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Discrete Series Whittaker Functions of $SU(n,1)$ *and* $Spin(2n,1)$

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Abstract. The Mellin transform of Whittaker functions gives the archimedean factor of the automorphic L-functions. Hence it is very important to obtain explicit formulae of Whittaker functions. In this paper, we obtain explicit formulae of discrete series Whittaker functions of $SU(n,1)$ and $Spin(2n,1)$ $(n \geq 2)$.

§0. Introduction

Let G be a real connected semisimple Lie group with finite center, $G = KAN$ be its Iwasawa decomposition, η be a non-degenerate character of N (i.e.the differential of η is non-trivial on every root space corresponding to simple roots). The space of Whittaker functions $C^\infty(G/N;\eta)$ is defined by

$$C^\infty(G/N;\eta) = \{\phi : G \overset{C^\infty}{\to} \mathbb{C}; \phi(gn) = \eta(n)^{-1}\phi(g) \quad \text{for any } g \in G, n \in N\}.$$

For a G-module (π, V), a realization of (π, V) in $C^\infty(G/N;\eta)$ is called a Whittaker model of (π, V). Notice that determination of a Whittaker model of (π, V) is equivalent to that of an intertwining operator ι from (π, V) to $C^\infty(G/N;\eta)$. For any $v \in V$, $\iota(v)(g) \in C^\infty(G/N;\eta)$ is called a Whittaker function corresponding to v.

The Mellin transform of Whittaker functions gives the archimedean factor of the automorphic L-functions. Hence it is very important to obtain explicit formulae of Whittaker functions. Recently, Hayata, Iida, Koseki,

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Miyazaki, Oda, Tsuzuki and Yamashita obtained explicit formulae of spherical functions, Whittaker functions and Shintani functions of some groups and some representations (cf.[K-O], [M-O1], [M-O2], [O1], [O2], [Y1], [Y2]). The author obtained explicit formulae of discrete series Whittaker functions of $SU(n,1)$ and $Spin(2n,1)$ ($n \geq 2$).

The significance of these cases is in that the explicit formulae are calculated for non-quasi-split groups $SU(n,1)$ and $Spin(2n,1)$.

Main results

We will explain the main results of this article.

In this paper, E_{ij} is a matrix $(\delta_{ik}\delta_{jl})_{kl}$ and $F_{ij} := E_{ij} - E_{ji}$.

Let π_Λ be the discrete series representation of G whose Harish-Chandra parameter is Λ, and let π_Λ^* be its contragredient representation. The space $\pi_{\Lambda,K}$ of K-finite vectors in π_Λ becomes a $(\mathfrak{g}_{\mathbb{C}}, K)$-module. Let π_Λ^∞ be the C^∞-globalization of $\pi_{\Lambda,K}$ and let $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_{\Lambda,K}^*, C^\infty(G/N;\eta))$ be the space of the intertwining operators as a $(\mathfrak{g}_{\mathbb{C}}, K)$-module and $\mathrm{Hom}_G(\pi_\Lambda^{*\infty}, C^\infty(G/N;\eta))$ be the space of the intertwining operators as a continuous G-module.

In [Y1], Yamashita proved the following linear isomorphism:

$$\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_{\Lambda,K}^*, C^\infty(G/N;\eta)) \simeq \mathrm{Ker}(\mathcal{D}_{\lambda,\eta}).$$

(For the definition of this differential operator $\mathcal{D}_{\lambda,\eta}$, see §1.3). Then, the determination of intertwining operators reduces to solving a differential equation $\mathcal{D}_{\lambda,\eta}\phi = 0$.

We know that $SU(1,1) \simeq Spin(2,1) \simeq SL(2,\mathbb{R})$ and Whittaker functions of $SL(2,\mathbb{R})$ are well known (cf.[J-L] and [J]). Therefore we investigate $SU(n,1)$ and $Spin(2n,1)$ case for $n \geq 2$.

$SU(n,1)$ **case** ($n \geq 2$)

Let $\{e_i\}$ be the usual basis of the dual space of a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$ (cf. §2.2). The set of Harish-Chandra parameters of discrete series of $SU(n,1)$ is $\Xi = \bigcup_{i=1}^{n+1} \Xi_i$, where

$$\Xi_k = \left\{ \Lambda = \sum_{i=1}^{n} \Lambda_i e_i;\ \ \Lambda_1 > \cdots > \Lambda_{k-1} > 0 > \Lambda_k > \cdots > \Lambda_n \ \ (\Lambda_i \in \mathbb{Z}) \right\},$$

and the corresponding Blattner parameters are denoted by

$$\begin{aligned}\Xi_k \ni \Lambda \Leftrightarrow \lambda &= \sum_{i=1}^{n} \lambda_i e_i \\ &= \sum_{i=1}^{k-1}(\Lambda_i + k + i - n - 1)e_i + \sum_{i=k}^{n}(\Lambda_i + k + i - n - 2)e_i.\end{aligned}$$

Let

$$\begin{cases} X_i = E_{i,n} - E_{i,n+1} - E_{n,i} - E_{n+1,i} \quad (1 \leq i \leq n-1), \\ Y_i = \sqrt{-1}(E_{i,n} - E_{i,n+1} + E_{n,i} + E_{n+1,i}) \quad (1 \leq i \leq n-1), \\ W = \sqrt{-1}(E_{n,n} - E_{n+1,n+1} - E_{n,n+1} + E_{n+1,n}) \end{cases}$$

be a basis of $\mathfrak{n}$.

We define our preferred character η of N by

$$\eta\left(\exp\left(\sum_{i=1}^{n-1}(x_i X_i + y_i Y_i) + wW\right)\right) = e^{\sqrt{-1}y_{n-1}\xi} \qquad (x_i, y_i, w \in \mathbb{R},\ \ \xi \in \mathbb{R}_{>0}). \tag{0.1}$$

Since $\phi \in \mathrm{Ker}\mathcal{D}_{\lambda,\eta}$ is a V_λ-valued function, we can write $\phi(g) = \sum_Q c(Q;g)Q$ by means of the Gel'fand-Zetlin basis of V_λ. For details on Gel'fand-Zetlin basis, we refer to §2.3. The main theorem for $SU(n,1)$ is as follows.

Theorem A (Lemma 3.2.1(1), Proposition 3.2.3, §§6.2 and 6.3).

(1) *Let ζ be any non-degenerate character of N. Then*

$$\dim \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\Lambda,K}, C^\infty(G/N;\zeta)) = \{0\} \qquad (\text{if } \Lambda \in \Xi_1 \cup \Xi_{n+1}),$$

$$\begin{aligned}&\dim \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\Lambda,K}, C^\infty(G/N;\zeta)) \\ &\quad = 2 \sum_{\substack{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k-2} \geq \mu_{k-2} \geq \lambda_{k-1}, \\ \lambda_k \geq \mu_{k-1} \geq \lambda_{k+1} \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-2} \geq \lambda_n}} \dim V^A_{n-2}(\mu_1, \ldots, \mu_{n-2}) \\ &\qquad\qquad (\text{ if } \Lambda \in \Xi_k,\ \ 2 \leq k \leq n),\end{aligned}$$

$$\begin{aligned}&\dim \mathrm{Hom}_G(\pi^{*\infty}_\Lambda, C^\infty(G/N;\zeta)) \\ &\quad = \frac{1}{2}\dim \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\Lambda,K}, C^\infty(G/N;\zeta)),\end{aligned}$$

where, $V^A_{n-2}(\mu_1,\dots,\mu_{n-2})$ *is the irreducible* $U(n-2)$*-module with highest weight* $(\mu_1,\dots,\mu_{n-2})$.

(2) *Let* η *be defined by (0.1) and* $\Lambda\in\Xi_k$ $(2\le k\le n)$. *Then* $\phi\in\mathrm{Ker}\mathcal{D}_{\lambda,\eta}$ *is completely determined by* $c(Q;a)$*'s* $(a\in A)$ *for* Q *satisfying*

$$\begin{aligned}q_{1,n-1}&=q_{1,n-2},\dots,q_{k-2,n-1}=q_{k-2,n-2},\\ q_{k-1,n-1}&=\lambda_{k-1},\\ q_{k-1,n-2}&=q_{k,n-1},\dots,q_{n-2,n-2}=q_{n-1,n-1}.\end{aligned}$$

(3) *For* Q *which satisfies the conditions in (2), the explicit formula of* $c(Q;a)$ *is*

$$\begin{aligned}c(Q;a)=a&^{-\sum_{i=1}^{k-1}\lambda_i+\sum_{i=k}^{n}\lambda_i+\sum_{i=1}^{k-2}q_{i,n-1}-\sum_{i=k}^{n-1}q_{i,n-1}-n+\frac12}\\ &\times\left\{c_1(Q)W_{0,\lambda_{k-1}+n-2k+2}\left(\frac{2\xi}{a}\right)\right.\\ &\qquad\left.+c_2(Q)M_{0,|\lambda_{k-1}+n-2k+2|}\left(\frac{2\xi}{a}\right)\right\},\end{aligned}$$

where, $c_1(Q),c_2(Q)$ *are arbitrary constants and* $W_{\alpha,\beta}(t),M_{\alpha,\beta}(t)$ *are Whittaker's confluent hypergeometric functions (* [W-W] *). Moreover,* $c(Q;a)$ *corresponds to an element of* $\mathrm{Hom}_G(\pi^{*\infty}_\Lambda,C^\infty(G/N;\eta))$ *if and only if* $c_2(Q)=0$.

$Spin(2n,1)$ **case** $(n\ge 2)$

Let $\{e_i\}$ be the usual basis of the dual space of a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$ (cf. §4.2). The set of Harish-Chandra parameters of discrete series of $Spin(2n,1)$ is $\Xi=\Xi_1\cup\Xi_2$, where

$$\Xi_1=\left\{\Lambda=\sum_{i=1}^n\Lambda_ie_i;\ \ \Lambda_1>\dots>\Lambda_n>0,\ \ \Lambda_i\in\frac12\mathbb{Z},\Lambda_i-\Lambda_{i+1}\in\mathbb{Z}\right\},$$

$$\begin{aligned}\Xi_2=\Bigg\{\Lambda=\sum_{i=1}^n\Lambda_ie_i;\ \ &\Lambda_1>\dots>\Lambda_{n-1}>-\Lambda_n>0,\\ &\Lambda_i\in\frac12\mathbb{Z},\Lambda_i-\Lambda_{i+1}\in\mathbb{Z}\Bigg\},\end{aligned}$$

and corresponding Blattner parameters are

$$\Xi_1 \ni \Lambda \Leftrightarrow \lambda = \sum_{i=1}^{n} \lambda_i e_i = \sum_{i=1}^{n} \left(\Lambda_i - i + n + \frac{1}{2}\right) e_i,$$

$$\Xi_2 \ni \Lambda \Leftrightarrow \lambda = \sum_{i=1}^{n} \lambda_i e_i = \sum_{i=1}^{n} \left(\Lambda_i - i + n + \frac{1}{2}\right) e_i - e_n.$$

Let

$$X_i = F_{2n,i} + \sqrt{-1} F_{2n+1,i} \quad (i = 1, \ldots, 2n-1)$$

be a basis of $\mathfrak{n}$. We define our preferred character η of N by

$$\eta\left(\exp\left(\sum_{i=1}^{2n-1} x_i X_i\right)\right) = e^{\sqrt{-1} x_{2n-1}\xi} \quad (x_i \in \mathbb{R},\ \ \xi \in \mathbb{R}_{>0}). \tag{0.2}$$

Since $\phi \in \mathrm{Ker}\mathcal{D}_{\lambda,\eta}$ is V_λ-valued function, we can write $\phi(g) = \sum_Q c(Q; g) Q$ by means of the Gel'fand-Zetlin basis of V_λ (cf. §4.3).

Theorem B (Proposition 5.2.3, §§6.2 and 6.3).

(1) *Let ζ be any non-degenerate character of N. If $\Lambda \in \Xi_1 \cup \Xi_2$, then*

$$\begin{aligned} &\dim \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\Lambda,K}, C^\infty(G/N;\zeta)) \\ &= 2 \sum_{\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-2} \geq \mu_{n-2} \geq \lambda_{n-1} \geq \mu_{n-1} \geq |\lambda_n|} \dim V^D_{2n-2}(\mu_1, \ldots, \mu_{n-1}), \\ &\dim \mathrm{Hom}_G(\pi^{*\infty}_\Lambda, C^\infty(G/N;\zeta)) = \frac{1}{2} \dim \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\Lambda,K}, C^\infty(G/N;\zeta)), \end{aligned}$$

where $V^D_{2n-2}(\mu_1, \ldots, \mu_{n-1})$ is the irreducible Spin$(2n-2)$-module with highest weight $(\mu_1, \ldots, \mu_{n-1})$.

(2) *Let η be defined by (0.2). Then $\phi \in \mathrm{Ker}\mathcal{D}_{\lambda,\eta}$ is completely determined by $c(Q; a)$'s for Q satisfying*

$$\begin{aligned} \lambda_1 \geq\, & q_{1,2n-2} = q_{1,2n-3} \geq \lambda_2 \geq \cdots \geq \lambda_{n-2} \\ & \geq q_{n-2,2n-2} = q_{n-2,2n-3} \geq \lambda_{n-1} \geq q_{n-1,2n-2} = |q_{n-1,2n-3}| \geq |\lambda_n|. \end{aligned}$$

(3) *For Q which satisfies the above conditions in (2), the explicit formula of $c(Q;a)$ is*

$$c(Q;a)=\alpha(Q)a^{-n+1-\sum_{i=1}^{n-1}\lambda_i-|\lambda_n|+\sum_{i=1}^{n-1}q_{i,2n-2}}e^{\mathrm{sgn}q_{n-1,2n-3}\frac{\xi}{a}},$$

where, $\alpha(Q)$ is an arbitrary constant and $\mathrm{sgn}x:=\frac{|x|}{x}$ $(x\in\mathbb{R}_{\neq 0})$. *Further, $c(Q;a)$ corresponds to an element of* $\mathrm{Hom}_G(\pi_\Lambda^{*\infty},C^\infty(G/N;\eta))$ *if and only if $\alpha(Q)=0$ for Q satisfying* $\mathrm{sgn}q_{n-1,2n-3}>0$.

Our dimension formula Theorem A(1) and Theorem B(1) follows from results of Chang ([C]) and Matumoto ([M2]).

Let us give an interpretation of our results. Let $Z_M(\eta)$ be the centralizer of η in M. In our case, for example $G=SU(n,1)$ and η is non-degenerate, $Z_M(\eta)$ is isomorphic to $U(n-2)$ modulo the center. By the right action of $Z_M(\eta)$ on $\mathrm{Ker}\mathcal{D}_{\lambda,\eta}$ (cf.§1.4), $\mathrm{Ker}\mathcal{D}_{\lambda,\eta}$ is decomposed into irreducible $Z_M(\eta)$-modules. The dimension formula in Theorem A(1) describes irreducible $Z_M(\eta)$-modules which occur in this decomposition. Their highest weights are controlled by compact simple roots in the following sense. The compact simple roots of the positive system Δ_k^+ which corresponds to Ξ_k are $e_1-e_2,\ldots,e_{k-2}-e_{k-1},e_k-e_{k+1},\ldots,e_{n-1}-e_n$. The representation of $Z_M(\eta)$ with highest weight $(\mu_1,\ldots,\mu_{n-2})$ enters $\mathrm{Ker}\mathcal{D}_{\lambda,\eta}$ if and only if $(\mu_1,\ldots,\mu_{n-2})$ separates the set of compact simple roots:

$$\begin{array}{ccccccccccc}
e_1 & & e_2 & & \cdots & & e_{k-2} & & e_{k-1} & & \\
\circ & - & \circ & - & \cdots & - & \circ & - & \circ & - & \\
\lambda_1 & & \lambda_2 & & \cdots & & \lambda_{k-2} & & \lambda_{k-1} & & \\
 & \leftrightarrow & & \leftrightarrow & & \leftrightarrow & & \leftrightarrow & & & \\
 & \mu_1 & & \mu_2 & \cdots & \mu_{k-3} & & \mu_{k-2} & & &
\end{array}$$

$$\begin{array}{cccccccccccccc}
 & e_{n+1} & & e_k & & e_{k+1} & & \cdots & & e_{n-1} & & e_n \\
- & \bullet & - & \circ & - & \circ & - & \cdots & - & \circ & - & \circ \\
 & & & \lambda_k & & \lambda_{k+1} & & \cdots & & \lambda_{n-1} & & \lambda_n . \\
 & & & & \leftrightarrow & & \leftrightarrow & & \leftrightarrow & & \leftrightarrow & \\
 & & & & \mu_{k-1} & & \mu_k & \cdots & \mu_{n-3} & & \mu_{n-2} &
\end{array}$$

The author expects that analogous dimension formula is valid for quasi-large discrete series representations of other semisimple Lie groups.

