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Fractal Geometry of Self-avoiding Processes

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Abstract. We study a family of self-avoiding walks on the 2- and 3-dimensional Sierpinski gasket, respectively and give the complete classification of their limits.

1. Introduction

If you take the 'continuum limit' of random walks on fractals, you get diffusions on fractals, which have been intensively studied in recent years. Then how about the continuum limit of self-avoiding walks? In [5], we studied the continuum limit of a family of self-avoiding walks on the 2-dimensional pre-Sierpinski gasket with path probability proportional to $\exp\{-\beta L(w)\}$, where $\beta > 0$ is a parameter and $L(w)$ is the number of the steps. It has been shown that for any values of β, the continuum limit exists, with an appropriate time-scale transformation. We observe a phase transition, that is, a drastic change in the limit process according to the change in β. There is a critical value β_c such that for $\beta = \beta_c$ the limit process is almost surely self-avoiding and the Hausdorff dimension of the path is almost surely $\left(\log \dfrac{7-\sqrt{5}}{2}\right) / (\log 2)$. With $\beta > \beta_c$, we have a constant speed motion along a line as the limit, which is self-avoiding but uninteresting. With $\beta < \beta_c$, the limit is a 'Peano curve' motion with its path filling all over the Sierpinski gasket, and intersecting with itself infinitely many times. In [6], we considered a parallel model with a parameter β on the 3-dimensional Sierpinski gasket. It has been shown that for $\beta = \beta_c^{(3)}$, the critical value, a non-trivial self-avoiding limit exists and the lower bound of the path Hausdorff dimension has been given.

In this paper we consider another families of self-avoiding walks, which we call the branching models, with different parametrizaions from the previous models and give the classification of their continuum limits in terms

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of self-avoiding property and the path Hausdorff dimension. On the 2-dimensional Sierpinski gasket, the branching model has a parameter p. In contrast with the previous model, the limit process is almost surely self-avoiding for any value of p (no phase transition). The path Hausdorff dimension is evaluated and it takes all the values between 1 and $\dfrac{\log 3}{\log 2}$ (the latter is the Hausdorff dimension of the gasket itself) as we vary p from 0 to 1. On the 3-dimensional Sierpinski gasket, we start with a two-parameter family of self-avoiding walks. One parameter, r, controls the self-avoiding property. The parameter space is divided into two regions. If $r \le (34)^{-\frac{1}{4}}$, the limit is almost surely self-avoiding. If $r > (34)^{-\frac{1}{4}}$, the limit is almost surely self-intersecting but not space-filling. The path Hausdorff dimension is also evaluated and it is controled by the other parameter. The difficulty of the problem in 3 dimensions lies in the fact that while a self-avoiding process is allowed to go through a triangle at most once, it can go through a tetrahedron twice. This property makes the study of the self-avoiding property in the 3-dimensional case more complex, in the sense that we are obliged to study dynamical systems of more than one dimension. As for the branching model, we can reduce the problem to a solvable dynamical system, which leads to the complete classification of self-avoiding property.

The structure of this paper is as follows. In Section 2, we give definitons and notations in a form common to both the 2-dimensional and 3-dimensional gaskets. Section 3 is devoted to the branching model of the self-avoiding processes on the 2-dimentional gasket. In Setion 4, which is the main part of this paper, we give the classification of the continuum limit processes of the branching model on the 3-dimensional gasket. In Section 5, we relate the results obtained here to our previous results in [5] and [6].

2. Notations and Definitions

We start with the definitions of the finite Sierpinski gaskets. In the following $d = 2, 3$. For $d = 2$, let $O = (0, 0)$, $a = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $b = (1, 0)$, $G_0 = \{O, a, b\}$ and let F_0 be the set of all the points on the perimeter of ΔOab. For $d = 3$, let $O = (0, 0, 0)$, $a = (\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3})$, $b = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, $c = (1, 0, 0)$, and let $G_0 = \{O, a, b, c\}$. Let F_0 be the set of all the points on the edges of the tetrahedron $Oabc$.

Let us define two sequences of sets recursively by,

$$G_{n+1} = \frac{1}{2}\{\bigcup_{y \in G_0} (G_n + y)\},$$

$$F_{n+1} = \frac{1}{2}\{\bigcup_{y \in G_0} (F_n + y)\}\,,\ n \in \mathbf{Z}_+ \stackrel{\mathrm{def}}{=} \mathbf{N} \cup \{0\}\,,$$

where, $A + y = \{x + y \mid x \in A\}\,,\ y \in \mathbf{R}^d$, and $k\,A = \{k\,x \mid x \in A\}\,,\ k \in \mathbf{R}$. Let $F = \overline{\bigcup_{n=0}^{\infty} F_n}$. F is the finite Sierpinski gasket. For $d = 2$, we define $\mathcal{T}_n$ to be the set of closed triangles in $\mathbf{R}^2$ which are the translations of $2^{-n}\Delta Oab$, with edges lying in F and for $d = 3$, the set of closed tetrahedrons in $\mathbf{R}^3$ which are the translations of the tetrahedron $Oabc$ scaled by 2^{-n}, with edges lying in F .

Let

$$C = \{w \in C([0,\infty) \to F) \mid w(0) = O\,,\ \lim_{t\to\infty} w(t) = a\}\,.$$

C is a complete separable metric space with the metric

$$d(v, w) = \sup_{t\in[0,\infty)} |v(t) - w(t)|\,,\quad v \in C\,,\ w \in C\,,$$

where $|\cdot|$ denotes the Euclidean metiric in $\mathbf{R}^d$.

For $w \in C$, define $L(w) \in \mathbf{R}_+ \cup \{\infty\}$, the arrival time at a by

$$L(w) = \inf\{0 \le t \le \infty \mid w(t) = a\}.$$

We define W^n, the set of self-avoiding paths on F_n starting from O and ending at a, to be the set of paths $w \in C$ such that, $L(w) \in \mathbf{Z}_+$, and

$$\begin{aligned}
&w(t) = a,\ t \ge L(w)\,,\\
&|w(i) - w(i+1)| = 2^{-n},\ i = 0, 1, 2, \cdots, L(w) - 1\,,\\
&\overline{w(i)w(i+1)} \subset F_n\,,\ i = 0, 1, 2, \cdots, L(w) - 1\,,\\
&w(i) \ne w(j)\,,\ i, j = 0, 1, 2 \cdots, L(w)\,,\ i \ne j\,,
\end{aligned}$$

and for $t \notin \mathbf{Z}_+$ satisfying $i < t < i + 1$ for some $i \in \mathbf{Z}_+$,

$$w(t) = (i + 1 - t)\,w(i) + (t - i)\,w(i + 1).$$

$w \in W^n$ is self-avoiding in the sense that

$$w(t) \neq w(s)\,,\ 0 \leq t < s \leq L(w)\,.$$

In this paper, we use the word 'self-avoiding' in this sense.

