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Mixed Hodge Structures of Siegel Modular Varieties and Siegel Eisenstein Series

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Abstract. In this paper we study the mixed Hodge structure on the middle degree cohomology of the Siegel modular variety of level n. We attach some global automorphic forms to its highest weight quotient space and also show a vanishing of the next weight quotient. As an appendix, we also consider the universal family of abelian varieties over the moduli space and treat its middle degree mixed Hodge structure similar to the above case.

1. Introduction

The purpose of this paper is to give a description of some graded quotients associated with the weight filtration on the mixed Hodge structure defined for the middle degree cohomology group of Siegel modular variety. Here is a more precise statement of the main result.

Let H_g be the Siegel upper half space of degree g. The Siegel modular group $\Gamma_g = Sp(g, \mathbf{Z})$ acts on it properly discontinuously as usual by

$$Z \mapsto \gamma\langle Z\rangle = (AZ+B)(CZ+D)^{-1}, \text{ for } Z \in \mathrm{H}_g, \ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g.$$

Moreover the principal congruence subgroup of level $n \geq 3$ of $\Gamma_g : \Gamma_g(n) = \ker(\Gamma_g \to Sp(g, \mathbf{Z}/n\mathbf{Z}))$, acts freely. Then the quotient space $V_g(n) = \Gamma_g(n)\backslash \mathrm{H}_g$ becomes to be a smooth (open) algebraic variety over $\mathbf{C}$ of dimention $N = \frac{1}{2}g(g+1)$, and is known to be quasi-projective because there exists a projective minimal compactification: Baily-Borel-Satake compactification ([3], [20]).

The cohomology group $\mathrm{H}^i(V_g(n), \mathbf{Q})$ has the mixed Hodge structure by Deligne [5, 6]. Let $\{W_k\}$ be the weight filtration. Then we have the following main result.

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Main theorem. *Let $V_g(n)^*$ be the minimal compactification of a Siegel modular variety, and $N = \frac{1}{2}g(g+1)$ be its dimension. Then for $g \geq 2$, we have that*
(i) $\dim Gr^W_{2N}\mathrm{H}^N(V_g(n), \mathbf{C}) =$ *the number of 0-dimensional cusps in* $V_g(n)^*$.
(ii) $Gr^W_{2N-1}\mathrm{H}^N(V_g(n), \mathbf{C}) = \{0\}$.

The main ingredients of the proof are Poincaré residue map of the mixed Hodge structures, toroidal compactification of Siegel modular variety and some regularity results on the Eisenstein series proved by Shimura [22].

Let us explain the contents of this paper. We review the mixed Hodge structure of the cohomology groups of a non-singular algebraic variety in §2. We explain about the edge component of mixed Hodge structure on $\mathrm{H}^i(V, \mathbf{C})$ in §3. The Poincaré residue map is also explained in this section. After brief review on the toroidal compactification of a Siegel modular variety in §4, we rewrite in §5 a Fourier-Jacobi expansion of Siegel modular forms of weight $g+1$ in the local coordinates of a smooth compactification of $V_g(n)$. There we describe the image of a modular form by the Poincaré residue map in terms of the constant term of its Fourier expansion. In §6, we give two examples of the smooth compactification in case of $g = 2,\ 3$ as in Namikawa [16] and Nakamura [15], and explain about degenerate boundary coordinates. In §7, we review the holomorphic Siegel Eisenstein series. We restate the main result in §8, and give a proof for it. There we use theorems of Shimura [22] for the holomorphy of the Eisenstein series of low weight. In the last section §9, as an appendix, we remark some results about the weight filtrations in the case of the universal family of principally polarized abelian varieties which is similar with the case of Siegel modular varieties.

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2. Mixed Hodge structures

We recall briefly the theory of mixed Hodge structures for a smooth algebraic variety defined by P. Deligne. The references are [5], [6].

Given a smooth quasi-projective algebraic variety V over $\mathbf{C}$ of dimension N, it can be imbedded into a smooth projective variety $\widetilde{V}$ as a Zariski open subset, and $D = \widetilde{V} - V$ is a finite union of smooth irreducible divisors

$\{D_i\}_{i\in I}$ which have at most simply normal crossings (theorem of Nagata and the resolution of singularities by Hironaka). Choose such an embedding: $j = V \hookrightarrow \widetilde{V}$. Then $\{\mathrm{H}^i(V,\mathbf{C})\}_{i\in\mathbf{Z}}$ make the mixed Hodge structures defined as follows.

Definition 2.1. A mixed Hodge structure (abbreviated M.H.S.) is triple data: $\{H_{\mathbf{Z}}, W_\bullet, F^\bullet\}$;
(i) $H_{\mathbf{Z}}$ is a $\mathbf{Z}$-module of finite type,
(ii) A finite increasing filtration $W_\bullet$ on $H_{\mathbf{Q}} = H_{\mathbf{Z}} \otimes \mathbf{Q}$,
(iii) A finite decreasing filtration $F^\bullet$ on $H_{\mathbf{C}} = H_{\mathbf{Z}} \otimes \mathbf{C}$;
with following requirement. Denote also $W_\bullet$ the naturally induced filtration on $H_{\mathbf{C}}$ and define $F^p(Gr_k^W H_{\mathbf{C}})$ to be the image of $F^pH_{\mathbf{C}} \cap W_kH_{\mathbf{C}} \to Gr_k^W H_{\mathbf{C}}$, then for all $k \in \mathbf{Z}$, $\{Gr_k^W H_{\mathbf{Q}}, F^\bullet\}$ is the pure $\mathbf{Q}$-Hodge structure of weight k. $W_\bullet$ and $F^\bullet$ are called as weight and Hodge filtration respectively.

Write $H^{p,q} = Gr_F^p Gr_{\overline{F}}^q Gr_k^W H_{\mathbf{C}}$, ($\overline{F}^\bullet$ is the complex conjugation to $F^\bullet$). Then the definition means that

$$\text{(i)}\quad H^{p,q} = 0, \text{ if } \ p+q \neq k, \text{ and}$$

$$\text{(ii)}\quad Gr_k^W H_{\mathbf{C}} = \bigoplus_{p+q=k} H^{p,q} : \text{direct sum decomposition}, \ H^{p,q} = \overline{H}^{q,p}.$$

(When these properties are satisfied, F and $\overline{F}$ are said to be k-opposite to each other.)

For a smooth (open) algebraic variety V, the mixed Hodge structures on $\mathrm{H}^*(V,\mathbf{C})$ are obtained in the following manner. First for the holomorphic de Rham complex $\Omega_V^\bullet$, there are isomorphisms (Grothendieck [10]):

$$\mathrm{H}^*(V,\mathbf{C}) \simeq \mathbf{H}^*(V,\Omega_V^\bullet) \simeq \mathbf{H}^*(\widetilde{V}, j_*\Omega_V^\bullet).$$

We take the subcomplex $\Omega_{\widetilde{V}}^\bullet(logD)$ of $j_*\Omega_V^\bullet$, the logarithmic de Rham differential complex as follows. The sheaf $\Omega_{\widetilde{V}}^1(logD)$ is the locally free $\mathcal{O}_{\widetilde{V}}$-module generated by sections $\frac{dz_i}{z_i}$ $(1 \leq i \leq l)$, dz_j $(l+1 \leq j \leq N)$ at a point where D is defined locally by $\{z_1 \cdots z_l = 0\}$ with a local coordinates $\{z_i\}_{1\leq i\leq N}$ of $\widetilde{V}$. Setting $\Omega_{\widetilde{V}}^k(logD) = \bigwedge^k \Omega_{\widetilde{V}}^1(logD)$, we obtain the complex $\Omega^\bullet_{\widetilde{V}}(logD)$, and the natural inclusion $\Omega^\bullet_{\widetilde{V}}(logD) \hookrightarrow j_*\Omega^\bullet_V$ is a quasi-isomorphism of complexes (Deligne [5] (3.1.8)). Therefore we get isomorphisms:

$$\mathrm{H}^\bullet(V, \mathbf{C}) \simeq \mathbf{H}^\bullet(\widetilde{V}, j_*\Omega^\bullet_V) \simeq \mathbf{H}^\bullet(\widetilde{V}, \Omega^\bullet_{\widetilde{V}}(logD)).$$

Thus we can identify the singular cohomology of V with the hypercohomology of the logarithmic differential complex on $\widetilde{V}$ (Deligne [5], (3.1.5)).

