we know that $\omega = g_d \cdots g_1$ is in the center of $\pi_1(\mathbf{P}^2 - C \cup L_{\eta'_0}; b_0)$ ([O2]). Thus we get

$$(\star) \qquad \omega g_j = g_j \omega, \quad j = 1, \dots, d$$

in $\pi_1(\mathbf{P}^2 - C \cup L_{\eta'_0}; b_0)$. Conversely in the group $\langle g_1, \dots, g_d; R_1, \dots, R_\ell, (\star) \rangle$, we have the equality:

$$g_j^{-1} h'_\sharp(g_j) = g_j^{-1} \omega g_j^{-\sigma_\infty} \omega^{-1} \overset{(\star)}{=} g_j^{-1} g_j^{-\sigma_\infty} = e.$$

Thus we can replace $R(\infty)$ by $(\star)$ □

(C) Milnor fiber

Consider the affine hypersurface $V(C) = \{(x, y, z) \in \mathbf{C}^3; F(x, y, z) = 1\}$ where $F(X, Y, Z) = Z^d f(X/Z, Y/Z)$. The restriction of Hopf fibration to $V(C)$ is d-fold cyclic covering over $\mathbf{P}^2 - C$. Thus we have an exact sequence:

$$(2.6) \qquad 1 \to \pi_1(V(C)) \to \pi_1(\mathbf{P}^2 - C) \to \mathbf{Z}/d\mathbf{Z} \to 1$$

Comparing with Hurewicz homomorphism, we get

Proposition (2.7) ([O2]). *If C is irreducible, $\pi_1(V(C))$ is isomorphic to the commutator group $\mathcal{D}(\pi_1(\mathbf{P}^2 - C))$ of $\pi_1(\mathbf{P}^2 - C)$.*

§3. Cyclic transforms of plane curves

(A) Cyclic transforms

Let $C \subset \mathbf{P}^2$ be a projective curve of degree d. Fixing a line at infinity L_∞, we assume that the affine curve $C^a := C \cap \mathbf{C}^2$ is defined by $f(x, y) = 0$ in $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$. We assume that $f(x, y)$ is written with mutually distinct non-zero $\alpha_1, \dots, \alpha_k$ as

$$(\sharp) \qquad f(x, y) = \prod_{i=1}^{k} (y^a - \alpha_i x^b)^{\nu_i} + (\text{lower terms}), \qquad \gcd(a, b) = 1$$

Here (lower term) implies that it is a linear combination of monomials $x^\alpha y^\beta$ with $a\alpha + b\beta < kab$. This implies that $\deg_y f(x, y) = d'$, $\deg_x f(x, y) = d''$

where $d' := a\sum_{i=1}^k \nu_i$, $d'' := b\sum_{i=1}^k \nu_i$ and $d = \max(d', d'')$ and both pencils $\{x = \eta\}_{\eta\in\mathbf{C}}$ and $\{y = \delta\}_{\delta\in\mathbf{C}}$ are admissible. Note that the assumption ($\sharp$) does not change by the change of coordinates of the type $(x, y) \mapsto (x + \alpha, y + \beta)$.

(1) If $a = b = 1$, then $d = d' = d''$ and $L_\infty \cap C = \{[1; \alpha_i; 0]; i = 1, \dots, k\}$. In particular, if $\nu_i = 1$ for each i, L_∞ is generic for C and thus L_∞ intersects transversely with C.

(2) If $a > b$ (respectively $a < b$), we have $d = d'$, $C \cap L_\infty = \{\rho_\infty := [1; 0; 0]\}$ (resp. $d = d''$, $C \cap L_\infty = \{\rho'_\infty := [0; 1; 0]\}$) and C has a singularity at ρ_∞ (resp. at ρ'_∞). The local equation of C at ρ_∞ (resp. ρ'_∞) takes the form:

$$(3.1)\qquad \begin{cases} \prod_{i=1}^k(\zeta^a - \alpha_i\xi^{a-b})^{\nu_i} + (\text{higher terms}) = 0, \\ \qquad \zeta = Y/X, \xi = Z/X,\ a > b \\ \prod_{i=1}^k({\zeta'}^{b-a} - \alpha_i{\xi'}^b)^{\nu_i} + (\text{higher terms}) = 0, \\ \qquad \zeta' = Z/Y, \xi' = X/Y,\ a < b \end{cases}$$

Here (higher terms) is defined similarly. For instance, in the first equality it is a linear combinations of monomilas $\zeta^\alpha\xi^\beta$ with $(a-b)\alpha + a\beta > ka(a-b)$. Now we consider the horizontal pencil $M_\eta = \{y = \eta\}$, $\eta \in \mathbf{C}$ and let $D = M_\beta$ be a generic pencil line. As β is generic, $D \cap C^a$ is d'' distinct points in $\mathbf{C}^2$. For an integer $n \geq 2$, we consider the n-fold cyclic covering $\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2$, defined by

$$\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2, \quad \varphi_n(x, y) = (x, (y - \beta)^n + \beta)$$

which is branched along D. Let $\mathcal{C}_n(C; D)^a = \varphi_n^{-1}(C^a)$ and let $\mathcal{C}_n(C; D)$ be the closure of $\mathcal{C}_n(C; D)^a$ in $\mathbf{P}^2$. We call $\mathcal{C}_n(C; D)$ the *cyclic transform of order n with respect to the line D*. To avoid the confusion, we denote the source space of φ_n by $\widetilde{\mathbf{C}^2}$ and the coordinates of $\widetilde{\mathbf{C}^2}$ by $(\tilde{x}, \tilde{y})$. Thus the line $\{\tilde{y} = \beta\}$ is equal to $\varphi_n^{-1}(D)$ and we denote it by $\widetilde{D}$. We denote the line at infinity $\mathbf{P}^2 - \widetilde{\mathbf{C}^2}$ by $\widetilde{L}_\infty$. Let $f^{(n)}(\tilde{x}, \tilde{y})$ be the defining polynomial of $\mathcal{C}_n(C; D)^a$. As $f^{(n)}(\tilde{x}, \tilde{y}) = f(\tilde{x}, (\tilde{y} - \beta)^n + \beta)$, $f^{(n)}(\tilde{x}, \tilde{y})$ takes the form:

$$(3.2)\qquad f^{(n)}(x, y) = \prod_{i=1}^k (\tilde{y}^{na} - \alpha_i\tilde{x}^b)^{\nu_i} + (\text{lower terms}).$$

Observer that $f^{(n)}(\tilde{x}, \tilde{y})$ also satisfies ($\sharp$).

(B) Singularities of $\mathcal{C}_\mathbf{n}(\mathbf{C};\mathbf{D})$

Let $\mathbf{a}_1,\dots,\mathbf{a}_s$ be the singular points of C^a and put $L_\infty\cap C=\{\mathbf{a}_\infty^1,\dots,\mathbf{a}_\infty^\ell\}$ and $\mathcal{C}_n(C;D)\cap\widetilde{L}_\infty=\{\tilde{\mathbf{a}}_\infty^i; i=1,\dots,\tilde{\ell}\}$ where $\widetilde{L}_\infty$ is the line at infinity of the projective compactification of the source space $\widetilde{\mathbf{C}^2}$ of φ_n. Note that $\ell=k$ if $a=b=1$ and $\ell=1$ otherwise. Note also that $\tilde{\ell}=kb$ or 1 according to $na=b$ or $na\neq b$. $\mathcal{C}_n(C;D)\cap\widetilde{L}_\infty$ is either $\{[1;0;0]\}$ if $na>b$ or $\{[0;1;0]\}$ if $na<b$. It is obvious that for each $i=1,\dots,s$, $\mathcal{C}_n(C;D)$ has n-copies of singularities $\mathbf{a}_{i,1},\dots,\mathbf{a}_{i,n}$ which are locally isomorphic to $\mathbf{a}_i$. We denote the local Milnor number at $\mathbf{a}\in C$ by $\mu(C;\mathbf{a})$. First we recall the modified Plücker's formula for the topological Euler characteristics (see, for instance,[O2]):

$$\chi(C)=3d-d^2+\sum_{j=1}^{s}\mu(C;\mathbf{a}_j)+\sum_{i=1}^{\tilde{\ell}}\mu(C;\mathbf{a}_\infty^i) \tag{3.3.1}$$

Proposition (3.3.2). *If the branching locus D is a generic pencil line, the topological types of $(\widetilde{\mathbf{C}^2},\mathcal{C}_n(C;D)^a)$ and $(\mathbf{P}^2,\mathcal{C}_n(C;D))$ do not depend on the choice of a generic β.*

Proof. By an easy computation, we have $\chi(\mathcal{C}_n(C;D)^a)=n(\chi(C^a)-d'')+d''$ which is independent of the choice of β. As $\chi(\mathcal{C}_n(C;D))=\chi(\mathcal{C}_n(C;D)^a)+\tilde{\ell}$, $\chi(\mathcal{C}_n(C;D))$ is also independent of a generic β. On the other hand, the Milnor number of $\mathcal{C}_n(C;D)$ at $\mathbf{a}_{i,j}$ is equal to that of C at $\mathbf{a}_i$. Therefore by the modified Plücker's formula, the sum $\sum_{i=1}^{\tilde{\ell}}\mu(\mathcal{C}_n(C;D);\tilde{\mathbf{a}}_\infty^i)$ is also independent of β. This implies, by the upper semi-continuity[1] of the Milnor number the independentness of each $\mu(\mathcal{C}_n(C;D);\tilde{\mathbf{a}}_\infty^i)$. The assertion results immediately from this observation. □

Note that $\mathcal{C}_n(C;D)$ has further singularities, if the branching line D is not generic.

(C) Main results of this section

Let G be an arbitrary group. We denote the commutator subgroup and the center of G by $\mathcal{D}(G)$ and $\mathcal{Z}(G)$ respectively. The main result of this section is :

[1]This easily follows from the mapping degree characterization of Milnor number ([M]).

THEOREM (3.4). *Assume that* ($\sharp$) *is satisfied and* D *is a generic horizontal pencil line.*
(1) *The canonical homomorphism* $\varphi_{n\sharp} : \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C;D)^a) \to \pi_1(\mathbf{C}^2 - C^a)$ *is an isomorphism.*
(2-a) *Assume* $a \geq b$ *(so* $\deg \mathcal{C}_n(C;D) = nd$*). Then there is a surjective homomorphism* $\Phi_n : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) \to \pi_1(\mathbf{P}^2 - C)$ *which gives the following commutative diagram.*

$$\begin{array}{ccc} \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) & \xrightarrow{\Phi_n} & \pi_1(\mathbf{P}^2 - C) \\ \uparrow \widetilde{\iota}_\sharp & & \uparrow \iota_\sharp \\ \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C;D)^a) & \xrightarrow{\varphi_{n\sharp}} & \pi_1(\mathbf{C}^2 - C^a) \end{array}$$

where $\widetilde{\iota}_\sharp$ *and* $\iota_\sharp$ *are induced by the respective inclusions and the kernel of* Φ_n *is normally generated by the class of* $\omega' := \varphi_{n\sharp}^{-1}(\omega)$ *where* ω^{-1} *is a lasso for* L_∞ *and* ω'^{-n} *is a lasso for the line at infinity* $\widetilde{L}_\infty$ *of* $\widetilde{\mathbf{C}^2}$.
(2-b) *Assume that* $na \leq b$ *(so* $\deg \mathcal{C}_n(C;D) = \deg C^a = d$*). Then* $\widetilde{\omega} := \varphi_{n\sharp}^{-1}(\omega)$ *is a lasso for* $\widetilde{L}_\infty$ *and we have an isomorphism:* $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) \cong \pi_1(\mathbf{P}^2 - C)$.

COROLLARY (3.4.1). *Assume that* $a \geq b$ *and* L_∞ *is central for* C. *Then*
(1) $\widetilde{L}_\infty$ *is central for* $\mathcal{C}_n(C;D)$ *and there is a canonical central extension of groups*

$$1 \to \mathbf{Z}/n\mathbf{Z} \xrightarrow{\iota} \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) \xrightarrow{\Phi_n} \pi_1(\mathbf{P}^2 - C) \to 1$$

(i.e., $\iota(\mathbf{Z}/n\mathbf{Z}) \subset \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)))$*) and* $\mathbf{Z}/n\mathbf{Z}$ *is generated by* $\omega' = \varphi_{n\sharp}^{-1}(\omega)$.
(2) *The restriction of* Φ_n *gives an isomorphism of commutator groups*

$$\Phi_n : \mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D))) \to \mathcal{D}(\pi_1(\mathbf{P}^2 - C))$$

and the following exact sequences of the centers and the first homology groups:

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C;D))) & \xrightarrow{\Phi_n} & \mathcal{Z}(\pi_1(\mathbf{P}^2 - C)) & \to & 1 \\ 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & H_1(\mathbf{P}^2 - \mathcal{C}_n(C;D)) & \xrightarrow{\overline{\Phi}_n} & H_1(\mathbf{P}^2 - C) & \to & 1 \end{array}$$

Proof of Theorem (3.4). Taking the change of coordinates $(x,y)\mapsto (x,y+\beta)$, we may assume $D=\{y=0\}$ for simplicity. We first prove the assertion (1). We consider the horizontal pencil $\{M_\eta,\eta\in\mathbf{C}\}$ where $M_\eta^a=\{y=\eta\}$. Let $\Delta_\varepsilon=\{\eta\in\mathbf{C};|\eta|\le\varepsilon\}$, $E(\varepsilon)=\cup_{\eta\in\Delta_\varepsilon}(M_\eta^a-C^a\cap M_\eta^a)$ and $E(\varepsilon)^*=E(\varepsilon)-D$. As $M_0=D$ is a generic pencil line, $E(\varepsilon)$ and $E(\varepsilon)^*$ are homeomorphic to the products $(M_\varepsilon^a-C^a\cap M_\varepsilon^a)\times\Delta_\varepsilon$ and $(M_\varepsilon^a-C^a\cap M_\varepsilon^a)\times\Delta_\varepsilon^*$ respectively for a sufficiently small $\varepsilon>0$. Thus we have the isomorphism $\pi_1(E(\varepsilon)^*)=\pi_1(M_\varepsilon^a-C^a\cap M_\varepsilon^a)\times\mathbf{Z}$ so that the canonical homomorphism $\iota_\sharp:\pi_1(M_\varepsilon^a-C^a\cap M_\varepsilon^a)\to\pi_1(E(\varepsilon)^*)$ is the canonical injection $g\mapsto(g,0)$. Let τ be the generator of $\mathbf{Z}$ represented by a lasso for the branch locus D and let $\rho_1,\dots,\rho_{d''}$ be the generators of $\pi_1(M_\varepsilon^a-C^a\cap M_\varepsilon^a)$. Then τ commutes with every ρ_i and the monodromy relations for $\rho_1,\dots,\rho_{d''}$ in $\pi_1(\mathbf{C}^2-C^a)$ and in $\pi_1(\mathbf{C}^2-C^a\cup D)$ are the same. Therefore by Proposition (2.3), we can see that $\pi_1(\mathbf{C}^2-C^a\cup D)\cong\pi_1(\mathbf{C}^2-C^a)\times\mathbf{Z}$ and the canonical homomorphism associated with the inclusion map $a_\sharp:\pi_1(\mathbf{C}^2-C^a\cup D)\to\pi_1(\mathbf{C}^2-C^a)$ is the first projection under this identification. For simplicity, we denote $\mathcal{C}_n(C;D)$ by $\mathcal{C}_n(C)$ hereafter. We have the following exact sequence of the covering:

$$1\to\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C)^a\cup\widetilde{D})\xrightarrow{\varphi_{n\sharp}}\pi_1(\mathbf{C}^2-C^a\cup D)\to\mathbf{Z}/n\mathbf{Z}\to 1$$

As a subgroup of $\pi_1(\mathbf{C}^2-C^a\cup D)\cong\pi_1(\mathbf{C}^2-C^a)\times\mathbf{Z}$, $\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C)^a\cup\widetilde{D})$ can be identified with $\pi_1(\mathbf{C}^2-C^a)\times n\mathbf{Z}$ by $\varphi_{n\sharp}$. Note that $\varphi_{n\sharp}^{-1}(e\times n)$ is represented by a lasso $\widetilde{\tau}$ for $\widetilde{D}$. Let us consider a subgroup $H:=\varphi_{n\sharp}^{-1}(\pi_1(\mathbf{C}^2-C^a)\times\{e\})\subset\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C)^a\cup\widetilde{D})$. Now we consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C)^a\cup\widetilde{D}) & \supset & H & \xrightarrow{\widetilde{a}_\sharp} & \pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C)^a) \\
 & & \downarrow{\scriptstyle\varphi_{n\sharp}} & & \downarrow{\scriptstyle\varphi_{n\sharp}} \\
 & & \pi_1(\mathbf{C}^2-C^a\cup D) & \xrightarrow{a_\sharp} & \pi_1(\mathbf{C}^2-C^a)
\end{array}$$

where $\widetilde{a}$ and a are respective inclusion map. As $\widetilde{a}_\sharp:\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C)^a\cup\widetilde{D})\to\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C)^a)$ is surjective and $\varphi_{n\sharp}^{-1}(n\mathbf{Z})$ is included in the kernel of $\widetilde{a}_\sharp$, the restriction $\widetilde{a}_\sharp:H\to\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C)^a)$ is surjective. On the other hand, as the composition $\varphi_{n\sharp}\circ\widetilde{a}_\sharp:H\to\pi_1(\mathbf{C}^2-C^a)$ is equal to $a_\sharp\circ\varphi_{n\sharp}$, it is obviously bijective. Thus we conclude: $\widetilde{a}_\sharp:H\to\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C)^a)$ and

$\varphi_{n\sharp}:\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}_n(C)^a)\to\pi_1(\mathbf{C}^2-C^a)$ are isomorphisms. This proves the assertion (1).