Let us explain the contents of this paper. In §1, we recall well known facts on discrete series representations and the method, due to Yamashita [Y1], to investigate embeddings of discrete series into the space of Whittaker functions. In §2, we review the structure of $SU(n,1)$, parametrize discrete series of $SU(n,1)$ and realize irreducible representations of $K \simeq U(n)$. Yamashita's differential operator $\mathcal{D}_{\lambda,\eta}$ is explicitly computed and explicit representations of the radial part of the minimal K-type Whittaker functions (if they exist) are given in §3. The $Spin(2n,1)$ case is computed in §§4, 5. §4 corresponds to §2 and §5 to §3. In §6, we prove the dimension formula of Theorem A(1) and Theorem B(1). Here we use the fact, due to H.Matumoto [M1], that, for a quasi-large $(\mathfrak{g}_{\mathbb{C}}, K)$-module V, the dimension of the intertwining space $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(V, \mathcal{A}(G/N;\eta))$ coincides with the Bernstein degree $c(V)$ of V. Chang calculated the characteristic cycles of discrete series for $\mathbb{R}$-rank one matrix groups (cf.[C]). His result implies the explicit value of $c(V)$. Finally, we have Theorem A and B.

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§1. Discrete series representations of semisimple Lie groups

1.1. Parametrization of discrete series representations

In this subsection, we review some basic results on discrete series representations for real semisimple Lie groups. For general theory, see [K, Ch.IX], for example.

Let G be a real connected semisimple Lie group with finite center, K be a maximal compact subgroup of G and $\mathfrak{g}$, $\mathfrak{k}$ be their Lie algebras respectively. For any real vector space $\mathfrak{l}$, we denote its complexification $\mathfrak{l} \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathfrak{l}_{\mathbb{C}}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and θ the corresponding Cartan involution. Throughout in this paper, we assume that G has discrete series. In this case, $\mathfrak{g}$ has a compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$. Let Δ be the root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$. For an $\alpha \in \Delta$, let $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ be the corresponding root space. Since $\mathfrak{t} \subset \mathfrak{k}$, for any $\alpha \in \Delta$, $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ is contained either in $\mathfrak{k}_{\mathbb{C}}$ or in $\mathfrak{p}_{\mathbb{C}}$. A root α is called compact (resp. noncompact) if $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{k}_{\mathbb{C}}$ (resp.

$\mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{p}_{\mathbb{C}}$). We will denote the set of all compact roots (resp. noncompact roots) by Δ_c (resp. Δ_n). Δ_c is a root subsystem of Δ and is identified with the root system of $\mathfrak{k}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$. We denote by W and W_c the Weyl groups of Δ and Δ_c respectively, and denote by $B(\ ,\ )$ the Killing form of $\mathfrak{g}_{\mathbb{C}}$, which induces, through its restriction to $\mathfrak{t}_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}$, a W-invariant nondegenerate bilinear form $(\ ,\ )$ on $\mathfrak{t}_{\mathbb{C}}$ and on its dual space $\mathfrak{t}_{\mathbb{C}}^*$ in the canonical way.

Now we parametrize discrete series representations.

Let Ξ be the set of $\Lambda \in \mathfrak{t}_{\mathbb{C}}^*$ which is regular and $\Lambda + \rho$ is K-integral. Here ρ denotes half the sum of all positive roots in Δ with respect to some positive system. The above condition is independent of the choice of a positive system of Δ, and Ξ is W-stable.

We fix a positive system Δ_c^+ of Δ_c. Then $\Xi_c^+ := \{\Lambda \in \Xi; (\Lambda, \alpha) \geq 0 \text{ for } {}^\forall\alpha \in \Delta_c^+\}$ parametrizes discrete series representations of G. We call $\Lambda \in \Xi_c^+$ (resp.$\lambda := \Lambda + \rho - 2\rho_c$) the Harish-Chandra parameter (resp.the Blattner parameter) of a discrete series representation π_Λ.

1.2. Iwasawa decomposition and the space of Whittaker functions

Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$, Σ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$, and Σ^+ be a positive system of Σ. We call $\dim_{\mathbb{R}} \mathfrak{a}$ the real rank of G. For any $\beta \in \Sigma$, we denote the corresponding root space by $\mathfrak{g}_\beta$, and set $\mathfrak{n} := \sum_{\beta \in \Sigma^+} \mathfrak{g}_\beta$, $A := \exp \mathfrak{a}$ and $N := \exp \mathfrak{n}$. We have the well-known Iwasawa decomposition:

$$G = KAN, \qquad \mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}. \tag{1.2.1}$$

Further, we denote the centralizer of $\mathfrak{a}$ in K (resp. in $\mathfrak{k}$) by M (resp. $\mathfrak{m}$).

For a non-degenerate unitary character η of N (i.e. the differential of η is non-trivial on every root space corresponding to simple roots), the space of Whittaker functions $C^\infty(G/N; \eta)$ is defined by

$$C^\infty(G/N; \eta) = \{\phi : G \overset{C^\infty}{\to} \mathbb{C}; \phi(gn) = \eta(n)^{-1}\phi(g) \quad \text{for any } x \in G, n \in N\}. \tag{1.2.2}$$

By the usual semi-norm system, $C^\infty(G/N;\eta)$ is equipped with a structure of Fréchet space. Through left translation and its differential

$$L_x\phi(g) = \phi(x^{-1}g), \qquad L_X\phi(g) = \frac{d}{dt}\phi(\exp(-tX)g)\mid_{t=0}, \tag{1.2.3}$$

$C^\infty(G/N;\eta)$ has continuous G-module and $\mathfrak{g}_{\mathbb{C}}$-module structures.

For a $(\mathfrak{g}_{\mathbb{C}},K)$-module (π,V), we denote by $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi,C^\infty(G/N;\eta))$ the space of intertwining operators as a $(\mathfrak{g}_{\mathbb{C}},K)$-module. Let π^∞ be the C^∞-globalization of (π,V) and let $\mathrm{Hom}_G(\pi^\infty,C^\infty(G/N;\eta))$ be the space of the intertwining operators as a continuous G-module.

1.3. Description of embeddings of discrete series representations into the space of Whittaker functions

In this subsection, we review the method, developed by Yamashita in [Y1], of describing embeddings of discrete series representations into the space of Whittaker functions.

For a finite dimensional continuous representation (τ,V) of K and a unitary character η of N, define

$$\begin{aligned} C^\infty_\tau(K\backslash G/N;\eta) &= \Big\{\phi : G \overset{C^\infty}{\to} V; \phi(kgn) = \eta^{-1}(n)\tau(k)\phi(g) \\ &\qquad\qquad \text{for all} \quad n\in N, k\in K, g\in G\Big\}. \end{aligned} \tag{1.3.1}$$

Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form on $\mathfrak{g}$. We define a K-homomorphism $\nabla_{\tau,\eta} : C^\infty_\tau(K\backslash G/N;\eta) \to C^\infty_{\tau\otimes\mathrm{Ad}_{\mathbb{C}}}(K\backslash G/N;\eta)$ by

$$\nabla_{\tau,\eta}\phi(g) := \sum_i L_{X_i}\phi(g)\otimes X_i, \tag{1.3.2}$$

where $\mathrm{Ad}_{\mathbb{C}}$ denotes the adjoint representation of K on $\mathfrak{p}_{\mathbb{C}}$. It is easy to see that $\nabla_{\tau,\eta}\phi$ is independent of the choice of a basis $\{X_i\}$.

Let (τ_μ,V_μ) denote the irreducible representation of K with highest weight μ.

Now suppose λ is a Blattner parameter. The tensor product $\tau_\lambda \otimes \mathrm{Ad}_{\mathbb{C}}$ decomposes into two K-submodules:

$$(\tau_\lambda \otimes \mathrm{Ad}_{\mathbb{C}}, V_\lambda \otimes \mathfrak{p}_{\mathbb{C}}) \simeq (\tau_\lambda^+, V_\lambda^+) \oplus (\tau_\lambda^-, V_\lambda^-), \tag{1.3.2}$$

where $\tau_\lambda^\pm = \bigoplus_{\alpha \in \Delta_n^+} [u_\lambda(\pm\alpha)]\tau_{\lambda\pm\alpha}$ with $u_\lambda(\pm\alpha) = 0$ or 1. Let $P_\lambda : V_\lambda \otimes \mathfrak{p}_{\mathbb{C}} \to V_\lambda^-$ be the projection operator along this decomposition. We define a first order G-homogeneous differential operator $\mathcal{D}_{\lambda,\eta} : C^\infty_{\tau_\lambda}(K\backslash G/N;\eta) \to C^\infty_{\tau_\lambda^-}(K\backslash G/N;\eta)$ by

$$\mathcal{D}_{\lambda,\eta}\phi(g) := P_\lambda(\nabla_{\lambda,\eta}\phi(g)) \quad (\phi \in C^\infty_{\tau_\lambda}(K\backslash G/N;\eta), g \in G). \tag{1.3.3}$$

(Here, $\nabla_{\lambda,\eta} = \nabla_{\tau_\lambda,\eta}$.)

DEFINITION 1.3.1.
The Blattner parameter λ of a discrete series representation π_Λ is said to be *far from the wall* provided that

$$\lambda - \sum_{\alpha \in Q} \beta \quad \text{is} \quad \Delta_c^+ - \text{dominant for any subset } Q \text{ of } \Delta_n^+. \tag{1.3.5}$$

Notice that the longest element w_0 of the Weyl group W_c induces a bijection $\Lambda \mapsto -w_0\Lambda$ on the set Ξ_c^+ of Harish-Chandra parameters, and that $\pi_{-w_0\Lambda}$ is unitary equivalent to the contragredient representation π_Λ^* of π_Λ. For later convenience, we deal with embeddings of π_Λ^* instead of those of π_Λ.

Under these preparations, we can state the embedding theorem due to Yamashita.

THEOREM 1.3.2 (Yamashita [Y1,Theorem 2.4]).
If the Blattner parameter $\lambda = \Lambda + \rho - 2\rho_c$ of a discrete series representation π_Λ is far from the wall, then we have a linear isomorphism

$$\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\Lambda,K}, C^\infty(G/N;\eta)) \simeq \mathrm{Ker}(\mathcal{D}_{\lambda,\eta}). \tag{1.3.6}$$

This isomorphism is given by:

$$\begin{gathered} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\Lambda,K}, C^\infty(G/N;\eta)) \ni \iota \mapsto F^{[\iota]} \in \mathrm{Ker}(\mathcal{D}_{\lambda,\eta}), \\ \iota(v^*)(g) = \langle v^*, F^{[\iota]}(g)\rangle, \\ (g \in G, v^* \textit{ is a minimal } K\textit{-type vector of } \pi^*_{\Lambda,K}). \end{gathered} \tag{1.3.7}$$

1.4. The cases for $\mathbb{R}$-rank one groups

For $\mathbb{R}$-rank one groups, we can simplify the calculation for the solutions of $\mathcal{D}_{\lambda,\eta}$.

The positive system Σ^+ of a $\mathbb{R}$-rank one Lie group G consists of only one element $\{\beta\}$ or two elements $\{\beta, 2\beta\}$. The set of differential of unitary characters of N can be identified with $\sqrt{-1}\mathfrak{g}_\beta^*$.

Lemma 1.4.1.
Suppose G is $\mathbb{R}$-rank one and $\mathfrak{g} \not\simeq \mathfrak{sl}(2,\mathbb{R})$. Then for any $0 \neq X \in \mathfrak{g}_\beta$, $\mathfrak{g}_\beta - \{0\} = \mathrm{Ad}(M)\mathbb{R}_{>0}X$.

Proof. M acts on $\mathfrak{n}$ by the adjoint action, and $-B(Y_1, \theta Y_2)$ $(Y_1, Y_2 \in \mathfrak{g}_\beta)$ defines a M-invariant inner product on $\mathfrak{g}_\beta$. Set $S_{|X|}(\mathfrak{g}_\beta) := \{Y \in \mathfrak{g}_\beta; -B(Y, \theta Y) = -B(X, \theta X)\}$. Then, counting $\dim M - \dim Z_M(X)$ explicitly ($Z_M(X)$ is the centralizer of X in M), the M-orbit $\mathrm{Ad}(M)X$ is open in $S_{|X|}(\mathfrak{g}_\beta)$ and compact, closed. It follows that $\mathrm{Ad}(M)X$ is a connected component of $S_{|X|}(\mathfrak{g}_\beta)$. On the other hand, if $\dim \mathfrak{g}_\beta \geq 2$ i.e. if $\mathfrak{g} \not\simeq \mathfrak{sl}(2,\mathbb{R})$, then $S_{|X|}(\mathfrak{g}_\beta)$ is connected. This implies that $\mathrm{Ad}(M)X = S_{|X|}(\mathfrak{g}_\beta)$ and the lemma follows. $\square$

For a unitary character η of N and every $m \in M$, let η^m be a unitary character of N such that:

$$\eta^m(n) := \eta(m^{-1}nm) \quad \text{for any} \quad n \in N. \tag{1.4.1}$$

Corollary 1.4.2.
Suppose G is $\mathbb{R}$-rank one and $\mathfrak{g} \not\simeq \mathfrak{sl}(2,\mathbb{R})$.

(1) *For any two non-degenerate unitary characters η_1, η_2 of N, there exists an element $m \in M$ and $c \in \mathbb{R}_{>0}$ given by $d\eta_1^m(X) = d\eta_2(cX)$ for any $X \in \mathfrak{g}_\beta$, where d denotes the differential of these characters.*

(2)

$$\begin{aligned} &C^\infty(G/N;\eta) \ni \phi(g) \\ &\qquad \mapsto \phi^m(g) := \phi(gm) \in C^\infty(G/N;\eta^m) \qquad (x \in G) \end{aligned} \tag{1.4.2}$$

gives a continuous G-module and $\mathfrak{g}_\mathbb{C}$-module isomorphism for every $m \in M$.

(3) *For every $m \in M$, there exists a bijection:*

$$\begin{aligned}(1.4.3)\qquad \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\Lambda,K}, C^\infty(G/N;\eta)) &\simeq \mathrm{Ker}(\mathcal{D}_{\lambda,\eta}) \ni \phi(g) \\ &\mapsto \phi^m(g) := \phi(gm) \in \mathrm{Ker}(\mathcal{D}_{\lambda,\eta^m}) \\ &\simeq \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\Lambda,K}, C^\infty(G/N;\eta^m)).\end{aligned}$$

PROOF. (1) : Direct consequences of Lemma 1.4.1. (2) : trivial. We can show (3) by (2) and Theorem 1.3.2. □

We will explicitly calculate all the elements of $\mathrm{Ker}(\mathcal{D}_{\lambda,\eta})$ for $G = SU(n,1)$ and $Spin(2n,1)$.