With this identification the Hodge and weight filtrations on $\mathrm{H}^\bullet(V, \mathbf{C})$ are induced by the following filtrations on the complex. For any complex $C^\bullet$ in abelian category, define a subcomplex $\sigma_{\geq p}$ as $(\sigma_{\geq p}C^\bullet)^i = C^i$ if $i \geq p$, and $= 0$ if $i < p$. (This filtration is called as the stupid filtration.) Applying this to $\Omega^\bullet_{\widetilde{V}}(logD)$, we have a filtration on complex: $F^\bullet = \{F^p\Omega^\bullet_{\widetilde{V}}(logD)\}_{p\in\mathbf{Z}}$,

$$F^p\Omega^\bullet_{\widetilde{V}}(logD) = \sigma_{\geq p}\Omega^\bullet_{\widetilde{V}}(logD).$$

On the other hand, the weight filtration $W_\bullet = \{W_p\Omega^\bullet_{\widetilde{V}}(logD)\}_{p\in\mathbf{Z}}$ is defined by

$$W_p\Omega^k_{\widetilde{V}}(logD) = \begin{cases} 0 \quad (p < 0) \\ \Omega^k_{\widetilde{V}}(logD) \quad (p > k) \\ \Omega^{k-p}_{\widetilde{V}} \wedge \Omega^p_{\widetilde{V}}(logD) \quad (0 \leq p \leq k). \end{cases}$$

For each p, one has inclusions $F^p\Omega^\bullet_{\widetilde{V}}(logD) \hookrightarrow \Omega^\bullet_{\widetilde{V}}(logD)$, $W_p\Omega^\bullet_{\widetilde{V}}(logD) \hookrightarrow \Omega^\bullet_{\widetilde{V}}(logD)$ and these induce maps between hypercohomologies. Define the Hodge and weight filtration on $\mathrm{H}^\bullet(V, \mathbf{C}) = \mathbf{H}^\bullet(\widetilde{V}, \Omega^\bullet_{\widetilde{V}}(logD)))$ as follows.

(1.1) $F^p\mathrm{H}^i(V, \mathbf{C}) =$ image of $\mathbf{H}^i(\widetilde{V}, F^p\Omega^\bullet_{\widetilde{V}}(logD)) \to \mathbf{H}^i(\widetilde{V}, \Omega^\bullet_{\widetilde{V}}(logD))$

(1.2) $W_{p+i}\mathrm{H}^i(V, \mathbf{C}) =$ image of $\mathbf{H}^i(\widetilde{V}, W_p\Omega^\bullet_{\widetilde{V}}(logD)) \to \mathbf{H}^i(\widetilde{V}, \Omega^\bullet_{\widetilde{V}}(logD))$

Observe that though the weight filtration is constructed over $\mathbf{C}$ in the above, it can be defined over $\mathbf{Q}$ (Deligne [5], (3.2.4)). Write $D = \bigcup_{i\in I} D_i = \widetilde{V} - V$ with a finite (ordered) index set I. Each D_i is a smooth irreducible divisor which is also projective, for $\widetilde{V}$ is projective. We fix one orientation for each D_i. Consider the disjoint union of all m-fold intersections of $\{D_i\}$,

$$D^{[m]} = \coprod_{\{i_1<\cdots<i_m\}\subset I} D_{i_1} \cap \cdots \cap D_{i_m} : \text{ disjoint union.}$$

It is a complex manifold of dimension of $N - m$. We set $D^{[0]} = \widetilde{V}$. We consider the Poincaré residue maps $Res_{[m]}$:

$$Res_{[m]} : W_m \Omega^{\bullet}_{\widetilde{V}}(log D) \to i_{m*}\Omega^{\bullet}_{D^{[m]}}[-m],$$

where $i_m : D^{[m]} \to \widetilde{V}$ are natural maps and, for any complex $C^{\bullet}$, define $C^{\bullet}[m]$ as $(C^{\bullet}[m])^i = C^{i+m}$. With local coordinates on $\widetilde{V}$ it is given by

$$\omega \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_m}}{z_{i_m}} \mapsto \omega|_{D_{i_1} \cap \cdots \cap D_{i_m}},$$

for holomorphic differential forms ω on $\widetilde{V}$. Here the order of components $D_{i_j} = \{z_{i_j} = 0\}$ is taken to be increasing. Moreover it should be noted that we must consider the contribution of orientation to the target complex (Deligne [5], (3.1.4), (3.1.5)), but we omit the explicit suitable notation. $Res_{[m]}$ becomes a morphism of complexes. It is surjective, trivial on $W_{m-1}\Omega^{\bullet}_{\widetilde{V}}(log D)$, and induces an isomorphism of complexes:

$$Res_{[m]} : Gr^W_m \Omega^{\bullet}_{\widetilde{V}}(log D) \simeq i_{m*}\Omega^{\bullet}_{D^{[m]}}[-m],$$

(Deligne [7], [5]). Hence there is an isomorphism of hypercohomologies:

$$(1.3) \qquad \begin{aligned} Res_{[m]} : \mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^{\bullet}_{\widetilde{V}}(log D)) &\simeq \mathbf{H}^i(\widetilde{V}, i_{m*}\Omega^{\bullet}_{D^{[m]}}[-m]) \\ &\simeq \mathrm{H}^{i-m}(D^{[m]}, \mathbf{C})(-m). \end{aligned}$$

The last term defines a pure Hodge structure of weight $i + m$ by classical Hodge theory. ($(-m)$ means the $(-m)$-th Tate twist.) Then it follows that a spectral sequence for hypercohomology of filtered complex

$${}_F E_1^{p,i-p} = \mathbf{H}^i(\widetilde{V}, Gr^p_F Gr^W_m \Omega^{\bullet}_{\widetilde{V}}(log D)) \Rightarrow \mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^{\bullet}_{\widetilde{V}}(log D))$$

degenerates at E_1- term because, by the above residue map, this is identified with de Rham-Hodge spectral sequence for $\mathrm{H}^*(D^{[m]}, \mathbf{C})$. And the induced filtration on the right hand term becomes to be $i+m$-opposite to its complex conjugation (Deligne [5], (3.2.6), (3.2.7)). On the other hand, the left hand side of the isomorphism (1.3) gives an E_1-term of a spectral sequence

$${}_W E_1^{-m,m+i} = \mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^{\bullet}_{\widetilde{V}}(log D)) \Rightarrow \mathbf{H}^i(\widetilde{V}, \Omega^{\bullet}_{\widetilde{V}}(log D)) \simeq \mathrm{H}^i(V, \mathbf{C}).$$

It is shown that each of its differentials d_1 (the connecting homomorphisms of a long exact sequence of hypercohomologies induced from a short exact sequence $0 \to Gr^W_{m-1} \to W_m/W_{m-2} \to Gr^W_m \to 0$) is strictly compatible with the filtration on ${}_W E_1^{-m,i+m}$ induced above ([5], (3.2.8)). Then it is proved that a unique filtration is defined on ${}_W E_2$-term from the one on ${}_W E_1$. (There are different types of filtrations on ${}_W E_2$ which canonically induced from the filtration $F^\bullet$ on the complex $\Omega^\bullet_{\widetilde{V}}(logD)$, but now these are coincident with each other.) Also we get $d_j = 0,\ j \geq 2$, hence above spectral sequence ${}_W E_r$ degenerates at ${}_W E_2$-term ([5], (3.2.9), (3.2.10)). Thus we obtain a filtration on $Gr^W_{m+i}\mathrm{H}^i(V, \mathbf{C}) = {}_W E_2$ which is $i+m$-opposite to its complex conjugation, which is proved to coincide with the filtration defined by (1.1), (1.2). Therefore $\{\mathrm{H}^\bullet(V, \mathbf{C}), W_\bullet, F^\bullet\}$ defines a M.H.S. ([5], (3.2.5)). Together with [5, (3.2.13)] and [6, (7.2.8)] we have the following.