To describe the large-scale (decimated) behavior of each function $w \in C$, we define the 'hitting times' $T_i^n(w)$, $n \in \mathbf{Z}_+$, $i \in \mathbf{Z}_+$. Let $T_0^n(w) = 0$, and recursively,

$$T_i^n(w) = \inf\left\{t > T_{i-1}^n(w) \mid w(t) \in G_n \setminus \{w(T_{i-1}^n(w))\}\right\}\,,\ i \geq 1\,,$$

if the right hand side is finite, otherwise, $T_i^n(w) = \infty$. $T_i^n(w)$ is the time when w hits the points in G_n for the i-th time on condition that if w hits the same point more than once on end, we count it as 'once'. Noting that $w(t) \to a$ as $t \to \infty$, and writing $w(\infty) = a$, we obtain an integer $M = M(n, \omega)$ and a sequence $\{T_i^n(w)\}_{i=0,1,\cdots}$ such that $w(T_m^n(w)) = a$, $m \geq M$, $w(T_{M-1}^n(w)) \neq a$. Let $S_i^n(w) = T_i^n(w) - T_{i-1}^n(w)$, $n = 0, 1, \cdots$, $i = 1, 2, \cdots, M$, and call it the crossing time of the i-th $\mathcal{T}_n$ triangle/tetrahedron.

For $n \in \mathbf{Z}_+$, we define a 'decimation' map $Q_n :\ C \to C$, by

$$(Q_n w)(i) = w\left(T_i^n(w)\right)\,,$$

$$(Q_n w)(t) = (i + 1 - t)\,(Q_n w)(i) + (t - i)\,(Q_n w)(i + 1), \quad i < t < i + 1\,,$$

for $i = 0, 1, \cdots$. $Q_n w$ shows the behavior of w on the scale of 2^{-n}. Note that if $k \leq n$, we have $Q_k \circ Q_n = Q_k$.

3. The Branching Model on the 2-dimensional Sierpinski Gasket

We start with the definition of the branching model, as a family of probability measures on C. Let $a = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $a_1 = (\frac{1}{2}, 0)$, $a_2 = (\frac{3}{4}, \frac{\sqrt{3}}{4})$, and $l_1 = \overline{Oa}$, $l_2 = \overline{Oa_1} \cup \overline{a_1 a_2} \cup \overline{a_2 a}$. For $w \in C$ let

$$N_{n,i}(w) = \sharp\{\Delta \in \mathcal{T}_n \mid \{(Q_{n+1}w)(t) \mid 0 \leq t < \infty\} \cap \Delta \text{ is similar to } l_i\}\,,$$
$$i \in \{1, 2\},\ n \in \mathbf{Z}_+.$$

Define V^n by

$$V^n = \{w \in W^n \mid N_{k,1}(w) + N_{k,2}(w) = L(Q_k w), \quad k = 1, \cdots, n-1\ \}\,.$$

Note that

$$Q_m V^n = V^m,\ m \le n. \tag{3.1}$$

Let $0 < p < 1$ be a parameter and define a probability measure $P_1(p)$ on C by,

$$P_1(p)[w] = \begin{cases} p, & \text{if } w \in V^1 \text{ and } N_{0,1}(w) = 1, \\ 1-p, & \text{if } w \in V^1 \text{ and } N_{0,2}(w) = 1, \end{cases}$$

and for any Borel subset A of C,

$$P_1(p)[A] = 0 \text{ if } A \cap V^1 = \emptyset.$$

For $n = 2, 3, \cdots$, we define $P_n(p)$ recursively by,

$$P_n(p)[w] = P_{n-1}(p)[Q_{n-1}w]\, p^{N_{n-1,1}(w)}\, (1-p)^{N_{n-1,2}(w)}, w \in V^n,$$

and for any Borel subset A,

$$P_n(p)[A] = 0 \text{ if } A \cap V^n = \emptyset.$$

From the definition and (3.1), we have the 'self-similarity',

$$Q_m P_n(p) = P_m(p), \tag{3.2}$$

for $m < n$, where $Q_m P_n(p)$ is the image measure of $P_n(p)$ induced by Q_m.

Let $\Omega = C \times C \times C \times \cdots$ and $\mathcal{B}$ the Borel field on Ω. By virtue of (3.2) and Kolmogorov's extension theorem, for each p, there is a probability measure $P(p)$ on $(\Omega, \mathcal{B})$ such that

$$P(p)[\ \{\omega = \{w_k\}_{k=1}^{\infty} \in \Omega \mid Q_m w_n = w_m,\ n \ge m\}]\ =\ 1,$$

and

$$y_n\, P(p) = P_n(p),$$

where $y_n\, P(p)$ denotes the image measure of $P(p)$ induced by the natural projection y_n from Ω to the n-th C in the product $\Omega = C \times C \times C \times \cdots$. We regard each y_n as an F-valued process on $(\Omega, P(p))$.

Let $v \in V^m$. Then for $n > m$, under the conditional probability $P(p)[\ \cdot\ \mid y_m = v\ ]$, $\{S_i^m(y_n)\}$, $i = 1, \cdots, L(v)$, are i.i.d. and independent of v since for all $\Delta \in \mathcal{T}_m$, $\Delta \cap F$ has the identical structure, and from the definition the 2^{-n}-scale behaviors of the path contained in different elements of $\mathcal{T}_m$ are independent. Moreover, under the conditional probability above, we see that for each m and i, $\{S_i^m(y_{m+r})\}$, $r = 0, 1, 2, \cdots$, is a supercritical branching process with

$$E^{P(p)}[S_i^m(y_{m+r})] = \lambda_p{}^r\,, \quad \lambda_p = 2p + 3(1-p)\,.$$

Our 'continuum limit' corresponds to the limit distribution of y_n as n tends to infinity. To obtain a non-trivial limit, we introduce a time-scale transformation $U_n(\alpha) : \ C \to C$, $\alpha \in (0, \infty)$, $n \in \mathbf{N}$. For $w \in C$, define,

$$(U_n(\alpha)w)(t) = w(\alpha^n t).$$

Let

$$X_n(\omega) = U_n(\lambda_p)y_n(\omega)\,, \quad \omega \in \Omega.$$

X_n has a p-dependence, though we do not write it explicitly. We have,

PROPOSITION 3.1. *Let $v \in V^m$. Under the conditional probability $P(p)[\ \cdot \ \mid y_m = v\ ]$, the following holds:*

(1) For each i,

$S_i^m(X_n)$ converges a.s. as $n \to \infty$ to a random variable S_i^{*m}. $S_i^{*m} > 0$ a.s..