§2. Parametrization of discrete series representations of $SU(n,1)$

2.1. Structure of $SU(n,1)$

First, we review the structure of $SU(n,1)$. As in §0, E_{ij} is a matrix $(\delta_{ik}\delta_{jl})_{kl}$. The group $SU(n,1)$ is defined by

$$(2.1.1)\qquad G = SU(n,1) = \{g \in SL(n+1,\mathbb{C}); {}^t\bar{g} I_{n,1} g = I_{n,1}\}$$
$$\left(I_{n,1} = \begin{pmatrix} 1_n & 0 \\ 0 & -1 \end{pmatrix}\right)$$

and its Lie algebra $\mathfrak{g}$ and a maximal compact subgroup K are:

$$\begin{aligned}(2.1.2)\qquad \mathfrak{g} = \mathfrak{su}(n,1) &= \{X \in \mathfrak{sl}(n+1,\mathbb{C})\ ;\ {}^t\bar{X} I_{n,1} + I_{n,1} X = 0\}, \\ K &= G \cap U(n+1) \\ &= \left\{\begin{pmatrix} k & 0 \\ 0 & (\det k)^{-1} \end{pmatrix}\ ;\ k \in U(n)\right\} \simeq U(n).\end{aligned}$$

The orthocomplement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$ (with respect to the Killing form) is

$$(2.1.3)\qquad \mathfrak{p} = \left\{\begin{pmatrix} 0_n & \vec{z} \\ {}^t\bar{\vec{z}} & 0 \end{pmatrix}; \vec{z} \in \mathbb{C}^n\right\}$$

(0_n is $n \times n$-zero matrix) and we fix a maximal abelian subspace $\mathfrak{a} = \mathbb{R}H$ of $\mathfrak{p}$, where $H := E_{n,n+1} + E_{n+1,n}$. Let f be an element of $\mathfrak{a}^*$ (the linear dual space of $\mathfrak{a}$) defined by $f(H) = 1$. A positive root system in $\Sigma(\mathfrak{a}, \mathfrak{g})$ is $\{f, 2f\}$, and the corresponding positive root spaces are

$$\mathfrak{g}_f = \left\{ \begin{pmatrix} 0_{n-1} & \vec{z} & -\vec{z} \\ -{}^t\overline{\vec{z}} & 0 & 0 \\ -{}^t\overline{\vec{z}} & 0 & 0 \end{pmatrix} ; \vec{z} \in \mathbb{C}^{n-1} \right\}, \tag{2.1.4}$$

$$\mathfrak{g}_{2f} = \left\{ \sqrt{-1} \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & x & -x \\ 0 & x & -x \end{pmatrix} ; x \in \mathbb{R} \right\}.$$

The centralizer M and $\mathfrak{m}$ of $\mathfrak{a}$ in K and $\mathfrak{k}$ are

$$M = \left\{ \begin{pmatrix} k & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & l \end{pmatrix} ; k \in U(n-1), l \in U(1), l^2 \cdot \det k = 1 \right\}, \tag{2.1.5}$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} X & 0 & 0 \\ 0 & -\frac{1}{2}\mathrm{tr}X & 0 \\ 0 & 0 & -\frac{1}{2}\mathrm{tr}X \end{pmatrix} ; X \in \mathfrak{u}(n-1) \right\},$$

respectively. We define a basis $\{X_i, Y_i, W\}$ of $\mathfrak{n}$ by

$$\begin{cases} X_i = E_{i,n} - E_{i,n+1} - E_{n,i} - E_{n+1,i} \quad (1 \le i \le n-1), \\ Y_i = \sqrt{-1}(E_{i,n} - E_{i,n+1} + E_{n,i} + E_{n+1,i}) \quad (1 \le i \le n-1), \\ W = \sqrt{-1}(E_{n,n} - E_{n+1,n+1} - E_{n,n+1} + E_{n+1,n}), \end{cases} \tag{2.1.6}$$

and the complexified Iwasawa decomposition of elements of $\mathfrak{p}_{\mathbb{C}}$ are

$$\begin{cases} E_{i,n+1} = \frac{1}{2}(-X_i + \sqrt{-1}Y_i) + E_{i,n} \quad (1 \le i \le n-1), \\ E_{n+1,i} = \frac{1}{2}(-X_i - \sqrt{-1}Y_i) - E_{n,i} \quad (1 \le i \le n-1), \\ E_{n,n+1} = \frac{1}{2}(H + \sqrt{-1}W + E_{n,n} - E_{n+1,n+1}), \\ E_{n+1,n} = \frac{1}{2}(H - \sqrt{-1}W - E_{n,n} + E_{n+1,n+1}). \end{cases} \tag{2.1.7}$$

Here, $E_{i,n}, E_{n,i}$ and $E_{n,n} - E_{n+1,n+1}$ are elements of $\mathfrak{k}_{\mathbb{C}}$.

2.2. Parametrization of discrete series of $SU(n,1)$ (cf.[BS])

In this subsection, we will parametrize discrete series of $SU(n,1)$.

We choose

$$\mathfrak{t} := \left\{ \sqrt{-1}\sum_{i=1}^{n+1} a_i E_{i,i} \ ; \ a_i \in \mathbb{R},\ \sum_{i=1}^{n+1} a_i = 0 \right\} \tag{2.2.1}$$

as a compact Cartan subalgebra of $\mathfrak{g}$ and fix it. The root systems Δ and Δ_c of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ are

$$\begin{aligned} \Delta &= \{e_i - e_j;\ \ 1 \le i \ne j \le n+1\}, \\ \Delta_c &= \{e_i - e_j;\ \ 1 \le i \ne j \le n\}, \end{aligned} \tag{2.2.2}$$

respectively. Here $e_i\left(\sqrt{-1}\sum_{i=1}^{n+1} a_i E_{i,i}\right) = \sqrt{-1}a_i$ $(i = 1, \dots, n+1)$. We fix one of the positive systems of Δ_c:

$$\Delta_c^+ = \{e_i - e_j;\ \ 1 \le i < j \le n\}.$$

There are $n+1$ different positive systems $\Delta_1^+, \dots, \Delta_{n+1}^+$ of Δ which contain Δ_c^+. Their simple roots are :

$$\begin{aligned} \Delta_1^+ &\Longleftrightarrow \Pi_1 = \{e_{n+1} - e_1, e_1 - e_2 \dots, e_{n-1} - e_n\}, \\ \Delta_2^+ &\Longleftrightarrow \Pi_2 = \{e_1 - e_{n+1}, e_{n+1} - e_2, \dots, e_{n-1} - e_n\}, \\ &\cdots \\ \Delta_k^+ &\Longleftrightarrow \Pi_k \\ &\quad = \{e_1 - e_2, \dots, e_{k-1} - e_{n+1}, e_{n+1} - e_k, \dots, e_{n-1} - e_n\}, \\ &\cdots \\ \Delta_{n+1}^+ &\Longleftrightarrow \Pi_{n+1} = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n - e_{n+1}\}. \end{aligned} \tag{2.2.3}$$

Since $\sum_{i=1}^{n+1} e_i \equiv 0$ on $\mathfrak{t}_{\mathbb{C}}$, we identify $\mathfrak{t}_{\mathbb{C}}^*$ with $\sum_{i=1}^{n} \mathbb{C}e_i$ by $e_i \mapsto e_i$ $(i = 1, \dots, n)$ and $e_{n+1} \mapsto -\sum_{i=1}^{n} e_i$. This identification is compatible with the isomorphism $K \ni \begin{pmatrix} k & 0 \\ 0 & (\det k)^{-1} \end{pmatrix} \mapsto k \in U(n)$. Then, the set of Harish-Chandra parameters is denoted by $\bigcup_{k=1}^{n+1} \Xi_k$, where

$$(2.2.4) \quad \Xi_k = \Big\{ \Lambda = \sum_{i=1}^{n} \Lambda_i e_i; \\ \Lambda_1 > \cdots > \Lambda_{k-1} > 0 > \Lambda_k > \cdots > \Lambda_n \ (\Lambda_i \in \mathbb{Z}) \Big\},$$

and the corresponding Blattner parameters are

$$(2.2.5) \qquad \Xi_k \ni \Lambda \Leftrightarrow \lambda = \sum_{i=1}^{n} \lambda_i e_i = \sum_{i=1}^{k-1} (\Lambda_i + k + i - n - 1) e_i \\ + \sum_{i=k}^{n} (\Lambda_i + k + i - n - 2) e_i.$$

2.3. Realization of finite dimensional representations of K

Irreducible representations of $K \simeq U(n)$ are parametrized by n-tuple of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \Leftrightarrow \lambda_1 e_1 + \cdots + \lambda_n e_n$. We denote the corresponding irreducible representation by $(\tau_\lambda, V_\lambda)$. We realize $(\tau_\lambda, V_\lambda)$ by means of the Gel'fand-Zetlin basis (cf. [G-Z1]).

The Gel'fand-Zetlin basis of $(\tau_\lambda, V_\lambda)$ is a set $GZ(\lambda) := \{Q\}$, where Q's are diagrams of shapes

$$(2.3.1) \qquad Q = (q_{ij}) = \begin{pmatrix} q_{1,n} \ q_{2,n} \ \cdots\cdots \ q_{n-1,n} \ q_{n,n} \\ q_{1,n-1} \ q_{2,n-1} \cdots \ q_{n-1,n-1} \\ \cdots\cdots \\ \cdots \\ q_{1,2} \ q_{2,2} \\ q_{1,1} \end{pmatrix}$$

which satisfy

$$\begin{cases} q_{i,j} - q_{i,j-1} \in \mathbb{Z}_{\geq 0}, \\ q_{i,j-1} - q_{i+1,j} \in \mathbb{Z}_{\geq 0}, \\ q_{i,n} = \lambda_i \ (1 \leq i \leq n). \end{cases}$$

The actions of $E_{ij} \in \mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u}(n) \otimes_{\mathbb{R}} \mathbb{C}$ are given by

$$\begin{aligned}
\tau_\lambda(E_{j,j+1})Q &= \sum_{i=1}^{j} a_{i,j}(Q)\sigma_{i,j}Q, \\
\tau_\lambda(E_{j+1,j})Q &= \sum_{i=1}^{j} b_{i,j}(Q)\tau_{i,j}Q, \\
\tau_\lambda(E_{jj})Q &= \left(\sum_{i=1}^{j} q_{i,j} - \sum_{i=1}^{j-1} q_{i,j-1}\right) Q,
\end{aligned} \tag{2.3.2}$$

where

$$a_{i,j}(Q) = \sqrt{\left|\frac{\prod_{k=1}^{j+1}(q_{k,j+1}-q_{i,j}-k+i)\prod_{k=1}^{j-1}(q_{k,j-1}-q_{i,j}-k+i-1)}{\prod_{\substack{k=1\\k\neq i}}^{j}(q_{k,j}-q_{i,j}-k+i)\prod_{\substack{k=1\\k\neq i}}^{j}(q_{k,j}-q_{i,j}-k+i-1)}\right|}, \tag{2.3.3}$$

$$b_{i,j}(Q) = \sqrt{\left|\frac{\prod_{k=1}^{j+1}(q_{k,j+1}-q_{i,j}-k+i+1)\prod_{k=1}^{j-1}(q_{k,j-1}-q_{i,j}-k+i)}{\prod_{\substack{k=1\\k\neq i}}^{j}(q_{k,j}-q_{i,j}-k+i)\prod_{\substack{k=1\\k\neq i}}^{j}(q_{k,j}-q_{i,j}-k+i+1)}\right|}, \tag{2.3.4}$$

$$\begin{aligned}
\sigma_{ij} &: \quad q_{i,j} \mapsto q_{i,j}+1 \text{ and the other } q_{k,l} \mapsto q_{k,l}, \\
\tau_{ij} &: \quad q_{i,j} \mapsto q_{i,j}-1 \text{ and the other } q_{k,l} \mapsto q_{k,l}.
\end{aligned}$$

The actions of general $E_{k,l}$'s are determined by those of bracket products of $E_{j,j+1}$'s and $E_{j+1,j}$'s.

2.4. Irreducible decomposition of $V_\lambda \otimes \mathfrak{p}_\mathbb{C}$

The adjoint representation of K on $\mathfrak{p}_\mathbb{C}$ is decomposed into two irreducible components $\mathfrak{p}_\mathbb{C}^\pm$, where

$$\mathfrak{p}_\mathbb{C}^+ := \bigoplus_{i=1}^n \mathbb{C}E_{i,n+1}, \qquad \mathfrak{p}_\mathbb{C}^- := \bigoplus_{i=1}^n \mathbb{C}E_{n+1,i}. \tag{2.4.1}$$

We denote these representations on $\mathfrak{p}_\mathbb{C}^\pm$ by $(\mathrm{Ad}_\mathbb{C}^\pm, \mathfrak{p}_\mathbb{C}^\pm)$. The highest weights of $(\mathrm{Ad}_\mathbb{C}^+, \mathfrak{p}_\mathbb{C}^+)$ and $(\mathrm{Ad}_\mathbb{C}^-, \mathfrak{p}_\mathbb{C}^-)$ are $(2,1,\dots,1)$ and $(-1,\dots,-1,-2)$, respectively.

Lemma 2.4.1.
If λ is far from the wall, then the irreducible decomposition of $(\tau_\lambda \otimes \mathrm{Ad}_\mathbb{C}^\pm, V_\lambda \otimes \mathfrak{p}_\mathbb{C}^\pm)$ is

$$(\tau_\lambda, V_\lambda) \otimes (\mathrm{Ad}_\mathbb{C}^\pm, \mathfrak{p}_\mathbb{C}^\pm) \simeq \bigoplus_{k=1}^n (\tau_k^\pm, V_k^\pm),$$

where $(\tau_k^\pm, V_k^\pm) = (\tau_{\lambda \pm e'_k}, V_{\lambda \pm e'_k})$ $\left(e'_k = \sum_{i=1}^n e_i + e_k \right)$.

Proof. This follows immediately from Weyl's character formula. □

Let $P_k^\pm : V_\lambda \otimes \mathrm{Ad}_\mathbb{C}^\pm \to V_k^\pm$ be the projection operators. $(\tau_\lambda, V_\lambda)$ is a $U(n)$-submodule of the irreducible $U(n+1)$-modules $V_{\tilde\lambda}$ $(\tilde\lambda = (\lambda_1+1, \lambda_1, \dots, \lambda_n))$ and $V_{\hat\lambda}$ $(\hat\lambda = (\lambda_1, \dots, \lambda_n, \lambda_n - 1))$. Similarly, V_k^+ and V_k^- $(k = 1, \dots, n)$ are $U(n)$-submodules of the irreducible $U(n+1)$-modules $V_{\tilde{\tilde\lambda}}$ and $V_{\hat{\hat\lambda}}$, whose highest weights are $\tilde{\tilde\lambda} = \tilde\lambda + \sum_{i=1}^{n+1} e_i$ and $\hat{\hat\lambda} = \hat\lambda - \sum_{i=1}^{n+1} e_i$, respectively. The corresponding embeddings are given by

$$GZ(\lambda) \ni Q \mapsto \tilde{Q} := \begin{pmatrix} \lambda_1 + 1 & \lambda_1 & \dots & \lambda_n \\ & Q & & \end{pmatrix} \in GZ(\tilde\lambda),$$
$$GZ(\lambda) \ni Q \mapsto \hat{Q} := \begin{pmatrix} \lambda_1 & \dots & \lambda_n & \lambda_n - 1 \\ & Q & & \end{pmatrix} \in GZ(\hat\lambda),$$

$$\iota_k^+ : GZ(\lambda + e'_k) \ni P \mapsto \begin{pmatrix} \lambda_1+2 & \lambda_1+1 & \dots & \lambda_n+1 \\ & & P & \end{pmatrix} \in GZ(\tilde{\tilde{\lambda}}) \qquad (1 \le k \le n),$$

$$\iota_k^- : GZ(\lambda - e'_k) \ni P \mapsto \begin{pmatrix} \lambda_1-1 & \dots & \lambda_n-1 & \lambda_n-2 \\ & & P & \end{pmatrix} \in GZ(\hat{\hat{\lambda}}) \qquad (1 \le k \le n),$$

respectively. Set

$$\begin{aligned} \tilde{\sigma}_{k,n} : GZ(\tilde{\lambda}) \ni \tilde{Q} = (q_{i,j}) &\mapsto \tilde{\sigma}_{k,n}\tilde{Q} = (\tilde{q}_{i,j}) \in GZ(\tilde{\tilde{\lambda}}), \\ \tilde{q}_{i,j} &= q_{i,j} + 1 \quad ((i,j) \neq (k,n)), \\ \tilde{q}_{k,n} &= q_{k,n} + 2, \\ \hat{\tau}_{k,n} : GZ(\hat{\lambda}) \ni \hat{Q} = (q_{i,j}) &\mapsto \hat{\tau}_{k,n}\hat{Q} = (\hat{q}_{i,j}) \in GZ(\hat{\hat{\lambda}}), \\ \hat{q}_{i,j} &= q_{i,j} - 1 \quad ((i,j) \neq (k,n)), \\ \hat{q}_{k,n} &= q_{k,n} - 2. \end{aligned}$$

Using the theory of tensor operators (cf.[Kr]), we can write down $\iota_k^+ \circ P_k^+(Q \otimes E_{n,n+1})$ and $\iota_k^- \circ P_k^-(Q \otimes E_{n+1,n})$ $(Q \in GZ(\lambda))$ explicitly. For notational convenience, $\iota_k^\pm \circ P_k^\pm$ are also denoted by $P_k^\pm$.