Theorem 2.2. (Deligne)
(i) *A spectral sequence* ${}_F E_1^{p,q} = \mathbf{H}^q(\widetilde{V}, Gr^p_F \Omega^\bullet_{\widetilde{V}}(logD)) \Rightarrow \mathrm{H}^{p+q}(V, \mathbf{C})$ *degenerates at* ${}_F E_1$*-terms.*
(ii) *A spectral sequence* $\mathbf{H}^i(\widetilde{V}, Gr^W_p \Omega^k_{\widetilde{V}}(logD)) \Rightarrow \mathbf{H}^i(\widetilde{V}, \Omega^k_{\widetilde{V}}(logD))$ *degenerates at* E_2*-terms.*
(iii) *There is an isomorphism of spectral sequences*

$$Gr^i_F E_r(\mathbf{R}\Gamma(\Omega^\bullet_{\widetilde{V}}(logD)), W) \simeq E_r(\mathbf{R}\Gamma(Gr^i_F \Omega^\bullet_{\widetilde{V}}(logD)), W),$$

here $\mathbf{R}\Gamma(K^\bullet)$ *is a filtered complex with filtration* $W_\bullet$ *which is derived from an acyclic bi-filtered resolution* $K'^\bullet$ *of a bi-filtered complex* $K^\bullet$*.*

2. Residue map on edge parts

In this section we define a certain homomorphism from subspaces of a weight quotient space of $H^i(V)$ to the cohomology groups $H^j(D^{[m]})$, which is induced from the Poincaré residue map. This construction is necessary in the proof of the main result (§7).

We begin with isomorphisms:

$$\begin{aligned} &Gr^W_{m+i}\mathrm{H}^i(V, \mathbf{C}) \\ &\quad = \mathrm{H}(\mathbf{H}^{i-1}(\widetilde{V}, Gr^W_{m+1}\Omega^\bullet_{\widetilde{V}}(logD)) \xrightarrow{d_1} \mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^\bullet_{\widetilde{V}}(logD)) \\ &\qquad \xrightarrow{d_1} \mathbf{H}^{i+1}(\widetilde{V}, Gr^W_{m-1}\Omega^\bullet_{\widetilde{V}}(logD))) \end{aligned}$$

$$= \mathrm{H}(\mathrm{H}^{i-m-2}(D^{[m+1]}, \mathbf{C})(-m-1) \xrightarrow{d_1} \mathrm{H}^{i-m}(D^{[m]}, \mathbf{C})(-m) \xrightarrow{d_1} \mathrm{H}^{i-m+2}(D^{[m-1]}, \mathbf{C})(-m+1)).$$

Here $H(* \to * \to *)$ means the cohomology of the 3-terms complexes. The first isomorphism comes from E_2-terms of the spectral sequence in (ii) of Theorem(1.2). The second one is obtained on passing to the targets of the Poincaré residue map $Res_{[m]}$ in §2.

By the first isomorphism, $Gr^W_{m+i}\mathrm{H}^i(V, \mathbf{C})$ is regarded as a subquotient space of $\mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^\bullet_{\widetilde{V}}(logD))$.

Lemma 3.1. *Let $\{F^\bullet\}$ be the Hodge filtration on $Gr^W_{m+i}\mathrm{H}^i(V, \mathbf{C})$, then $F^i Gr^W_{m+i}\mathrm{H}^i(V, \mathbf{C})$ injects into the $\mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^\bullet_{\widetilde{V}}(logD))$.*

Proof. By Theorem(1.2), we have

$$\begin{aligned} Gr^i_F Gr^W_{m+i}\mathbf{H}^i(\widetilde{V}, \Omega^\bullet_{\widetilde{V}}(logD)) &\simeq Gr^i_{F\,W}E_2(\mathbf{R}\Gamma(\Omega^\bullet_{\widetilde{V}}(logD)), W) \\ \simeq {}_WE_2(Gr^i_F\mathbf{R}\Gamma(\Omega^\bullet_{\widetilde{V}}(logD)), W) &\simeq {}_WE_2(\mathbf{R}\Gamma(\Omega^i_{\widetilde{V}}(logD)), W), \end{aligned}$$

last of which equals the cohomology of the next complex;
$\simeq \mathrm{H}(\mathbf{H}^{i-1}(\widetilde{V}, Gr^W_{m+1}\Omega^i_{\widetilde{V}}(logD)[-i])$

$$\xrightarrow{d_1} \mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^i_{\widetilde{V}}(logD)[-i]) \xrightarrow{d_1} \mathbf{H}^{i+1}(\widetilde{V}, Gr^W_{m-1}\Omega^i_{\widetilde{V}}(logD)[-i])).$$

Since the first term of above 3-term complex is zero space, we know that the space $Gr^i_F Gr^W_{m+i}\mathbf{H}^i(\widetilde{V}, \Omega^\bullet_{\widetilde{V}}(logD))$ can be seen as a subspace of $\mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^i_{\widetilde{V}}(logD))$. Also we have that

$$\mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^i_{\widetilde{V}}(logD)) = Gr^i_F\mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^\bullet_{\widetilde{V}}(logD)).$$

On the other hand, we have $F^{i+1}\mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^\bullet_{\widetilde{V}}(logD)) = \{0\}$. In fact, since the Poincaré residue map $Res_{[m]}$ is compatible with the Hodge filtration on $\mathrm{H}^i(V, \mathbf{C})$, we get

$$F^j\mathbf{H}^i(\widetilde{V}, Gr^W_m \Omega^\bullet_{\widetilde{V}}(logD)) \simeq F^{j-m}\mathrm{H}^{i-m}(D^{[m]}, \mathbf{C})(-m).$$

Here the Hodge types of $\mathrm{H}^{i-m}(D^{[m]}, \mathbf{C})(-m)$ is only $\{(p+m, q+m)\}$, where $p+q = i-m$, and $p,\ q \geq 0$. Therefore for $j > i$, considering above Hodge type, we obtain that $F^{j-m}\mathrm{H}^{i-m}(D^{[m]}, \mathbf{C})(-m) = \{0\}$. Hence combining this with above, the lemma follows. □

By this lemma, we can consider a restriction of the Poincaré residue map on this edge subspace and this restriction map gives a isomorphism of $F^i Gr^W_{m+i}\mathrm{H}^i(V, \mathbf{C})$ into its image. By an abuse of notation, we denote this homomorphism

$$Res_{[m]} : F^i Gr^W_{m+i}\mathrm{H}^i(V, \mathbf{C}) \to \mathrm{H}^{i-m}(D^{[m]}, \mathbf{C})(-m).$$

by the same symbol as the Poincaré residue map on the complexes. Moreover the following lemma shows that the domain of above map is a subquotient of the space of global sections of $\Omega^i_{\widetilde{V}}(logD)$.

LEMMA 3.2.

$$Gr^i_F Gr^W_{m+i}\mathrm{H}^i(V, \mathbf{C}) \simeq Gr^W_{m+i}\mathrm{H}^0(\widetilde{V}, \Omega^i_{\widetilde{V}}(logD)).$$

PROOF. This results from Theorem (1.2). □

4. Toroidal compactification of $V_g(n)$

In this section, we recall the construction of toroidal compactification of a quotient variety of Hermitian symmetric space. Main references are, for example, Ash, Mumford, Rapoport, Tai [1], Namikawa [16], [17], Nakamura [15].