We consider now the fundamental groups $\pi_1(\mathbf{P}^2-\mathcal{C}_n(C))$ and $\pi_1(\mathbf{P}^2-C)$. First we consider the easy case : $na\le b$ (Case (2-b)). In this case, $d=d''$, $C\cap L_\infty=\{\rho'_\infty=[0,1,0]\}$ and $\deg_x f(x,y)=\deg_{\tilde{x}} f^{(n)}(\tilde{x},\tilde{y})=d$. Take a generic horizontal pencil line $M_{\eta_0}:=\{y=\eta_0\}$ with $\eta_0\ne 0$, a base point $b_0\in M^a_{\eta_0}$ and generators $g_1,\dots,g_d$ of $\pi_1(M^a_{\eta_0}-M^a_{\eta_0}\cap C^a;b_0)$ as before. Let $\omega=g_d\cdots g_1$. We can assume that ω is homotopic to a big circle as in Proposition (2.3). Take $\widetilde{\eta}_0\in\mathbf{C}$ so that $\widetilde{\eta}_0^n=\eta_0$. We also take a base point $\widetilde{b}_0\in\widetilde{M}^a_{\widetilde{\eta}_0}$ so that $\varphi_n(\widetilde{b}_0)=b_0$. By the definition, the pencil line $\widetilde{M}_{\widetilde{\eta}_0}$ is generic and $\varphi_n:\widetilde{M}^a_{\widetilde{\eta}_0}-\widetilde{M}^a_{\widetilde{\eta}_0}\cap\mathcal{C}^a_n(C;D)\to M^a_{\eta_0}-M^a_{\eta_0}\cap C^a$ is homeomorphism which is simply given by $(u,\widetilde{\eta}_0)\to(u,\eta_0)$. Thus we can take the pull-back $\widetilde{g}_j$ of g_j for $j=1,\dots,d$ as generators of $\pi_1(\widetilde{M}^a_{\widetilde{\eta}_0}-\widetilde{M}^a_{\widetilde{\eta}_0}\cap\mathcal{C}^a_n(C;D))$. Let $\widetilde{\omega}=\widetilde{g}_d\cdots\widetilde{g}_1$. Then $\varphi_{n,\sharp}(\widetilde{\omega})=\omega$. Thus the assertion (2-b) follows from

$$\begin{aligned}\pi_1(\mathbf{P}^2-\mathcal{C}_n(C);\widetilde{b}_0)&\cong\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{C}^a_n(C;D);b_0)/\mathcal{N}(\widetilde{\omega})\\&\cong\pi_1(\mathbf{C}^2-C^a;b_0)/\mathcal{N}(\varphi_{n,\sharp}(\widetilde{\omega}))\\&\cong\pi_1(\mathbf{P}^2-C;b_0)\quad\text{as }\varphi_{n,\sharp}(\widetilde{\omega})=\omega\end{aligned}$$

where $\mathcal{N}(g)$ is the normal subgroup normally generated by g.

Now we consider the non-trivial case $a\ge b$ (Case (2-a)). Then $d=d'$ and $\deg f(x,y)=\deg_y f(x,y)$ and $nd=\deg f^{(n)}(\tilde{x},\tilde{y})=\deg_{\tilde{y}} f^{(n)}(\tilde{x},\tilde{y})$. Now we consider the vertical pencil $L_\eta=\{x=\eta\}$ for the computation of the monodromy relations for $\pi_1(\mathbf{C}^2-C^a)$. Take a generic pencil line L_{η_0} and let $C^a\cap L_{\eta_0}=\{\xi_1,\dots,\xi_d\}$. Now we take $R>0$ sufficiently large so that $C^a\cap L_{\eta_0}\subset\{\Im y>-R\}$ and $f(x,-R)$ has distinct d'' roots. We can assume that $\beta=-R$ by Proposition (3.3.2). Taking a change coordinates $(x,y)\mapsto(x,y+R)$, we may assume from the beginning that

$$D=\{y=0\},\quad C^a\cap L_{\eta_0}\subset\{y\in\mathbf{C};\Im y>0\}$$

We take the base point b_0 on the imaginary axis near the base point B_0 of the pencil as in §2 so that $\{|y|\le|b_0|/2\}\supset C^a\cap L_{\eta_0}$ and we take a system of generators $g_1,\dots,g_d$ of $\pi_1(L^a_{\eta_0}-C^a;b_0)$ represented as $g_j=[\mathcal{L}\circ\sigma_j\circ\mathcal{L}^{-1}]$ where $\mathcal{L}$ is the segment from b_0 to $b_0/2$ and σ_j is a loop in $\{\Im y>0\}\cap\{|y|\le|b_0|/2\}$ starting from $b_0/2$ and $\omega=g_d\cdots g_1$ is homotopic to the big circle

$\Omega : t \mapsto \exp(2\pi t i) b_0$. See the left side of Figure (3.4.A). Then by Proposition (2.3), we have

$$\pi_1(\mathbf{P}^2 - C) \cong \pi_1(\mathbf{C}^2 - C^a; b_0)/\mathcal{N}(\omega) \tag{3.4.2}$$

Now we consider the fundamental groups $\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a)$ and $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ using the pencil $\widetilde{L}_\eta = \{\tilde{x} = \eta\}$ in the source space $\widetilde{\mathbf{C}^2}$ of φ_n. We identify $\widetilde{L}^a_{\eta 0}$ with $\mathbf{C}$ by $\tilde{y}$-coordinate. Then by the definition of $\mathcal{C}_n(C)$, the intersection of $\mathcal{C}_n(C)^a \cap \widetilde{L}_{\eta 0}$ is n-th roots of ξ_j, for $j = 1, \dots, d$. As we have assumed $\Im \xi_j > 0$, $\mathcal{C}_n(C)^a \cap \widetilde{L}_{\eta 0}$ consists of nd points. So $\widetilde{L}_{\eta 0}$ is a generic line for $\mathcal{C}_n(C)$. Consider the conical region

$$D_j := \{(\eta_0, \tilde{y}) \in \widetilde{L}_{\eta 0}; 2\pi j/2n < \arg \tilde{y} < \pi(2j+1)/2n\}, \quad j = 0, \dots, n-1$$

is biholomorphic onto $\mathcal{H} = \{(\eta_0, y) \in L^a_{\eta 0}; \Im y > 0\}$ by φ_n. Thus the intersection $\widetilde{L}^a_{\eta 0} \cap \mathcal{C}_n(C)^a \cap D_j$ consists of d-points which correspond bijectively to those $L^a_{\eta 0} \cap C^a$. Let $b_0^{(j)} \in D_j, j = 0, \dots, n-1$ be the inverse image of the base point b_0 by φ_n and we may assume $\widetilde{b_0} = b_0^{(0)}$ for example. (As a complex number, $b_0^{(j)}$ is an n-th root of b_0 for $j = 0, \dots, n-1$.) Let $\widetilde{\omega}$ be the class of the big circle: $\widetilde{\omega} : [0,1] \to \widetilde{L}^a_{\eta 0}$, $\widetilde{\omega}(t) = \widetilde{b}_0 \exp(2\pi t i)$. We take the pull-back $g_1^{(j)}, \dots, g_d^{(j)}$ of $g_1, \dots, g_d$, in each conical region D_j. They gives a system of free generators of $\pi_1(D_j - \mathcal{C}_n(C)^a \cap \widetilde{L}^a_{\eta 0}; b_0^{(j)})$. Let ℓ_j be the arc : $t \mapsto e^{it} b_0^{(0)}$, $0 \le t \le 2j\pi/n$ which connects $b_0^{(0)}$ to $b_0^{(j)}$. We associate $g_i^{(j)}$ an element $g_{i,j}$ of $\pi_1(\widetilde{L}^a_{\eta 0} - \mathcal{C}_n(C)^a \cap \widetilde{L}^a_{\eta 0}; b_0^{(0)})$ by the change of the base point: $g_i^{(j)} \mapsto g_{i,j} := \ell_j g_i^{(j)} \ell_j^{-1}$. Thus $\{g_{i,j}; 1 \le i \le d,\ 0 \le j \le n-1\}$ is a system of free generators of $\pi_1(\widetilde{L}^a_{\eta 0} - \mathcal{C}_n(C)^a \cap \widetilde{L}^a_{\eta 0}; b_0^{(0)})$. See the right side of Figure (3.4.A).

Let $\omega_j = g_{d,j} \cdots g_{1,j}$ for $j = 0, \dots, n-1$. Then it is easy to see that

$$\widetilde{\omega} = \omega_{n-1} \cdots \omega_0 \tag{3.4.3}$$

and by Proposition (2.3), we have

$$\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C); b_0^{(0)}) = \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a; b_0^{(0)})/\mathcal{N}(\widetilde{\omega}) \tag{3.4.4}$$

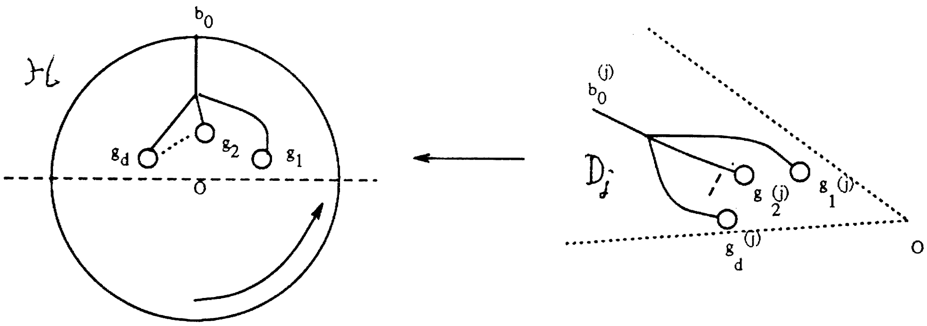


Figure (3.4.A)

Now we examine the isomorphism: $\varphi_{n\sharp} : \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a; b_0^{(0)}) \to \pi_1(\mathbf{C}^2 - C^a; b_0)$ more carefully. Note first that $\varphi_n(\ell_j)$ is j-times the big circle Ω: $t \mapsto b_0 \exp(2\pi t i)$, $0 \le t \le 1$. Thus it is homotopic to ω^j. Therefore we obtain

$$\varphi_{n\sharp}(g_{i,j}) = \omega^j g_i \omega^{-j}, \quad \varphi_{n\sharp}(\omega_j) = \omega \tag{3.4.5}$$

This implies that $\omega' = \omega_1 = \cdots = \omega_n$ and

$$\varphi_{n\sharp}(\widetilde{\omega}) = \omega^n \tag{3.4.6}$$

Thus the assertion follows immediately from the isomorphisms:

$$\begin{aligned}\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C); b_0^{(0)}) &\cong \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a; b_0^{(0)})/\mathcal{N}(\widetilde{\omega})\\ &\cong \pi_1(\mathbf{C}^2 - C^a; b_0)/\mathcal{N}(\varphi_{n\sharp}(\widetilde{\omega}))\\ &\cong \pi_1(\mathbf{C}^2 - C^a; b_0)/\mathcal{N}(\omega^n)\end{aligned}$$

In fact, by this isomorphism and (3.4.2) we have the canonical surjective homomorphism:

$$\Phi_n : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C); b_0^{(0)}) \to \pi_1(\mathbf{P}^2 - C; b_0)$$

which is defined by $\Phi_n(g_{i,j}) = g_i$. It is obvious that Φ_n makes the diagram in (2) of Theorem (3.4) commutative. This completes the proof of Theorem (3.4). □

Proof of Corollary (3.4.1). Assume that L_∞ is central. Then $\omega \in \mathcal{Z}(\pi_1(\mathbf{C}^2 - C^a; b_0))$. As $\varphi_{n\sharp}$ is an isomorphism, $\omega' \in \mathcal{Z}(\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C); b_0^{(0)}))$. Thus the normal subgroup $\mathcal{N}(\omega')$ of $\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C); b_0^{(0)})$ is simply the cyclic group $\langle \omega' \rangle$ generated by ω'. We consider the Hurewicz image of ω' in $H_1(\mathbf{P}^2 - \mathcal{C}_n(C))$. Suppose that C has r irreducible components C_j of degree d_j, $j = 1, \dots, r$. Then it is obvious that $\mathcal{C}_n(C)$ consists of r irreducible components $\mathcal{C}_n(C_1), \dots, \mathcal{C}_n(C_r)$ of degree $nd_1, \dots, nd_r$ respectively. For any fixed j, d_j-elements of $\{g_{1,j}, \dots, g_{d,j}\}$ are lassos for $\mathcal{C}_n(C_j)$. Thus ω' corresponds to the class $[\omega'] = (d_1, \dots, d_r)$ of $H_1(\mathbf{P}^2 - \mathcal{C}_n(C)) \cong \mathbf{Z}^r/(nd_1, \dots, nd_r)$. Thus $[\omega']$ has order n in the first homology group. As $\omega'^n = e$ already in $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$, $\mathrm{order}(\omega') = n$ and the kernel of Φ_n is a cyclic group of order n generated by ω'. This proves the first assertion (1).