PROPOSITION 2.4.2 ([Kr, Proposition 4.3]).
For $Q \in GZ(\lambda)$,

$$\begin{aligned} P_k^+(Q \otimes E_{n,n+1}) &= a_{k,n}(\tilde{Q})\tilde{\sigma}_{k,n}\tilde{Q}, \\ P_k^-(Q \otimes E_{n+1,n}) &= b_{k,n}(\hat{Q})\hat{\tau}_{k,n}\hat{Q}. \end{aligned} \tag{2.4.4}$$

§3. The differential equation $\mathcal{D}_{\lambda,\eta}\phi = 0$

3.1. The explicit formula of $P_k^\pm(\nabla_{\lambda,\eta}^\pm \phi) = 0$

In this subsection, we write down the equation $\mathcal{D}_{\lambda,\eta}\phi = 0$.

Using the Gel'fand-Zetlin basis, we can write $\phi \in C_{\tau_\lambda}^\infty(K\backslash G/N;\eta)$ as

$$\phi(g) = \sum_{Q \in GZ(\lambda)} c(Q;g)Q. \tag{3.1.1}$$

Since $\left\{\frac{E_{i,n+1}+E_{n+1,i}}{2\sqrt{n+1}}, \sqrt{-1}\frac{E_{i,n+1}-E_{n+1,i}}{2\sqrt{n+1}} \ \ (1 \le i \le n)\right\}$ forms an orthonormal basis of $\mathfrak{p}$,

$$\begin{aligned}\nabla_{\lambda,\eta}\phi(g) &= \sum_{i=1}^{n} L_{\frac{E_{i,n+1}+E_{n+1,i}}{2\sqrt{n+1}}}\phi(g) \otimes \frac{E_{i,n+1}+E_{n+1,i}}{2\sqrt{n+1}} \\ &\quad + \sum_{i=1}^{n} L_{\sqrt{-1}\frac{E_{i,n+1}-E_{n+1,i}}{2\sqrt{n+1}}}\phi(g) \otimes \sqrt{-1}\frac{E_{i,n+1}-E_{n+1,i}}{2\sqrt{n+1}} \\ &= \frac{1}{2(n+1)}\sum_{i=1}^{n}\left(L_{E_{n+1,i}}\phi(g)\otimes E_{i,n+1} + L_{E_{i,n+1}}\phi(g)\otimes E_{n+1,i}\right).\end{aligned}$$

We define $\nabla^{\pm}_{\lambda,\eta} : C^{\infty}_{\tau_\lambda}(K\backslash G/N;\eta) \to C^{\infty}_{\tau_\lambda\otimes \mathrm{Ad}^{\pm}_{\mathbb{C}}}(K\backslash G/N;\eta)$ by

$$\nabla^{+}_{\lambda,\eta}\phi(g) := \sum_{i=1}^{n} L_{E_{n+1,i}}\phi(g)\otimes E_{i,n+1}, \tag{3.1.2}$$

$$\nabla^{-}_{\lambda,\eta}\phi(g) := \sum_{i=1}^{n} L_{E_{i,n+1}}\phi(g)\otimes E_{n+1,i}. \tag{3.1.3}$$

Let $R(\mathcal{D}_{\lambda,\eta})$ and $R(\nabla^{\pm}_{\lambda,\eta})$ be the radial A-part of $\mathcal{D}_{\lambda,\eta}$ and $\nabla^{\pm}_{\lambda,\eta}$, respectively. To determine $\phi(g) \in \mathrm{Ker}\mathcal{D}_{\lambda,\eta}$, it is sufficient to calculate $\phi|_A \in \mathrm{Ker}R(\mathcal{D}_{\lambda,\eta})$.

Assume that $\eta \in \hat{N}$ is given by

$$\begin{aligned}\eta\left(\exp\left(\sum_{i=1}^{n-1}(x_iX_i + y_iY_i) + wW\right)\right) \\ = e^{\sqrt{-1}y_{n-1}\xi} \quad (x_i, y_i, w \in \mathbb{R},\ \ \xi \in \mathbb{R}_{>0}).\end{aligned} \tag{3.1.4}$$

Because of Corollary 1.4.2(1) and (3), it suffices to calculate $\phi|_A \in \mathrm{Ker}R(\mathcal{D}_{\lambda,\eta})$ only for this character.

Next, we introduce a coordinate system of A by

$$\mathbb{R}_{>0} \ni a \mapsto \exp((\log a)H) \in A.$$

Then, by (2.1,7), (3.1.2) and (3.1.3), we have

LEMMA 3.1.1.

$$-2R(\nabla^{+}_{\lambda,\eta})\phi(a) = \left(a\frac{d}{da} - \tau_\lambda(E_{n,n} - E_{n+1,n+1})\right)\phi(a)\otimes E_{n,n+1}$$

$$- 2\sum_{i=1}^{n-1}\tau_\lambda(E_{n,i})\phi(a)\otimes E_{i,n+1} - \frac{\xi}{a}\phi(a)\otimes E_{n-1,n+1}, \tag{3.1.5}$$

$$-2R(\nabla^{-}_{\lambda,\eta})\phi(a) = \left(a\frac{d}{da} + \tau_\lambda(E_{n,n} - E_{n+1,n+1})\right)\phi(a)\otimes E_{n+1,n}$$

$$+ 2\sum_{i=1}^{n-1}\tau_\lambda(E_{i,n})\phi(a)\otimes E_{n+1,i} + \frac{\xi}{a}\phi(a)\otimes E_{n+1,n-1}. \tag{3.1.6}$$

For any $Q \in GZ(\lambda)$,

$$\begin{aligned}\tau_\lambda(E_{n,i})&Q\otimes E_{i,n+1}\\ &= (\tau_\lambda\otimes \mathrm{Ad}^{+}_{\mathbb{C}})(E_{n,i}E_{i,n})(Q\otimes E_{n,n+1})\\ &\quad - \tau_\lambda(E_{n,i}E_{i,n})Q\otimes E_{n,n+1} - Q\otimes E_{n,n+1}.\end{aligned}$$

Hence we have

$$\begin{aligned}\sum_{i=1}^{n-1}&\tau_\lambda(E_{n,i})Q\otimes E_{i,n+1}\\ &= \sum_{i=1}^{n-1}(\tau_\lambda\otimes \mathrm{Ad}^{+}_{\mathbb{C}})(E_{n,i}E_{i,n})(Q\otimes E_{n,n+1})\\ &\quad - \sum_{i=1}^{n-1}\tau_\lambda(E_{n,i}E_{i,n})Q\otimes E_{n,n+1} - (n-1)Q\otimes E_{n,n+1}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_{i=1}^{n-1}&\tau_\lambda(E_{i,n})Q\otimes E_{n+1,i}\\ &= -\sum_{i=1}^{n-1}(\tau_\lambda\otimes \mathrm{Ad}^{+}_{\mathbb{C}})(E_{i,n}E_{n,i})(Q\otimes E_{n+1,n})\\ &\quad + \sum_{i=1}^{n-1}\tau_\lambda(E_{i,n}E_{n,i})Q\otimes E_{n+1,n} + (n-1)Q\otimes E_{n+1,n}.\end{aligned}$$

Let C_n and C_{n-1} be the Casimir elements of $\mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{gl}(n-1,\mathbb{C})$, respectively. Since the Killing form B of $\mathfrak{gl}(n,\mathbb{C})$ is given by $B(X,Y) = 2n\mathrm{tr}(XY)$, C_n and C_{n-1} are :

$$\begin{aligned}2nC_n &= \sum_{i=1}^{n} E_{i,i}^2 + 2\sum_{1\le i<j\le n} E_{i,j}E_{j,i} - \sum_{i=1}^{n}(n+1-2i)E_{i,i}\\ &= \sum_{i=1}^{n} E_{i,i}^2 + 2\sum_{1\le i<j\le n} E_{j,i}E_{i,j} + \sum_{i=1}^{n}(n+1-2i)E_{i,i},\end{aligned}$$

$$\begin{aligned}2(n-1)C_{n-1} &= \sum_{i=1}^{n-1} E_{i,i}^2 + 2\sum_{1\le i<j\le n-1} E_{i,j}E_{j,i} - \sum_{i=1}^{n-1}(n-2i)E_{i,i}\\ &= \sum_{i=1}^{n-1} E_{i,i}^2 + 2\sum_{1\le i<j\le n-1} E_{j,i}E_{i,j} + \sum_{i=1}^{n-1}(n-2i)E_{i,i}.\end{aligned}$$

Then, it follows:

$$\begin{aligned}\sum_{i=1}^{n-1} E_{n,i}E_{i,n} &= nC_n - (n-1)C_{n-1} - \frac{1}{2}E_{n,n}^2 - \frac{1}{2}\sum_{i=1}^{n-1}E_{i,i} + \frac{1}{2}(n-1)E_{n,n},\\ \sum_{i=1}^{n-1} E_{i,n}E_{n,i} &= nC_n - (n-1)C_{n-1} - \frac{1}{2}E_{n,n}^2 + \frac{1}{2}\sum_{i=1}^{n-1}E_{i,i} - \frac{1}{2}(n-1)E_{n,n}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\tau_\lambda(2nC_n)Q &= \left\{\sum_{i=1}^{n}\lambda_i^2 + \sum_{i=1}^{n}(n+1-2i)\lambda_i\right\}Q,\\ \tau_\lambda(2(n-1)C_{n-1})Q &= \left\{\sum_{i=1}^{n-1}q_{i,n-1}^2 + \sum_{i=1}^{n-1}(n-2i)q_{i,n-1}\right\}Q.\end{aligned}$$

Using these formulae and Proposition 2.4.2, we have the following equalities:

LEMMA 3.1.2.

For any $Q \in GZ(\lambda)$,

$$(3.1.7)\qquad P_k^+\left(\sum_{i=1}^{n-1}\tau_\lambda(E_{n,i})Q\otimes E_{i,n+1}\right) = a_{k,n}(\tilde{Q})\left\{-\sum_{\substack{i=1\\ i\neq k}}^{n}\lambda_i+\sum_{i=1}^{n-1}q_{i,n-1}-k+1\right\}\tilde{\sigma}_{k,n}\tilde{Q},$$

$$(3.1.8)\qquad P_k^-\left(\sum_{i=1}^{n-1}\tau_\lambda(E_{i,n})Q\otimes E_{n+1,i}\right) = b_{k,n}(\hat{Q})\left\{-\sum_{\substack{i=1\\ i\neq k}}^{n}\lambda_i+\sum_{i=1}^{n-1}q_{i,n-1}+n-k\right\}\hat{\tau}_{k,n}\hat{Q}.$$

From

$$\begin{aligned} Q\otimes E_{n-1,n+1} &= (\tau_\lambda\otimes\mathrm{Ad}_{\mathbb{C}}^+)(E_{n-1,n})(Q\otimes E_{n,n+1}) \\ &\quad -\tau_\lambda(E_{n-1,n})Q\otimes E_{n,n+1}, \\ Q\otimes E_{n+1,n-1} &= -(\tau_\lambda\otimes\mathrm{Ad}_{\mathbb{C}}^-)(E_{n,n-1})(Q\otimes E_{n+1,n}) \\ &\quad +\tau_\lambda(E_{n,n-1})Q\otimes E_{n+1,n}, \end{aligned}$$

we have :

LEMMA 3.1.3.
For any $Q \in GZ(\lambda)$,

$$(3.1.9)\qquad P_k^+(Q\otimes E_{n-1,n+1}) = \sum_{j=1}^{n-1}\frac{a_{j,n-1}(\tilde{Q})a_{k,n}(\sigma_{j,n-1}\tilde{Q})}{\lambda_k-q_{j,n-1}-k+j}\sigma_{j,n-1}\tilde{\sigma}_{k,n}\tilde{Q},$$

$$(3.1.10)\qquad P_k^-(Q\otimes E_{n+1,n-1}) = \sum_{j=1}^{n-1}\frac{b_{j,n-1}(\hat{Q})b_{k,n}(\tau_{j,n-1}\hat{Q})}{\lambda_k-q_{j,n-1}-k+j+1}\tau_{j,n-1}\hat{\tau}_{k,n}\hat{Q}.$$

The next proposition follows from Proposition 2.4.2, Lemma 3.1.1, 3.1.2, and 3.1.3.

PROPOSITION 3.1.4.
For $1 \leq k \leq n$, we have:

$$
\begin{aligned}
(3.1.11)\ & P_k^+(R(\nabla_{\lambda,\eta}^+)\phi(a)) = 0 \\
\iff & \sum_{Q \in GZ(\lambda)} a_{k,n}(\tilde{Q}) \\
& \times \left(a\frac{d}{da} + \sum_{i=1}^{n} \lambda_i - 2\lambda_k - \sum_{i=1}^{n-1} q_{i,n-1} + 2k - 2 \right) c(Q;a)\tilde{\sigma}_{k,n}\tilde{Q} \\
& - \frac{\xi}{a} \sum_{j=1}^{n-1} \sum_{\tau_{j,n-1}Q \in GZ(\lambda)} \\
& \times \frac{a_{k,n}(\tilde{Q}) a_{j,n-1}(\tau_{j,n-1}\tilde{Q})}{\lambda_k - q_{j,n-1} - k + j + 1} c(\tau_{j,n-1}Q;a)\tilde{\sigma}_{k,n}\tilde{Q} = 0,
\end{aligned}
$$

$$
\begin{aligned}
(3.1.12)\ & P_k^-(R(\nabla_{\lambda,\eta}^-)\phi(a)) = 0 \\
\iff & \sum_{Q \in GZ(\lambda)} b_{k,n}(\hat{Q}) \\
& \times \left(a\frac{d}{da} - \sum_{i=1}^{n} \lambda_i + 2\lambda_k + \sum_{i=1}^{n-1} q_{i,n-1} + 2n - 2k \right) c(Q;a)\hat{\tau}_{k,n}\hat{Q} \\
& + \frac{\xi}{a} \sum_{j=1}^{n-1} \sum_{\sigma_{j,n-1}Q \in GZ(\lambda)} \\
& \times \frac{b_{k,n}(\hat{Q}) b_{j,n-1}(\sigma_{j,n-1}\hat{Q})}{\lambda_k - q_{j,n-1} - k + j} c(\sigma_{j,n-1}Q;a)\hat{\tau}_{k,n}\hat{Q} = 0.
\end{aligned}
$$

These equations are the explicit representations of $P_k^{\pm}(R(\nabla_{\lambda,\eta}^{\pm})\phi(a)) = 0$, which we needed.

3.2. The explicit formulae of $c(Q;a)$

If $\Lambda \in \Xi_k$, then $\mathcal{D}_{\lambda,\eta}\phi(g) = 0$ is equivalent to

$$
\begin{aligned}
P_1^-(\nabla_{\lambda,\eta}^-\phi(g)) &= \cdots = P_{k-1}^-(\nabla_{\lambda,\eta}^-\phi(g)) \\
&= P_k^+(\nabla_{\lambda,\eta}^+\phi(g)) = \cdots = P_n^+(\nabla_{\lambda,\eta}^+\phi(g)) = 0.
\end{aligned}
$$

By (3.1.11), we have

$$
\begin{aligned}
(3.2.1) \qquad & P_l^+(R(\nabla_{\lambda,\eta}^+)\phi(a)) = 0 \\
& \quad \Longrightarrow c(Q;a) = 0 \quad \text{for } Q = (q_{ij}) \\
& \qquad\quad \text{satisfying } q_{l,n-1} = \lambda_l, q_{l-1,n-2} > \lambda_l.
\end{aligned}
$$

Moreover, we can show that $P_l^+(R(\nabla_{\lambda,\eta}^+)\phi(a)) = 0$ implies

$$
\begin{cases} c(Q;a) = 0 \text{ for any } Q \text{ satisfying } q_{l-1,n-2} > \lambda_l & (\text{if } 2 \le l \le n-1), \\ c(Q;a) = 0 \text{ for any } Q \quad (\text{if } l = 1), \end{cases}
$$

by (3.2.1) and recursive usage of (3.1.11). Similarly, $P_l^-(\nabla_{\lambda,\eta}^-\phi(a)) = 0$ implies

$$
\begin{cases} c(Q;a) = 0 \text{ for any } Q \text{ satisfying } q_{l-1,n-2} < \lambda_l & (\text{if } 2 \le l \le n-1), \\ c(Q;a) = 0 \text{ for any } Q \quad (\text{if } l = n). \end{cases}
$$

Consequently,

LEMMA 3.2.1.

(1) *If* $\Lambda \in \Xi_1 \cup \Xi_{n+1}$, *then* $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_{\Lambda,K}^*, C^\infty(G/N;\eta)) = \{0\}$.