(2) $\{S_i^{*m}\}$, $i = 1, \cdots, L(v)$, are i.i.d. random variables.

(3) Let S^* be a random variable equal in law to S_1^{*0}. S_i^{*m} is equal in law to $\lambda_p{}^{-m}S^*$. The characteristic function of S^*, $g(t) = E^{P(p)}[\exp(tS^*)]$, $t \in \mathbf{C}$, is finite and is the unique solution to

$$g(\lambda_p t) = f(g(t)),$$

where $f(x) = px^2 + (1-p)x^3$, and $g'(0) = 1$.

(1), (2) and (3) with $\Re(t) \leq 0$ are obtained from the limit theorem for supercritical branching processes ([9], [1]). For (3) with a general $t \in \mathbf{C}$, we need closer study used in [6].

THEOREM 3.2. *For any $0 < p < 1$, X_n converges a.s. in C as $n \to \infty$ to a process $X^{(p)}$. $X^{(p)}$ is almost surely self-avoiding. The Hausdorff dimension of the path $\{X^{(p)}(t) \mid t \geq 0\}$ is almost surely equal to* $\dfrac{\log \lambda_p}{\log 2}$.

As we change the value of the parameter p in $(0,1)$, the Hausdorff dimension takes all the value in $(1, \dfrac{\log 3}{\log 2})$.

PROOF. The convergence of crossing times in any scale (any m) stated in Proposition 3.1 combined with the fact that for any $n, n' \geq m$ and $j = 0, 1, \cdots, L(y_m)$,

$$X_n(T_j^m(X_n)) = X_{n'}(T_j^m(X_{n'})), \tag{3.3}$$

leads to the convergence of X_n to $X^{(p)}$ as $n \to \infty$. Since the argument is standard (see [2] and [5]) we will not go into the details here.

To prove the self-avoiding property, we classify possible self-intersections as follows.

(1) There are $t_1 \geq 0$ and $t_0 > 0$ such that

$$X^{(p)}(t) = X^{(p)}(t_1), \quad \text{for} \quad t_1 \leq t \leq t_1 + t_0 < L(X^{(p)}).$$

(2) There are t_1, t_2 and t_3, $t_1 < t_3 < t_2$, such that

$$X^{(p)}(t_1) = X^{(p)}(t_2), \quad X^{(p)}(t_3) \neq X^{(p)}(t_1).$$

First, we consider (1). Let $\{\Delta^{(k)}\}$, $k = 0, 1, \cdots$, be a sequence of triangles such that $\Delta^{(k)} \in \mathcal{T}_k$, $X^{(p)}(t_1) \in \Delta^{(k)}$, $k = 0, 1, \cdots$. (1) implies that $X^{(p)}$ stays in all $\{\Delta^{(k)}\}$ at least for time t_0. This possibility is excluded from Proposition 3.1 (3), with $s > 0$, which shows the distribution of S^* has a tail that decreases at least as fast as an exponential (see [5]).

Next we show that (2) occurs with probability zero. From the fact that $X^{(p)}$ is the limit of X_n, which is self-avoiding, it follows that $X^{(p)}$ cannot go 'inside' any triangle more than once, or cannot enter and exit from a triangle through the same vertex more than once. That is, for any $k \in \mathbf{Z}_+$ and any $\Delta \in \mathcal{T}_k$,

$$\begin{aligned} P[\ & X^{(p)}(s_1),\ X^{(p)}(s_3) \in \Delta^c \ \text{ and } X^{(p)}(s_2),\ X^{(p)}(s_4) \in \Delta \setminus G_k, \\ & \text{for some } s_1 < s_2 < s_3 < s_4\ ] = 0. \end{aligned} \tag{3.4}$$

$$(3.5) \qquad \begin{aligned} P[\ X^{(p)}(s_1) = X^{(p)}(s_2) \in \Delta \cap G_k, \quad \text{and } X^{(p)}(t) \in \Delta \setminus G_k, \\ s_1 < t < s_2, \quad \text{for some } s_1 < s_2\ ] = 0. \end{aligned}$$

(3.4) and (3.5) imply that the only possibility of (2) is the case that two parts of the path passing through a pair of neighboring triangles meet each other at the junction. In this case $X^{(p)}(t_1)$ belongs to $\bigcup_{k=0}^{\infty} G_k$. Let $z \in G_K \backslash G_{K-1}$ for some K and let $\{\Delta_1^k\}$ and $\{\Delta_2^k\}$, $k = K, K+1, \dots$, be two sequences of triangles satisfying $\Delta_1^k \cap \Delta_2^k = \{z\}$, and $\Delta_i^k \in \mathcal{T}_k$, $\Delta_i^{k+1} \subset \Delta_i^k$, for $i = 1, 2$. Let u and v be the other two vertices of Δ_1^K. Assume (2) occurs with $z = X^{(p)}(t_1)$. Then this situation is restated as follows: There exist two sequences of triangles, $\{\Delta_1^{*k}\}$ and $\{\Delta_2^{*k}\}$, $k = K, K+1, \dots$, and s_i^k, r_i^k, $s_i^k < t_i < r_i^k$, $i = 1, 2$, such that $(\Delta_1^{*k}, \Delta_2^{*k}) = (\Delta_1^k, \Delta_2^k)$ for all $k \geq K$, or $(\Delta_1^{*k}, \Delta_2^{*k}) = (\Delta_2^k, \Delta_1^k)$ for all $k \geq K$ and $X^{(p)}(t) \in \Delta_i^{*k}$, for $s_i^k < t < r_i^k$, $i = 1, 2$. From the fact that X_n converges to $X^{(p)}$ a.s., we see that for any $N > K$, there exists a.s. $n_N(\omega) > N$ such that for any $n > n_N$,