As Φ_n is surjective, the commutator subgroup $\mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C; D)))$ by Φ_n is mapped surjectively onto the commutator subgroup $\mathcal{D}(\pi_1(\mathbf{P}^2 - C))$. On the other hand, the kernel $\mathbf{Z}/n\mathbf{Z}$ is injectively mapped to the first homology group $H_1(\mathbf{P}^2 - \mathcal{C}_n(C))$. Thus $\mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))) \cap \mathbf{Z}/n\mathbf{Z} = \{e\}$. Therefore Φ_n induces an isomorphism of the commutator groups. The sequence

$$1 \to \mathbf{Z}/n\mathbf{Z} \to \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))) \xrightarrow{\Psi'_n} \mathcal{Z}(\pi_1(\mathbf{P}^2 - C))$$

is clearly exact. We show the surjectivity of Ψ'_n. Take $h' \in \mathcal{Z}(\pi_1(\mathbf{P}^2 - C))$ and choose $h \in \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ so that $\Phi_n(h) = h'$. For any $g \in \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$, the image of the commutator $hgh^{-1}g^{-1}$ by Φ_n is trivial. Thus we can write $hgh^{-1}g^{-1} = \omega'^a$ for some $0 \le a \le n-1$. As $[\omega']$ has order n in first homology, this implies that $a = 0$ and thus $hg = gh$ for any g. Therefore h is in the center. The last exact sequence of the assertion (2) follows by a similar argument. This completes the proof of Corollary (3.4.1). □

Remark (3.5). (1) We remark that the rational map $\varphi'_n : \mathbf{P}^2 \to \mathbf{P}^2$ which is associated with φ_n is defined by $\varphi'_n([X; Y; Z]) = [XZ^{n-1}; Y^n; Z^n]$ and thus φ'_n is not defined at $\rho_\infty := [1; 0; 0] \in \mathcal{C}_n(C)$ and $\varphi'_n(\widetilde{L}_\infty - \{\rho_\infty\}) = \rho'_\infty = [0; 1; 0]$.
(2) In the case of $na > b > a$, there does not exist a surjective homomorphism $\Phi_n : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) \to \pi_1(\mathbf{P}^2 - C)$ in general. For example, take C' a smooth curve of degree d' and let $C = \mathcal{C}_2(C'; D')$ a generic two fold covering with respect to a generic line $D' := \{x = \alpha\}$. Then we take a

covering $\mathcal{C}_3(C;D)$ of degree 3 with respect to a generic $D := \{y = \beta\}$. Then we know that $\deg C = 2d'$ and $\deg \mathcal{C}_3(C;D) = 3d'$ and therefore $\pi_1(\mathbf{P}^2 - \mathcal{C}_3(C;D)) = \mathbf{Z}/3d'\mathbf{Z}$ and $\pi_1(\mathbf{P}^2 - \mathcal{C}_2(C';D')) = \mathbf{Z}/2d'\mathbf{Z}$. Thus there does not exist any surjective homomorphism.

(D) Generic cyclic covering

Now we consider the generic case:

$$(3.6) \qquad f(x,y) = \prod_{i=1}^{d}(y - \alpha_i x) + (\text{lower terms}),$$
$$\alpha_1, \dots, \alpha_d \in \mathbf{C}^*, \quad \alpha_i \neq \alpha_j \ (i \neq j)$$

This is always the case if we choose the line at infinity L_∞ to be generic and then generic affine coordinates (x,y). Take positive integers $n \geq m \geq 1$ and we denote $\mathcal{C}_n(C;D)$ by $\mathcal{C}_n(C)$ and $\mathcal{C}_m(\mathcal{C}_n(C;D);D')$ by $\mathcal{C}_{m,n}(C)$ where $D = \{y = \beta\}$ and $D' = \{x = \alpha\}$ with generic α, β. Note that $\mathcal{C}_n(C) = \mathcal{C}_{1,n}(C)$. The topology of the complement of $\mathcal{C}_{m,n}(C)$ depends only on C and m, n. We will refer $\mathcal{C}_n(C)$ and $\mathcal{C}_{m,n}(C)$ as *a generic n-fold* (respectively *a generic* (m,n)*-fold*) *covering transform* of C. They are defined in $\mathbf{C}^2$ by

$$\mathcal{C}_n(C)^a = \{(\tilde{x},\tilde{y}) \in \mathbf{C}^2; f(\tilde{x},\tilde{y}^n) = 0\},$$
$$\mathcal{C}_{m,n}(C)^a = \{(\tilde{x},\tilde{y}) \in \mathbf{C}^2; f(\tilde{x}^m,\tilde{y}^n) = 0\}$$

taking a change of coordinate $(x,y) \mapsto (x+\alpha, y+\beta)$ if necessary. If $n > m$, $\mathcal{C}_{m,n}(C)$ has only one singularity at $\rho_\infty = [1;0;0]$ and the local equation takes the following form:

$$\prod_{i=1}^{d}(\zeta^n - \alpha_i \xi^{n-m}) + (\text{higher terms}) = 0, \quad \zeta = Y/X, \xi = Z/X$$

Therefore $\mathcal{C}_{m,n}(C)$ is locally $d \times \gcd(m,n)$ irreducible components at ρ_∞. $(\mathcal{C}_{m,n}(C), \rho_\infty)$ is topologically equivalent to the germ of a Brieskorn singularity $B((n-m)d, nd)$ where $B(p,q) := \{\xi^p - \zeta^q\} = 0$. In the case $m = n$, we have no singularity at infinity. By Theorem (3.4) and Corollary (3.4.1), we have the following.

THEOREM (3.7). *Let $\mathcal{C}_n(C)$ and $\mathcal{C}_{m,n}(C)$ be as above. Then the canonical homomorphisms*

$$\pi_1(\widetilde{\widetilde{\mathbf{C}^2}} - \mathcal{C}_{m,n}(C)^a) \xrightarrow{\varphi_{m\sharp}} \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{C}_n(C)^a) \xrightarrow{\varphi_{n\sharp}} \pi_1(\mathbf{C}^2 - C^a)$$

and $\Phi_m : \pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C)) \to \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))$ are isomorphisms. There exist canonical central extensions of groups where the diagrams are commutative.

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}/n\mathbf{Z} & \xrightarrow{\iota} & \pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C)) & \xrightarrow{\Phi_{m,n}} & \pi_1(\mathbf{P}^2 - C) & \to & 1 \\ & & \downarrow \mathrm{id} & \circlearrowleft & \cong \downarrow \Phi_m & \circlearrowleft & \downarrow \mathrm{id} & & \\ 1 & \to & \mathbf{Z}/n\mathbf{Z} & \xrightarrow{\iota'} & \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) & \xrightarrow{\Phi_n} & \pi_1(\mathbf{P}^2 - C) & \to & 1 \end{array}$$

The kernel $\operatorname{Ker}\Phi_n$ *(respectively* $\operatorname{Ker}\Phi_{m,n}$*) is generated by an element ω' (resp. $\omega'' = \Phi_m^{-1}(\omega')$) in the center such that ω'^n (resp. ω''^n) is a lasso for $\widetilde{L}_\infty$ (resp. for $\widetilde{\widetilde{L}}_\infty$). The restriction of $\Phi_{m,n}$, Φ_m and Φ_n give an isomorphism of the respective commutator groups*

$$\Phi_{m,n,\mathcal{D}} : \mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C))) \xrightarrow{\Phi_{m,\mathcal{D}}} \mathcal{D}(\pi_1(\mathbf{P}^2 - \mathcal{C}_n(C))) \xrightarrow{\Phi_{n,\mathcal{D}}} \mathcal{D}(\pi_1(\mathbf{P}^2 - C))$$

and exact sequences of the centers and the first homology groups:

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & \mathcal{Z}(\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C))) & \xrightarrow{\Phi_{m,n}} & \mathcal{Z}(\pi_1(\mathbf{P}^2 - C)) & \to & 1 \\ 1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & H_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C)) & \xrightarrow{\overline{\Phi}_{m,n}} & H_1(\mathbf{P}^2 - C) & \to & 1 \end{array}$$

Let $\{\mathbf{a}_1, \dots, \mathbf{a}_s\}$ be singular points as before. Then $\mathcal{C}_n(C)$ (respectively $\mathcal{C}_{m,n}(C)$) has n copies (resp. nm copies) of $\mathbf{a}_i$ for each $i = 1, \dots, s$ and one singularity at $\rho_\infty := [1; 0; 0]$ except the case $n = m$. The curve $\mathcal{C}_{n,n}(C)$ has no singularity at infinity. The similar assertion for $\mathcal{C}_{n,n}(C)$ is obtained independently by Shimada [Sh].

COROLLARY (3.7.1). (1) *$\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C))$ is abelian if and only if $\pi_1(\mathbf{P}^2 - C)$ is abelian.*
(2) *Assume that C is irreducible. Then the fundamental groups $\pi_1(V(\mathcal{C}_{m,n}(C)))$ and $\pi_1(V(C))$ of the respective Milnor fibers $V(\mathcal{C}_{m,n}(C))$ of $\mathcal{C}_{m,n}(C)$ and $V(C)$ of C are isomorphic.*

Proof. The assertion (1) follows from Theorem (3.7). The assertion (2) is immediate from Proposition (2.7) and Theorem (3.7). □

The following is also an immediate consequence of Theorem (3.7) and Corollary (2.5).

Corollary (3.7.2). *$\widetilde{\widetilde{L}}_\infty$ is central for $\mathcal{C}_{m,n}(C)$ i.e., $\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C) \cup \widetilde{\widetilde{L}}_\infty)$ is isomorphic to the fundamental group of the generic affine complement of $\mathcal{C}_{m,n}(C)$.*

(E) Homologically injectivity condition of the center

The following is useful to produce Zariski pairs from a given Zariski pair (See §5). First we consider the following condition for a group G:

$$\text{(H.I.C)} \qquad \mathcal{Z}(G) \cap \mathcal{D}(G) = \{e\}$$

This is equivalent to the injectivity of the composition: $\mathcal{Z}(G) \hookrightarrow G \to H_1(G) := G/\mathcal{D}(G)$. When this condition is satisfied, we say that G satisfies *homological injectivity condition of the center* (or (H.I.C)-condition in short).

Theorem (3.8). *Let $C = C_1 \cup \cdots \cup C_r$ and $C' = C'_1 \cup \cdots \cup C'_r$ be projective curves with the same number of irreducible components and assume that* $\mathrm{degree}(C_i) = \mathrm{degree}(C'_i) = d_i$ *for $i = 1, \dots, r$ and assume that $\pi_1(\mathbf{P}^2 - C')$ satisfies (H.I.C)-condition. Assume that $\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C))$ and $\pi_1(\mathbf{P}^2 - \mathcal{C}_{m,n}(C'))$ are isomorphic for some integer m, n with $1 \le m \le n$. Then $\pi_1(\mathbf{P}^2 - C)$ and $\pi_1(\mathbf{P}^2 - C')$ are also isomorphic.*

Proof. We may assume that $m = 1$ by Theorem (3.7). Suppose that $\alpha : \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) \to \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C'))$ is an isomorphism. This induces isomorphisms of the respective commutator subgroups, centers and the first homology groups. We consider the exact sequences given by Corollary (3.4.1):

$$\begin{array}{ccccccccc}
1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C)) & \xrightarrow{\Phi_n} & \pi_1(\mathbf{P}^2 - C) & \to & 1 \\
 & & & & \downarrow \alpha & & & & \\
1 & \to & \mathbf{Z}/n\mathbf{Z} & \to & \pi_1(\mathbf{P}^2 - \mathcal{C}_n(C')) & \xrightarrow{\Phi'_n} & \pi_1(\mathbf{P}^2 - C') & \to & 1
\end{array}$$

Let ω' and ω'' be the generator of the kernels of Φ_n and Φ'_n respectively. As $[\omega'] = [(d_1, \dots, d_r)] \in H_1(\mathbf{P}^2 - \mathcal{C}_n(C)) = \mathbf{Z}^r/(nd_1, \dots, nd_r)$ in the notation of (2.1) and $[\omega']$ has order n, the homology class $[\alpha(\omega')]$ corresponding to $\alpha(\omega')$ has also order n in $H_1(\mathbf{P}^2 - \mathcal{C}_n(C'))$, thus $[\alpha(\omega')]$ is also anihilated by n. Therefore it is homologous to $[(ad_1, \dots, ad_r)] \in H_1(\mathbf{P}^2 - \mathcal{C}_n(C'))$ for some $a \in \mathbf{Z}$. This implies $[\Phi'_n(\alpha(\omega'))] = 0 \in H_1(\mathbf{P}^2 - C')$ and thus $\Phi'_n(\alpha(\omega')) \in \mathcal{D}(\pi_1(\mathbf{P}^2 - C'))$. Therefore $\Phi'_n(\alpha(\omega')) \in \mathcal{D}(\pi_1(\mathbf{P}^2 - C')) \cap \mathcal{Z}(\pi_1(\mathbf{P}^2 - C'))$. By the (H.I.C)-condition, this implies that $\Phi'_n(\alpha(\omega')) = e$. Thus by the above exact sequence, $\alpha(\omega') = (\omega'')^\beta$ for some $\beta \in \mathbf{N}$ with $\gcd(\beta, n) = 1$. Thus the restriction of α to $\mathrm{Ker}(\Phi_n) \cong \mathbf{Z}/n\mathbf{Z}$ is an isomorphism onto $\mathrm{Ker}(\Phi'_n) \cong \mathbf{Z}/n\mathbf{Z}$. Thus it induces an isomorphism : $\bar{\alpha} : \pi_1(\mathbf{P}^2 - C) \to \pi_1(\mathbf{P}^2 - C')$. □

Remark (3.9). (1) Take a non-generic line $D = \{y = \beta\}$ for C and consider the corresponding cyclic covering branched along D, $\varphi_n : \mathbf{C}^2 \to \mathbf{C}^2$. Then the assertions in Theorem (3.4) and Corollary (3.4.1) for the pull back $C' = \varphi_n^{-1}(C)$ may fail in general. For example, we can take the quartic defined by (5.1.1) in §5. Then L_∞ is central for C and $\pi_1(\mathbf{P}^2 - C) = \mathbf{Z}/4\mathbf{Z}$. Take $D = \{y = 0\}$ and consider $\varphi_2 : \mathbf{C}^2 \to \mathbf{C}^2$, $\varphi_2(x, y) = (x, y^2)$. Then the pull back Z_4 of C is a so called Zariski's three cuspidal quartic and $\pi_1(\mathbf{P}^2 - Z_4)$ ia a finite non-abelian group of order 12 ([Z1],[O5]). See also §5.
(2) We do not have any example of a plane curve C such that $\pi_1(\mathbf{P}^2 - C)$ does not satisfy the (H.I.C)-condition.

§4. Jung transforms of plane curves

Let C be a projective curve of degree d in $\mathbf{P}^2$ and let $f(x, y) = 0$ be the defining polynomial of C with respect to the affine space $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$. In this section, we introduce another operation which produces a projective curve $\mathcal{J}_n(C)$ of degree nd.

(A) Jung transform of degree n

First for any integer $n \geq 2$ we consider the following automorphism of $\mathbf{C}^2$ ([J]).

$$J_n : \mathbf{C}^2 \to \mathbf{C}^2, \quad J_n(x, y) = (x + y^n, y). \tag{4.1}$$

The inverse of J_n is given by $J_n^{-1}(x, y) = (x - y^n, y)$. Let $\mathcal{J}_n(C; L_\infty)$ be the projective closure of $J_n^{-1}(C^a)$. We call $\mathcal{J}_n(C; L_\infty)$ an *Jung transform* of C of degree n. By the definition, $\mathcal{J}_n(C; L_\infty)$ is birationally equivalent to C and the affine complements $\mathbf{C}^2 - C^a$ and $\mathbf{C}^2 - \mathcal{J}_n(C; L_\infty)^a$ are biholomorphic. We denote the source space of J_n by $\widetilde{\mathbf{C}^2}$, the line at infinity by $\widetilde{L}_\infty$ and the affine coordinates by $(\tilde{x}, \tilde{y})$ as in §3. By the definition, $\mathcal{J}_n(C; L_\infty)$ is defined in $\widetilde{\mathbf{C}^2}$ by

$$f^{(n)}(\tilde{x}, \tilde{y}) = f(\tilde{x} + \tilde{y}^n, \tilde{y}). \tag{4.2}$$

We say that J_n or the affine coordinates (x, y) is *an admissible* for C if $[1; 0; 0] \notin C$. We call $\mathcal{J}_n(C; L_\infty)$ an *admissible Jung transform* of C of degree n if J_n is admissible. Note that the admissibility of $\mathcal{J}_n$ implies that $\deg f^{(n)}(\tilde{x}, \tilde{y}) = nd$. Finally we call $\mathcal{J}_n(C; L_\infty)$ a *generic* Jung transform of C of degree n, if L_∞ is generic with respect to C and J_n is admissible for C. In this case, we denote $\mathcal{J}_n(C; L_\infty)$ simply by $\mathcal{J}_n(C)$.