4.1. Minimal compactification

First we recall the Baily-Borel-Satake minimal compactification. The Siegel upper half space H_g is analytically isomorphic to $\mathcal{D}_g = \{\tau = {}^t\tau \in M_g(\mathbf{C});\ 1 - \tau\bar{\tau} > 0\}$ by $\mathrm{H}_g \ni z \mapsto (z - \sqrt{-1}\mathbf{1}_g)(z + \sqrt{-1}\mathbf{1}_g)^{-1}$. $\mathcal{D}_g$ is a bounded symmetric domain in $\mathbf{C}^{\frac{1}{2}g(g+1)}$. We set $\mathcal{D}'_g = \{\tau = {}^t\tau \in M_g(\mathbf{C});\ 1 - \tau\bar{\tau} \geq 0\} \supset \mathcal{D}_g$, which is a union of $\mathcal{D}_g$ and its boundary components. We take the only rational boundaries defined over $\mathbf{Q}$. Then we set $\mathcal{D}^*_g = \mathcal{D}_g \cup \{$rational boundary components of $\mathcal{D}_g\}$. According to this, we also define for the Siegel upper half space

$$\mathrm{H}^*_g = \mathrm{H}_g \cup \{\text{rational boundary components of } \mathrm{H}_g\}.$$

The action of $Sp(g,\mathbf{Q})$ on H_g extends to H_g^* and we make the quotient $V_g(n)^* = \Gamma_g(n)\backslash \mathrm{H}_g^*$. Then $V_g(n)^*$ becomes a compact Hausdorff space by defining a suitable topology on H_g^* (called Satake topology). This is the minimal compactification of $V_g(n)$, which is a projective variety with singularities on $V_g(n)^* - V_g(n)$ (Satake [20]).

4.2. Rational boundary components

We fix some symbols to denote rational boundary components of H_g.

Symbols :

$\{F_\alpha\}$; $\Gamma_g(n)$-equivalence classes of rational boundary components of H_g. $F_\alpha \simeq \mathrm{H}_{g_0}$ $(0 \le \exists g_0 \le g)$.

$P_\alpha = \{g \in Sp(g,\mathbf{R});\ gF_\alpha = F_\alpha\}$; maximal $\mathbf{Q}$-parabolic subgroups associated to F_α.

$W_\alpha \subset P_\alpha$; a unipotent radical.

$U_\alpha \subset W_\alpha$; the center of W_α, $\simeq \{b \in M_{g_1}(\mathbf{R})\ ;{}^t b = b\} = \mathcal{Q}_{g_1}, g_1 = g - g_0$.

Ω_α; a self dual open cone, $\simeq \{b \in M_{g_1}(\mathbf{R}); {}^t b = b, b > 0\} = \mathcal{Q}_{g_1}^+$.

Among $\{F_\alpha\}$, the standard boundaries $F_{g_0}^{st}$ can be chosen for each g_0. The maximal $\mathbf{Q}$-parabolic subgroup P_{g_0} associated to this standard boundary is given as follows:

$$P_{g_0} = \left\{ \begin{pmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \in Sp(g,\mathbf{R}) \;\middle|\; \begin{array}{c} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in Sp(g_0,\mathbf{R}), \\ u \in GL(g_1,\mathbf{R}) \end{array} \right\},$$

$$W_{g_0} = \left\{ \begin{pmatrix} \mathbf{1}_{g_0} & & 0 & n \\ {}^t m & \mathbf{1}_{g_1} & {}^t n & b \\ 0 & 0 & \mathbf{1}_{g_0} & -n \\ 0 & 0 & 0 & \mathbf{1}_{g_1} \end{pmatrix} \in P_{g_0} \;\middle|\; {}^t nm + b = {}^t mn + {}^t b \right\},$$

$$U_{g_0} = \left\{ \begin{pmatrix} \mathbf{1}_{g_0} & & 0 & 0 \\ & \mathbf{1}_{g_1} & 0 & b \\ 0 & 0 & \mathbf{1}_{g_0} & 0 \\ 0 & 0 & 0 & \mathbf{1}_{g_1} \end{pmatrix} \in W_{g_0} \;\middle|\; {}^t b = b \right\} = \mathcal{Q}_{g_1}.$$

All rational boundaries F_α are transformed into one $F_{g_0}^{st} \simeq \mathrm{H}_{g_0}$ by the action of $Sp(g,\mathbf{Z}) = \Gamma_g$; $F_\alpha = \gamma_\alpha F_{g_0}^{st}$, $\exists \gamma_\alpha \in \Gamma_g$. Under this situation we first construct a partial compactification in direction of a rational boundary component F_α.

4.3. Partial compactification

Fix $F = F^{st}_{g_0}$, a standard rational boundary and we consider specially the partial compactification for it. Let $P_{g_0\mathbf{Z}} = P_{\mathbf{Z}} = P_{g_0} \cap \Gamma_g$, $U_{g_0\mathbf{Z}} = U_{\mathbf{Z}} = U_{g_0} \cap \Gamma_g$, $P_{g_0}(n) = P(n) = P_{g_0} \cap \Gamma_g(n)$, $U_{g_0}(n) = U(n) = U_{g_0} \cap \Gamma_g(n)$, $U_{\mathbf{C}} = U_{\mathbf{Z}} \otimes \mathbf{C}$. We make a next map;

$$e : \mathrm{H}_g \to \mathrm{H}_{g_0} \times V_{g_0g_1} \times (U_{\mathbf{Z}} \backslash U_{\mathbf{C}}),\ Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \mapsto (z_1, z_2, \mathbf{e}\left(\frac{z_3}{n}\right)),$$

here $\mathbf{e}(z) = \exp(2\pi\sqrt{-1}z) = (\exp(2\pi\sqrt{-1}z_{k,l}))_{k,l}$, $V_{g_0g_1}$ is the space of $g_0 \times g_1$-matrices with coefficients in $\mathbf{C}$. $T_{g_1} = U_{\mathbf{Z}} \backslash U_{\mathbf{C}}$ is a complex torus of dimension $\frac{1}{2}g_1(g_1+1)$, $\simeq (\mathbf{C}^\times)^{\frac{1}{2}g_1(g_1+1)}$. It can be seen that the above map e factors through $U(n)\backslash\mathrm{H}_g$. Then, the image $T^\circ_{g_0,g_1}$ of e is an open subset of $T_{g_0g_1} = \mathrm{H}_{g_0} \times V_{g_0g_1} \times T_{g_1}$, and $U(n)\backslash\mathrm{H}_g$ is isomorphic to this image. We identify them, thus consider $T^\circ_{g_0,g_1}$ in $T_{g_0g_1}$.

For the third factor of complex torus T_{g_1}, there exists a toroidal embedding as following. We remark that $U_{\mathbf{Z}} \simeq \mathcal{Q}_{g_1\mathbf{Z}} \simeq \mathrm{Hom}_{alg-grp}(\mathbf{G}_m, T_{g_1}) \simeq \pi_1(T_{g_1})$, where $\mathcal{Q}_{g_1\mathbf{Z}}$ is the $\mathbf{Z}$-lattice of symmetric integral matrices in $\mathcal{Q}_{g_1}$. Take $\widehat{\mathcal{Q}}_{g_1}$ to be a dual real vector space of $\mathcal{Q}_{g_1}$, and denote by $\langle\ ,\ \rangle : \widehat{\mathcal{Q}}_{g_1} \times \mathcal{Q}_{g_1} \to \mathbf{R}$ the natural pairing. Then the dual lattice M of $\mathcal{Q}_{g_1\mathbf{Z}}$ is defined by

$$M = \{\hat{y} \in \widehat{\mathcal{Q}}_{g_1}; \langle \hat{y}, y\rangle \in \mathbf{Z} \text{ for } \forall y \in \mathcal{Q}_{g_1}\}.$$

Let $\mathcal{Q}^+_{g_1}$ be the set of positive definite real quadratic forms in $\mathcal{Q}_{g_1}$. By $\overline{\mathcal{Q}}^+_{g_1}$ we denote the rational closure in the space of nonnegative real quadratic forms which is, by definition, the convex hull of the set of nonnegative integral quadratic forms. The group $GL(g_1, \mathbf{Z})$ operates on $\mathcal{Q}_{g_1}$ as $y \mapsto uy^tu$ for $u \in GL(g_1, \mathbf{Z})$, and the action preserves $\mathcal{Q}^+_{g_1}$ and $\overline{\mathcal{Q}}^+_{g_1}$. Every element of $\overline{\mathcal{Q}}^+_{g_1}$ can be transformed by a unimodular integral matrix u to $uy^tu = \begin{pmatrix} 0 & 0 \\ 0 & y' \end{pmatrix}$; $y' > 0$ (Namikawa [16]).