$$d(X_n, X^{(p)}) < 2^{-N},$$

and

$$X_n(T_j^N(X_n)) = X_N(T_j^N(X_N)).$$

Thus for an arbitrary $N > K$,

$$\begin{aligned} &P[\ X^{(p)}(t_1) = X^{(p)}(t_2) = z,\ X^{(p)}(t_3) \neq z, \quad \text{for some } t_1 < t_3 < t_2\ ] \\ &\leq P[X_N(s_1) \in \Delta_1^N,\ X_N(s_2) = u,\ X_N(s_3) = v, \\ &\ \text{for some } s_1, s_2, s_3, \text{ with } s_2 < s_1 < s_3 \text{ or } s_3 < s_1 < s_2] \\ &\leq P[X_N(s_1) \in \Delta_1^N, \quad \text{for some } s_1, \text{ with } s_2 < s_1 < s_3 \text{ or } s_3 < s_1 < s_2 \mid \\ &X_K(s_2) = u,\ X_K(s_3) = v, \quad \text{for some } s_2, s_3] \\ &\times P[X_K(s_2) = u,\ X_K(s_3) = v, \quad \text{for some } s_2, s_3] \\ &\leq (1-p)^{N-K} \cdot P[X_K(s_2) = u,\ X_K(s_3) = v, \quad \text{for some } s_2, s_3]. \end{aligned}$$

Here we used (3.3). Since N is arbitrary, we see

$$P[\ X^{(p)}(t_1) = X^{(p)}(t_2) = z,\ X^{(p)}(t_3) \neq z, \quad \text{for some } t_1 < t_3 < t_2\ ] = 0.$$

Summing up for all $z \in \bigcup_{k=1}^{\infty} G_k$, which is countable, we see that the probability of (2) is zero.

The shape of the path $\{w(t) \mid t \geq 0\}$ falls in the category of random fractals studied by Mauldin and Williams ([8]), Falconer ([3]) and Graf ([4]).

Since the calculation of the Hausdorff dimension of the path for this model is a straightforward application of Theorem 1.1 of [8], we will omit the proof here. Instead, we will see in more detail in the 3-dimensional case. □

REMARK. From the proof of self-avoiding property above, it follows immedeately that for any $n, i \in \mathbf{Z}_+$,

$$S_i^{*n}(\omega) = S_i^n(X^{(p)}(\omega)), \text{a.s.}.$$

In other words,

$$Q_n X^{(p)}(\omega) = X_n(\omega)\,, \quad \text{a.s.}.$$

Here for simplicity, we dealt with self-avoiding walks limited on V^n. It is also possible to construct self-avoiding processes with varying path Hausdorff dimension from path measures supported on the whole W^n.

We remark that Mandelbrot introduced a one-parameter family of random fractals 'squig' ([7]) as a substitute for self-avoiding walk on a 2-dimensional regular triangular lattice. Its Hausdorff dimension ranges from 1 to $\dfrac{\log 3}{\log 2}$.

4. The Branching Model on the 3-dimensional Sierpinski Gasket

In this section we study the continuum limit of the branching model of the self-avoiding walks on the 3-dimensional pre-Sierpinski gasket. Let $c_1 = \overline{Oa}$, $c_2 = \overline{Oa} \cup \overline{bc}$. For $w \in C$ and $A \subset \mathbf{R}^3$, let

$$M_{n,i}(A)(w) = \sharp\{\Delta \in \mathcal{T}_n \mid \{(Q_n w)(t) \mid t \geq 0\} \cap A \cap \Delta \text{ is similar to } c_i\}\,, \\ i = 1, 2,\ n \in \mathbf{Z}_+.$$

If $A = F$, we just denote $M_{n,i}(w)$. Define V^n by

$$V^n = \{w \in W^n \mid M_{k,1}(w) + 2M_{k,2}(w) = L(Q_k w),\ k = 1, \cdots, n\}.$$

Note that

$$Q_m V^n = V^m, \quad m \leq n.$$

We define V^n_{bc}, a set of self-avoiding paths from b to c, by replacing O and a in the definition of V^n by b and c, respectively. Let

$$\begin{aligned}\tilde{V}^n &= \{w = (w_1, w_2) \mid w_1 \in V^n,\ w_2 \in V^n_{bc}, \\ &\qquad \{w_1(t) \mid t \geq 0\} \cap \{w_2(t) \mid t \geq 0\} = \emptyset\}.\end{aligned}$$

For $x > 0$ and $y > 0$, we define

$$\begin{aligned}\phi_1(x, y) &= \sum_{w \in V^1} x^{M_{1,1}(w)} y^{M_{1,2}(w)} = x^2 + 2x^3 + 2x^4 + 4x^3 y + 6x^2 y^2, \\ \phi_2(x, y) &= \sum_{w \in \tilde{V}^1} x^{M_{1,1}(w)} y^{M_{1,2}(w)} = x^4 + 4x^3 y + 22y^4.\end{aligned}$$

Define a probability measure $P_1(x, y)$ on C by,

$$P_1(x, y)[w] = \begin{cases} \phi_1(x, y)^{-1}\, x^{M_{1,1}(w)} y^{M_{1,2}(w)}, & \text{if } w \in V^1, \\ 0, & \text{otherwise.} \end{cases}$$

For $n = 2, 3, \cdots$, we define $P_n(x, y)$ recursively by,

$$P_n(x, y)[w] = \begin{cases} P_{n-1}(x, y)[Q_{n-1}w]\ \phi_1(x, y)^{-M_{n-1,1}(Q_{n-1}w)} & \\ \times \phi_2(x, y)^{-M_{n-1,2}(Q_{n-1}w)}\ x^{M_{n,1}(w)}\ y^{M_{n,2}(w)}, & w \in V^n, \\ 0, & \text{otherwise.} \end{cases}$$

From the definition, we have the self-similarity;

$$Q_m P_n(x, y) = P_m(x, y),$$

for $m < n$.