(B) Singularities of $\mathcal{J}_{\mathbf{n}}(\mathbf{C}; \mathbf{L}_\infty)$

We consider the singularities of an admissible Jung transform $\mathcal{J}_n(C; L_\infty)$. Let $\mathbf{a}_1, \dots, \mathbf{a}_s$ be the singular points of C^a and let $\{\mathbf{a}_\infty^1, \dots, \mathbf{a}_\infty^k\} = C \cap L_\infty$ be the points at infinity. Let r_i be the number of local irreducible components of C at $\mathbf{a}_\infty^i$. As J_n is biholomorphic, the singularities of $\mathcal{J}_n(C; L_\infty)$ in $\mathbf{C}^2$ corresponds bijectively to $\mathbf{a}_1, \dots, \mathbf{a}_s$. Let $f(x, y) = f_d(x, y) + f_{d-1}(x, y) + \cdots + f_0$ be the homogeneous decomposition of f. By admissibility, we can write $f_d(x, y) = \prod_{i=1}^k (x - \alpha_i y)^{\nu_i}$ where $\alpha_1, \dots, \alpha_d \in \mathbf{C}$ are mutually distinct and $\sum_{i=1}^k \nu_i = d$. We may assume that $\mathbf{a}_\infty^i = (\alpha_i; 1; 0)$ in the homogeneous coordinates. Then the homogeneous polynomial which defines $\mathcal{J}_n(C; L_\infty)$ is

$$\begin{aligned} F^{(n)}(X, Y, Z) := \prod_{i=1}^k &(XZ^{n-1} + Y^n - \alpha_i Y Z^{n-1})^{\nu_i} \\ &+ \sum_{j=1}^d Z^{jn} f_{d-j}(XZ^{n-1} + Y^n, YZ^{n-1}) \end{aligned} \tag{4.3}$$

Thus $\deg \mathcal{J}_n(C; L_\infty) = nd$ and $\rho_\infty := [1; 0; 0]$ is the only intersection of $\mathcal{J}_n(C; L_\infty)$ with the line at infinity $\widetilde{L}_\infty$ and ρ_∞ is a singular point of $\mathcal{J}_n(C; L_\infty)$. The number of local irreducible components of $\mathcal{J}_n(C; L_\infty)$

at ρ_∞ is $\sum_{i=1}^k r_i$ and the local Milnor number $\mu(\mathcal{J}_n(C;L_\infty);\mathbf{a}_\infty)$ can be computed using the modified Plücker's formula :

$$(4.4)\quad \begin{aligned}\chi(\mathcal{J}_n(C;L_\infty)) &= 3nd - n^2d^2 + \sum_{i=1}^{s}\mu(C;\mathbf{a}_i) + \mu(\mathcal{J}_n(C;L_\infty);\mathbf{a}_\infty)\\ &= \chi(C) - k + 1\end{aligned}$$

Thus the Milnor number $\mu(\mathcal{J}_n(C;L_\infty);\mathbf{a}_\infty)$ is independent of the choice of the admissible affine coordinate (x,y) of $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$. As the space of the admissible affine coordinates are connected and a μ-constant family of plane curves are topologically equivalent to each other, we have:

PROPOSITION (4.5). *The topological type of the pair* $(\mathbf{P}^2, \mathcal{J}_n(C;L_\infty))$ *depend only on* C *and* L_∞ *and it does not depend on the choice of the admissible affine coordinates* (x,y)*. If* L_∞ *is generic, the topological type of the pair* $(\mathbf{P}^2, \mathcal{J}_n(C;L_\infty))$ *does not depend on* L_∞*.*

Let us study the structure of the singularity $\rho_\infty \in \mathcal{J}_n(C)$ of a generic admissible Jung transform of degree n in detail. Let $\zeta = Y/X, \xi = Z/X$ be affine coordinates centered at ρ_∞ of the affine space $\mathbf{P}^2 - \{X = 0\}$. Then local defining polynomial takes the following form:

$$(4.6)\quad h(\zeta,\xi) = \prod_{i=1}^{d}(\xi^{n-1} + \zeta^n - \alpha_i\zeta\xi^{n-1}) + \sum_{j=1}^{d}\xi^{jn} f_{d-j}(\xi^{n-1}+\zeta^n, \zeta\xi^{n-1})$$

$\mathcal{J}_n(C)$ has d irreducible components at ρ_∞. Consider an admissible toric modification

$$\pi : \mathbf{C}^2 \to \mathbf{C}^2, \quad \pi(u,v) = (\zeta,\xi), \quad \zeta = uv^{n-1},\ \xi = uv^n.$$

Then the defining polynomial changes into

$$\pi^* h(u,v) = v_1^{dn(n-1)}(-1)^{d(n-1)}\left(\prod_{i=1}^{d}(u_1 + \alpha_i v_1^{n-1}) + (\text{higher terms})\right)$$

where $u_1 := u+1, v_1 := v$ are local coordinates at $(u,v) = (-1,0)$. Thus we see that the Newton boundary of $\pi^* h$ in (u_1,v_1) is non-degenerate. Thus the resolution complexity $\varrho(\mathcal{J}_n(C);\rho_\infty)$ is two for $n\ge 3$. See [Le-Oka] for the definition of the resolution complexity. The Milnor number is given by $\mu(\mathcal{J}_n(C);\rho_\infty)=d^2(n^2-1)-d(3n-2)+1$. (In the case of $n=2$, the resolution complexity $\varrho(\mathcal{J}_n(C);\rho_\infty)$ is 1.) The germ $(\mathcal{J}_n(C);\rho_\infty)$ is topologically determened by the first term of (4.6) and it is equivalent to $B(n-1,n;d):=\{(\xi^{n-1}+\zeta^n)^d-(\zeta\xi^{n-1})^d=0\}$.

(C) Main results of this section

Now we state the main result of this section.

THEOREM (4.7). *Assume that L_∞ is central for C and let $J_n:\widetilde{\mathbf{C}^2}\to\mathbf{C}^2$ be an admissible Jung transform of degree n of C. Then $\widetilde{L}_\infty$ is central for $\mathcal{J}_n(C;L_\infty)$ and there exists a unique surjective homomorphism $\Psi_n:\pi_1(\mathbf{P}^2-\mathcal{J}_n(C;L_\infty))\to\pi_1(\mathbf{P}^2-C)$ which gives the following commutative diagram*

$$\begin{array}{ccc}\pi_1(\mathbf{P}^2-\mathcal{J}_n(C;L_\infty)) & \xrightarrow{\Psi_n} & \pi_1(\mathbf{P}^2-C)\\ \uparrow{\scriptstyle\widetilde{\iota}_\sharp} & & \uparrow{\scriptstyle\iota_\sharp}\\ \pi_1(\widetilde{\mathbf{C}^2}-\mathcal{J}_n(C;L_\infty)^a) & \xrightarrow{J_{n\sharp}} & \pi_1(\mathbf{C}^2-C^a)\end{array}$$

where $\widetilde{\iota}_\sharp$ and $\iota_\sharp$ are associated with the respective inclusion maps. Ψ_n has the following property.

(1) The kernel of Ψ_n is a cyclic group of order n which is a subgroup of the center. So we have a central exactension of groups:

$$1\to\mathbf{Z}/n\mathbf{Z}\xrightarrow{\alpha}\pi_1(\mathbf{P}^2-\mathcal{J}_n(C;L_\infty))\xrightarrow{\Psi_n}\pi_1(\mathbf{P}^2-C)\to 1$$

The image $\alpha(\mathbf{Z}/n\mathbf{Z})$ is generated by $\widetilde{\iota}_\sharp(\omega')$ where $\omega':=J_{n\sharp}^{-1}(\omega)$, ω is a lasso for L_∞ in the base space $\mathbf{P}^2\supset C$, and ω'^n is a lasso for the line at infinity $\widetilde{L}_\infty$.

(2) The restriction of Ψ_n gives an isomorphism $\Psi_n:\mathcal{D}(\pi_1(\mathbf{P}^2-\mathcal{J}_n(C;L_\infty)))\to\mathcal{D}(\pi_1(\mathbf{P}^2-C))$ and the following exact sequences of the centers and the first homology groups:

$$1\to\mathbf{Z}/n\mathbf{Z}\to\mathcal{Z}(\pi_1(\mathbf{P}^2-\mathcal{J}_n(C;L_\infty)))\xrightarrow{\Psi_n}\mathcal{Z}(\pi_1(\mathbf{P}^2-C))\to 1$$

$$1\to\mathbf{Z}/n\mathbf{Z}\to H_1(\mathbf{P}^2-\mathcal{J}_n(C;L_\infty))\xrightarrow{\overline{\Psi}_n}H_1(\mathbf{P}^2-C)\to 1$$

PROOF. First we note that $[1;0;0]\notin C$ by admissibility and

$$J_{n\sharp}:\pi_1(\widetilde{\mathbf{C}^2}-\mathcal{J}_n(C;L_\infty)^a)\to\pi_1(\mathbf{C}^2-C^a) \tag{4.7.1}$$

is an isomorphism as J_n is an automorphism of $\mathbf{C}^2$. We consider the pencil $L_\eta = \{Y = \eta Z\}, \eta \in \mathbf{C}$ in the original affine space $\mathbf{C}^2$. The base point B_0 of the pencil is $[1;0;0]$. We fix a generic η_0 with $|\eta_0|$ large enough and we take generators $g_1, \dots, g_d$ of $\pi_1(L^a_{\eta_0} - L^a_{\eta_0} \cap C^a; b_0)$ as before so that

$$(4.7.2) \qquad g_d \cdots g_1 = \omega, \quad \pi_1(\mathbf{P}^2 - C; b_0) \cong \pi_1(\mathbf{C}^2 - C; b_0)/\langle \omega \rangle$$

where ω is in the center of $\pi_1(\mathbf{C}^2 - C; b_0)$ and ω^{-1} is a lasso for L_∞. We choose base point $\widetilde{b}_0$ of $\widetilde{\mathbf{C}^2} - \mathcal{J}_n(C; L_\infty)$ so that $J_n(\widetilde{b}_0) = b_0$. In $\widetilde{\mathbf{C}^2}$, we consider the pencil $\widetilde{M}_\xi = \{\tilde{x} = \xi\}$. We may assume that $\widetilde{b}_0 \in \widetilde{M}_{\xi_0}$ and $\widetilde{M}_{\xi_0}$ is generic for $\mathcal{J}_n(C; L_\infty)$. By the definition, $J_n(\tilde{x}, \tilde{y}) = (\tilde{x} + \tilde{y}^n, \tilde{y})$. Thus $J_n(\widetilde{M}_\xi) = M_\xi$ where M_ξ is a rational curve defined by $\{x - y^n = \xi\}$. Note that $M_\xi \cap \mathbf{C}^2$ is isomorphic to a line $\mathbf{C}$ and $M_{\xi_0} \cap C^a$ consists of nd distinct points. Let $\widetilde{\omega}$ be the class of a big disk $\partial \widetilde{\Delta}$ (counter-clockwise oriented) in $\widetilde{M}_{\xi_0}$ where $\widetilde{\Delta} = \{(\xi_0, \widetilde{y}); |\widetilde{y}| \le |\widetilde{y}_0|\}$ where $\widetilde{b}_0 = (\xi_0, \widetilde{y}_0)$. By Proposition (2.3) and (4.7.1), we have

$$(4.7.3) \qquad \begin{aligned} \pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty); \widetilde{b}_0) &= \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{J}_n(C; L_\infty)^a; \widetilde{b}_0)/\langle \widetilde{\omega} \rangle \\ &= \pi_1(\mathbf{C}^2 - C^a; b_0)/\langle J_{n\sharp}(\widetilde{\omega}) \rangle \end{aligned}$$

Thus we need to know the image $J_{n\sharp}(\widetilde{\omega})$. Let $\omega' = J_{n\sharp}^{-1}(\omega) \in \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{J}_n(C; L_\infty); \widetilde{b}_0)$.

LEMMA (4.7.4). *$J_{n\sharp}(\widetilde{\omega}) = \omega^n$, $\omega'^n = \widetilde{\omega}$ and the order of $\widetilde{\iota}_\sharp(\omega')$ in $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty))$ is n.*

Assuming this for a moment, we complete the proof of Theorem (4.7). As $J_{n\sharp}$ is an isomorphism, $\omega' \in \mathcal{Z}(\pi_1(\widetilde{\mathbf{C}^2} - \mathcal{J}_n(C; L_\infty); \widetilde{b}_0))$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty); \widetilde{b}_0) \cong \pi_1(\mathbf{C}^2 - C^a; b_0)/\langle \omega^n \rangle$ by (4.7.3). Combining this with (4.7.2), we get a central extension

$$1 \to \langle \widetilde{\iota}_\sharp(\omega') \rangle \to \pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty); \widetilde{b}_0) \xrightarrow{\Psi_n} \pi_1(\mathbf{P}^2 - C; b_0) \to 1$$

where Ψ_n is the quotient homomorphism which is associated with the above identification. This proves (1). The assertion (2) can be proved by the exact same way as in the proof of Corollary (3.4.1). □

Proof of Lemma (4.7.4). The main difficulty is that the image of a pencil line $\widetilde{M}_\xi$ is not a pencil line but it is a smooth rational curve M_ξ and it is not so easy to see how ω' and $\widetilde{\omega}$ are related. First observe that the base point of the family of rational curves $M_\xi,\ \xi \in \mathbf{C}$ is $[1;0;0]$. We take $R > 0$ sufficiently large so that L_η is generic for C for any η with $|\eta| \ge R, \eta \ne \infty$. As we are going to study the behavior of M_ξ and L_η for $|\xi|, |\eta| \to \infty$, it is convenient to take another affine coordinates $s = X/Y, t = Z/Y$ for the affine space $\mathbf{P}^2 - \{Y = 0\}$. We identify $L_\eta,\ \eta \ne 0$ with $\mathbf{P}^1$ by the rational mapping $s : L_\eta \cong \mathbf{P}^1$. Note that B_0 corresponds to $\infty \in \mathbf{P}^1$ and $L_\eta - \{B_0\}$ is identified with $\mathbf{C} = \mathbf{P}^1 - \{\infty\}$. In this affine coordinates, L_η is defined by $L_\eta = \{t = \eta^{-1}\}$. We choose a positive number S so that $D_S(\eta) \supset L_\eta \cap C$ for any $|\eta| \ge R$ where

$$D_S(\eta) := \{(s,t) \in L_\eta; t = \eta^{-1},\ |s| \le S\}.$$

We can assume that $|\eta_0| \ge R$ and ω which is represented by a loop $\mathcal{L}\circ\Omega\circ\mathcal{L}^{-1}$ where $\mathcal{L}$ is a line segment on the imaginary axis connecting b_0 and $2Si \in \partial D_{2S}(\eta_0)$ and $\Omega = \partial D_{2S}(\eta_0)$ as before. To show the assertion, we look at the behavior of M_ξ when $\xi \to \infty$. Put $\xi_0 = 1/\varepsilon_0^n$. M_{ξ_0} is defined by

$$M_{\xi_0} = \{[X;Y;Z] \in \mathbf{P}^2; \xi_0^{-1}(XZ^{n-1} - Y^n) = Z^n\}$$

and $M_{\xi_0} \cap \{Y = 0\} = \{[1;0;0], [\xi_0;0;1]\}$. In the affine space $\mathbf{P}^2 - \{Y = 0\}$, we have $M_{\xi_0} \cap \{Y \ne 0\} = \{(s,t) \in \mathbf{C}^2; t^n = \varepsilon_0^n(st^{n-1} - 1)\}$. The affine equation can be rewritten as

$$(4.7.5)\qquad M_{\xi_0} \cap \{Y \ne 0\} = \{(s,t) \in \mathbf{C}^2; (t/\varepsilon_0)^n = -1 + \varepsilon_0^{n-1} s\,(t/\varepsilon_0)^{n-1}\}$$

Thus we see that $\lim_{\varepsilon_0 \to 0} t/\varepsilon_0 = \theta_j$ for some $j = 0, \dots, n-1$ where $\theta_j = \exp((2j+1)\pi i/n)$. Thus the curve M_{ξ_0} behaves approximately like the union of n lines $L_{\xi_{0,0}} \cup \cdots \cup L_{\xi_{0,n-1}}$ outside of B_0 where $\xi_{0,j}^{-1} = \varepsilon_0\theta_j$ when $\varepsilon_0 \to 0$. To see this assertion more precisely, we consider the projection

$$\varphi_{\xi_0} : M_{\xi_0} \to L_\infty \cong \mathbf{P}^1, \quad \varphi_{\xi_0}([X;Y;Z]) = [X;Y], \quad \varphi_{\xi_0}(s,t) = (s,0)$$

By an easy computation, we see that φ_{ξ_0} is an n-fold covering branched over

$$\Sigma(\varphi_{\xi_0}) := \{s \in \mathbf{C}; s^n = \varepsilon_0^{-n(n-1)} n^n/(n-1)^{n-1}\} \cup \{[1;0;0]\}.$$

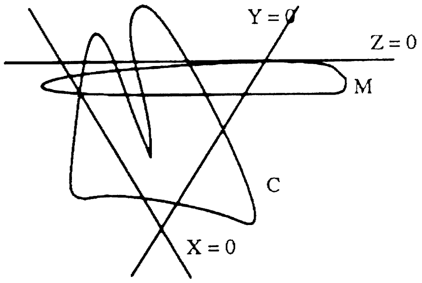


Figure (4.7.A) ($n = 2,\ M = M_{\xi_0}$)

Here $s \in \mathbf{C}$ corresponds to $[s;1;0]$. Note that $|\Sigma(\varphi_{\xi_0})| = n+1$ and each point of $\Sigma(\varphi_{\xi_0})$ goes to infinity when $\varepsilon_0 \to 0$. Thus $\Sigma(\varphi_{\xi_0}) \cap D_{2S} = \emptyset$ as long as $|\varepsilon_0|$ is small enough where $D_{2S} := \{(s,0) \in L_\infty^a; |s| \le 2S\}$.