On $\overline{\mathcal{Q}}^+_{g_1} (\simeq \overline{\Omega}_{g_0})$, we consider a $GL(g_1, \mathbf{Z})$-admissible cone decomposition $\Sigma_{g_1} = \{\sigma\}$ which satisfies following properties:

(1) each $\sigma \in \Sigma_{g_1}$ is a rational convex cone, namely, generated by finite number of semipositive integral quadratic forms,

(2) $\sigma \in \Sigma_{g_1}$, $\tau \prec \sigma$ (τ is a face of σ) $\Rightarrow \tau \in \Sigma_{g_1}$, $\sigma, \tau \in \Sigma_{g_1} \Rightarrow \sigma \cap \tau \in \Sigma_{g_1}$,

(3) the decomposition is invariant under the action of $GL(g_1, \mathbf{Z})$,

(4) there are only a finite number of classes of σ's modulo $GL(g_1, \mathbf{Z})$,
(5) $\bigcup_{\sigma\in\Sigma_{g_1}} \sigma = \overline{\mathcal{Q}}^+_{g_1}$.

Later we glue all partial compactifications into one $\widetilde{V}_g(n)$. For this purpose, we have to assume that the family of cone decompositions $\{\Sigma^{F_\alpha}_{g_1}\}$, each of which is associated to a rational boundary F_α, satisfies next compatibility conditions:
(6) if $F_\alpha = \gamma F_\beta$ with $\gamma \in \Gamma_g(n)$, then $\Sigma^{F_\alpha}_{g_1} = \gamma\Sigma^{F_\beta}_{g_1}$, via the natural isomorphism $\gamma : \overline{\Omega}_\alpha \to \overline{\Omega}_\beta$.
(7) if $g_1' < g_1$, for natural embedding $\overline{\mathcal{Q}}^+_{g_1'} \to \overline{\mathcal{Q}}^+_{g_1}$: $y' \mapsto \begin{pmatrix} 0 & 0 \\ 0 & y' \end{pmatrix}$, the restriction of $\overline{\mathcal{Q}}^+_{g_1}$ to $\overline{\mathcal{Q}}^+_{g_1'}$ is the cone decomposition $\Sigma^{F_\beta}_{g_1'}$.

A family of cone decompositions for each rational boundaries satisfying from (1) to (7) is called $\Gamma_g(n)$-admissible collection. For given admissible Σ_{g_1} for $\overline{\mathcal{Q}}^+_{g_1}$, we can construct an affine torus embeddings $\{\mathcal{T}_\sigma\}$ of T_{g_1} to every $\sigma \in \Sigma_{g_1}$. Set the dual cone of σ to be $\hat{\sigma} = \{\hat{y} \in \widehat{\mathcal{Q}}_{g_1}; \langle \hat{y}, y\rangle \geq 0 \;\; for \;\; \forall y \in \sigma\}$. $T_{g_1} = Spec\ \mathbf{C}[M] = Spec\ \mathbf{C}[z_{ij}, z_{ij}^{-1}; 1 \leq i \leq j \leq g_1]$. Then we get an embedding

$$T_{g_1} \hookrightarrow \mathcal{T}_\sigma = Spec\ \mathbf{C}[z^A;\ A \in \hat{\sigma} \cap M],$$

where $z^A = \prod_{1\leq i\leq j\leq g_1} z_{ij}^{A_{ij}}$, and $\hat{\sigma} \cap M$ is a sub-semigroup of M.

From the property (1), $\mathcal{T}_\sigma$ becomes an algebraic scheme. And from (2) $\{\mathcal{T}_\sigma\}_{\sigma\in\Sigma_{g_1}}$ can be glued with each other (i.e. for $\sigma' \prec \sigma$ ($\hat{\sigma} \prec \hat{\sigma}'$), use natural open embedding $\mathcal{T}_{\sigma'} \subset \mathcal{T}_\sigma$). Then we get a torus embedding $\mathcal{T}_{g_1}$;

$$T_{g_1} \hookrightarrow \mathcal{T}_{g_1} = \bigcup_{\sigma\in\Sigma_{g_1}} \mathcal{T}_\sigma \quad \text{(gluing)}.$$

The scheme $\mathcal{T}_{g_1}$ is not necessary of finite type, but locally of finite type. Here the natural action of T_{g_1} on its image in $\mathcal{T}_{g_1}$ by the product of torus extends to all over $\mathcal{T}_{g_1}$. Each of the T_{g_1}-orbits in $\mathcal{T}_{g_1}$ is in one-to-one correspondence to cones $\sigma \in \Sigma_{g_1}$ (Namikawa [16, Theorem(4.6)], [17, Prop.(6.12)]):

$$\begin{aligned} \Sigma_{g_1} \ni \sigma \quad &\leftrightarrow \quad \mathcal{O}(\sigma) \in \{T_{g_1}\text{-orbits } \subset \mathcal{T}_{g_1}\}. \\ \mathcal{O}(\sigma) \quad &= \quad \{\lim_{t\to\infty} t^\eta z;\ z \in T_{g_1}\}. \end{aligned}$$

Here $\eta \in \sigma^\circ \cap \overline{\mathcal{Q}}^+_{g_1 Z}$, $t^\eta z = (t^{\eta_{ij}} z_{ij})_{i,j}$, and η is considered as an element of one parameter subgroup $\mathcal{Q}_{g_1 Z}$. In this correspondence, one get also $\sigma' \prec$

$\sigma \Leftrightarrow \mathcal{O}(\sigma) \subset \overline{\mathcal{O}(\sigma')}$. Besides, for $\{0\} \in \Sigma_{g_1}$, we have $\{\widehat{0}\} \cap \widehat{\mathcal{Q}}_{g_1 Z} = \widehat{\mathcal{Q}}_{g_1 Z}$. Therefore $\mathcal{T}_{\{0\}} = T_{g_1}$ is the T_{g_1}-orbit corresponding to $\{0\} \in \Sigma_{g_1}$, and $T_{g_1} \hookrightarrow \mathcal{T}_{g_1}$ defines a Zariski open subset.