In a parallel way to the 2-dimensional case, for each x, y, we see there is a probability measure $P(x, y)$ on $(\Omega, \mathcal{B})$, where $\Omega = C \times C \times C \times \cdots$ and $\mathcal{B}$ is the Berel field on Ω, such that

$$P(x, y)[\ \{\omega = \{w_k\}_{k=1}^{\infty} \mid Q_m w_n = w_m,\ n \geq m\}]\ =\ 1,$$

and

$$y_n\, P(x, y) = P_n(x, y),$$

where y_n denotes the natural projection from Ω to the n-th C in the product $\Omega = C \times C \times C \times \cdots$. We regard each y_n as an F-valued process on $(\Omega, \mathcal{B}, P(x, y))$.

If for some $\Delta \in \mathcal{T}_m$, $\Delta \cap \{Q_m w(t) \mid t \geq 0\}$ is similar to c_1 or c_2, then let us call w 'crosses ' Δ once, or twice, respectively. On the 3-dimensional Sierpinski gasket, unlike the 2-dimensional Sierpinski gasket, a self-avoiding path is allowed to cross a tetrahedron twice. When we study crossing times of the tetrahedrons, we need to consider the tetrahedrons the path crosses only once and those the path crosses twice separately. Let $v \in V^m$. Assume v crosses two distinct elements of $\mathcal{T}_m$, Δ_1 and Δ_2 . Then for $n \geq m$, $\mathbf{M}_n(\Delta_1)(y_n) = (M_{n,1}(\Delta_1)(y_n), M_{n,2}(\Delta_1)(y_n))$ and $\mathbf{M}_n(\Delta_2)(y_n) = (M_{n,1}(\Delta_2)(y_n), M_{n,2}(\Delta_2)(y_n))$ are independent under the conditional probability $P(x,y)[\ \cdot \mid y_m = v\ ]$. Next we compare $\mathbf{M}_n(\Delta_1)(y_n)$ with the numbers of crossed tetrahedrons at different scales. Fix any $u \in V^1$ such that $\Delta'_i \cap \{u(t) \mid t \geq 0\}$ is similar to c_i, $i = 1, 2$, respectively, for some Δ'_1, $\Delta'_2 \in \mathcal{T}_1$. Let $\tilde{\mathbf{M}}_k(i) = (\tilde{M}_{k,1}(i), \tilde{M}_{k,2}(i))$ be a random vector equal in law to $\mathbf{M}_k(\Delta'_i)(y_k)$ under $P(x,y)[\ \cdot \mid \{w \mid Q_1 w = u\}\ ]$, $i = 1,\ 2,\ k \geq 1$. Then $\mathbf{M}_n(\Delta_1)(y_n)$ under $P(x,y)[\ \cdot \mid y_m = v\ ]$, $n \geq m$, is equal in law to $\tilde{\mathbf{M}}_{n-m+1}(i)$, where $i = 1$ or 2 according to whether $\Delta_1 \cap \{v(t) \mid t \geq 0\}$, is similar to c_1 or c_2. Thus under $P(x,y)[\ \cdot \mid \{w \mid y_m = v\}\ ]$ we see that for each $\Delta \in \mathcal{T}_m$ that v crosses, $\mathbf{M}_{m+r}(\Delta)(y_{m+r})$ $r = 0, 1, 2, \cdots$, is a two-type branching process. Define the mean matrix $A(x,y) = \{a(x,y)_{ij}\}$ by

$$\begin{aligned} a(x,y)_{ij} &= E[\tilde{M}_{2,j}(i)] \\ &= \phi_i(x,y)^{-1} \frac{\partial \phi_i(x_1,x_2)}{\partial x_j} x_j \bigg|_{x_1=x, x_2=y}, \quad i,j = 1,2. \end{aligned}$$

From the definition, we see the elements satisfy the following:

$$a(x,y)_{ij} > 0,\ i,j = 1,2, \tag{4.1}$$

$$2 < a(x,y)_{11} + a(x,y)_{12} < 4, \tag{4.2}$$

$$a(x,y)_{21} + a(x,y)_{22} = 4. \tag{4.3}$$

(4.1) allows us to use Frobenius' theorem, which implies that $A(x,y)$ has a positive eigenvalue, $\rho(x,y)$ which is the largest in absolute value of all the eigenvalues. Furthermore, (4.1), (4.2) and (4.3) imply

$$2 < \rho(x,y) < 4. \tag{4.4}$$

From the limit theorem for multi-type supercritical branching processes (Theorem 1 in Section 6 of Chapter V, [1]), we have the following proposition.

PROPOSITION 4.1. *Assume $v \in V^m$ crosses $\Delta \in \mathcal{T}_m$. There are random variables $M_k^{(m)}(\Delta)$, such that under the conditional probability, $P(x,y)[\ \cdot \ \mid y_m = v\ ]$,*

$$\rho(x,y)^{-r}\mathbf{M}_{m+r}(\Delta)(y_{m+r}) \to (u_1,u_2)M_k^{(m)}(\Delta)\,, \quad a.s. \quad r \to \infty\,,$$

$k = 1,2$ according to v crosses Δ once or twice, respectively. Here (u_1,u_2) is the eigenvector corresponding to $\rho(x,y)$, satisfying $u_1,u_2 > 0$. Furtheremore, there are random variables M_1^* and M_2^* such that $M_k^* > 0$, a.s., $M_k^{(m)}(\Delta)$ equals in law to $\rho(x,y)^{-m}M_k^*$, $k = 1,2$, and

$$g_k(t) = E[\ exp(tM_k^*)\ ], \quad k = 1,2$$

is finite for any $t \in \mathbf{C}$.

Let

$$X_n(\omega) = U_n(\rho(x,y))y_n(\omega),$$

and

$$r = \frac{y}{x}.$$

We have,

THEOREM 4.2. *For any $x > 0$, $y > 0$, X_n converges a.s. in C as $n \to \infty$ to a process $X^{(x,y)}$. If $r \le 34^{-\frac{1}{4}}$, $X^{(x,y)}$ is self-avoiding a.s. If $r > 34^{-\frac{1}{4}}$, $X^{(x,y)}$ is self-intersecting a.s.*

PROOF. The almost sure convergence of X_n to $X^{(x,y)}$ is obtained by a standard argument using the convergence of the crossing times stated above (See [6]) . We will focus here on the self-avoiding property. As in the 2-dimensional case, the possibilities of self-intersection are classified into type (1) and type (2).