(2) *In order to solve* $R(\mathcal{D}_{\lambda,\eta})\phi(a) = 0$ *(* $\Lambda \in \Xi_k$, $2 \le k \le n$*), we have only to calculate* $c(Q;a)$ *for* Q *satisfying*

$$
\begin{aligned}
(3.2.2) \qquad & \lambda_1 \ge q_{1,n-1} \ge q_{1,n-2} \ge \lambda_2 \ge \dots \\
& \dots \lambda_{k-2} \ge q_{k-2,n-1} \ge q_{k-2,n-2} \ge \lambda_{k-1} \\
& \quad \ge q_{k-1,n-1} \ge \lambda_k \ge q_{k-1,n-2} \ge q_{k,n-1} \ge \lambda_{k+1} \ge \dots \\
& \quad\quad \dots \ge \lambda_{n-1} \ge q_{n-2,n-2} \ge q_{n-1,n-1} \ge \lambda_n.
\end{aligned}
$$

Suppose $\Lambda \in \Xi_k$. In order to solve $\mathcal{D}_{\lambda,\eta}\phi = 0$, we eliminate the difference terms of equations $P_1^-(\nabla_{\lambda,\eta}^-\phi) = \dots = P_{k-1}^-(\nabla_{\lambda,\eta}^-\phi) = 0$ and $P_k^+(\nabla_{\lambda,\eta}^+\phi) = \dots = P_n^+(\nabla_{\lambda,\eta}^+\phi) = 0$.

LEMMA 3.2.2.

(1) *If* $P_k^+(\nabla_{\lambda,\eta}^+\phi) = \cdots = P_n^+(\nabla_{\lambda,\eta}^+\phi) = 0$, *then for any* $l; p_1, \ldots, p_l; j_1, \ldots, j_{l-1}$ *which satisfy* $1 \leq l \leq n-k+1, k \leq p_1 < \cdots < p_l \leq n, 1 \leq j_1 < \cdots < j_{l-1} \leq n-1$, *we have*

$$\left(\prod_{i=1}^{l} a_{p_i,n}(\tilde{Q})\right) \times \Biggl\{ \Biggl(a\frac{d}{da} + \sum_{i=1}^{n} \lambda_i - 2\sum_{i=1}^{l}(\lambda_{p_i} - p_i) - \sum_{i=1}^{n-1} q_{i,n-1} + 2\sum_{i=1}^{l-1}(q_{j_i,n-1} - j_i) - 2l \Biggr) c(Q;a) - \frac{\xi}{a} \sum_{\substack{j=1 \\ j\neq j_1,\ldots,j_{l-1}}}^{n-1} \frac{\prod_{i=1}^{l-1}(q_{j_i,n-1} - q_{j,n-1} + j - j_i)}{\prod_{i=1}^{l}(\lambda_{p_i} - q_{j,n-1} - p_i + j + 1)} \times a_{j,n-1}(\tau_{j,n-1}\tilde{Q}) c(\tau_{j,n-1}Q;a) \Biggr\} = 0.$$

We call the above equation $(3.2.3)_{p_1,\ldots,p_l;j_1,\ldots,j_{l-1}}$.

(2) *If* $P_1^-(\nabla_{\lambda,\eta}^-\phi) = \cdots = P_{k-1}^-(\nabla_{\lambda,\eta}^-\phi) = 0$, *then for any* $l; p_1, \ldots, p_l; j_1, \ldots, j_{l-1}$ *which satisfy* $1 \leq l \leq k-1, 1 \leq p_1 < \cdots < p_l \leq k-1, 1 \leq j_1 < \cdots < j_{l-1} \leq n-1$,

$$\left(\prod_{i=1}^{l} b_{p_i,n}(\hat{Q})\right) \times \Biggl\{ \Biggl(a\frac{d}{da} - \sum_{i=1}^{n} \lambda_i + 2\sum_{i=1}^{l}(\lambda_{p_i} - p_i) + \sum_{i=1}^{n-1} q_{i,n-1} - 2\sum_{i=1}^{l-1}(q_{j_i,n-1} - j_i) + 2n \Biggr) c(Q;a) + \frac{\xi}{a} \sum_{\substack{j=1 \\ j\neq j_1,\ldots,j_{l-1}}}^{n-1} \frac{\prod_{i=1}^{l-1}(q_{j_i,n-1} - q_{j,n-1} + j - j_i)}{\prod_{i=1}^{l}(\lambda_{p_i} - q_{j,n-1} - p_i + j)}$$

$$\left. \times\, b_{j,n-1}(\sigma_{j,n-1}\hat{Q})c(\sigma_{j,n-1}Q;a) \right\} = 0.$$

We call the above equation $(3.2.4)_{p_1,\dots,p_l;j_1,\dots,j_{l-1}}$.

PROOF. We prove these formulae by induction on l. If $l = 1$, then these formulae are coefficients of $\tilde{\sigma}_{p_1,n}\tilde{Q}$ and $\hat{\tau}_{p_1,n}\hat{Q}$ in equations (3.1.11) and (3.1.12), respectively. If these formulae hold for some l, then, by eliminating a difference term, we can check that they are true for $l+1$. □

This lemma implies that, if all $c(Q;a)$'s are given for Q satisfying

$$(3.2.5)\qquad \begin{aligned} &\lambda_1 \geq q_{1,n-1} = q_{1,n-2} \geq \lambda_2 \geq \dots \\ &\qquad \dots \lambda_{k-2} \geq q_{k-2,n-1} = q_{k-2,n-2} \geq \lambda_{k-1} = q_{k-1,n-1}, \\ &\lambda_k \geq q_{k-1,n-2} = q_{k,n-1} \geq \lambda_{k+1} \geq \dots \\ &\qquad\qquad \dots \geq \lambda_{n-1} \geq q_{n-2,n-2} = q_{n-1,n-1} \geq \lambda_n, \end{aligned}$$

then all the other $c(Q;a)$'s are uniquely determined.

Let us find the explicit formulae of $c(Q;a)$'s.

Suppose $Q = (q_{i,j})$ satisfies $q_{k-1,n-1} > \lambda_k$ and the other $q_{i,j}$'s satisfy (3.2.5). Let $l, p_i (1 \leq i \leq l), p'_i (1 \leq p'_i \leq n-k-l+1)$ and $j_i (1 \leq j_i \leq l-1)$ be integers determined by

$$(3.2.6)\qquad \begin{cases} k+1 \leq p_1 < \dots < p_{l-1} \leq n, \ \ q_{p_i-1,n-1} \neq \lambda_{p_i}, \\ k+1 \leq p'_1 < \dots < p'_{n-k-l+1} \leq n, \ \ q_{p'_i-1,n-1} = \lambda_{p'_i}, \\ p_l = k, \\ j_i = p_i - 1, \ \ 1 \leq i \leq l-1. \end{cases}$$

Then, we have

$$\begin{cases} a_{p_i,n}(\tilde{Q}) \neq 0 \ \ (1 \leq i \leq l), \\ c(\tau_{p'_i-1,n-1}Q;a) = 0 \ \ (1 \leq i \leq n-k-l+1), \\ c(\tau_{j,n-1}Q;a) = 0 \ \ (1 \leq j \leq k-2), \end{cases}$$

$$-2\sum_{i=1}^{l}(\lambda_{p_i} - p_i) + 2\sum_{i=1}^{l-1}(q_{j_i,n-1} - j_i) - 2l = -2\sum_{i=k}^{n}\lambda_i + 2\sum_{i=k}^{n-1} q_{i,n-1} + 2k - 2,$$

$$\frac{\prod_{i=1}^{l-1}(q_{j_i,n-1}-q_{k-1,n-1}+k-1-j_i)}{\prod_{i=1}^{l}(\lambda_{p_i}-q_{k-1,n-1}-p_i+k-1+1)}=\frac{\prod_{i=k}^{n-1}(q_{i,n-1}-q_{k-1,n-1}+k-i-1)}{\prod_{i=k}^{n}(\lambda_i-q_{k-1,n-1}-i+k)}.$$

Finally, equation $(3.2.3)_{p_1,\dots,p_l;j_i,\dots,j_{l-1}}$ is written as follows.

$$\begin{aligned}(3.2.7)\qquad & c(\tau_{k-1,n-1}Q;a)\\ &=\frac{a}{\xi}\frac{1}{a_{k-1,n-1}(\tau_{k-1,n-1}\tilde{Q})}\\ &\quad\times\frac{\prod_{i=k}^{n}(\lambda_i-q_{k-1,n-1}-i+k)}{\prod_{i=k}^{n-1}(q_{i,n-1}-q_{k-1,n-1}-i+k-1)}\\ &\quad\times\left(a\frac{d}{da}+\sum_{i=1}^{k-1}\lambda_i-\sum_{i=k}^{n}\lambda_i-\sum_{i=1}^{k-1}q_{i,n-1}\right.\\ &\qquad\left.+\sum_{i=k}^{n-1}q_{i,n-1}+2k-2\right)c(Q;a).\end{aligned}$$

(Notice that, by our assumption, $a_{k-1,n-1}(\tau_{k-1,n-1}Q)\neq 0$ holds.) Similarly, if Q satisfies $q_{k-1,n-1}<\lambda_{k-1}$ and the other $q_{i,j}$'s satisfy (3.2.5), then equation $(3.2.4)_{p_1,\dots,p_l;j_i,\dots,j_{l-1}}$ is

$$\begin{aligned}(3.2.8)\qquad & c(\sigma_{k-1,n-1}Q;a)\\ &=-\frac{a}{\xi}\frac{1}{b_{k-1,n-1}(\sigma_{k-1,n-1}\hat{Q})}\\ &\quad\times\frac{\prod_{i=1}^{k-1}(\lambda_i-q_{k-1,n-1}-i+k-1)}{\prod_{i=1}^{k-2}(q_{i,n-1}-q_{k-1,n-1}-i+k-1)}\\ &\quad\times\left(a\frac{d}{da}+\sum_{i=1}^{k-1}\lambda_i-\sum_{i=k}^{n}\lambda_i-\sum_{i=1}^{k-2}q_{i,n-1}\right.\\ &\qquad\left.+\sum_{i=k-1}^{n-1}q_{i,n-1}+2n-2k+2\right)c(Q;a).\end{aligned}$$

($b_{k-1,n-1}(\sigma_{k-1,n-1}Q)\neq 0$ by same reason.)

Therefore, $c(Q;a)$ for Q satisfying (3.2.5) is a solution of the following single equation;

$$\left\{\left(a\frac{d}{da}+A-q_{k-1,n-1}+2k-2\right)\right.$$
$$\left.\times\left(a\frac{d}{da}+A+q_{k-1,n-1}+2n-2k+2\right)-\frac{\xi^2}{a^2}\right\}c(Q;a)=0,$$

where $A=\sum_{i=1}^{k-1}\lambda_i-\sum_{i=k}^{n}\lambda_i-\sum_{i=1}^{k-2}q_{i,n-1}+\sum_{i=k}^{n-1}q_{i,n-1}$. Set $t=\frac{2\xi}{a}$ and $c(Q;\frac{2\xi}{t})=t^{A+n-\frac{1}{2}}f(Q;t)$. Then $f(Q;t)$ satisfies

$$\left\{t^2\frac{d^2}{dt^2}-\frac{t^2}{4}-(q_{k-1,n-1}+n-2k+2)^2+\frac{1}{4}\right\}f(Q;t)=0.$$

This is the so-called Whittaker's differential equation.

We have shown the following proposition :

PROPOSITION 3.2.3.

(1) *If* $\Lambda\in\Xi_k$ *(*$2\leq\Lambda\leq n$*), then*

(3.2.9)
$$\begin{aligned}&\dim\operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\Lambda,K},C^\infty(G/N;\eta))\\&\qquad\leq 2\sum_{\substack{\lambda_1\geq\mu_1\geq\lambda_2\geq\cdots\geq\lambda_{k-2}\geq\mu_{k-2}\geq\lambda_{k-1},\\ \lambda_k\geq\mu_{k-1}\geq\lambda_{k+1}\geq\cdots\geq\lambda_{n-1}\geq\mu_{n-2}\geq\lambda_n}}\dim V^A_{n-2}(\mu_1,\ldots,\mu_{n-2}),\end{aligned}$$

where $V^A_{n-2}(\mu_1,\ldots,\mu_{n-2})$ *is the irreducible* $U(n-2)$*-module with highest weight* $(\mu_1,\ldots,\mu_{n-2})$*.*

(2) *The explicit formula of* $c(Q;a)$ *for* Q *which satisfies (3.2.5) is*

(3.2.10)
$$\begin{aligned}c(Q;a)=\;&a^{-\sum_{i=1}^{k-1}\lambda_i+\sum_{i=k}^{n}\lambda_i+\sum_{i=1}^{k-2}q_{i,n-1}-\sum_{i=k}^{n-1}q_{i,n-1}-n+\frac{1}{2}}\\&\times\left\{c_1(Q)W_{0,q_{k-1,n-1}+n-2k+2}\left(\frac{2\xi}{a}\right)\right.\\&\left.\qquad+c_2(Q)M_{0,|q_{k-1,n-1}+n-2k+2|}\left(\frac{2\xi}{a}\right)\right\},\end{aligned}$$

where, $c_1(Q),c_2(Q)$ *are arbitrary constants and* $W_{\alpha,\beta}(t),M_{\alpha,\beta}(t)$ *are Whittaker's confluent hypergeometric functions (cf.*[W-W]*).*

As a matter of fact, the equal sign in (3.2.9) holds and we will prove it in §6.2.

§4. Parametrization of discrete series representations of $Spin(2n,1)$

4.1. Structure of $Spin(2n,1)$

We review the structure of $G = Spin(2n,1)$. As in §0, $F_{ij} = E_{ij} - E_{ji}$. The group $Spin(2n,1)$ is the connected two-fold linear cover of $SO_0(2n,1)$ and its maximal compact subgroup K is isomorphic to $Spin(2n)$. The Lie algebra $\mathfrak{g} = \mathfrak{o}(2n,1)$ of G is given by

$$\mathfrak{g} = \left\{ \begin{pmatrix} X & \sqrt{-1}\vec{v} \\ -\sqrt{-1}{}^t\vec{v} & 0 \end{pmatrix}; X \in \mathfrak{o}(2n), \vec{v} \in \mathbb{R}^{2n} \right\},$$

and its Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}; X \in \mathfrak{o}(2n) \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0_{2n} & \sqrt{-1}\vec{v} \\ -\sqrt{-1}{}^t\vec{v} & 0 \end{pmatrix}; \vec{v} \in \mathbb{R}^{2n} \right\}.$$

Then $G = Spin(2n,1) = K \exp \mathfrak{p}$. Fix a maximal abelian subspace $\mathfrak{a} = \mathbb{R}H$ in $\mathfrak{p}$ where $H := \sqrt{-1}F_{2n+1,2n}$. Let f be an element of $\mathfrak{a}^*$ defined by $f(H) = 1$. A positive system in $\Sigma(\mathfrak{a},\mathfrak{g})$ is $\{f\}$, and the corresponding root space is

$$\mathfrak{n} = \mathfrak{g}_f = \sum_{i=1}^{2n-1} \mathbb{R}(F_{2n,i} + \sqrt{-1}F_{2n+1,i}).$$

We denote $X_i = F_{2n,i} + \sqrt{-1}F_{2n+1,i}$ $(1 \leq i \leq 2n-1)$. The centralizer M of $\mathfrak{a}$ in K is isomorphic to $Spin(2n-1)$ and its Lie algebra $\mathfrak{m}$ is

$$\mathfrak{m} = \left\{ \begin{pmatrix} X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; X \in \mathfrak{o}(2n-1) \right\}.$$

The Iwasawa decomposition of elements of $\mathfrak{p}$ are

$$\text{(4.1.1)} \qquad \begin{cases} \sqrt{-1}F_{2n+1,i} = X_i - F_{2n,i} & (1 \leq i \leq 2n-1), \\ \sqrt{-1}F_{2n+1,2n} = H. \end{cases}$$

Here, $F_{2n,i}$ $(i = 1, \ldots, 2n-1)$ are elements of $\mathfrak{k}$.

4.2. Parametrization of discrete series of $Spin(2n,1)$ (cf.[BS])

In this subsection, we parametrize discrete series of $Spin(2n,1)$.