Then we set $T_{g_0 g_1} \hookrightarrow \mathcal{X}_{g,g_1} = \mathrm{H}_{g_0} \times V_{g_0 g_1} \times \mathcal{T}_{g_1}$, and define the partial compactification $(U(n)\backslash \mathrm{H}_g)_{\Sigma_{g_1}}$ of $T^{\circ}_{g_0,g_1} \subset T_{g_0 g_1}$ in the direction of $F^{st}_{g_0}$ as

$$(U(n)\backslash \mathrm{H}_g)_{\Sigma_{g_1}} = \text{ the interior of the closure of } T^{\circ}_{g_0,g_1} \text{ in } \mathcal{X}_{g,g_1}.$$

(Namikawa [16, Prop.(6.3)]). For this we have the following proposition (Namikawa [16, Prop.(6.6), (6.9)]):

PROPOSITION 4.1. *$\overline{P(n)} = P(n)/U(n)$ acts properly discontinuously on $(U(n)\backslash \mathrm{H}_g)_{\Sigma_{g_1}}$. Moreover, if level $n \geq 3$, then this action is without fixed points.*

Therefore the quotient $\overline{P(n)}\backslash(U(n)\backslash \mathrm{H}_g)_{\Sigma_{g_1}}$ has a structure of normal analytic space. (We remark that $P(n)(\hookrightarrow GL(g_1, \mathbf{Z}))$ is considered as a subgroup of finite index.) On the other hand, let

$$\mathcal{O}_{\Sigma_{g_0}} = \bigcup_{\sigma \cap \mathcal{Q}^{+}_{g_1} \neq \phi, \sigma \in \Sigma_{g_1}} \mathrm{H}_{g_0} \times V_{g_0 g_1} \times \mathcal{O}(\sigma) \subset \mathcal{X}_{g,g_1},$$

then $\mathcal{O}_{\Sigma_{g_0}} \subset (U(n)\backslash \mathrm{H}_g)_{\Sigma_{g_1}}$ and one can see that the quotient $\overline{P(n)}\backslash \mathcal{O}_{\Sigma_{g_0}}$ is a closed subset in the above normal analytic space. By the reduction theory of Siegel, we have the following:

For any point $p \in \mathcal{O}_{\Sigma_{g_0}}$, there is a neighborhood Y of p in $(U(n)\backslash \mathrm{H}_g)_{\Sigma_{g_1}}$ such that if $z_1 = M \cdot z_2$, $\exists M \in \Gamma_g(n)$ for z_1, $z_2 \in e^{-1}(Y) \cap \mathrm{H}_{g_1}$ then $M \in P_{g_0}(n)$.

This means, near $\overline{P(n)}\backslash \mathcal{O}_{\Sigma_{g_0}}$, the variety $V_g(n) = \Gamma_g(n)\backslash \mathrm{H}_g$ is locally isomorphic to $\overline{P(n)}\backslash(U(n)\backslash \mathrm{H}_g)_{\Sigma_{g_1}}$. Therefore we can glue these in a neighbourhood of each boundary point, and put an analytic structure on these neighbourhoods from the one on $\overline{P(n)}\backslash(U(n)\backslash \mathrm{H}_g)_{\Sigma_{g_1}}$. The above procedure (partial compactification, and gluing pieces nearby boundary orbits) for all $\Gamma_g(n)$-equivalent classes of rational boundaries, is canonically compatible by the properties of $\Gamma_g(n)$-admissible family of cone decompositions. Finally we obtain a toroidal compactification $\widetilde{V}_g(n)$ of $V_g(n)$, with underlying set

$$\widetilde{V}_g(n) = \bigcup_{0 \leq g_0 \leq g} \bigcup_{\{F_{g_0}\} \ mod \ \Gamma_g(n)} \overline{P_{g_0}(n)}\backslash \mathcal{O}_{\Sigma_{g_0}}, \qquad (\ \overline{P_g}\backslash \mathcal{O}_g = V_g(n)\ ).$$

It is shown that $\widetilde{V}_g(n)$ is a compact normal space, and by choosing certain suitable cone decompositions, it becomes a smooth and projective variety over **C**. Moreover $\widetilde{V}_g(n) - V_g(n) = D = \bigcup D_i$ is a finite union of smooth irreducible divisors with simply normal crossing.

4.4. The Map from $\widetilde{V}_g(n)$ to $V_g(n)^*$

As a set, $V_g(n)^*$ is a disjoint union of modular varieties of lower dimension $= V_g(n) \amalg V^{(g-1)} \amalg \cdots \amalg V^{(0)}$. Here $V^{(i)} = \coprod_{\{F_i\}\ mod\ \Gamma_g(n)} \Gamma_i' \backslash F_i$ is the i-th rational boundary component ($F_i \simeq \mathrm{H}_i,\ 0 \le i \le g-1$). Moreover, $V_g(n)^*$ is a projective over **C**. We have a holomorphic map from $\widetilde{V}_g(n)$ to $V_g(n)$,

$$\pi : \widetilde{V}_g(n) \to V_g(n)^*.$$

The restriction of π over a rational boundary component $\Gamma_i' \backslash F_i$ comes from the naturally extended map:

$$(U(n)\backslash \mathrm{H}_g)_{\Sigma_{g_1}} \overset{p_{F_{g_0}}}{\to} V_g(n)^*,$$

and the inverse image of a rational boundary is given by

$$p_{F_{g_0}}^{-1}(\Gamma_{g_0}'\backslash F_{g_0}) = \mathcal{O}_{\Sigma_{g_0}}$$

(Namikawa [16], §6). Then π gives an identity on the open stratum $V_g(n)$ and

$$\pi^{-1}(V^{(g_0)}) = \coprod_{\{F_{g_0}\}\ mod\ \Gamma_g(n)} \overline{P_{g_0}(n)}\backslash \mathcal{O}_{\Sigma_{g_0}}.$$

Later we consider the intersections of the boundary divisors D_i of $\widetilde{V}_g(n)$ inside $\pi^{-1}(V^{(g_0)})$. A description of these intersections in local coordinates can be given from the local structure of

$$\mathcal{O}_{\Sigma_{g_0}} = \bigcup \mathrm{H}_{g_0} \times V_{g_0 g_1} \times \mathcal{O}(\sigma)$$

or more precisely from the local structure of

$$\bigcup_{\sigma \cap \mathcal{Q}_{g_1}^+ \neq \phi, \sigma \in \Sigma_{g_1}} \mathcal{O}(\sigma).$$

Details are discussed in §6.

5. Poincaré residue maps on the space of holomorphic Siegel modular forms

From now on we set $V = V_g(n)$, and $N = \dim V = \frac{1}{2}g(g+1)$. The space of holomorphic Siegel modular forms of weight $g+1$ is identified with a subspace of $\mathrm{H}^N(V, \mathbf{C})$. By use of the Poincaré residue map, we study the weight filtrations on this subspace of $\mathrm{H}^N(V, \mathbf{C})$. In §3, it is shown that the Poincaré residue map defined for the edge part of $Gr^W_{m+N}\mathrm{H}^N(V, \mathbf{C})$, can be transferred to the space $\Gamma(\widetilde{V}, \Omega^N_{\widetilde{V}}(logD))$ (*cf.* Lemma (3.1) and (3.2)). For $\omega \in \Gamma(\widetilde{V}, \Omega^N_{\widetilde{V}}(logD))$, denote the pull-back of ω to H_g by

$$\omega_0 = f(z_1, \cdots, z_N) dz_1 \wedge \cdots \wedge dz_N.$$

in the coordinates of H_g.

Lemma 5.1. *If* $g \geq 2$, $\Gamma(\widetilde{V}, \Omega^N_{\widetilde{V}}(logD)) \cong M_{g+1}(\Gamma_g(n))$. *Here* $M_{g+1}(\Gamma_g(n))$ *is the space of holomorphic Siegel modular forms on* H_g *of weight* $g+1$ *for* $\Gamma_g(n)$,

$$M_{g+1}(\Gamma_g(n)) = \{f : \mathrm{H}_g \to \mathbf{C};\ \textit{holomorphic}$$
$$\mid f(\gamma\langle Z\rangle) = \det(CZ+D)^{g+1} f(Z) \ \textit{for} \ \ \forall\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n)\}.$$

Proof. This is a standard fact (e.g. Chai-Faltings [4, Chap. V]). But for our purpose, we here review its proof.

We put $\omega_s = \bigwedge_{1\leq i\leq j\leq g} dz_{ij}$ where $Z = (z_{i,j})_{i,\ j=1,\cdots,g} \in \mathrm{H}_g$. For $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n)$, it is well known that $\gamma^*\omega_s = \det(CZ+D)^{-(g+1)}\omega_s$, see for example Maass [13, §3, p.23]. From the $\Gamma_g(n)$-invariance of $\omega_0 = f(Z)\omega_s$, $f(z)$ is a holomorphic modular form of weight $g+1$. Thus the inclusion $\Gamma(\widetilde{V}, \Omega^N_{\widetilde{V}}(logD)) \subset M_{g+1}(\Gamma_g(n))$ is shown. The converse inclusion is proved as followings.