(1) There are $t_1 \ge 0$ and $t_0 > 0$ such that

$$X^{(x,y)}(t) = X^{(x,y)}(t_1), \quad \text{for} \ \ t_1 \le t \le t_1 + t_0 < L(X^{(x,y)}).$$

(2) There are t_1, t_2 and t_3, $t_1 < t_2 < t_3$, such that

$$X^{(x,y)}(t_1) = X^{(x,y)}(t_3), \quad X^{(x,y)}(t_2) \ne X^{(x,y)}(t_1).$$

Type (1) is proved to occur with probability 0 in a similar way to the previous secton. To study type (2), more consideration is needed here in the 3-dimensional case. It is furthere classified into three cases.
(2a) Two parts of the path crossing neighbouring tetrahedrons meet at the junction.
(2b) The path goes in or out from a tetrahedron through the same vetex more than once.
(2c) The path crosses a tetrahedron twice. The two parts of the path corresponding to the first and the second crossing intersect each other in the tetrahedron.

(2a) and (2b) are ruled out just in a similar way to the 2-dimensional case and in its course it is also shown that $X^{(x,y)}$ can cross a tetrahedron at most twice a.s.. The main difference between the 2- and the 3-dimensional case lies in (2c). Let us restate type (2c) more precisely.
(2c) There exist a positive integer K and a sequence of tetrahedrons $\{\Delta^{(k)}\}$, $k = K, K+1, \cdots$, such that
(i) $\Delta^{(k)} \in \mathcal{T}_k$, $\Delta^{(k+1)} \subset \Delta^{(k)}$,
(ii) $X^{(x,y)}$ crosses each $\Delta^{(k)}$ twice,
(iii) the two parts of the path within $\Delta^{(k+1)}$ (the first and the second crossings) originate from distinct parts of the path within $\Delta^{(k)}$.

The condition (iii) is necessary to ensure the inequality $t_1 < t_3$ in (2). For a $\Delta \in \mathcal{T}_m$, $m = 1, 2, \cdots$, let B_n, $n = 1, 2, \cdots, \infty$, be the event that there exists a sequence $\{\Delta^{(k)}\}$, $k = m, m+1, \cdots, m+n$, satisfying (i), (ii), (iii) and $\Delta^{(m)} = \Delta$. Let q_n be the probability of B_n^c under the condition that $X^{(x,y)}$ crosses Δ twice. Note that from the self-similarity of $X^{(x,y)}$ which comes from that of $P_n(x, y)$, and the identical structure of $\Delta \cap F$ with $\Delta \in \mathcal{T}^m$, q_n is independent of m and the choice of Δ. Let $q_0 = 0$. $\{q_n\}$, $n = 0, 1, 2, \cdots$, is increasing, thus, $\lim_{n\to\infty} q_n = q_\infty$ exists. $\{q_n\}$ satisfies the recursion,

$$q_{n+1} = h(q_n), \quad n \in \mathbf{Z}_+ ,$$

$$h(t) = \phi_2(x, y)^{-1}\phi_3(x, y, yt),$$

where

$$\begin{aligned}\phi_3(x, y_1, y_2) \\ = \sum_{w=(w_1,w_2)\in\tilde{V}^1} x^{M_{1,1}(w)} y_1^{M_{1,2}(w_1)+M_{1,2}(w_2)} y_2^{M_{1,2}(w)-M_{1,2}(w_1)-M_{1,2}(w_2)}\end{aligned}$$

$$= 2y_2^4 + 8y_1y_2^3 + 12y_1^2y_2^2 + 4x^3y_2 + x^4.$$

If $r > 34^{-\frac{1}{4}}$, the equation $q = h(q)$ has two solutions in $[0,1]$, 1 and q^*, with $0 < q^* < 1$. It is easily seen that $q_\infty = \lim_{n\to\infty} q_n = q^*$ and thus the probability that (2c) occurs is strictly positive. If $r \leq 34^{-\frac{1}{4}}$, the only solution to the equation above in $[0,1]$ is 1, thus (2c) occurs with probability zero.

We further show that if the probability of the occurrence of (2c) is positive, it is equal to one. Take any $\Delta \in \mathcal{T}_1$ and let s_1 and s_2 be the probabilities that Δ contains no infinite sequence satisfying (i), (ii) and (iii) for any K, under the conditions that $X^{(x,y)}$ passes Δ once and twice, respectively. That means that none of the elements of $\mathcal{T}_2$ within Δ contains such infinite sequences, either. This leads to the equations s_1 and s_2 should satisfy;

$$s_i = \Phi_i(s_1, s_2), \quad i = 1, 2, \tag{4.5}$$

where

$$\Phi_i(s_1, s_2) = \phi_i(x,y)^{-1}\phi_i(s_1x, s_2y), \quad i = 1, 2.$$

$\Phi_1(s_1, s_2)$ and $\Phi_2(s_1, s_2)$ are polynomials of s_1 and s_2 with positive coefficients and without constant terms. From this combined with the fact that $s_1 = s_2 = 0$ and $s_1 = s_2 = 1$ satisfy (4.5), it follows that

$$s_i > \Phi_i(s_1, s_2), \quad i = 1, 2,$$

for any $(s_1, s_2) \in [0,1]^2 \setminus \{(0,0),\ (1,1)\}$, which means that $s_1 = s_2 = 0$ and $s_1 = s_2 = 1$ are the only solutions to (4.5). Thus if $s_i \neq 1$, it follows that the probability of the occurrence of (2c) is equal to one. From (iii) and the almost sure positivity of M_i^*, it follows that t_1, t_2 and t_3 do satisfy $t_1 < t_2 < t_3$ (strict inequality) almost surely, which means that the path is really self-intersecting. □

Next we consider the path Hausdorff dimension. We start with a general theorem. Let J be a non-empty compact subset of $\mathbf{R}^\mathbf{d}$ such that $J = cl(int(J))$. Let $D = \{1, 2, \cdots, N\}$ and $D^* = \bigcup_{n=0}^{\infty} D^n$, where D^0 stands for $\{\emptyset\}$. For σ, $\tau \in D^*$, $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$, $\tau = (\tau_1, \tau_2, \cdots, \tau_m)$, denote $\sigma * \tau = (\sigma_1, \cdots, \sigma_n, \tau_1, \cdots, \tau_m)$, $|\sigma| = n$, $t(\sigma) = \sigma_n$, $\sigma|k = (\sigma_1, \sigma_2, \cdots, \sigma_k)$, $1 \leq k \leq n$, and $\sigma|0 = \emptyset$.