We choose

$$\mathfrak{t} := \sum_{i=1}^{n} \mathbb{R} F_{2i,2i-1} \tag{4.2.1}$$

as a compact Cartan subalgebra of $\mathfrak{g}$ and fix it. The root system Δ (resp. Δ_c) of $\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathfrak{k}_{\mathbb{C}}$) with respect to $\mathfrak{t}_{\mathbb{C}}$ is

$$\begin{aligned} &\Delta = \{\pm e_i \pm e_j;\ \ 1 \le i < j \le n\} \cup \{\pm e_i; 1 \le i \le n\}, \\ &(\text{resp. } \Delta_c = \{\pm e_i \pm e_j;\ \ 1 \le i < j \le n\}), \end{aligned} \tag{4.2.2}$$

where $e_i(\sqrt{-1}F_{2j,2j-1}) = \delta_{ij}$ $(1 \le i, j \le n)$. We fix one of the positive systems of Δ_c:

$$\Delta_c^+ = \{e_i \pm e_j;\ \ 1 \le i < j \le n\}.$$

There are two different positive systems Δ_1^+, Δ_2^+ of Δ which contain Δ_c^+:

$$\begin{aligned} \Delta_1^+ &= \{e_i \pm e_j; 1 \le i < j \le n\} \cup \{e_i; 1 \le i \le n\}, \\ \Delta_2^+ &= \{e_i \pm e_j; 1 \le i < j \le n\} \cup \{e_i; 1 \le i \le n-1\} \cup \{-e_n\}. \end{aligned}$$

The set of Harish-Chandra parameters is denoted by $\Xi_1 \cup \Xi_2$, where

$$\begin{aligned} \Xi_1 &= \Bigg\{ \Lambda = \sum_{i=1}^{n} \Lambda_i e_i;\ \ \Lambda_1 > \cdots > \Lambda_n > 0, \\ &\qquad\qquad \Lambda_i \in \frac{1}{2}\mathbb{Z}, \Lambda_i - \Lambda_{i+1} \in \mathbb{Z} \Bigg\}, \\ \Xi_2 &= \Bigg\{ \Lambda = \sum_{i=1}^{n} \Lambda_i e_i;\ \ \Lambda_1 > \cdots > \Lambda_{n-1} > -\Lambda_n > 0, \\ &\qquad\qquad \Lambda_i \in \frac{1}{2}\mathbb{Z}, \Lambda_i - \Lambda_{i+1} \in \mathbb{Z} \Bigg\}, \end{aligned} \tag{4.2.3}$$

and the corresponding Blattner parameters are

$$\begin{aligned} \Xi_1 \ni \Lambda \Leftrightarrow \lambda &= \sum_{i=1}^{n} \lambda_i e_i = \sum_{i=1}^{n} \left(\Lambda_i + i - n + \frac{1}{2} \right) e_i, \\ \Xi_2 \ni \Lambda \Leftrightarrow \lambda &= \sum_{i=1}^{n} \lambda_i e_i = \sum_{i=1}^{n} \left(\Lambda_i + i - n + \frac{1}{2} \right) e_i - e_n. \end{aligned}$$

4.3. Realization of finite dimensional representations of K

Irreducible representations of $K \simeq Spin(2n)$ are parametrized by n-tuple of positive half integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \Leftrightarrow \lambda_1 e_1 + \cdots + \lambda_n e_n$ satisfying (1) all $\lambda_i \in \frac{1}{2}\mathbb{Z}$ or all $\lambda_i \in \mathbb{Z}$, (2) $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|$. We denote the corresponding irreducible representations by $(\tau_\lambda, V_\lambda)$. To explicitly calculate, we realize $(\tau_\lambda, V_\lambda)$ by means of the Gel'fand-Zetlin basis. (cf.[G-Z2]).

The Gel'fand-Zetlin basis of $(\tau_\lambda, V_\lambda)$ is a set $GZ(\lambda) := \{Q\}$, where Q's are diagrams of shapes

$$(4.3.1) \qquad Q = (q_{ij}) = \begin{pmatrix} q_{1,2n-1} \;\; q_{2,2n-1} \;\; \cdots\cdots \;\; q_{n-1,2n-1} \;\; q_{n,2n-1} \\ q_{1,2n-2} \;\; q_{2,2n-2} \cdots \; q_{n-1,2n-2} \\ q_{1,2n-3} \;\; q_{2,2n-3} \cdots \; q_{n-1,2n-3} \\ \cdots\cdots \\ \cdots \\ q_{1,4} \;\; q_{2,4} \\ q_{1,3} \;\; q_{2,3} \\ q_{1,2} \\ q_{1,1} \end{pmatrix}$$

which satisfy

$$\begin{cases} \text{all } q_{i,j} \in \frac{1}{2}\mathbb{Z} \text{ or all } q_{i,j} \in \mathbb{Z}, \\ q_{i,2j+1} \geq q_{i,2j} \geq q_{i+1,2j+1} \quad (i = 1, \ldots, j-1), \\ q_{j,2j+1} \geq q_{j,2j} \geq |q_{j+1,2j+1}|, \\ q_{i,2j} \geq q_{i,2j-1} \geq q_{i+1,2j} \quad (i = 1, \ldots, j-1), \\ q_{j,2j} \geq q_{j,2j-1} \geq -q_{j,2j}, \\ q_{i,2n-1} = \lambda_i. \end{cases}$$

The actions of F_{ij} are given by

$$(4.3.2) \qquad \tau_\lambda(F_{2j+1,2j})Q = \sum_{i=1}^{j} a_{i,2j-1}(Q)\sigma_{i,2j-1}Q$$

$$
\begin{aligned}
&\qquad - \sum_{i=1}^{j} A_{i,2j-1}(Q)\tau_{i,2j-1}Q,\\
\tau_\lambda(F_{2j+2,2j+1})Q &= \sum_{i=1}^{j} b_{i,2j}(Q)\sigma_{i,2j}Q\\
&\qquad - \sum_{i=1}^{j} B_{i,2j}(Q)\tau_{i,2j}Q + \sqrt{-1}c_{2j}(Q)Q,
\end{aligned}
$$

where

$$
a_{i,2j-1}(Q) = \sqrt{\left|\frac{\prod_{k=1}^{j-1}(l_{k,2j-2}^2 - l_{i,2j-1}^2 - l_{k,2j-2} - l_{i,2j-1})\prod_{k=1}^{j}(l_{k,2j}^2 - l_{i,2j-1}^2 - l_{k,2j} - l_{i,2j-1})}{4\prod_{\substack{k=1\\k\neq i}}^{j}(l_{k,2j-1}^2 - l_{i,2j-1}^2)\{l_{k,2j-1}^2 - (l_{i,2j-1}+1)^2\}}\right|},
$$

$$
A_{i,2j-1}(Q) = a_{i,2j-1}(\tau_{i,2j-1}Q),
$$

$$
b_{i,2j}(Q) = \sqrt{\left|\frac{\prod_{k=1}^{j}(l_{k,2j-1}^2 - l_{i,2j}^2)\prod_{k=1}^{j+1}(l_{k,2j+1}^2 - l_{i,2j}^2)}{l_{i,2j}^2(4l_{i,2j}^2 - 1)\prod_{\substack{k=1\\k\neq i}}^{j}(l_{k,2j}^2 - l_{i,2j}^2)\{(l_{k,2j}-1)^2 - l_{i,2j}^2\}}\right|},
$$

$$
B_{i,2j}(Q) = b_{i,2j}(\tau_{i,2j}Q),
$$

$$
c_{2j}(Q) = \frac{\prod_{k=1}^{j} l_{k,2j-1}\prod_{k=1}^{j+1} l_{k,2j+1}}{\prod_{k=1}^{j} l_{k,2j}(l_{k,2j}-1)},
$$

$$
\begin{aligned}
l_{k,2j-1} &:= q_{k,2j-1} + j - k,\\
l_{k,2j} &:= q_{k,2j} + j + 1 - k,\\
\sigma_{ij} &: \ q_{i,j} \mapsto q_{i,j}+1 \text{ and the other } q_{k,l} \mapsto q_{k,l},\\
\tau_{ij} &: \ q_{i,j} \mapsto q_{i,j}-1 \text{ and the other } q_{k,l} \mapsto q_{k,l}.
\end{aligned}
$$

The actions of other $F_{k,l}$'s are determined by those of bracket products of $F_{2j+1,2j}$'s and $F_{2j+2,2j+1}$'s.

4.4. Irreducible decomposition of $V_\lambda \otimes \mathfrak{p}_{\mathbb{C}}$

$(\mathrm{Ad}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}})$ is an irreducible K-module and its highest weight is $(1, 0, \ldots, 0)$.

Lemma 4.4.1.
If λ is far from the wall, then the irreducible decomposition of $(\tau_\lambda \otimes \mathrm{Ad}_{\mathbb{C}}, V_\lambda \otimes \mathfrak{p}_{\mathbb{C}})$ is

$$(\tau_\lambda, V_\lambda) \otimes (\mathrm{Ad}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}) \simeq \bigoplus_{k=1}^{n} (\tau_{\lambda+e_k}, V_{\lambda+e_k}) \oplus \bigoplus_{k=1}^{n} (\tau_{\lambda-e_k}, V_{\lambda-e_k}).$$

Let $P_k^\pm : V_\lambda \otimes \mathrm{Ad}_{\mathbb{C}}^\pm \to V_{\lambda \pm e_k}$ be the projection operators. Notice that $(\tau_\lambda, V_\lambda)$ and $V_{\lambda \pm e_k}$ $(1 \le k \le n)$ are $Spin(2n)$-submodules of the irreducible representation $V_{\tilde{\lambda}}$ of $Spin(2n+1)$ whose highest weight is $\tilde{\lambda} = \sum_{i=1}^{n-1} (\lambda_i + 1)e_i + (|\lambda_n| + 1)e_n$. The corresponding embeddings are given by

$$\iota : GZ(\lambda) \ni Q \mapsto \tilde{Q} = \begin{pmatrix} \lambda_1 + 1 & \ldots & \lambda_{n-1} + 1 & |\lambda_n| + 1 \\ & & Q & \end{pmatrix} \in GZ(\tilde{\lambda}),$$

$$\iota_k^\pm : GZ(\lambda \pm e_k) \ni P \mapsto \begin{pmatrix} \lambda_1 + 1 & \ldots & \lambda_{n-1} + 1 & |\lambda_n| + 1 \\ & & P & \end{pmatrix} \in GZ(\tilde{\lambda})$$

$$(1 \le k \le n).$$

Using the theory of tensor operators, we can write down $\iota_k^\pm \circ P_k^\pm(Q \otimes F_{2n+1,2n})$ $(Q \in GZ(\lambda))$ explicitly. As in §2.4, $\iota_k^\pm \circ P_k^\pm$'s are also denoted by $P_k^\pm$.

Proposition 4.4.2.
For $Q \in GZ(\lambda)$,

$$\begin{aligned} P_k^+(Q \otimes F_{2n+1,2n}) &= a_{k,2n-1}(\tilde{Q}) \sigma_{k,2n-1} \tilde{Q}, \\ P_k^-(Q \otimes F_{2n+1,2n}) &= -A_{k,2n-1}(\tilde{Q}) \tau_{k,2n-1} \tilde{Q}. \end{aligned} \tag{4.4.1}$$

Proof. The proof of this proposition is just similar to that of Proposition 2.4.2 ($\mathfrak{gl}(n, \mathbb{C})$ case). We can apply the argument of $\mathfrak{gl}(n, \mathbb{C})$ case by Kraljević ([Kr, §4]) to this $\mathfrak{o}(n, \mathbb{C})$ case. □

§5. The differential equation $\mathcal{D}_{\lambda,\eta}\phi = 0$

5.1. The explicit formula of $P_k^{\pm}(\nabla_{\lambda,\eta}\phi) = 0$

In this subsection, we will write down the equation $\mathcal{D}_{\lambda,\eta}\phi = 0$.

Using the Gel'fand-Zetlin basis, we can write $\phi \in C^{\infty}_{\tau_\lambda}(K\backslash G/N;\eta)$ as

$$\phi(g) = \sum_{Q\in GZ(\lambda)} c(Q;g)Q. \tag{5.1.1}$$

Since $\left\{\sqrt{\frac{-1}{2(2n-1)}}F_{2n+1,i}\ \ (1\le i\le 2n)\right\}$ forms an orthonormal basis of $\mathfrak{p}$,

$$\nabla_{\lambda,\eta}\phi(g) = \sum_{i=1}^{2n} L_{\sqrt{\frac{-1}{2(2n-1)}}F_{2n+1,i}}\phi(g)\otimes\sqrt{\frac{-1}{2(2n-1)}}F_{2n+1,i}. \tag{5.1.2}$$

Let $R(\mathcal{D}_{\lambda,\eta})$ and $R(\nabla_{\lambda,\eta})$ be the radial A-part of $\mathcal{D}_{\lambda,\eta}$ and $\nabla_{\lambda,\eta}$, respectively. To determine $\phi(g)\in \mathrm{Ker}\mathcal{D}_{\lambda,\eta}$, it is sufficient to calculate $\phi|_A \in \mathrm{Ker}R(\mathcal{D}_{\lambda,\eta})$.

Assume that $\eta\in\hat{N}$ is given by

$$\eta\left(\exp\left(\sum_{i=1}^{2n-1} x_iX_i\right)\right) = e^{\sqrt{-1}x_{2n-1}\xi}\quad (x_i\in\mathbb{R},\ \ \xi\in\mathbb{R}_{>0}). \tag{5.1.3}$$

Because of Corollary 1.4.2(1) and (3), it suffices to calculate $\phi|_A \in \mathrm{Ker}R(\mathcal{D}_{\tau_\lambda,\eta})$ only for this character.

Next, we introduce a coordinate system of A by

$$\mathbb{R}_{>0}\ni a\mapsto \exp((\log a)H)\in A.$$

Then, by (4.1.1) and (5.1.2), we have

PROPOSITION 5.1.1.

$$\begin{aligned} &2(2n-1)\sqrt{-1}R(\nabla_{\lambda,\eta})\phi(a)\\ &\quad = a\frac{d}{da}\phi(a)\otimes F_{2n+1,2n} - \sum_{i=1}^{2n-1}\tau_\lambda(F_{2n,i})\phi(a)\otimes F_{2n+1,i}\\ &\qquad - \sqrt{-1}\frac{\xi}{a}\phi(a)\otimes F_{2n+1,2n-1}. \end{aligned} \tag{5.1.4}$$

For any $Q \in GZ(\lambda)$,

$$2\tau_\lambda(F_{2n,i})Q \otimes F_{2n+1,i} = -(\tau_\lambda \otimes \mathrm{Ad}_{\mathbb{C}})(F_{2n,i})^2(Q \otimes F_{2n+1,2n}) \\ + \tau_\lambda(F_{2n,i})^2 Q \otimes F_{2n+1,2n} - Q \otimes F_{2n+1,2n}.$$

Hence we have

$$\sum_{i=1}^{2n-1} \tau_\lambda(F_{2n,i})Q \otimes F_{2n+1,i} \\ = -\frac{1}{2}\sum_{i=1}^{2n-1}(\tau_\lambda \otimes \mathrm{Ad}_{\mathbb{C}})(F_{2n,i})^2(Q \otimes F_{2n+1,2n}) \\ + \frac{1}{2}\sum_{i=1}^{2n-1}\tau_\lambda(F_{2n,i})^2 Q \otimes F_{2n+1,2n} - \frac{2n-1}{2} Q \otimes F_{2n+1,2n}.$$

Let C_m be the Casimir element of $\mathfrak{o}(m,\mathbb{C})$. Since the Killing form B of $\mathfrak{o}(m,\mathbb{C})$ is given by $B(X,Y) = (m-2)\mathrm{tr}(XY)$, C_{2n} and C_{2n-1} are $-2(2n-2)C_{2n} = \sum_{1\le i<j\le 2n} F_{j,i}^2$ and $-2(2n-3)C_{2n-1} = \sum_{1\le i<j\le 2n-1} F_{j,i}^2$. Then, it follows : $\sum_{i=1}^{2n-1} F_{2n,i}^2 = -2(2n-2)C_{2n} + 2(2n-3)C_{2n-1}$. On the other hand, for any $Q \in GZ(\lambda)$,

$$\tau_\lambda(-2(2n-2)C_{2n})Q = \left\{-\sum_{i=1}^{n}\lambda_i^2 - 2\sum_{i=1}^{n}(n-i)\lambda_i\right\}Q,$$
$$\tau_\lambda(-2(2n-3)C_{2n-1})Q = \left\{-\sum_{i=1}^{n-1}q_{i,2n-2}^2 - \sum_{i=1}^{n-1}(2n-1-2i)q_{i,2n-2}\right\}Q.$$

Using these formulae and Proposition 4.4.2, we have the following equalities:

Lemma 5.1.2.
For any $Q \in GZ(\lambda)$,

$$P_k^+\left(\sum_{i=1}^{2n-1}\tau_\lambda(F_{2n,i})Q \otimes F_{2n+1,i}\right) \tag{5.1.5}$$

$$= a_{k,2n-1}(\tilde{Q})(\lambda_k - k + 1)\sigma_{k,2n-1}\tilde{Q},$$

$$P_k^- \left(\sum_{i=1}^{2n-1} \tau_\lambda(F_{2n,i})Q \otimes F_{2n+1,i} \right) = -A_{k,2n-1}(\tilde{Q})(-\lambda_k - 2n + k + 1)\tau_{k,2n-1}\tilde{Q}. \tag{5.1.6}$$

From

$$\begin{aligned} Q \otimes F_{2n+1,2n-1} = \tau_\lambda(F_{2n,2n-1})Q \otimes F_{2n+1,2n} \\ - (\tau_\lambda \otimes \mathrm{Ad}_{\mathbb{C}})(F_{2n,2n-1})(Q \otimes F_{2n+1,2n}), \end{aligned}$$

we have :

LEMMA 5.1.3.
For any $Q \in GZ(\lambda)$,

$$\begin{aligned} P_k^+(Q \otimes F_{2n+1,2n-1}) = &- \sum_{j=1}^{n-1} \frac{a_{k,2n-1}(\sigma_{j,2n-2}\tilde{Q})b_{j,2n-2}(\tilde{Q})}{l_{k,2n-1} - l_{j,2n-2}} \sigma_{j,2n-2}\sigma_{k,2n-1}\tilde{Q} \\ &+ \sum_{j=1}^{n-1} \frac{a_{k,2n-1}(\tau_{j,2n-2}\tilde{Q})B_{j,2n-2}(\tilde{Q})}{l_{k,2n-1} + l_{j,2n-2} - 1} \tau_{j,2n-2}\sigma_{k,2n-1}\tilde{Q} \\ &- \frac{\sqrt{-1}}{l_{k,2n-1}} a_{k,2n-1}(\tilde{Q})c_{2n-2}(\tilde{Q})\sigma_{k,2n-1}\tilde{Q}, \\ P_k^-(Q \otimes F_{2n+1,2n-1}) = &- \sum_{j=1}^{n-1} \frac{A_{k,2n-1}(\sigma_{j,2n-2}\tilde{Q})b_{j,2n-2}(\tilde{Q})}{l_{k,2n-1} + l_{j,2n-2}} \sigma_{j,2n-2}\tau_{k,2n-1}\tilde{Q} \\ &+ \sum_{j=1}^{n-1} \frac{A_{k,2n-1}(\tau_{j,2n-2}\tilde{Q})B_{j,2n-2}(\tilde{Q})}{l_{k,2n-1} - l_{j,2n-2} + 1} \tau_{j,2n-2}\tau_{k,2n-1}\tilde{Q} \\ &- \frac{\sqrt{-1}}{l_{k,2n-1}} A_{k,2n-1}(\tilde{Q})c_{2n-2}(\tilde{Q})\tau_{k,2n-1}\tilde{Q}. \end{aligned}$$

The next proposition follows from Proposition 4.4.2, Lemma 5.1.1, 5.1.2, and 5.1.3.

PROPOSITION 5.1.4.
For $1 \leq k \leq n$, we have:

$$
\begin{aligned}
(5.1.7)\quad & P_k^+(R(\nabla_{\lambda,\eta})\phi(a)) = 0 \Longleftrightarrow \\
& \sum_{Q\in GZ(\lambda)} a_{k,2n-1}(\tilde{Q})\left(a\frac{d}{da} - \lambda_k + k - 1\right) c(Q;a)\sigma_{k,2n-1}\tilde{Q} \\
& - \frac{\xi}{a}\sum_{Q\in GZ(\lambda)} \frac{a_{k,2n-1}(\tilde{Q})c_{2n-2}(\tilde{Q})}{l_{k,2n-1}} c(Q;a)\sigma_{k,2n-1}\tilde{Q} \\
& + \frac{\sqrt{-1}\xi}{a}\sum_{j=1}^{n-1}\sum_{\tau_{j,2n-2}Q\in GZ(\lambda)} \frac{a_{k,2n-1}(\tilde{Q})b_{j,2n-2}(\tau_{j,2n-2}\tilde{Q})}{l_{k,2n-1} - l_{j,2n-2} + 1} \\
& \qquad \times c(\tau_{j,2n-2}Q;a)\sigma_{k,2n-1}\tilde{Q} \\
& - \frac{\sqrt{-1}\xi}{a}\sum_{j=1}^{n-1}\sum_{\sigma_{j,2n-2}Q\in GZ(\lambda)} \frac{a_{k,2n-1}(\tilde{Q})B_{j,2n-2}(\sigma_{j,2n-2}\tilde{Q})}{l_{k,2n-1} + l_{j,2n-2}} \\
& \qquad \times c(\sigma_{j,2n-2}Q;a)\sigma_{k,2n-1}\tilde{Q} \\
& = 0,
\end{aligned}
$$

$$
\begin{aligned}
(5.1.8)\quad & P_k^-(R(\nabla_{\lambda,\eta})\phi(a)) = 0 \Longleftrightarrow \\
& \sum_{Q\in GZ(\lambda)} A_{k,2n-1}(\tilde{Q})\left(a\frac{d}{da} + \lambda_k - k + 2n - 1\right) c(Q;a)\tau_{k,2n-1}\tilde{Q} \\
& + \frac{\xi}{a}\sum_{Q\in GZ(\lambda)} \frac{A_{k,2n-1}(\tilde{Q})c_{2n-2}(\tilde{Q})}{l_{k,2n-1}} c(Q;a)\tau_{k,2n-1}\tilde{Q} \\
& - \frac{\sqrt{-1}\xi}{a}\sum_{j=1}^{n-1}\sum_{\tau_{j,2n-2}Q\in GZ(\lambda)} \frac{A_{k,2n-1}(\tilde{Q})b_{j,2n-2}(\tau_{j,2n-2}\tilde{Q})}{l_{k,2n-1} + l_{j,2n-2} - 1} \\
& \qquad \times c(\tau_{j,2n-2}Q;a)\tau_{k,2n-1}\tilde{Q} \\
& + \frac{\sqrt{-1}\xi}{a}\sum_{j=1}^{n-1}\sum_{\sigma_{j,2n-2}Q\in GZ(\lambda)} \frac{A_{k,2n-1}(\tilde{Q})B_{j,2n-2}(\sigma_{j,2n-2}\tilde{Q})}{l_{k,2n-1} - l_{j,2n-2}} \\
& \qquad \times c(\sigma_{j,2n-2}Q;a)\tau_{k,2n-1}\tilde{Q} \\
& = 0.
\end{aligned}
$$

These equations are the explicit representations of $P_k^{\pm}(R(\nabla_{\lambda,\eta}^{\pm})\phi(a)) = 0$, which we needed.

5.2. The explicit formulae of $c(Q;a)$

If $\Lambda \in \Xi_1$, then

$$\mathcal{D}_{\lambda,\eta}\phi(g) = 0 \Leftrightarrow P_1^-(\nabla_{\lambda,\eta}\phi(g)) = \cdots = P_n^-(\nabla_{\lambda,\eta}\phi(g)) = 0,$$

and if $\Lambda \in \Xi_2$, then

$$\begin{aligned}&\mathcal{D}_{\lambda,\eta}\phi(g) = 0\\ &\qquad \Leftrightarrow P_1^-(\nabla_{\lambda,\eta}\phi(g)) = \cdots = P_{n-1}^-(\nabla_{\lambda,\eta}\phi(g)) = P_n^+(\nabla_{\lambda,\eta}\phi(g)) = 0.\end{aligned}$$

By (5.1.8), we have

$$\begin{aligned}(5.2.1)\qquad & P_l^-(R(\nabla_{\lambda,\eta})\phi(a)) = 0\\ &\quad \Longrightarrow c(Q;a) = 0\\ &\quad \text{for } Q = (q_{ij}) \text{ satisfying } q_{l-1,2n-2} = \lambda_l, q_{l-1,2n-3} < \lambda_l.\end{aligned}$$

Moreover, we can show that $P_l^-(R(\nabla_{\lambda,\eta})\phi(a)) = 0$ implies

$$c(Q;a) = 0 \text{ for any } Q \text{ satisfying } q_{l-1,2n-3} < \lambda_l \quad (2 \le l \le n-1)$$

by (5.2.1) and recursive usage of (5.1.8). Similarly, if $\Lambda \in \Xi_1$, then $P_n^-(\nabla_{\lambda,\eta}\phi(a)) = 0$ implies

$$c(Q;a) = 0 \text{ for any } Q \text{ satisfying } |q_{n-1,2n-3}| < \lambda_n,$$

and if $\Lambda \in \Xi_2$, then $P_n^+(\nabla_{\lambda,\eta}\phi(a)) = 0$ implies

$$c(Q;a) = 0 \text{ for any } Q \text{ satisfying } |q_{n-1,2n-3}| < -\lambda_n.$$

Consequently,

LEMMA 5.2.1.

In order to solve $R(\mathcal{D}_{\lambda,\eta})\phi(a) = 0$, we have only to calculate $c(Q; a)$ for Q satisfying

$$\begin{aligned}\lambda_1 \geq & q_{1,2n-2} \geq q_{1,2n-3} \geq \lambda_2 \geq \cdots \geq \lambda_{n-2} \\ & \geq q_{n-2,2n-2} \geq q_{n-2,2n-3} \geq \lambda_{n-1} \geq q_{n-1,2n-2} \\ & \geq |q_{n-1,2n-3}| \geq |\lambda_n|.\end{aligned} \tag{5.2.2}$$

Suppose $\Lambda \in \Xi_1$. In order to solve $\mathcal{D}_{\lambda,\eta}\phi = 0$, we eliminate the difference terms of equations $P_1^-(\nabla_{\lambda,\eta}\phi) = \cdots = P_n^-(\nabla_{\lambda,\eta}\phi) = 0$.

LEMMA 5.2.2.
Suppose $l; p_1, \ldots, p_l; j_1, \ldots, j_{l-1}$ satisfy $1 \leq l \leq n, 1 \leq p_1 < \cdots < p_l \leq n, 1 \leq j_1 < \cdots < j_{l-1} \leq n-1$, and $P_{p_1}^-(\nabla_{\lambda,\eta}\phi) = \cdots = P_{p_l}^-(\nabla_{\lambda,\eta}\phi) = 0$. Then,

$$\begin{aligned}&\left(\prod_{i=1}^{l} A_{p_i,2n-1}(\tilde{Q})\right) \times \\ &\left\{\left(a\frac{d}{da} + \sum_{i=1}^{l}(\lambda_{p_i} - p_i) - \sum_{i=1}^{l-1}(q_{j_i,2n-2} - j_i) + 2n - 1\right) c(Q; a)\right. \\ &+ \frac{\xi}{a}\frac{\prod_{i=1}^{l-1} l_{j_i,2n-2}}{\prod_{i=1}^{l} l_{p_i,2n-1}} c_{2n-2}(\tilde{Q})c(Q; a) \\ &- \frac{\sqrt{-1}\xi}{a}\sum_{j=1}^{n-1}\frac{\prod_{i=1}^{l-1}(l_{j_i,2n-2} + l_{j,2n-2} - 1)}{\prod_{i=1}^{l}(l_{p_i,2n-1} + l_{j,2n-2} - 1)} b_{j,2n-2}(\tau_{j,2n-2}\tilde{Q})c(\tau_{j,2n-2}Q; a) \\ &+ \frac{\sqrt{-1}\xi}{a}\sum_{\substack{j=1 \\ j\neq j_1,\ldots,j_{l-1}}}^{n-1}\frac{\prod_{i=1}^{l-1}(l_{j_i,2n-2} - l_{j,2n-2})}{\prod_{i=1}^{l}(l_{p_i,2n-1} - l_{j,2n-2})} \\ &\left.\times B_{j,2n-2}(\sigma_{j,2n-2}\tilde{Q})c(\sigma_{j,2n-2}Q; a)\right\} = 0.\end{aligned}$$

We call the above equation $(5.2.3)_{p_1,\dots,p_l;j_1,\dots,j_{l-1}}$.

The proof of this lemma is just similar to the proof of Lemma 3.2.2.

This lemma and similar computation for the case $\Lambda \in \Xi_2$ implies that, if $\Lambda \in \Xi_1 \cup \Xi_2$ and all $c(Q;a)$'s are given for Q satisfying

$$\begin{aligned}\lambda_1 \geq{} & q_{1,2n-2} = q_{1,2n-3} \geq \lambda_2 \geq \cdots \geq \lambda_{n-2} \\ & \geq q_{n-2,2n-2} = q_{n-2,2n-3} \geq \lambda_{n-1} \geq q_{n-1,2n-2} \\ & = |q_{n-1,2n-3}| \geq |\lambda_n|,\end{aligned} \tag{5.2.4}$$

then all the other $c(Q;a)$'s are uniquely determined.

Let us find the explicit formulae of $c(Q;a)$'s.

Suppose $\Lambda \in \Xi_1$ and $Q = (q_{i,j})$ satisfies (5.2.4). Let $l, p_i (1 \leq i \leq l), p_i' (1 \leq p_i' \leq n-l-1)$ and $j_i (1 \leq j_i \leq l-1)$ be integers determined by

$$\begin{cases} 1 \leq p_1 < \cdots < p_{l-1} \leq n-1, \ \ q_{p_i,2n-2} \neq \lambda_{p_i}, \\ 1 \leq p_1' < \cdots < p_{n-l-1}' \leq n-1, \ \ q_{p_i',2n-2} = \lambda_{p_i'}, \\ p_l = n, \\ j_i = p_i \ \ (1 \leq i \leq l-1). \end{cases} \tag{5.2.5}$$

Then, we have

$$\begin{cases} A_{p_i,2n-1}(\tilde{Q}) \neq 0 \ \ (1 \leq i \leq l), \\ c(\sigma_{p_i',2n-2}Q;a) = 0 \ \ (1 \leq i \leq n-l-1), \\ c(\tau_{j,2n-2}Q;a) = 0 \ \ (1 \leq j \leq n-1), \end{cases}$$

$$\frac{\prod_{i=1}^{l-1} l_{j_i,2n-2}}{\prod_{i=1}^{l} l_{p_i,2n-1}} c_{2n-2}(Q) = \operatorname{sgn} q_{n-1,2n-3},$$

$$\sum_{i=1}^{l}(\lambda_{p_i} - p_i) - \sum_{i=1}^{l-1}(q_{j_i,2n-2} - j_i) = \sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n-1} q_{i,2n-2} - n.$$

Finally, equation $(5.2.3)_{p_1,\dots,p_l;j_i,\dots,j_{l-1}}$ is written as follows.

$$\left(a\frac{d}{da} + n - 1 + \sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n-1} q_{i,2n-2} + \operatorname{sgn} q_{n-1,2n-3}\frac{\xi}{a}\right) c(Q;a) = 0.$$

It follows that $c(Q;a) = const.a^{-n+1-\sum\limits_{i=1}^{n}\lambda_i+\sum\limits_{i=1}^{n-1}q_{i,2n-2}}e^{\mathrm{sgn}q_{n-1,2n-3}\frac{\xi}{a}}$. The $\Lambda \in \Xi_2$ case can be calculated similarly.

PROPOSITION 5.2.3.

(1) *If* $\Lambda \in \Xi_1 \cup \Xi_2$, *then*

$$\begin{aligned}(5.2.6)\quad & \dim \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi^*_{\Lambda,K}, C^\infty(G/N;\eta)) \\ & \leq 2 \sum_{\substack{\lambda_1\geq\mu_1\geq\lambda_2\geq\dots \\ \geq\lambda_{n-2}\geq\mu_{n-2}\geq\lambda_{n-1}\geq\mu_{n-1}\geq|\lambda_n|}} \dim V^D_{2n-2}(\mu_1,\dots,\mu_{n-1}),\end{aligned}$$

where $V^D_{2n-2}(\mu_1,\dots,\mu_{n-1})$ *is the irreducible* $Spin(2n-2)$*-module with highest weight* $(\mu_1,\dots,\mu_{n-1})$.