Let $\Delta_j(\xi_0), j = 1,\dots,n$ be the connected components of $\varphi_{\xi_0}^{-1}(D_{2S}) \cap M_{\xi_0}$. Here we may assume that $\Delta_j(\xi_0)$ is sufficiently near to $D_{2S}(\xi_{0,j})$ so that $\Delta_j(\xi_0)$ contains exactly d points of $M_{\xi_0} \cap C^a$ in its interior which are sufficiently near $L_{\xi_{0,j}} \cap C^a$. Let $\Omega_j := \partial\Delta_j(\xi_0)$. Then by the above observation, Ω_j is free homotopic to $\partial D_{2S}(\xi_{0,j}) \subset L^a_{\xi_{0,j}}$ in $\mathbf{C}^2 - C^a$ by the homotopy $H : \Delta_j \times [0,1] \to \mathbf{C}^2 - C^a$ which is defined by $H(s,t,\tau) = (1-\tau)(s,t) + \tau(s, \varepsilon_0\theta_j)$. Recall that $\partial D_{2S}(\xi_{0,j})$ is free homotopic to a bracelet of L_∞. Therefore Ω_j is also free homotopic to a bracelet of L_∞. We have assumed that $b_0 \in M_{\xi_0} \cap L_{\eta_0}$. Thus we can choose a point $b_j \in \partial\Delta_j(\xi_0)$ and a simple path ℓ_j from b_0 to b_j in M_{ξ_0} so that $\ell_j \cap \ell_k = \{b_0\}$ and the following property is satisfied. Let

$$\omega_j := [\ell_j \circ \Omega_j \circ \ell_j^{-1}] \in \pi_1(\mathbf{C}^2 - C; b_0),$$
$$\widetilde{\omega}_j = J_n^{-1}(\widetilde{\omega}_j) \in \pi_1(\widetilde{\mathbf{C}^2} - \mathcal{J}_n(C; L_\infty); \widetilde{b}_0).$$

Then $\omega_n \cdots \omega_1$ is homotopic to the counter-clockwise oriented big circle $\Omega := \{(x,y) \in M_{\xi_0}; |y| = |y(b_0)|\}$ in M_{ξ_0} and

$$\text{(4.7.6)} \qquad \widetilde{\omega} = \widetilde{\omega}_n \cdots \widetilde{\omega}_1 \in \pi_1(\mathbf{C}^2 - \mathcal{J}_n(\widetilde{C}; L_\infty); \widetilde{b}_0)$$

Figure (4.7.B) shows these loops in $M^a_{\xi_0} \cong \mathbf{C}$. On the other hand, $\omega_j = \omega$ as Ω_j is free homotopic to Ω in $\mathbf{C}^2 - C^a$. As $J_{n\sharp}$ is an isomorphism, we

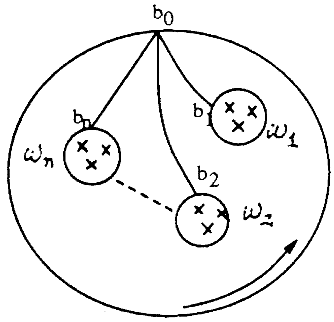


Figure (4.7.B) ($d = 3$)

conclude that $\omega' = \widetilde{\omega}_1 = \cdots = \widetilde{\omega}_n$ and $J_{n\sharp}(\widetilde{\omega}) = \omega^n$. The order of $\widetilde{\omega}_j$ in $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty))$ is n by the exact same homological argument as in the proof of Theorem (3.4). This completes the proof of Lemma (4.7.4). □

(D) Corollaries

The proofs of the following Corollaries are given by the exact same way as those of Corollaries (3.7.1), (3.7.2) and Theorem (3.8).

COROLLARY (4.8). *Let $J_n : \mathbf{C}^2 \to \mathbf{C}^2$ be an admissible Jung transform of degree n with respect to a central line at infinity L_∞. Then we have the following.*
(1) *$\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty))$ is abelian if and only if $\pi_1(\mathbf{P}^2 - C)$ is abelian.*
(2) *Assume that C is irreducible. Then $\pi_1(V(\mathcal{J}_n(C; L_\infty))) \cong \pi_1(V(C))$ where $V(\mathcal{J}_n(C; L_\infty))$ and $V(C)$ are respective Milnor fibers of $\mathcal{J}_n(C; L_\infty)$ and C.*

COROLLARY (4.9). *Let $J_n : \mathbf{C}^2 \to \mathbf{C}^2$ be an admissible Jung transform of degree n with respect to a central line at infinity L_∞. Then $\widetilde{L}_\infty$ is central for $\mathcal{J}_n(C; L_\infty)$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty) \cup \widetilde{L}_\infty)$ is isomorphic to the fundamental group of a generic affine complement of $\mathcal{J}_n(C; L_\infty)$.*

COROLLARY (4.10). *Let $J_n : \mathbf{C}^2 \to \mathbf{C}^2$ be an admissible Jung transform of degree n with respect to a central line at infinity L_∞. Let $C =$*

$C_1 \cup \cdots \cup C_r$ and $C' = C_1' \cup \cdots \cup C_r'$ be projective curves with the same number of irreducible components and assume that $\mathrm{degree}(C_i) = \mathrm{degree}(C_i') = d_i$ *for $i = 1, \dots, r$. We assume that either $\pi_1(\mathbf{P}^2 - C)$ or $\pi_1(\mathbf{P}^2 - C')$ satisfies (H.I.C)-condition and that $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty))$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(C'))$ are isomorphic. Then $\pi_1(\mathbf{P}^2 - C)$ and $\pi_1(\mathbf{P}^2 - C')$ are isomorphic.*

Remark (4.11). (1) In the definition of an admissible Jung transform, we can take an affine automorphism

$$J_n' : \mathbf{C}^2 \to \mathbf{C}^2, \quad (x, y) \mapsto (x + h_n(y), y)$$

where $h_n(y)$ is an arbitrary polynomial of degree n. Let $\mathcal{J}_n'(C; L_\infty)$ be the closure of ${J_n'}^{-1}(C^a)$. Then the topological type of the pair $(\mathbf{P}^2, \mathcal{J}_n'(C; L_\infty))$ is equal to that of $(\mathbf{P}^2, \mathcal{J}_n(C; L_\infty))$.
(2) If $J_n : \mathbf{C}^2 \to \mathbf{C}^2$ is admissible but L_∞ is not necessarily central, there exists a surjective homomorphism $\Psi_n : \pi_1(\mathbf{P}^2 - \mathcal{J}_n(C; L_\infty)) \to \pi_1(\mathbf{P}^2 - C)$. In fact, assuming the admissibility $[1; 0; 0] \notin C$, J_n ca be extended a birational mapping $J_n' : \mathbf{P}^2 \to \mathbf{P}^2$ defined by $J_n'([X; Y; Z]) = [XZ^{n-1} + Y^n; YZ^{n-1}; Z^n]$. J_n' is well-defined on $\mathbf{P}^2 - \{[1; 0; 0]\}$ and $J_n'(\widetilde{L}_\infty - \{[1; 0; 0]\}) = [1; 0; 0]$. So $J_n' : \mathbf{P}^2 - \mathcal{J}_n(C; L_\infty)) \to \mathbf{P}^2 - C$ is well-defined. However $\mathrm{Ker}\,\Psi_n$ is not necessarily a cyclic group of order n. We will see an example in Theorem (6.7) in §6.

§5. Zariski's quartic and Zariski pairs

In this section, we apply the results of §3 and §4 to construct plane curves whose complement have interesting fundamental groups.

(A) Zariski's three cuspidal quartics

Let Z_4 be an irreducible quartic with three cusps. Such a curve is a rational curve. For example, we can take the following curve which is defined in $\mathbf{C}^2$ by the following equation ([O6]):

$$(5.1) \qquad Z_4^a = \{(x, y) \in \mathbf{C}^2; x^3(3x + 8) - 6x^2(y^2 - 1) - (y^2 - 1)^2 = 0\}$$

We call such a curve *a Zariski's three cuspidal quartic.* It is known that the fundamental group $\pi_1(\mathbf{C}^2 - Z_4)$ and $\pi_1(\mathbf{P}^2 - Z_4)$ have the following representations ([Z1],[O6]):

$$(5.2) \qquad \begin{cases} \pi_1(\mathbf{C}^2 - Z_4) & = \langle \rho, \xi; \{\rho, \xi\} = e, \ \rho^2 = \xi^2 \rangle \\ \pi_1(\mathbf{P}^2 - Z_4) & = \langle \rho, \xi; \{\rho, \xi\} = e, \ \rho^2 = \xi^2, \rho^4 = e \rangle \end{cases}$$

where ρ and ξ are lassos for C and $\{\rho,\xi\} := \rho\xi\rho\xi^{-1}\rho^{-1}\xi^{-1}$. The relation $\{\rho,\xi\} = e$ is equivalent to $\rho\xi\rho = \xi\rho\xi$. A lasso ω for L_∞ is given by $\rho^2\xi^2 (= \rho^4)$. Recall that ω^{-1} is a lasso for L_∞ and is contained in the center. A Zariski's three cuspidal quartic is the first example whose complement has a non-abelian finite fundamental group. We first recall the proof of the finiteness.

Lemma (5.3) ([Z1]). *Put*

$$G_1 = \langle \rho, \xi; \{\rho,\xi\} = e,\ \rho^2 = \xi^2, \rho^4 = e\rangle.$$

Then G_1 is a finite group of order 12 such that $\mathcal{D}(G_1) = \langle \rho^2\xi\rho\rangle \cong \mathbf{Z}/3\mathbf{Z}$, $\mathcal{Z}(G_1) = \langle \rho^2\rangle \cong \mathbf{Z}/2\mathbf{Z}$ and $H_1(G_1) \cong \mathbf{Z}/4\mathbf{Z}$ and it is generated by the class of ρ

Proof. Let $g \in G_1$. First, using the relations $\rho^4 = \xi^4 = e$, $\rho^2 = \xi^2$ and $\rho\xi\rho = \xi\rho\xi$, we can write g in one of the following expression: $\rho^\alpha,\ \rho^\alpha\xi,\ \rho^\alpha\xi\rho$ for $0 \le \alpha \le 3$. This observation already shows that $|G_1| \le 12$. Let G_1' be the subgroup of $\mathfrak{S}_{12}$ generated by $\sigma := (1,2,3,4)(5,6,7,8)(9,10,11,12)$ and $\tau := (1,5,3,7)(2,9,4,11)(6,10,8,12)$. It is easy to see that σ and τ satisfies the relations: $\sigma\tau\sigma = \tau\sigma\tau,\quad \sigma^2 = \tau^2\quad \sigma^4 = e$. Thus we have a homomorphism $\psi : G_1 \to G_1'$ defined by $\psi(\rho) = \sigma$ and $\psi(\xi) = \tau$. By an easy computation, we see that $\sigma\tau$ has order 6. As $\operatorname{order}\sigma = 4$, $\sigma \notin \langle\sigma\tau\rangle$. This implies that $|G_1'| \ge 12$. It is also easy to see $|G_1'| = 12$ directly. Thus we conclude that $|G_1| = |G_1'| = 12$ and ψ is an isomorphism. Taking abelianization of the above relations, we get that $\bar\rho = \bar\xi$, $4\bar\rho = 0$ i.e., $H_1(G_1)$ is a cyclic group of order 4 which is generated by $\bar\rho = \bar\xi$. This implies that $|\mathcal{D}(G_1)| = 3$. Let β be the commutator $[\rho,\xi]$. Then $\beta = \rho\xi\rho^{-1}\xi^{-1} = \rho\xi\rho^3\xi^3 = \rho^2\xi\rho$ and $\psi(\beta) = [\sigma,\tau] = (1,8,11)(2,12,5)(3,6,9)(4,10,7)$. Thus β has order 3 and therefore β generates the commutator subgroup. We can show by an easy computation that $\mathcal{Z}(G_1) = \langle\rho^2\rangle \cong \mathbf{Z}/2\mathbf{Z}$. □

We consider the Hurewicz exact sequence:

$$(5.4)\qquad 1 \to \mathcal{D}(G_1) \cong \mathbf{Z}/3 \xrightarrow{\iota_1} G_1 \xrightarrow{\psi} H_1(G_1) \cong \mathbf{Z}/4\mathbf{Z} \to 1$$

This sequence splits by taking the section $\bar\rho \mapsto \rho$ of ψ so that G_1 has a structure of a semi-direct product of $\mathbf{Z}/3\mathbf{Z}$ and $\mathbf{Z}/4\mathbf{Z}$. More precisely, the semi-direct structure is given by $\rho\beta\rho^{-1} = \beta^2$ as $\rho\beta\rho^{-1} = \rho(\rho^2\xi\rho)\rho^{-1} = \rho^3\xi = \beta^2$.