On a probability space $(\Omega, \mathcal{B}, P)$, consider a family of random subsets of $\mathbf{R^d}$,

$$\mathbf{J}(\omega) = \{J_\sigma(\omega) \mid \sigma \in D^*\},$$

and a family of random vectors

$$\mathbf{T}(\omega) = \{\mathbf{T}_\sigma(\omega) = (T_{\sigma*1}(\omega), \cdots, T_{\sigma*N}(\omega)) \mid \sigma \in D^*\}.$$

Assume $\mathbf{J}$ and $\mathbf{T}$ satisfy the following conditions:

(1) $J_\emptyset(\omega) = J$, for P-a.a. $\omega \in \Omega$. For $\sigma \in D^*$, $J_{\sigma*i}(\omega) \subset J_\sigma(\omega)$ and if $J_\sigma(\omega) \neq \emptyset$, then $J_\sigma(\omega)$ is similar to J, for P-a.a. ω.

(2) $int(J_{\sigma*i}(\omega)) \cap int(J_{\sigma*j}(\omega)) = \emptyset$, if $0 \leq i < j \leq N$, for all $\sigma \in D^*$ and P-a.a. ω.

(3) The random vectors $\mathbf{T}_\sigma(\omega) = (T_{\sigma*1}(\omega), \cdots, T_{\sigma*N}(\omega))$, $\sigma \in D^*$, are independent and $\mathbf{T}_\sigma$ is equal in law to $\mathbf{T}_\tau$ if $t(\sigma) = t(\tau)$. $T_\emptyset(\omega) = diam(J)$, for P-a.a. ω . Furthermore,

$$diam(J_\sigma(\omega)) = \prod_{k=0}^{|\sigma|} T_{\sigma|k}(\omega)\,, \quad P - a.a.\omega.$$

Let

$$K(\omega) = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in D^n} J_\sigma(\omega).$$

Define an $N \times N$ matrix $R(\beta) = \{R(\beta)_{ij}\}_{i,j=1,\cdots,N}$, for $\beta \geq 0$ by

$$R(\beta)_{ij} = E^P[\ T_{ij}^\beta\ ].$$

If $R(\beta)$ is irreducible, from Frobenius' theorem, there exists a positive eigenvalue $\lambda(\beta)$ which is the greatest in absolute value, and a corresponding eigenvector $\mathbf{x} = (x_1, \cdots, x_N)^T$ with $x_i > 0$, $i = 1, \cdots, N$. We have the following theorem as a combination of the results in [8] and [10].

THEOREM 4.3. *Suppose $R(0)$ is irreducible and $\lambda(0) > 1$. Then with positive probability, K is non-empty. If K is non-empty, K almost surely has Hausdorff dimension α, where α is determined by $\lambda(\alpha) = 1$.*

Theorem 4.3 is a slight (but necessary for our purpose) generalization of Theorem 3.11 in [10] in the sense that a condition in [10] that if $R(0)_{ij} > 0$ then $T_{ij} > 0$, for a.a. $\omega \in \Omega$ is removed. To obtain Theorem 4.3, we follow the proofs leading to Theorem 1.1 in [8], using the fact that $S_{\alpha,n} = \sum_{\sigma \in D^n} \prod_{m=0}^{n} T^{\alpha}_{\sigma|m} x_{t(\sigma)}$ converges as $n \to \infty$, instead of $\sum_{\sigma \in D^n} \prod_{m=0}^{n} T^{\alpha}_{\sigma|m}$, and changing subsequent formulas in [8] accordingly.

Now let us go back to our processes. Let J be the closed tetrahedron $Oabc$ and $D = \{1, 2, \cdots, 8\}$. On $(\Omega, \mathcal{B}, P)$, we define $\mathbf{J}$. First set $J_{\emptyset}(\omega) = J$, and then define J_σ, $\sigma \in D^n$, $n = 1, 2, \cdots$, recursively. Assume $J_\sigma(\omega)$, $\sigma \in D^n$, is defined and satisfies

$$J_\sigma(\omega) \in \mathcal{T}_n \cup \{\emptyset\},$$

$$int(J_\sigma(\omega)) \cap int(J_\tau(\omega)) = \emptyset, \text{ for any } \tau \in D^n, \tau \neq \sigma, \text{ a.a. } \omega \in \Omega.$$

Then we define $J_{\sigma * i}$, $i = 1, \cdots, 8$ as follows. Since it is shown in a similar way to the 2-dimensional case that $Q_n X^{(x,y)}(\omega) = X_n(\omega)$, a.s., (see Remark after Theorem 3.2), if $J_\sigma(\omega) \cap \{(Q_n X^{(x,y)})(\omega)(t) \mid t \geq 0\}$ is non-empty, it is almost surely similar to either c_1 or c_2. In the former case, there are $i \geq 0$ and u_1, $u_2 \in J_\sigma(\omega) \cap G_n$ such that $u_1 = X_n(\omega)(T^n_i(X_n(\omega)))$ and $u_2 = X_n(\omega)(T^n_{i+1}(X_n(\omega)))$. Denote the other two vertices of $J_\sigma(\omega)$ by u_3 and u_4 (the assignment is arbitrary). In the latter case, there are $j > i \geq 0$ and $u_1, \cdots, u_4 \in J_\sigma(\omega) \cap G_n$ such that $u_1 = X_n(\omega)(T^n_i(X_n(\omega)))$, $u_2 = X_n(\omega)(T^n_{i+1}(X_n(\omega)))$, $u_3 = X_n(\omega)(T^n_j(X_n(\omega)))$, and $u_4 = X_n(\omega)(T^n_{j+1}(X_n(\omega)))$. Denote the four $\mathcal{T}_{n+1}$ -tetrahedrons in $J_\sigma(\omega)$ by $\Delta_1, \cdots, \Delta_4$, so that they satisfy $u_i \in \Delta_i$, $i = 1, \cdots, 4$. Set $J_{\sigma * i} = \Delta_i$, if $\{(Q_{n+1} X^{(x,y)})(t) \mid t \geq 0\} \cap \Delta_i$ is similar to c_1, set $J_{\sigma *(i+4)} = \Delta_i$, if $\{(Q_{n+1} X^{(x,y)})(t) \mid t \geq 0\} \cap \Delta_i$ is similar to c_2. And for the rest of i's, set $J_{\sigma * i}(\omega) = \emptyset$. If $J_\sigma(\omega) \cap \{(Q_n X^{(x,y)})(\omega)(t) \mid t \geq 0\}$ is empty (including the case that $J_\sigma(\omega) = \emptyset$), or similar to neither c_1 or c_2, set $J_{\sigma * i}(\omega) = \emptyset$, $i = 1, \cdots, 8$. $J_{\sigma * i}$'s defined in this way satisfy

$$J_{\sigma * i} \subset J_\sigma,$$

$$J_{\sigma * i} \in \mathcal{T}_{n+1} \cup \{\emptyset\},$$

$$int(J_{\sigma * i}(\omega)) \cap int(J_{\sigma * j}(\omega)) = \emptyset, \text{ for } j \neq i, \text{ a.a. } \omega \in \Omega.$$