(2) *Suppose* η *is defined by (5.1.3). Then* $\phi \in \mathrm{Ker}\mathcal{D}_{\lambda,\eta}$ *is completely determined by* $c(Q;a)$*'s for* Q *satisfying*

$$\begin{aligned}\lambda_1 \geq & q_{1,2n-2} = q_{1,2n-3} \geq \lambda_2 \geq \dots \geq \lambda_{n-2} \\ & \geq q_{n-2,2n-2} = q_{n-2,2n-3} \geq \lambda_{n-1} \geq q_{n-1,2n-2} = |q_{n-1,2n-3}| \geq |\lambda_n|.\end{aligned}$$

(3) *For* Q *which satisfies the above conditions in (2), the explicit formula of* $c(Q;a)$ *is*

$$(5.2.7)\qquad c(Q;a) = \alpha(Q)a^{-n+1-\sum\limits_{i=1}^{n-1}\lambda_i-|\lambda_n|+\sum\limits_{i=1}^{n-1}q_{i,2n-2}}e^{\mathrm{sgn}q_{n-1,2n-3}\frac{\xi}{a}},$$

where, $\alpha(Q)$ *is an arbitrary constant.*

The equal sign in (5.2.6) holds, and we will prove it in §6.2.

§6. The dimension of the space of Whittaker models

In this section, we prove the explicit dimension formula of the space of Whittaker models, and the equal signs in (3.2.9) and (5.2.6) are shown.

6.1. The Gel'fand-Kirillov dimension and the Bernstein degree of finitely generated $U(\mathfrak{g}_{\mathbb{C}})$-modules

At first, we will recall the Gel'fand-Kirillov dimension and the Bernstein degree of finitely generated $U(\mathfrak{g}_{\mathbb{C}})$-modules.

Suppose $\mathfrak{g}_1$ is an arbitrary finite dimensional Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g}_1)$ is the universal enveloping algebra of $\mathfrak{g}_1$. Let $U_n(\mathfrak{g}_1) \subseteq U(\mathfrak{g}_1)$ be the subspace of $U(\mathfrak{g}_1)$ spanned by monomials which are products of at most n elements of $\mathfrak{g}_1$. Let V be a finitely generated $U(\mathfrak{g}_1)$-module. Choose a finite dimensional subspace V_0 of V that generates V as a $U(\mathfrak{g}_1)$-module. Set $V_n = U_n(\mathfrak{g}_1)V_0$, $M_n = V_n/V_{n-1}$ and $M = \mathrm{gr}V = \sum_{n=0}^{\infty} M_n$. M is a $\mathrm{gr}U(\mathfrak{g}_1) \simeq S(\mathfrak{g}_1)$-module. By a theorem of Hilbert-Serre, there exists a polynomial $\chi(x)$ over $\mathbb{Q}$ such that $\chi(n)$ is equal to $\sum_{k=0}^{n} \dim M_k$ for sufficiently large n. The degree and the leading coefficient of $\chi(x)$ are denoted by $\mathrm{Dim}V$ and $\frac{c(V)}{(\mathrm{Dim}V)!}$ ($c(V) \in \mathbb{Z}$), respectively. (For a graded $S(\mathfrak{g}_1)$-module N, we define $c(N)$ and $\mathrm{Dim}N$ similarly.) The integers $\mathrm{Dim}V$ and $c(V)$ are called the Gel'fand-Kirillov dimension and the Bernstein degree of V, respectively. Let d be any integer not smaller than $\mathrm{Dim}V$. We write

$$c_d(V) = \begin{cases} c(V) & \text{if } d = \mathrm{Dim}V, \\ 0 & \text{if } d > \mathrm{Dim}V. \end{cases}$$

Now, let V be an irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module and η be a non-degenerate character of N. In this case, since V admits an infinitesimal character, $\iota(v)(g)$ $(v \in V, \iota \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(V, C^\infty(G/N;\eta)))$ is a real analytic function on G. Then we have an isomorphism $\mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(V, C^\infty(G/N;\eta)) = \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(V, \mathcal{A}(G/N;\eta))$ ($\mathcal{A}$ denotes the space of real analytic functions).

THEOREM 6.1.1 ([M1, Corollary 2.2.2 and Theorem 6.2.1]).
Let $c_d(V)$, η and V be as above. Then

$$\dim \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(V, \mathcal{A}(G/N;\eta)) = c_d(V) \qquad (d = \dim \mathfrak{n}).$$

6.2. Characteristic cycles of $(\mathfrak{g}_\mathbb{C}, K)$-modules (cf.[V])

Since $\mathrm{gr}U(\mathfrak{g}_\mathbb{C}) \simeq S(\mathfrak{g}_\mathbb{C})$ is Noetherian and $M = \mathrm{gr}V$ is a finitely generated $S(\mathfrak{g}_\mathbb{C})$-module, there exists a sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of $S(\mathfrak{g}_\mathbb{C})$-submodules of M such that $M_i/M_{i-1} \simeq S(\mathfrak{g}_\mathbb{C})/Q_i$ ($^\exists Q_i \in \mathit{Spec}S(\mathfrak{g}_\mathbb{C})$) for any i. The characteristic cycle of M is the formal sum

$$\mathrm{Ch}(M) = \sum_{k=1}^{r} m(P_k, M)P_k,$$

where P_k's are minimal in $\{P \in \mathit{Spec}S(\mathfrak{g}_\mathbb{C}); P \supset \mathit{Ann}(M)\}$, and $m(P_k, M) = \#\{Q_i \in \mathit{Spec}S(\mathfrak{g}_\mathbb{C}); M_i/M_{i-1} \simeq S(\mathfrak{g}_\mathbb{C})/Q_i, Q_i = P_k\}$. By definition, $0 \to M_{i-1} \to M_i \to S(\mathfrak{g}_\mathbb{C})/Q_i \to 0$ is exact. It follows that $c_d(V) = \sum_{i=1}^{n} c_d(S(\mathfrak{g}_\mathbb{C})/Q_i)$. If $Q_i \subsetneq Q_j$, then there exists an element $x \in Q_j - Q_i$ and we have the following exact sequences;

$$0 \to S(\mathfrak{g}_\mathbb{C})/Q_i \xrightarrow{x} S(\mathfrak{g}_\mathbb{C})/Q_i \to S(\mathfrak{g}_\mathbb{C})/(Q_i + xS(\mathfrak{g}_\mathbb{C})) \to 0,$$
$$0 \to Q_j/(Q_i + xS(\mathfrak{g}_\mathbb{C})) \to S(\mathfrak{g}_\mathbb{C})/(Q_i + xS(\mathfrak{g}_\mathbb{C})) \to S(\mathfrak{g}_\mathbb{C})/Q_j \to 0.$$

Then $c_{\mathrm{Dim}(S(\mathfrak{g}_\mathbb{C})/Q_i)}(S(\mathfrak{g}_\mathbb{C})/Q_j) = 0$ and we have proved:

Lemma 6.2.1.
Let $d = \max_{\substack{Ann(M) \subset P_k \in SpecS(\mathfrak{g}_\mathbb{C}) \\ P_k : minimal}} \{\mathrm{Dim}(S(\mathfrak{g}_\mathbb{C})/P_k)\}$. *Then* $\mathrm{Dim}V = d$ *and*

$$c_d(V) = \sum_{\substack{Ann(M) \subset P_k \in SpecS(\mathfrak{g}_\mathbb{C}) \\ P_k : minimal}} m(P_k, M) c_d(S(\mathfrak{g}_\mathbb{C})/P_k).$$

In [C], Chang calculated $m(P_k, M)$ of discrete series representations for $\mathbb{R}$-rank one matrix groups.

Theorem 6.2.2 ([C, Theorem A.7, Theorem B.5]).
Let π_Λ *be a discrete series representation of* $G = SU(n,1)$ *or* $Spin(2n,1)$ *whose Harish-Chandra parameter is* Λ. *Then*

$$\mathrm{Ch}(\mathrm{gr}\pi^*_{\Lambda,K}) = m(P_{\pi^*_{\Lambda,K}}, \mathrm{gr}\pi^*_{\Lambda,K})P_{\pi^*_{\Lambda,K}},$$

where $P_{\pi^*_{\Lambda,K}}$ is the unique minimal prime ideal containing $Ann(\mathrm{gr}\pi^*_{\Lambda,K})$ and

$$
\begin{aligned}
&m(P_{\pi^*_{\Lambda,K}}, \mathrm{gr}\pi^*_{\Lambda,K}) \\
&= \begin{cases}
\displaystyle\sum_{\substack{\lambda_1\geq\mu_1\geq\lambda_2\geq\cdots\geq\lambda_{k-2}\geq\mu_{k-2}\geq\lambda_{k-1},\\ \lambda_k\geq\mu_{k-1}\geq\lambda_{k+1}\geq\cdots\geq\lambda_{n-1}\geq\mu_{n-2}\geq\lambda_n}} \dim V^A_{n-2}(\mu_1,\ldots,\mu_{n-2}) \\
\qquad (G = SU(n,1), \Lambda\in\Xi_k, 2\leq k\leq n), \\
\displaystyle\sum_{\lambda_1\geq\mu_1\geq\lambda_2\geq\cdots\geq\lambda_{n-2}\geq\mu_{n-2}\geq\lambda_{n-1}\geq\mu_{n-1}\geq|\lambda_n|} \dim V^D_{2n-2}(\mu_1,\ldots,\mu_{n-1}) \\
\qquad (G = Spin(2n,1), \Lambda\in\Xi_1\cup\Xi_2).
\end{cases}
\end{aligned}
$$

LEMMA 6.2.3.
If $\Lambda, \Lambda'\in\Xi_k$, then $c_d(S(\mathfrak{g}_{\mathbb{C}})/P_{\pi_{\Lambda,K}}) = c_d(S(\mathfrak{g}_{\mathbb{C}})/P_{\pi_{\Lambda',K}})$.

PROOF. We may assume that $\mu = \Lambda' - \Lambda$ is dominant integral.

Let E_μ be the irreducible $\mathfrak{g}_{\mathbb{C}}$-module with highest weight μ. Let V_0 be a finite dimensional subspace of $\pi_{\Lambda,K}$ that generates $\pi_{\Lambda,K}$ as a $U(\mathfrak{g}_{\mathbb{C}})$-module. Then $\pi_{\Lambda,K}\otimes E_\mu = U(\mathfrak{g}_{\mathbb{C}})(V_0\otimes E_\mu)$. Set $V_n = U_n(\mathfrak{g}_{\mathbb{C}})V_0$. For any $v\in V_n, e\in E_\mu$ and $X\in\mathfrak{g}_{\mathbb{C}}$, $X(v\otimes e) = Xv\otimes e + v\otimes Xe \equiv Xv\otimes e \pmod{V_n\otimes E_\mu}$. Therefore $Ann(\mathrm{gr}(\pi_{\Lambda,K}\otimes E_\mu)) = Ann(\mathrm{gr}\pi_{\Lambda,K})$ and $\mathcal{V}(\pi_{\Lambda,K}\otimes E_\mu) = \mathcal{V}(\pi_{\Lambda,K})$ holds for their associated varieties. We know that $\pi_{\Lambda',K}$ is an irreducible submodule of $\pi_{\Lambda,K}\otimes E_\mu$. Then $\mathcal{V}(\pi_{\Lambda',K})\subseteq\mathcal{V}(\pi_{\Lambda,K})$. We can show the inverse inclusion by "down" translation, and we have $\mathcal{V}(\pi_{\Lambda',K}) = \mathcal{V}(\pi_{\Lambda,K})$. Since $\pi_{\Lambda,K}$ is a discrete series representation, $\mathcal{V}(\pi_{\Lambda,K})$ is a closed irreducible variety. By the Hilbert Nullstellensatz, $P_{\pi_{\Lambda,K}} = P_{\pi_{\Lambda',K}}$. Eventually, $c_d(S(\mathfrak{g}_{\mathbb{C}})/P_{\pi_{\Lambda,K}}) = c_d(S(\mathfrak{g}_{\mathbb{C}})/P_{\pi_{\Lambda',K}})$. □

We will prove the equal sign in (3.2.9) (the dimension formula of the $SU(n,1)$ case). By Theorem 6.1.1, Lemma 6.2.1 and Theorem 6.2.2, it suffices to prove $c_d(S(\mathfrak{g}_{\mathbb{C}})/P_{\pi^*_{\Lambda,K}}) = 2$.

If we read [Y1], [H-P] and [K-W] carefully, we notice that the condition "far from the wall" in Theorem 1.3.2 can be a little weakened. In our case $G = SU(n,1)$, Theorem 1.3.2 is also true for $\pi^*_{\Lambda,K}$ ($\Lambda\in\Xi_k$), if the Blattner parameter $\lambda = \Lambda+\rho-2\rho_c = (\lambda_1,\ldots,\lambda_n)$ is $\lambda_1 = \cdots = \lambda_{k-1} >$

$2k-n-1, \lambda_k = \cdots = \lambda_n < 2k-n-3$ (see [H-P,§9]). For this parameter λ, $R(\mathcal{D}_{\lambda,\eta})\phi(a) = 0$ is equivalent to

$$\begin{aligned}
&\left(a\frac{d}{da} + \lambda_{k-1} - \lambda_k + q_{k-1,n-1} + 2n - 2k + 2\right) c(Q;a) \\
&\qquad + \frac{\xi}{a}\sqrt{\frac{q_{k-1,n-1} - \lambda_k + 1}{\lambda_{k-1} - q_{k-1,n-1}}} c(\sigma_{k-1,n-1}Q;a) = 0 \\
&\qquad (\lambda_{k-1} > q_{k-1,n-1} \geq \lambda_k), \\
&\left(a\frac{d}{da} + \lambda_{k-1} - \lambda_k - q_{k-1,n-1} + 2k - 2\right) c(Q;a) \\
&\qquad + \frac{\xi}{a}\sqrt{\frac{\lambda_{k-1} - q_{k-1,n-1} + 1}{q_{k-1,n-1} - \lambda_k}} c(\tau_{k-1,n-1}Q;a) = 0 \\
&\qquad (\lambda_{k-1} \geq q_{k-1,n-1} > \lambda_k),
\end{aligned}$$

and we can easily check the compatibility of these equations by direct calculation. Then, for this parameter λ, the equal sign in (3.2.9) holds and we have shown $c_d(S(\mathfrak{g}_{\mathbb{C}})/P_{\pi^*_{\Lambda,K}}) = 2$. By Lemma 6.2.3, $c_d(S(\mathfrak{g}_{\mathbb{C}})/P_{\pi^*_{\Lambda,K}}) = 2$ for every parameter Λ. Similarly, we can prove that the equal sign in (5.2.6) holds ($Spin(2n,1)$ case).

6.3. Whittaker functions of moderate growth

For a $(\mathfrak{g}_{\mathbb{C}}, K)$-module (π, V), let (π^∞, V^∞) be its C^∞-globalization and we denote by $(\pi^{-\infty}, V^{-\infty})$ the continuous dual to (π^∞, V^∞) with respect to $U(\mathfrak{g}_{\mathbb{C}})$-topology. We denote the continuous intertwining space by $\mathrm{Hom}_G(\pi^\infty, C^\infty(G/N;\eta))$ and set

$$\mathrm{Wh}(\pi^{-\infty}) := \{\varphi \in V^{-\infty}; \pi'(X)\varphi = -\eta(X)\varphi, \ \ (X \in \mathfrak{n})\}.$$

There is a canonical isomorphism :

$$\begin{aligned}
\mathrm{Wh}(\pi^{-\infty}) \ni \varphi \mapsto f_\varphi \in \mathrm{Hom}_G(\pi^\infty, C^\infty(G/N;\eta)), \\
\langle \varphi, \pi(g^{-1})v \rangle = f_\varphi(v)(g), \ \ (v \in V^\infty, g \in G).
\end{aligned}$$

By a theorem of Wallach (cf.[W, §8.3]), if $\iota \in \mathrm{Hom}_G(\pi^\infty, C^\infty(G/N;\eta))$, then $\iota(v)(g)$ must be of moderate growth.

In $SU(n,1)$ case, the M-Whittaker function is not of moderate growth but the W-Whittaker function is. In $Spin(2n,1)$ case, since $\xi > 0$, $e^{\mathrm{sgn} q_{n-1,2n-3}\frac{\xi}{a}}$ is of moderate growth if and only if $q_{n-1,2n-3} < 0$. Then, for each case, the dimension of $\mathrm{Wh}(\pi_\Lambda^{*-\infty})$ is just the half of that of the $(\mathfrak{g}_\mathbb{C}, K)$-intertwining space. This is consistent with Matumoto's theorem (cf. [M2, Theorem 5.5.2]).

At last, we have proved Theorem A and B.

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