(B) Generic transforms of a Zariski's quartic

Let $\mathcal{C}_n(Z_4)$ (respectively $\mathcal{C}_{n,n}(Z_4)$) be a generic cyclic transform of degree n (resp. of (n,n)) of the Zariski's quartic Z_4 and let $\mathcal{J}_n(Z_4)$ be a generic Jung transform of degree n of the Zariski's quartic Z_4. The singularities of $\mathcal{C}_n(Z_4)$ (respectively of $\mathcal{C}_{n,n}(Z_4)$) are $3n$ cusps (resp. $3n^2$ cusps). $\mathcal{C}_n(Z_4)$ has one more singularity at $\rho_\infty \in L_\infty$ and $(\mathcal{C}_n(Z_4), \rho_\infty)$ is equal to $B((n-1)d, nd) := \{\zeta^{nd} - \xi^{d(n-1)}\} = 0\}$. On the other hand, $\mathcal{J}_n(Z_4)$ is a rational curve which has 3 cusps and one more singularity at infinity $\rho_\infty \in \mathcal{J}_n(Z_4) \cap L_\infty$. $(\mathcal{J}_n(Z_4), \rho_\infty)$ is topologically equal to $B(n-1, n; d) := \{(\xi^{n-1} + \zeta^n)^d - (\zeta\xi^{n-1})^d = 0\}$. By Corollary (3.4.1) and Theorem (4.7), we have the following:

Theorem (5.5). *The affine fundamental groups $\pi_1(\mathbf{C}^2 - \mathcal{C}_n(Z_4)^a)$, $\pi_1(\mathbf{C}^2 - \mathcal{J}_n(Z_4)^a)$ are isomorphic to $\pi_1(\mathbf{C}^2 - Z_4) \cong \langle \rho_n, \xi_n; \{\rho_n, \xi_n\} = e,\ \rho_n^2 = \xi_n^2 \rangle$.*
(1) The projective fundamental groups $\pi_1(\mathbf{P}^2 - \mathcal{C}_n(Z_4))$ and $\pi_1(\mathbf{P}^2 - \mathcal{J}_n(Z_4))$ are isomorphic to G_n where G_n is defined by $G_n := \langle \rho_n, \xi_n; \{\rho_n, \xi_n\} = e,\ \rho_n^2 = \xi_n^2, \rho_n^{4n} = e \rangle$. Moreover we have a central extension of groups:

$$(5.5.1) \qquad 1 \to \mathbf{Z}/n\mathbf{Z} \to G_n \xrightarrow{\Phi_n} G_1 \to 1$$

defined by $\Phi_n(\rho_n) = \rho$ and $\Phi_n(\xi_n) = \xi$ and $\operatorname{Ker}\Phi_n$ is generated by ρ_n^4. In particular, we have $|G_n| = 12n$, $\mathcal{D}(G_n) = \langle \beta_n \rangle \cong \mathbf{Z}/3\mathbf{Z}$ where $\beta_n = [\rho_n, \xi_n]$ and $\mathcal{Z}(G_n) = \langle \rho_n^2 \rangle \cong \mathbf{Z}/2n\mathbf{Z}$.
(2) The Hurewicz sequence $1 \to \mathcal{D}(G_n) \to G_n \to H_1(G_n) \to 1$ has a canonical cross section $\theta : H_1(G_n) \to G_n$ which is given by $\theta(\bar\rho_n) = \rho_n$. This gives G_n a structure of semi-direct product $\mathbf{Z}/3$ and $\mathbf{Z}/4n\mathbf{Z}$ which is determined by $\rho_n \beta_n \rho_n^{-1} = \beta_n^2$.
(3) G_n is identified with the subgroup of the permutation group $\mathfrak{S}_{12n}$ of $12n$ elements $\{x_i, y_j, z_k; 1 \le i, j, k \le 4n\}$ generated by two permutations: $\sigma_n = (x_1, \dots, x_{4n})(y_1, \dots, y_{4n})(z_1, \dots, z_{4n})$ and $\tau_n = (x_1, y_1, x_3, y_3, \dots, x_{4n-1}, y_{4n-1})(x_2, z_1, x_4, z_3, \dots, x_{4n}, z_{4n-1})(y_2, z_2, y_4, z_4, \dots, y_{4n}, z_{4n})$.

Proof. The assertions (1) and (2) is due to Theorem (3.7) and Theorem (4.7). We prove the assertion about the semi-direct structure in (2). Note that any element of G_n can be uniquely written as one of $\rho^i,\ \rho^i\xi_n,\ \rho^i\xi_n\rho_n$ for $0 \le i \le 4n-1$. Let $\beta_n = [\rho_n, \xi_n] \in \mathcal{D}(G_n)$. Then by an easy computation, we have $\beta_n = \rho_n^{4n-2}\xi_n\rho_n$, $\beta^2 = \rho_n^{4n-1}\xi_n$ and and $\rho_n\beta_n\rho_n^{-1} = \rho_n^{4n-1}\xi_n = \beta_n^2$. Finally we prove the assertion (3). It is easy to see that $\{\sigma_n, \tau_n\}$ satisfies the relations: $\{\sigma_n, \tau_n\} = e, \quad \sigma_n^2 = \tau_n^2, \quad \sigma_n^{4n} = e$. Thus we have a homomorphism $\phi : G_n \to \mathfrak{S}_{12n}$ which is defined by $\phi(\rho_n) = \sigma_n$ and $\phi(\xi_n) = \tau_n$. Let G_n' be the image. As we know $|G_n| = 12n$ and $\mathrm{ord}(\sigma_n) = 4n$, we have either $|G_n'| = 4n$ or $12n$. As $\tau_n \notin \langle\sigma_n\rangle$, we must have $|G_n'| = 12n$, which implies that $\phi : G_n \to G_n' \subset \mathfrak{S}_{12n}$ is an isomorphism. □

Remark (5.6). Composing the cyclic and Jung transformations, we can produce many different types of singularities with the same fundamental group. For example, there are at least 7 types of curves $C_i,\ i = 1, \dots, 7$ of degree 12 whose complements have the fundamental group G_3 as follows. (In the list, $\Sigma(C_i)$ is the singularities of C_i.)
1. $C_1 = \mathcal{C}_{1,3}(Z_4)$ and $\Sigma(C_1) = \{9B(2,3) + B(8,12)\}$. 2. $C_2 = \mathcal{C}_{2,3}(Z_4)$ and $\Sigma(C_2) = \{18B(2,3) + B(4,12)\}$. 3. $C_3 = \mathcal{C}_{3,3}(Z_4)$ and $\Sigma(C_3) = \{27B(2,3)\}$. 4. $C_4 = \mathcal{J}_3(Z_4)$ and $\Sigma(C_4) = \{3B(2,3) + B(2,3;4)\}$. 5. $C_5 = \mathcal{C}_3(\mathcal{J}_3(Z_4); D)$ where $D = \{\widetilde{x} = \alpha\}$ and $\Sigma(C_5) = \{9B(2,3) + 3B(4,8)\}$. 6. $C_6 = \mathcal{C}_2(\mathcal{J}_3(Z_4); D)$ where $D = \{\widetilde{x} = \alpha\}$ and $\Sigma(C_5) = \{6B(2,3) + B(4,28)\}$. 7. $C_7 = \mathcal{C}_3(\mathcal{J}_2(Z_4); D)$ where $D = \{\widetilde{x} = \alpha\}$ and $\Sigma(C_7) = \{9B(2,3) + B(4,24)\}$.

(C) Zariski pairs

Let C and C' be plane curves of the same degree and let $\Sigma(C) = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ and $\Sigma(C') = \{\mathbf{a}_1', \dots, \mathbf{a}_{m'}'\}$ be the singular points of C and C' respectively. Assume that L_∞ is generic for both of them. We say that $\{C, C'\}$ is a *Zariski pair* if (1) $m = m'$ and the germ of the singularity $(C, \mathbf{a}_j)$ is topologically equivalent to $(C', \mathbf{a}_j')$ for each j and (2) there exist neighborhoods $N(C)$ and $N(C')$ of C and C' respectively so that $(N(C), C)$ and $(N(C'), C')$ are homeomorphic and (3) the pair $(\mathbf{P}^2, C)$ is not homeomorphic to the pair $(\mathbf{P}^2, C')$ ([Ba]).

The assumption (2) is not necessary if C and C' are irreducible. For our purpose, we replace (3) by one of the following:
(Z-1) $\pi_1(\mathbf{P}^2 - C) \not\cong \pi_1(\mathbf{P}^2 - C')$,
(Z-2) $\pi_1(\mathbf{C}^2 - C^a) \not\cong \pi_1(\mathbf{C}^2 - C'^a)$, where $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$ and L_∞ is generic for C and C',
(Z-3) $\mathcal{D}(\pi_1(\mathbf{P}^2 - C)) \not\cong \mathcal{D}(\pi_1(\mathbf{P}^2 - C'))$.

We say that $\{C, C'\}$ is a *strong Zariski pair* if the conditions (1), (2) and the condition (Z-1) are satisfied. Similarly we say $\{C, C'\}$ is a *strong generic affine Zariski pair* (respectively *strong Milnor pair*) if the conditions (1), (2) and the condition (Z-2) (resp. (Z-3)) are satisfied.

If C and C' are irreducible curves satisfying (1) and (2), $\{C, C'\}$ is a strong Milnor pair if and only if the fundamental groups of the respective Milnor fibers $V(C)$ and $V(C')$ are not isomorphic by Proposition (2.7). The above three conditions (Z-1)$\sim$ (Z-3) are related by the following.

PROPOSITION (5.7). (1) *If $\{C, C'\}$ is a strong Milnor pair, $\{C, C'\}$ is a strong Zariski pair as well as a strong generic affine Zariski pair.*
(2) *Assume that C and C' are irreducible and assume that $\{C, C'\}$ is a strong Zariski pair and either $\pi_1(\mathbf{C}^2 - C^a)$ or $\pi_1(\mathbf{C}^2 - C'^a)$ satisfies* (H.I.C)-*condition. Then $\{C, C'\}$ is a strong generic affine Zariski pair.*

PROOF. The assertion (1) is immediate by Proposition (2.3). Assume that C and C' are irreducible and assume that $\pi_1(\mathbf{C}^2 - C'^a)$ satisfies (H.I.C)-condition and assume that $\phi : \pi_1(\mathbf{C}^2 - C) \cong \pi_1(\mathbf{C}^2 - C')$ is an isomorphism. Let ω, ω' be the generators of the respective kernels of the canonical homomorphisms: $\iota_\sharp : \pi_1(\mathbf{C}^2 - C) \to \pi_1(\mathbf{P}^2 - C)$ and $\iota'_\sharp : \pi_1(\mathbf{C}^2 - C'^a) \to \pi_1(\mathbf{P}^2 - C')$. As the homology class of ω is divisible by $d = \text{degree}(C)$, the homology class of $\phi(\omega)$ is also divisible by d and therefore $\iota'_\sharp(\phi(\omega)) \in \mathcal{D}(\pi_1(\mathbf{P}^2 - C')) \cap \mathcal{Z}(\pi_1(\mathbf{P}^2 - C'))$. By (H.I.C)-condition, $\phi(\omega) \in \text{Ker}(\iota'_\sharp)$ and thus $\phi(\omega) = \omega'^j$ for some $j \in \mathbf{Z}$. As $H_1(\mathbf{C}^2 - C) \cong H_1(\mathbf{C}^2 - C') \cong \mathbf{Z}$ and $[\omega] = d$, $[\omega'] = d$, we must have $j = \pm 1$. Thus ϕ induces an isomorphism of $\text{Ker}\, \iota_\sharp$ and $\text{Ker}\, \iota'_\sharp$ and therefore an isomorphism of $\pi_1(\mathbf{P}^2 - C) \cong \pi_1(\mathbf{P}^2 - C')$ by Proposition (2.3) and by Five Lemma. □

The results of §3,4 can be restated as follows.

THEOREM (5.8). *Let C, C' be projective curves and let $\mathcal{C}_{n,m}(C)$, $\mathcal{C}_{n,m}(C')$ (respectively $\mathcal{J}_n(C)$ and $\mathcal{J}_n(C')$) be the generic (n, m)-fold cyclic transforms (resp. generic Jung transform of degree n) of C and C' respectively.*
(1) *Assume that $\{C, C'\}$ is a strong affine Zariski pair (respectively strong Milnor pair). Then $\{\mathcal{C}_{n,m}(C), \mathcal{C}_{n,m}(C')\}$ is a strong affine Zariski pair (resp. strong Milnor pair).*

(2) *Assume that* $\{C, C'\}$ *is a strong Zariski pair. We assume also either* C *or* C' *satisfies (H.I.C)-condition. Then* $\{\mathcal{C}_{n,m}(C), \mathcal{C}_{n,m}(C')\}$ *is a strong Zariski pair.*

The same assertion holds for $\mathcal{J}_n(C)$ *and* $\mathcal{J}_n(C')$.

Proof. The assertion (1) is due to Theorem (3.7) and Theorem (4.7). The assertion (2) follows from Theorem (3.8) and Corollary (4.10). □

A well-known example is given by Zariski ([Z1]). Let Z_6 be a curve of degree 6 with 6 cusps which are on a conic and let Z_6' be a curve of degree 6 with 6 cusps which are not on a conic. In [O6], such examples are explicitly given. It is known that $\pi_1(\mathbf{P}^2 - Z_6)$ is isomorphic to the free product $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$ and $\pi_1(\mathbf{P}^2 - Z_6')$ is isomorphic to $\mathbf{Z}/6\mathbf{Z}$.

Example (5.9) (A new example of a Zariski pair). In (1) $\sim$ (4), we apply generic 2-covering or $(2,2)$-covering and generic Jung transform of degree 2 to the pair $\{Z_6, Z_6'\}$ to obtain three strong Zariski pairs of curves of degree 12:
(1) Take $\{\mathcal{C}_2(Z_6), \mathcal{C}_2(Z_6')\}$. Both curves have 12 cusps ($= B(2,3)$) and one $B(6,12)$ singularity at infinity. $\pi_1(\mathbf{P}^2 - \mathcal{C}_2(Z_6))$ is a central $\mathbf{Z}/2\mathbf{Z}$-extension of $\mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ and it is denoted by $G(3;2;4)$ in [O5]. $\pi_1(\mathbf{P}^2 - \mathcal{C}_2(Z_6'))$ is isomorphic to a cyclic group $\mathbf{Z}/12\mathbf{Z}$.
(2) Take $\{\mathcal{C}_{2,2}(Z_6), \mathcal{C}_{2,2}(Z_6')\}$. They have 24 cusps. The fundamental groups are as above.
(3) Take $\{\mathcal{J}_2(Z_6), \mathcal{J}_2(Z_6')\}$. Singularities are 6 cusps and one $B(6,18)$. The fundamental groups are as in (1).
(4) Take $\{\mathcal{C}_2(\mathcal{J}_2(Z_6)), \mathcal{C}_2(\mathcal{J}_2(Z_6'))\}$. Singularities are 12 cusps and two $B(6,6)$ singularities.
(5) We now propose a new strong Zariski pair $\{C_1, C_2\}$ of degree 12. First for C_1, we take the generic cyclic transform $\mathcal{C}_3(Z_4)$ of degree 3 of a Zariski's three cuspidal quartic. Recall that C_1 has 9 cusps and one $B(8,12)$ singularity at $\rho_\infty := [1;0;0]$. We have seen that $\pi_1(\mathbf{P}^2 - C_1)$ is G_3, a finite group of order 36. We will construct below another irreducible curve C_2 of degree 12 with 9 cusps and one $B(8,12)$ singularity at ρ_∞ such that $\pi_1(\mathbf{P}^2 - C_2) \cong G(3;2;4)$ where $G(3;2;4)$ is introduced in [O5] (see also §6) and it is a central extension of $\mathbf{Z}/3\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ by $\mathbf{Z}/2\mathbf{Z}$.

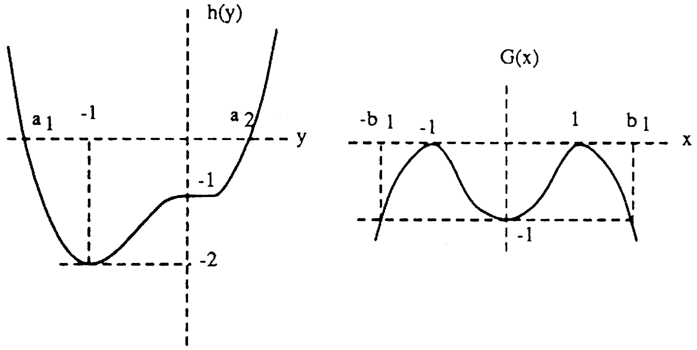


Figure (5.9.A)

(6) Take $\{\mathcal{C}_{3,3}(Z_4), \mathcal{C}_3(C_2; D)\}$ where $D = \{x = \alpha\}$ is generic. They are curves of degree 12 with 27 cusps. The fundamental groups $\pi_1(\mathbf{P}^2 - \mathcal{C}_{3,3}(Z_4))$ and $\pi_1(\mathbf{P}^2 - \mathcal{C}_3(C_2; D))$ are isomorphic to the case (5).