We fix a g_0, $0 \leq g_0 \leq g-1$, and denote by $\{F_0, F_1, \cdots, F_r\}$ all of the $\Gamma_g(n)$-equivalence classes of g_0-th rational boundaries ($\simeq \mathrm{H}_{g_0}$) of H_g. We take F_0 as the standard g_0-th rational boundary ($= F^{st}_{g_0}$) of H_g, and write corresponding maximal $\mathbf{Q}$-parabolic subgroup as $P_0 \subset Sp(g, \mathbf{R})$. Each

F_l transformed into F_0 by some $\gamma_l \in \Gamma_g = Sp(g, \mathbf{Z})$, and we fix $\gamma_l \in \Gamma_g$ such that $F_l = \gamma_l F_0$ for $l = 1, \cdots, r$. Then Fourier-Jacobi expansion of $f \in M_{g+1}(\Gamma_g(n))$ at F_l is

$$j(\gamma_l^{-1}, Z)^{g+1} f(Z) = a_0(z_1^l, z_2^l; F_l) + \sum_{\{T\}} a_T(z_1^l, z_2^l; F_l)\mathbf{e}(\frac{trTz_3^l}{n}),$$

where

$$\gamma_l^{-1}\langle Z\rangle = \begin{pmatrix} z_1^l & z_2^l \\ z_2^l & z_3^l \end{pmatrix}, \text{ with } z_1^l \in M_{g_0}(\mathbf{C}),\ \ z_3^l \in M_{g_1}(\mathbf{C}),\ \ g_1 = g - g_0,$$

and

$$j(\gamma, Z) = \det(CZ + D) \text{ for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here $\{T\}$ runs all the set of nonzero non-negative, half-integral symmetric matrices of degree g_0. Recall that for $g \geq 2$, the non-negativity of T is a consequence of the Koecher principle.

We make a toroidal compactification $\widetilde{V}_g(n)$ of $V_g(n)$ by taking a $\Gamma_g(n)$-admissible family of cone decompositions. We consider the map $\pi : \widetilde{V}_g(n) \to V_g(n)^*$. And for $V^{(l)} \subset V_g(n)^*$, let $m = m(l)$ be the greatest integer such that $\pi^{-1}(V^{(l)}) \cap i_m(D^{[m]}) \neq \phi$. (Here we concern ourselves with the maximally degenerate boundary in $\pi^{-1}(V^{(l)})$.) Then local coordinates system at a point of $\pi^{-1}(V^{(l)}) \cap i_m(D^{[m]})$ is written as $\{(z_i), (u_j), (q_k = \mathbf{e}(w_k/n))\}$. Here $z_i,\ 1 \leq i \leq \frac{1}{2}g_0(g_0+1)$ (resp. $u_j,\ 1 \leq j \leq g_0 g_1$) run those upper triangle coefficients of z_1^l (resp. z_2^l). For $1 \leq k \leq \frac{1}{2}g_1(g_1+1) = d$, w_k is a linear combination of uppertriangle coefficients of z_3^l. Now we can rewrite above Fourier-Jacobi expansion in this coordinates as follows.

$$j(\gamma_l^{-1}, Z) f(Z) = a_0((z_i), (u_j); F_l) + \sum_{\{T\}} a_T((z_i), (u_j); F_l) q_1^{t_1} \cdots q_d^{t_d}.$$

Note that for nonnegativity of T, all $t_n \geq 0, n = 1 \cdots d$, and $(t_1, \cdots, t_d) \neq (0, \cdots, 0)$.

Remark. For example, if $g_1 = 2$, writing

$$z_3^l = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}, \quad T = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_3 \end{pmatrix},$$

then

$$
\begin{aligned}
& \mathbf{e}\left(tr\begin{pmatrix} t_1 & t_2 \\ t_2 & t_3 \end{pmatrix}\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}\right) \\
= \ & \mathbf{e}(t_1(\tau_1+\tau_2) + t_3(\tau_2+\tau_3) - (t_1+t_3-2t_2)\tau_2) \\
= \ & q_1^{t_1} q_2^{t_1+t_3-2t_2} q_3^{t_3}.
\end{aligned}
$$

Now for the semi-positivity of T, we have that $t_1,\ t_3 \geq 0$ and $t_1 + t_3 - 2t_2 = (1,\ -1)T\begin{pmatrix} 1 \\ -1 \end{pmatrix} \geq 0$.

On the other hand, since we have that

$$
\wedge dz_{1,i}^l \wedge dz_{2,j}^l \wedge dz_{3,k}^l = \text{const.} \times \wedge dz_i \wedge du_j \wedge \frac{dq_k}{\prod_{k=1}^d q_k},
$$

the form ω is described with the above local coordinates as

$$
\omega = c_l \Bigg\{ (a_0((z_i),(u_j);F_l) + \sum_{\{T\}} a_T((z_i),(u_j);F_l) q_1^{t_1} \cdots q_d^{t_d} \Bigg\} \wedge dz_i \wedge du_j \wedge \frac{dq_k}{\prod q_k},
$$

where $c_l \neq 0$ is a constant which depends on γ_l. (Ash, Mumford, Rapoport, Tai [1] chap.4) If we take $g_0 = 0$ then it is shown that $\omega_0 = f(Z)\omega_s$ defines a meromorphic differential form at most with poles of order one on rational boundaries. This settles the proof of Lemma (5.1). □

Moreover by the definition of $Res_{[m]}$, we also conclude the following lemma.

LEMMA 5.2. *With local coordinates at one point of* $\pi^{-1}V^{(l)} \cap i_m(D^{[m]})$, *the image of* ω *by residue map:* $Res_{[m]}\ \omega \in F^{i-m}\mathrm{H}^i(D^{[m]}, \mathbf{C})(-m)$, *is described as*

$$
Res_{[m]}\ \omega = c_l \cdot a_0((z_i),(u_j);F_l) \wedge dz_i \wedge du_j.
$$

We rewrite above statement by Siegel's Φ-operator (cf. Maass [13, §13]). For $f \in M_k(\Gamma_g(n))$, Siegel's Φ-operator is defined as

$$\Phi f(Z_1) = \lim_{t\to\infty} f\begin{pmatrix} Z_1 & 0 \\ 0 & \sqrt{-1}\ t \end{pmatrix},\ Z_1 \in \mathrm{H}_{g-1}.$$

This is well defined and defines a holomorphic Siegel modular form of weight k for $\Gamma_{g-1}(n)$ on H_{g-1}.

In Fourier expansion $f(Z) = \sum_{\{T\}} a(T)\mathbf{e}(tr(TZ)/n)$,

$$\begin{aligned} \Phi f(Z_1) &= \sum_{\{T_1\}} a(T_1)\mathbf{e}(tr(T_1 Z_1)/n), \\ a(T_1) &= a\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix};\ T_1 \text{ is of rank } \le g-1. \end{aligned}$$

Iterating this, then

$$\Phi^j f(Z_j) = \sum_{\{T_j\}} a\begin{pmatrix} T_j & 0 \\ 0 & 0 \end{pmatrix}\mathbf{e}(tr(T_j Z_j)/n),\ \ Z_j \in \mathrm{H}_{g-j}.$$

Then each cusp component of $Res_{[d]}\,\omega$ is written as (0-th term of $\Phi^j(f)) \wedge dz_i \wedge du_j$ (cf. Chai-Faltings [4, Chap. V, Prop. 1.6]). Hence to show our main results we need to find a modular form of weight $g+1$ which remains non zero under the action of Siegel Φ-operator at one specified cusp, but vanishes at all other cusps. This is obtained by an Eisenstein series in §7.