J defined here satisfies the assumptions in Theorem 4.3. From the positivity of M_k^*'s in Proposition 4.1, it follows that K is almost surely non-empty (See Theorem 1.2 in [8]). Hence we have the following result.

THEOREM 4.4. $\{X^{(x,y)}(\omega)(t) \mid t \geq 0\}$ *almost surely has Hausdorff dimension* $d_H = \dfrac{\log \rho(x,y)}{\log 2}$.

PROOF. It is enough to show $\lambda(d_H) = 1$. In our case, if $T_{ij} \neq 0$ then $T_{ij} = \dfrac{1}{2}$, a.s.. Thus we have

$$
\begin{aligned}
&(\frac{1}{2})^{-\beta} R(\beta)_{i,j} \\
&\quad = \begin{cases}
\phi_1(x,y)^{-1}\,(x^2 + 2x^3 + 2x^4 + 2x^3 y), & 1 \leq i \leq 4, \quad j = 1,2, \\
\phi_1(x,y)^{-1}\,(x^3 + 2x^4 + 4x^3 y + 6x^2 y^2), & 1 \leq i \leq 4, \quad j = 3,4, \\
\phi_1(x,y)^{-1}\,(2x^3 y + 6x^2 y^2), & 1 \leq i \leq 4, \quad j = 5,6, \\
0, & 1 \leq i \leq 4, \quad j = 7,8, \\
\phi_2(x,y)^{-1}\,(x^4 + 3x^3 y), & 5 \leq i \leq 8, \quad 1 \leq j \leq 4, \\
\phi_2(x,y)^{-1}\,(x^3 y + 22y^4), & 5 \leq i \leq 8, \quad 5 \leq j \leq 8.
\end{cases}
\end{aligned}
$$

Since $(\frac{1}{2})^{-\beta} R(\beta) = R(0)$, and from the definitions, we see that $A(x,y)$ appeared in the beginning of this section and $R(\beta)$ are related as follows;

$$
\begin{aligned}
a(x,y)_{11} &= 2(R(0)_{11} + R(0)_{13}), \\
a(x,y)_{12} &= 2R(0)_{15}, \\
a(x,y)_{21} &= 4R(0)_{51}, \\
a(x,y)_{22} &= 4R(0)_{55}.
\end{aligned}
$$

From this we easily have

$$\lambda(\beta) = (\frac{1}{2})^{\beta} \rho(x,y). \ \square$$

A lower bound of the path Hausdorff dimension is obtained by considering only the tetrahedons $X^{(x,y)}$ crosses only once, which is a subset of K, and applying Theorem 1.1 in [8] to this lower bound set. We have

$$\rho(x,y) \geq \frac{2x^2 + 6x^3 + 8x^4}{x^2 + 2x^3 + (2 + 4r + 6r^2)x^4}\,.$$

For $r \leq 34^{-\frac{1}{4}}$, d_H can take any value between 1 and 2. We see that by taking x large enough and r small enough, we can have d_H as close to 2 as we want. (2 is the Hausdorff dimension of the 3-dimensional Sierpinski gasket .) For $r > 34^{-\frac{1}{4}}$, if we fix x and let $y \to \infty$, then $d_H \to 2$.

5. Concluding remarks

We have shown that on the 2-dimensional Sierpinski gasket, the continuum limit of the branching model is self-avoiding for all values of p and the path Hausdorff dimension can take any value in $(1, \dfrac{\log 3}{\log 2})$. On the 3-dimensional Sierpinski gasket, we have given the classification of the continuum limits according to the values of the parameters – the self-avoiding region and the self-intersecting region. The path Hausdorff dimension has been also given. We will state briefly the relation of our results obtained here and our previous results. In [5] and [6] we considered self-avoiding walks with path probability proportional to $\exp\{-\beta L(w)\}$, where $L(w)$ is the arrival time at a and $\beta > 0$. Their limit processes that are self-avoiding are included in the self-avoiding processes obtained as the limit of the branching models. In [5] we have proved the limit process which is self-avoiding is realized only at $\beta = \beta_c$ on the 2-dimensional Sierpinski gasket. The self-avoiding limit process in [5] coincides (up to the overall time-scaling) with the limit process of the branching model with $p = \dfrac{\sqrt{5}-1}{2}$. If we take the limit $p \to 1$, we get the straight line motion for $\beta > \beta_c$, and in the limit $p \to 0$, we get the Peano curve motion for $\beta < \beta_c$.

The limit self-avoiding process on the 3-dimensional Sierpinski gasket obtained in [6] is reproduced from the branching model if we set $(x, y) = (x_c, y_c)$, where (x_c, y_c) is the unique solution to

$$x = \phi_1(x, y),$$

$$y = \phi_2(x, y),$$

found in the domain $\{(x, y) \in \mathbf{R^2} \mid x > 0,\ y > 0,\ x^2 > y\ \}$. ($\phi_1(x, y)$ and $\phi_2(x, y)$ are defined in the beginning of Section 4.) This also means that the path Hausdorff dimension of the self-avoiding limit process, whose lower bound was given in [6], is evaluated here.

Things get extremely complicated and difficult as the dimension of the space increases (the d-dimensional Sierpinski gasket). The number of the self-avoiding paths increases exponentially and the study of the self-avoiding property boils down to the asymptotic bahavior of high-dimensional dynamical systems. While the complete classification as above may be difficult, as far as the construction of self-avoiding limit processes is concerned, it is possible in a similar fashion in an arbitrary dimension.

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