Construction of $\mathbf{C}_2$. Let us consider a family of affine curves $K^a(\tau) = \{(x, y) \in \mathbf{C}^2; h(y)^3 = \tau G(x)\}$ ($\tau \in \mathbf{C}^*$) where $h(y) = 3y^4 + 4y^3 - 1$, $G(x) = -(x^2 - 1)^2$. Let $K(\tau)$ be the projective compactification of $K^a(\tau)$. Let $a_1, \dots, a_4$ be the solution of $h(y) = 0$. Here we assume that a_1, a_2 are real roots with $a_1 < a_2$ and $a_3 = \overline{a_4}$. By a direct computation, we see that $K(\tau)$ has 8 cusp singularities at $\{A_1, A_1', \dots, A_4, A_4'\}$ where $A_i := (1, a_i)$, $A_i' := (-1, a_i)$ for $i = 1, \dots, 4$ and a $B(8, 12)$ singularity at $\rho_\infty = [1; 0; 0]$. Putting $\tau = 1$, $K(1)$ has one more cusp at $A_0 := (-1, 0)$. For C_2, we take $K(1)$. As $\pi_1(\mathbf{P}^2 - K(\tau)) = G(3; 2; 4)$ by [O5][2], $\pi_1(\mathbf{P}^2 - C_2)$ is not smaller than $G(3; 2; 4)$ as there exists a surjective morphism from $\pi_1(\mathbf{P}^2 - K(1))$ to $\pi_1(\mathbf{P}^2 - K(\tau)) = G(3; 2; 4)$. In fact, we assert that $\pi_1(\mathbf{P}^2 - C_2) = G(3; 2; 4)$.

Appendix: Proof of $\pi_1(\mathbf{P}^2 - C_2) = G(3; 2; 4)$. We use the pencil $L_\eta = \{x = \eta\}$ to compute the fundamental group. We use the same method which was used in [O5]. Note that the critical values of $H : \mathbf{C} \to \mathbf{C}$, $H(y) = h(y)^3$ is $\{0, -1, -8\}$. Let $\{a_1, \dots, a_4\}$ be the root of $h(y) = 0$ and let

[2]In [O5], we have only considered the curves of type $f(y) = g(x)$ with $\deg f = \deg g$. However the same assertion holds if $\deg f(y) \geq \deg g(x)$.

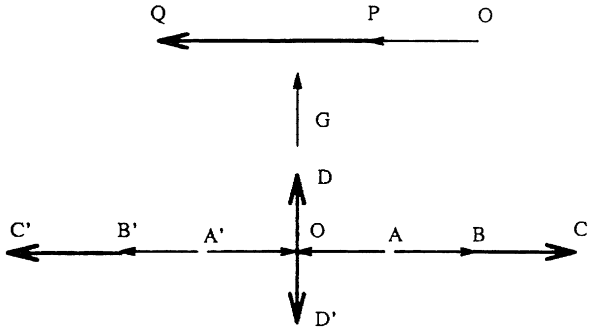


Figure (5.9.B) (Γ_G)

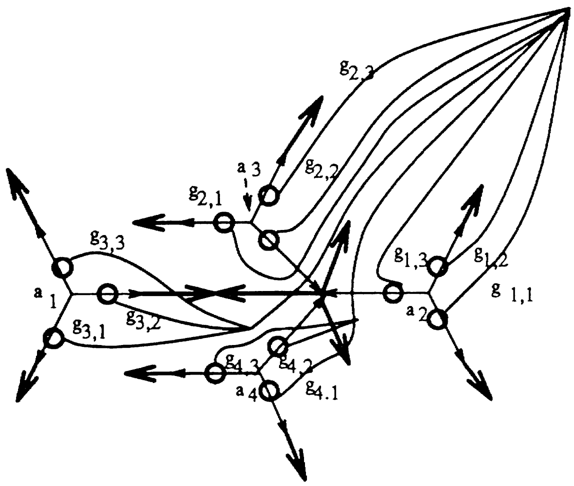


Figure (5.9.C)

$\{a_{i,j}; i = 1, \ldots, 4, j = 1, 2, 3\}$ be the roots of $H(y) = -1$. We assume that $a_1 < a_2$ are real solutions and a_3, a_4 are conjugate and $\Im(a_3) > 0$. Let O be the origin, $P = -1$ and $Q = -8$ in the complex plane and we consider the oriented thin line segment $\overline{OP}$ and the oriented thick line $\overline{PQ}$. We consider $\Gamma = \overline{OP} \cup \overline{PQ}$ as an oriented graph with vertices O, P, Q. Put $\Gamma_H = H^{-1}(\Gamma)$ and $\Gamma_G = G^{-1}(\Gamma)$ and we consider Γ_H and Γ_G. Let $b_1 > 0$ and $-b_1$ be the solution of $G(x)=-1$ and let $b_2>0, -b_2, b_3, \bar{b}_3$ be the roots of $G(x)=-8$. In the graph, $A=1, A'=-1, B=b_1, B'=-b_1$ and C, C', D, D' correspond to the roots of $G(x)=-8$. We move the pencil line L_η along Γ_G. Then the intersection $L_\eta\cap C_2$ moves along Γ_H. We take generators $g_{i,j}, 1\le i\le 4,\ 1\le j\le 3$ of $\pi_1(L_\varepsilon; b_0)$, ε small and $0<\varepsilon<1$, as in Figure (5.9.C). In a small circle centered at a_i, we have three intersections of $C_2\cap L_\varepsilon$ and in Figure (5.9.C), we find corresponding generators $g_{i,1}, g_{i,2}, g_{i,3}$. We obtain the monodromy relations by the deformation along Γ, turning counter-clockwise near the vertices. The real graphs of $h(y)$ and $G(x)$ (Figure (5.9.A)) and Figure (5.9.B) will be helpful to see the movement of the intersection $L_\eta\cap C_2$.

At $\eta=1$, we get the cusp relation: $R(1):\ g_{i,1}=g_{i,3},\ \{g_{i,1}g_{i,2}\}=e,\ i=1,\dots,4$. At $\eta=b_1$ and b_2, we obtain $R(2):\ g_{1,2}=g_{2,1}=g_{4,1}$ and $R(3):\ g_{3,1}=g_{2,1}$. At $\eta=0$, we obtain the relation $R(4):\ g_{1,3}^{-1}g_{4,2}g_{1,3}=g_{2,2},\ \{g_{1,3},g_{2,2}\}=e$. Finally at $\eta=b_3$ and $\bar{b}_3$, we obtain the relations: $R(5):\ g_{2,2}=g_{3,2}$ and $R(6):\ g_{3,2}=g_{4,2}$. Therefore we get $\rho:=g_{1,2}=g_{2,1}=g_{2,3}=g_{3,1}=g_{3,3}=g_{4,1}=g_{4,3}$ and $\xi:=g_{2,2}=g_{3,2}=g_{4,2}$. By R(4), we get $g_{1,3}\xi=\xi g_{1,3}$. Together with the relation $R(1)$, we get $g_{1,3}=\xi$. Thus we need two generators ρ,ξ and it has the braid relation: $\rho\xi\rho=\xi\rho\xi$. The vanishing relation of the big circle is given by $\omega=(g_{2,3}g_{2,2}g_{2,1})\cdot(g_{3,3}g_{3,2}g_{3,1})\cdot(g_{4,3}g_{4,2}g_{4,1})\cdot(g_{1,3}g_{1,2}g_{1,1})=(\rho\xi\rho)^4=e$. We know that $\langle\rho,\xi;\{\rho,\xi\}=(\rho\xi\rho)^4=e\rangle$ is $\mathbf{Z}/2\mathbf{Z}$ central extension of $\mathbf{Z}/3\mathbf{Z}*\mathbf{Z}/2\mathbf{Z}$ and we have denoted by $G(3;2;4)$. We can see easily that $\eta=-1,-b_1,-b_2$ do not give any further relations. □

§6. Non-atypical curves and some examples

Let $f(x,y)$ be a polynomial and we consider f as a mapping $f:\mathbf{C}^2\to\mathbf{C}$ and let $C_t^a:=f^{-1}(t)\subset\mathbf{C}^2$ and let C_t be the compactification of C_t^a. Let $\{\mathbf{a}_\infty^1,\dots,\mathbf{a}_\infty^k\}=L_\infty\cap C$ be the points at infinity as in §4. Recall that $\tau\in\mathbf{C}$ is not an atypical value if the embedded topological type of the germ $(C_t,\mathbf{a}_\infty^i)$ is stable at $t=\tau$. This is also equivalent to the local constancy of the Milnor number $\mu(C_t;\mathbf{a}_\infty^i)$ at $t=\tau$ for each $i=1,\dots,k$ ([H-L]). We say that C_τ^a is a *non-atypical* affine curve if τ is not atypical value. We say that a polynomial $f(x,y)$ is *non-atypical* if f has no atypical values. Assume that C_τ^a is a reduced affine curve and let $\{\mathbf{a}_1,\dots,\mathbf{a}_s\}$ be the singular points of C_τ^a. Define $\chi(C_\tau^a)':=\chi(C_\tau^a)-\sum_{i=1}^s\mu(C_\tau^a;\mathbf{a}_i)$. By the formula (3.3.1) and by the upper semi-continuity of the Milnor number, we have the following simple criterion for τ to be non-atypical value.

PROPOSITION (6.1). *Assume that* C_τ^a *is reduced. Then* τ *is not an atypical value if and only if* $\chi(C_\tau^a)' = \chi(C_{t_0}^a)$ *where* $C_{t_0}^a$ *is a smooth non-atypical curve. This is equivalent to* $\chi(C_\tau^a)' \le \chi(C_t^a)$ *for any* t *such that* C_t^a *is reduced.*

A projective curve C is called *non-atypical* if there exists a line L_∞ which passes through all singular points of C such that $C^a := C \cap (\mathbf{P}^2 - L_\infty)$ is a smooth non-atypical affine curve with respect to the affine space $\mathbf{C}^2 := \mathbf{P}^2 - L_\infty$. For a smooth non-atypical affine curve, we have the following result.

PROPOSITION (6.2). *Let* C_τ^a *be a non-atypical smooth affine curve. Then* $\pi_1(\mathbf{C}^2 - C_\tau^a) \cong \mathbf{Z}$.

PROOF. Let Σ be the finite set which is defined by the union of the critical values and the atypical values of f. Then $f : \mathbf{C}^2 - f^{-1}(\Sigma) \to \mathbf{C} - \Sigma$ is a locally trivial topological fibration. Let $D(\tau)$ be a small disk centered at τ such that $D(\tau) \cap \Sigma = \emptyset$ and let $D(\Sigma)$ be a domain containing Σ which is homeomorphic to a disk and $D(\Sigma) \cap D(\tau) = \emptyset$. We take a simple path L which joins $D(\Sigma)$ and $D(\tau)$. Let $D' = D(\tau) \cup D(\Sigma) \cup L$. We may assume that $D(\Sigma)$, $D(\Sigma) \cup L$ and D' are deformation retract of the base space $\mathbf{C}$. By the above fibration structure, we see that the following inclusions are homotopy equivalences.

$$\begin{aligned} &f^{-1}(D(\Sigma)) \hookrightarrow f^{-1}(D(\Sigma) \cup L) \hookrightarrow f^{-1}(D') \hookrightarrow \mathbf{C}^2, \\ &f^{-1}(D' - \{\tau\}) \hookrightarrow \mathbf{C}^2 - C_\tau^a \end{aligned}$$

As $f^{-1}(D(\tau) - \{\tau\})$ is diffeomorphic to the product $C_{\tau'}^a \times (D(\tau) - \{\tau\})$ where τ' is a point in the boundary $\partial D(\tau)$, we apply van Kampen theorem to $f^{-1}(D' - \{\tau\}) = f^{-1}(D(\Sigma) \cup L) \cup f^{-1}(D(\tau) - \{\tau\})$ and the assertion follows immediately. □

COROLLARY (6.2.1). *Let* C_τ *be a non-atypical projective curve. Then* $\pi_1(\mathbf{P}^2 - C_\tau) \cong \mathbf{Z}/d\mathbf{Z}$ *where* d *is the degree of* C_τ.

By Proposition (6.1), we have the following application for Jung transforms which are not necessarily central.

THEOREM (6.3). *Let $f(x,y)$ be the defining irreducible polynomial and let let $C_t^a := f^{-1}(t)$. Consider Jung map $J_n : \mathbf{C}^2 \to \mathbf{C}^2$, $(x,y) \mapsto (x+y^n, y)$ and let $f^{(n)}(x,y) = f(J_n(x,y))$ and $\widetilde{C}_t^a = J_n^{-1}(C_t^a) = {f^{(n)}}^{-1}(t)$.*
(i) *If $\tau \in \mathbf{C}$ is a non-atypical value for f, then τ is also a non-atypical value for $f^{(n)}$. Thus if C_τ^a is a smooth non-atypical curve, so is $\widetilde{C}_\tau^a$.*
(ii) *If f is a non-atypical polynomial, so is $f^{(n)}$.*

Example (6.4). We give some examples of non-atypical curves. Let C^a be an affine curve defined by $C^a = \{f(x,y) = 0\}$ and let $f(x,y) = f_0 + f_1(x,y) + \cdots + f_d(x,y)$ be the homogeneous decomposition of f. Let $f_d(x,y) = cx^r y^s \prod_{i=1}^k (y - \alpha_i x)^{\nu_i}$ where $\alpha_i \neq 0$ for $i = 1, \dots, k$ and mutually distinct.
1. (Generic line at infinity L_∞) Assume that $\nu_i = 1$ for any $i = 1, \dots, k$ and $\max(r,s) \leq 1$. Then C and L_∞ intersect transversely and f is a non-atypical polynomial.
2. Assume that $f(x,y)$ is convenient, $f(0,0) = 0$ and the outside faces of the Newton diagram $\Delta(f)$ are non-degenerate. Then the toric degeneracy $\nu_\infty^{tor}(f)$ is zero. Thus f is non-atypical ([L-O2,L-O3]).
3. We assume that C_0 is smooth at $[1;\alpha_i;0]$ for any $i = 1, \dots, k$. By a linear change of the affine coordinates, we assume that $f_d(x,y) = c\prod_{i=1}^k (y - \alpha_i x)^{\nu_i}$. For i with $\nu_i \geq 2$, the smoothness is equivalent to $f_{d-1}(1, \alpha_i) \neq 0$. Then C_t is smooth at infinity for any $t \in \mathbf{C}$ and f is non-atypical.
4. (One place at infinity) Assume that $f_d(x,y) = y^d$ and C_0 is locally irreducible at $[1;0;0]$. Then C_t is also irreducible at $[1;0;0]$ for any $t \in \mathbf{C}$ and f is non-atypical ([E], [A-O]).
5. Let $f(x,y)$ be a weighted homogeneous polynomial. Then 0 is only possible atypical value of f as $f : \mathbf{C}^2 - f^{-1}(0) \to \mathbf{C}^*$ is a fibration. If further the origin is an isolated singular point of $f^{-1}(0)$, f is non-atypical.

The following assertion can be proved by a standard argument.

PROPOSITION (6.5). *Let $f(x,y)$ be a non-atypical polynomial and let t_0 be a regular value and let B_1 be the first Betti number of the generic fiber $C_{t_0}^a := f^{-1}(t_0)$. Let $\Sigma = \{\rho_1, \dots, \rho_s\}$ be the critical points of f. Then*

$\sum_{i=1}^{s}\mu(f;\rho_i)=B_1$ *and the vanishing cycles at* $\rho_1,\dots,\rho_s$ *are linearly independent.*

Example (6.6). We consider the following simple polynomial $f(x,y):=q^q y^p - x^q$ with $p\ge q\ge 2$. The constant q^q is given by a technical reason. Let C_t be the projective closure of $f^{-1}(t)$ for $t\in \mathbf{C}$. Note that C_t is a rational curve if $\gcd(p,q)=1$. By the criterion (2) or (5) in the Example (6.4), f is a non-atypical polynomial. We denote the affine fundamental group $\pi_1(\mathbf{C}^2-C_0)$ by $G(p;q)$. If $p,q\ge 2$ and $(p,q)\ne(2,2)$, $G(p;q)$ is not commutative and it has the following representation ([O5]):

$$\begin{aligned}G(p;q)&=\langle \xi_0,\dots,\xi_{p-1};\xi_i=\xi_{q+i},\ i\le p-q-1,\\ &\qquad \xi_i=\omega\xi_{i+q-p}\omega^{-1},\ p-q\le i\le p-1\rangle\\ &=\langle \xi_i\ (i\in\mathbf{Z}),\omega;\ \omega=\xi_{p-1}\xi_{p-2}\cdots\xi_0,\\ &\qquad \xi_i=\xi_{q+i},\xi_{i+p}=\omega\xi_i\omega^{-1}, i\in\mathbf{Z},\rangle\end{aligned}\tag{6.6.1}$$

The second representation is useful for a systematical treatment of $G(p;q)$ ([O5]). We define $G(p;q;r)=G(p;q)/\langle\omega^r\rangle$. Thus $\pi_1(\mathbf{P}^2-C_0)\cong G(p;q;1):=G(p;q)/\langle\omega\rangle$ and we know

$$G(p;q;1)=\mathbf{Z}/p_1\mathbf{Z}*F(s-1),\quad s=\gcd(p,q),\ p=p_1 s$$

where $F(s-1)$ is a free group of rank $s-1$ ([O5]). The assertion for the case $\gcd(p,q)=1$ is also results from the simply connectedness of the Milnor fiber $V(f)$ ([O1]). Therefore L_∞ is not central in this case. We consider the Jung transform $\widetilde{C}_t$ of C_t of degree p. The affine equation is given by:

$$\widetilde{C}_t^a=\{(x,y)\in\mathbf{C}^2;\widetilde{f}(x,y)=t\},\quad \widetilde{f}(x,y)=q^q y^p-(x+y^p)^q$$

$\widetilde{C}_0$ has two singularities. The origin O is a non-degenerate singularity which consists of s cusps of type $y^{p_1}-x^{q_1}=0$ where $s=\gcd(p,q), p=p_1 s$ and $q=q_1 s$. So $\mu(\widetilde{C}_0;O)=(p-1)(q-1)$. Another singularity is $\mathbf{a}_\infty=[1;0;0]$ and $\mu(\widetilde{C}_0;\mathbf{a}_\infty)=pq(pq-4)+p+q+1$. $\mathbf{a}_\infty$ is also singular point of $\widetilde{C}_t$ with the same Milnor number. The local equation at $\mathbf{a}_\infty$ is given by

$$q^q\zeta^p\xi^{pq-p}-(\xi^{p-1}+\zeta^p)^q=t\xi^{pq},\quad \zeta=Y/X,\xi=Z/X$$

$\widetilde{C}_t$ has also s irreducible components at $\mathbf{a}_\infty$.

We know that $\pi_1(\mathbf{P}^2 - \widetilde{C}_t) = \mathbf{Z}/pq\mathbf{Z}$ for $t \neq 0$ by Corollary (6.2.1). Now for $\widetilde{C}_0$, we assert:

THEOREM (6.7). *We have an isomorphism,* $\pi_1(\mathbf{P}^2 - \widetilde{C}_0) \cong G(p;q;q)$ *and* $G(p;q;q)$ *is a* $\mathbf{Z}/s\mathbf{Z}$ *central extension of* $\mathbf{Z}/p_1\mathbf{Z} * \mathbf{Z}/q_1\mathbf{Z} * F(s-1)$ *where* $s = \gcd(p,q)$ *and* $p = sp_1$ *and* $q = sq_1$.

This example shows that Theorem (4.7) does not hold in general with a non-central line at infinity L_∞. It has been proved that the fundamental group of the complement of a projective curve of degree pq defined by $f_p^q - f_q^p = 0$ for generic homogeneous polynomials f_p and f_q of degree p and q respectively is isomorphic to $G(p;q;q)$ ([O4]). However such a curve has pq singularities of type $y^p - x^q = 0$. Our curve $\widetilde{C}$ has only two singularities. As a corollary, we have the following (cf.[Z2]).

COROLLARY (6.7.1). *Let* $p, q \geq 2$ *be positive integers with* $\gcd(p,q) = 1$. *Then there exists an irreducible rational plane curve* C *of degree* pq *with two irreducible singularities so that* $\pi_1(\mathbf{P}^2 - C) = \mathbf{Z}/p\mathbf{Z} * \mathbf{Z}/q\mathbf{Z}$.

PROOF OF THEOREM (6.7). We compute the fundamental group using the vertical pencil $L_\eta = \{x = \eta\}$. Put $h(\eta,t) := q^q t - (\eta+t)^q$. As $\widetilde{f}(x,y) = h(x,y^p)$, L_η is a singular pencil if and only if either $h(\eta,0) = 0$ or $h(\eta,t)$ has a non-simple solution. The first case occurs if and only if $\eta = 0$. The second case occurs if and only if the following equations have a common solution :

$$h(\eta,t) = q^q t - (\eta+t)^q = 0, \quad \frac{\partial h}{\partial t}(\eta,t) = q(q^{q-1} - (\eta+t)^{q-1}) = 0$$

Thus we have either $\eta = 0$ or

(6.7.2) $$\eta = (q-1)\gamma^j, \quad j = 0,\dots,q-2, \quad t = \gamma^j, \quad \text{where } \gamma = \exp 2\pi i/(q-1).$$

Thus we have $q-1$ singular pencil lines. Note that

(6.7.3) $$h(0,t) = t(q^q - t^{q-1}), \quad \widetilde{f}(0,y) = y^p(q^q - y^{p(q-1)})$$

(6.7.4) $$h(q-1,t) = (t-1)^2 h_{q-2}(t), \quad \widetilde{f}(q-1,y) = (y^p-1)^2 h_{q-2}(y^p)$$

where $h_{q-2}(t)$ is a polynomial of degree $q-2$ and the solutions of $h_{q-2}(t)=0$ are all simple and non-zero and there is no positive solution. In particular, $\widetilde{f}(q-1,y)=0$ has $\{\alpha^j; j=0,\dots,p-1\}$, $\alpha=\exp 2\pi i/p$, as solutions of multiplicity 2 and there are $q-2$ solutions in each angle region $\Omega_a=\{y\in \mathbf{C}; 2a\pi/p<\arg y<2(a+1)\pi/p\}$ for $a=0,\dots,p-1$. By the equalities $h(\eta,0)=-\eta^q$, $h(\eta,-\eta+q)=q^q(q-\eta-1)$, we can see easily that $h(\eta,t)=0$ has two positive real solutions for t, for a fixed $0<\eta<q-1$, which approach to 1 as $\eta\to q-1$ and no positive real solution for $\eta>q-1$. We take a small enough $\varepsilon>0$. Let

$$L_\varepsilon\cap\widetilde{C}=\{y_0,\dots,y_{p-1},y_{a,b}; 0\le a\le p-1, 0\le b\le q-2\}$$

where the intersection points are characterized by the above observation as

$$|y_a|\fallingdotseq(\varepsilon/q)^{q/p},\ \arg(y_a)=2\pi a/p$$
$$|y_{a,b}|\fallingdotseq q^{q/p(q-1)},\ \arg(y_{a,b})\fallingdotseq 2\pi(a/p+b/p(q-1))$$

and the strict equality $\arg(y_{a,0})=2a\pi/p$ for $b=0$. Taking above observation into consideration, we choose generators $g_a, g_{a,b}$, $0\le a\le p-1$, $0\le b\le q-2$ of $\pi_1(L_\varepsilon-L_\varepsilon\cap\widetilde{C})$ as in Figure (6.7.A). g_a and $g_{a,b}$ go around $\widetilde{C}_0$ at y_a and $y_{a,b}$ respectively. By the choice of generators, they satisfy the equality

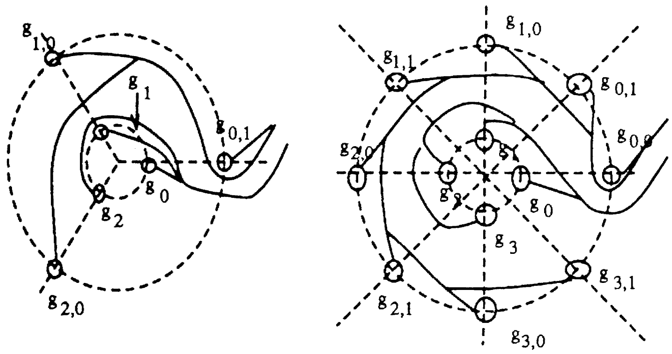
$$\omega=(g_{0,0}\cdots g_{0,q-2})\cdots(g_{p-1,0}\cdots g_{p-1,q-2})(g_{p-1}\cdots g_0) \tag{6.7.5}$$

where ω is represented by a big circle as before. We have q singular pencil lines L_η, $\eta=0$, $(q-1)\gamma^j$, $j=0,\dots,q-2$. It is convenient to introduce the elements $\theta:=g_{p-1}\cdots g_0$ and $g_{kp+j}:=\theta^k g_j\theta^{-k}$ for $0\le j\le p-1$ and $k\in\mathbf{Z}$. Then

$$\theta=g_{p-1}\cdots g_0,\quad g_{p+j}=\theta g_j\theta^{-1},\quad j\in\mathbf{Z}. \tag{R-1}$$

Recall that (R-1) implies (S) : $\theta=g_{j+p-1}g_{j+p-2}\cdots g_j$, $j\in\mathbf{Z}$. This can be proved by a two-side induction. See Proposition (2.6) in [O5]. Then the monodromy relation at $\eta=0$ can be simply written by (6.5.1) as

$$g_{j+q}=g_j,\quad \forall j\in\mathbf{Z}. \tag{R-2}$$



$(p = 3, q = 2)$ $(p = 4, q = 3)$

Figure (6.7.A)

Now we study the monodromy relation at $\eta = q - 1$, by moving the pencil line L_η from $\eta = \varepsilon$ to $\eta = q - 1$ along the real axis. By the above consideration, those intersections $\{y_a, y_{a,0}\}$ approaches to $\exp(2a\pi i/p)$ along the half line $\{y \in \mathbf{C}; \arg(y) = 2a\pi i/p\}$. The other intersections move in the open angle region Ω_a but their movement is topologically trivial. Thus we get $g_{a,0} = g_a$, $a = 0, \ldots, p-1$. Now we consider the monodromy relation at $\eta = (q-1)\gamma^b$ for $0 < b \le q-2$ where $\gamma = \exp 2\pi i/(q-1)$. We move the line L_η

(1) first, along the small circle $\varepsilon \exp(2\pi\tau i/(q-1))$ for $0 \le \tau \le b$, and then

(2) along the half line $\{\eta \in \mathbf{C}; \arg(\eta) = \arg(\gamma^b) = 2b\pi/(q-1)\}$ to $(q-1)\gamma^b$.

By the first movement, those p points $\{y_0, \ldots, y_{p-1}\}$ of $L_\eta \cap \widetilde{C}$ on the small circle $|y| \fallingdotseq (\varepsilon/q)^{q/p}$ are simply rotated by the angle $2bq\pi i/p(q-1)$. Thus y_a is transformed to y'_a which is approximately equal to $y_a\delta^{bq}$ where $\delta = \exp(2\pi i/p(q-1))$. The points $y_{a,c}$ move to $y'_{a,c}$ in the same angle region but this movement is sufficiently small. See Figure (6.7.A) for the case $p = 4, q = 3$ and $b = 1$. Note that $\gamma = \delta^p$ and $\gamma^{bq} = \gamma^b$ and therefore $\widetilde{f}(\eta\gamma^b, y\delta^{bq}) = \gamma^b\,(q^q y^p - (\eta + y^p)^q)$. Thus (6.7.4) can be read as $\widetilde{f}((q-1)\gamma^b, y) = \gamma^{-b}(y^p - \gamma^{bq})^2 h_{q-2}(\gamma^{-bq}y^p)$, $y'_a = y_a\delta^{bq}$ and $\arg(y'_{a,b}) = 2\pi((a+b)/p + b/p(q-1))$. Thus by the observation of the case $b = 0$, two points y'_a and $y'_{a+b,b}$ on the half line $\{y \in \mathbf{C}; \arg(y) = 2\pi((a+b)/p + b/p(q-1))\}$ approaches each other to the complex number $\gamma^a\delta^{bq} = \exp(2\pi i((a+b)/p +$

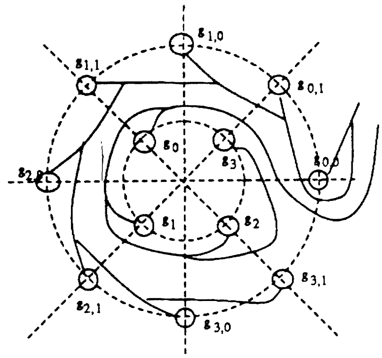


Figure (6.7.B) ($p = 4,\ q = 3, b = 1$)

$b/p(q-1)))$ along that line and other intersections moves topologically trivially. See Figure (6.7.B). Thus we obtain the following relation as the monodromy relation at $\eta := (q-1)\gamma^b$ is

(R-3) $$g_{a,b} = g_{a-b}, \quad 0 \le a \le p-1,\ 0 \le b \le q-2$$

By this relation, we can eliminate the generators $\{g_{a,b}; 0 \le a \le p-1, 0 \le b \le q-2\}$. Finally as the vanishing relation at infinity, $\omega = (g_{0,0} \cdots g_{0,q-1}) \cdots (g_{p-1,0} \cdots g_{p-1,q-1}) g_{p-1} \cdots g_0 = e$. Using (R-3), we can rewrite this as

$$\begin{aligned}\omega &= (g_0 g_{-1} \cdots g_{-q+2})(g_1 \cdots g_{3-q}) \cdots (g_{p-1} \cdots g_{p-q+1})(g_{p-1} \cdots g_0)\\ &\overset{\text{(R-2)}}{=} g_0 g_{-1} \cdots g_{-p(q-1)+1}\theta \overset{\text{(S)}}{=} \theta^q\end{aligned}$$

Therefore the vanishing relation $\omega = e$ implies

(R-4) $$\theta^q = e.$$

Thus we have proved that $\pi_1(\mathbf{P}^2 - \widetilde{C})$ is generated by $g_j,\ j \in \mathbf{Z}$, θ and the generating relations are (R-1), (R-2) and (R-4). Thus $\pi_1(\mathbf{P}^2 - \widetilde{C}) \cong G(p;q;q)$. We know that $\theta^{q_1} \in \mathcal{Z}(G(p;q;q))$ and $\mathrm{order}(\theta^{q_1}) = s$ and the quotient group $G(p;q;q)/\langle\theta^{q_1}\rangle$ is isomorphic to $\mathbf{Z}/p_1\mathbf{Z} * \mathbf{Z}/q_1\mathbf{Z} * F(s-1)$. See Proposition (2.5) and Theorem (2.12) of [O5], or the following appendix. □

Appendix. We recall the generators of $\mathbf{Z}/p_1\mathbf{Z}$, $\mathbf{Z}/q_1$ and $F(s-1)$. Let us write $1 = ap_1 + bq_1$ for some integers $a, b \in \mathbf{Z}$. Then $s = ap + bq$ and we have $g_{j+s\nu} = g_{j+\nu ap+\nu bq} = g_{j+\nu ap} = \theta^{\nu a} g_j \theta^{-\nu a}$ by (R-1) and (R-2). Therefore we can write θ as

$$\begin{aligned}\theta &= g_{p-1}\cdots g_0\\ &= (\theta^{(p_1-1)a} g_{s-1}\theta^{-(p_1-1)a})(\theta^{(p_1-1)a} g_{s-2}\theta^{-(p_1-1)a})\\ &\qquad \cdots(\theta^{(p_1-1)a} g_0 \theta^{-(p_1-1)a})\cdots(g_{s-1}\cdots g_0)\\ &= \theta^{ap_1}(\theta^{-a}\psi)^{p_1}, \quad \text{where} \quad \psi = g_{s-1}\cdots g_0\end{aligned}$$

Thus by (R-4), we have the relation

(R-5) $$(\theta^{-a}\psi)^{p_1} = e, \quad \psi = g_{s-1}\cdots g_0.$$

We put $\rho = \theta^{-a}\psi$. Then the above equality implies that $\rho^{p_1} = e$. The above cyclic groups $\mathbf{Z}/p_1\mathbf{Z}$ and $\mathbf{Z}/q_1\mathbf{Z}$ are generated by ρ and θ and $g_1, \ldots, g_{s-1}$ generate the free group $F(s-1)$.

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(Received June 14, 1995)

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