6. Structure of degenerate coordinates over an Satake rational boundary

Let $\pi : \widetilde{V}_g(n) \to V_g(n)^*$ is the natural morphism defined in §§ 4.4, and $D^{[m]}$ as in §1. In order to know explicitly the local defining equations of $D^{[m]}$, in this section we investigate the structure of $\pi^{-1}(V^{(g_0)}) \cap i_m(D^{[m]})$, where $V^{(g_0)}(\subset V_g(n)^*)$ is a union of g_0-th rational boundary components. Now $\widetilde{V}_g(n)$ can be taken to be a smooth projective variety. Indeed, as in Igusa [11], there is a $\Gamma_g(n)$-admissible family of cone decomposition: the central cone decompositions. When we make a toroidal compactification associated with these cone decompositions, it is a normalized blowing-up of Satake compactification at some ideals defining boundary components. This is non-singular projective variety over $\mathbf{C}$ if $g \le 3$ (in this case it is the same as the Delony-Voronoi compactification by Namikawa [16]). If $g \ge 4$, we take a suitable subdivision of the central cone decompositions and we can get a smooth projective compactification (Namikawa [17, (7.20), (7.26)]). As in §5, define the torus coordinates $\{q_k\}_{k=1,\dots,d}$, $d = \frac{1}{2}g_1(g_1+1)$, $g_1 = g - g_0$, for the cone decompositions of the space $\{z_3 = {}^t z_3 \in M_{g_1}(\mathbf{C})\}$. Then within $\pi^{-1}(V^{(g_0)})$, the intersections of divisors $i_m(D^{[m]})$ are defined as common zeros of m coordinates among $\{q_k\}_{k=1,\dots,d}$. Remark that inside $\pi^{-1}(V^{(g_0)})$ the coordinates $\{q_k\}$ are the only those which can determine the boundary components. Hence over the g_0-th rational boundaries the number of degenerated coordinates can be at most $d = \frac{1}{2}g_1(g_1+1)$ in the toroidal compactification. We get the following:

LEMMA 6.1. *$\pi^{-1}(V^{(g_0)}) \cap i_m(D^{[m]})$ is non-empty for only $m \le \frac{1}{2}g_1(g_1+1)$, $g_1 = g - g_0$. Especially the locus $i_N(D^{[N]})$, $N = \frac{1}{2}g(g+1)$ intersects with only $\pi^{-1}(V^{(0)})$.*

We have two examples which appear in Namikawa [16] and Nakamura [15].

Example 1. $g = 2$.

$V_2(n)^* = V_2(n) \amalg V^{(1)} \amalg V^{(0)}$. Each cone in the Delony-Voronoi (abbreviated D-V) decomposition Σ_2 of $\overline{\mathcal{Q}^+_{2,\mathbf{Z}}}$ is transformed into one of the followings by the action of $GL(2,\mathbf{Z})$.

$$\sigma_0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad \sigma_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix};\ \lambda \ge 0 \right\},$$
$$\sigma_2 = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix};\ \lambda_1, \lambda_2 \ge 0 \right\},$$
$$\sigma_3 = \left\{ \begin{pmatrix} \lambda_1+\lambda_2 & -\lambda_2 \\ -\lambda_2 & \lambda_2+\lambda_3 \end{pmatrix};\ \lambda_1, \lambda_2, \lambda_3 \ge 0 \right\}.$$

For $\overline{\mathcal{Q}^+_{0,\mathbf{Z}}}$, $\overline{\mathcal{Q}^+_{1,\mathbf{Z}}} \hookrightarrow \overline{\mathcal{Q}^+_{2,\mathbf{Z}}}$, we can restrict Σ_2 on $\overline{\mathcal{Q}^+_{0,\mathbf{Z}}}, \overline{\mathcal{Q}^+_{1,\mathbf{Z}}}$ then get $\Sigma_0 = \{\sigma_0\}$, $\Sigma_1 = \{\sigma_0, \sigma_1\}$. We have a bilinear form on $\mathcal{Q}_2$: $\mathcal{Q}_2 \times \mathcal{Q}_2 \to \mathbf{R}$: $(y, y') \mapsto tr(yy')$. Now $\sigma_3 \cap \overline{\mathcal{Q}^+_{2,\mathbf{Z}}}$ is generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

hence dual bases in $\hat{\sigma}_3 \cap M$ are

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}.$$

We take coordinates as

$$q_1 = \mathbf{e}\left(\frac{z_1 + z_2}{n}\right), q_2 = \mathbf{e}\left(\frac{-z_2}{n}\right), q_3 = \mathbf{e}\left(\frac{z_2 + z_3}{n}\right),$$

and construct affine torus embeddings:

$$\begin{aligned} \mathcal{T}_{\sigma_3} &= \mathit{Spec}\ \mathbf{C}[q_1, q_2, q_3] = \{t = (t_1, t_2, t_3)\} \\ &\supset \mathcal{T}_{\sigma_2} = \{t \in \mathcal{T}_{\sigma_3}; t_2 \neq 0\} \\ &\supset \mathcal{T}_{\sigma_1} = \{t \in \mathcal{T}_{\sigma_3}; t_1 \neq 0,\ t_2 \neq 0\} \\ &\supset \mathcal{T}_{\sigma_0} = \{t \in \mathcal{T}_{\sigma_3}; t_1 \neq 0,\ t_2 \neq 0,\ t_3 \neq 0\} \simeq T_2. \end{aligned}$$

Moreover the orbit for each $\{\sigma_i\}$ is written as

$$\begin{aligned} \mathcal{O}(\sigma_0) &= T_2, \\ \mathcal{O}(\sigma_1) &= \{(t_1, t_2, 0); t_1 \neq 0, t_2 \neq 0\}, \\ \mathcal{O}(\sigma_2) &= \{(0, t_2, 0); t_2 \neq 0\}, \\ \mathcal{O}(\sigma_3) &= \{(0, 0, 0)\}. \end{aligned}$$

We have for instance $\sigma_2 \prec \sigma_3 \Leftrightarrow \mathcal{O}(\sigma_3) \subset \overline{\mathcal{O}(\sigma_2)}$ etc.

Now we consider the structure of the D-V compactification over a Satake rational boundary.

(1) $\pi^{-1}(V_2(n)) = V_2(n)$.

(2) $\pi^{-1}(V^{(1)})$ $(g_0 = g_1 = 1)$,

$$\begin{aligned} \mathcal{O}_{\Sigma_1} &= \bigcup_{\sigma \cap \mathcal{Q}^+ \neq \phi, \sigma \in \Sigma_1} \mathrm{H}_1 \times \mathbf{C} \times \mathcal{O}(\sigma) \\ &= \mathrm{H}_1 \times \mathbf{C} \times \mathcal{O}(\sigma_1) = \mathrm{H} \times \mathbf{C} \times \{t_3 = 0\}. \end{aligned}$$

Therefore, in $\pi^{-1}(V^{(1)}) = \bigcup \overline{P_1(n)} \backslash \mathcal{O}_{\Sigma_1}$, the number of degenerating coordinates is ≤ 1.

(3) $\pi^{-1}(V^{(0)})$ $(g_0 = 0, g_1 = 2)$,

$$\begin{aligned} \mathcal{O}_{\Sigma_2} &= \bigcup_{\sigma \cap \mathcal{Q}_2^+ \neq \phi, \sigma \in \Sigma_2} \mathcal{O}(\sigma) \\ &= \bigcup_{\{\sigma_2\}} \mathcal{O}(\sigma_2) \cup \bigcup_{\{\sigma_3\}} \mathcal{O}(\sigma_3) \\ &= \bigcup \{t_1 = t_3 = 0\} \cup \bigcup \{t_1 = t_2 = t_3 = 0\}. \end{aligned}$$

Here, $\{\sigma_2\}$ and $\{\sigma_3\}$ denote the set of $GL(2, \mathbf{Z})$-transformations of σ_2 and σ_3 respectively. Hence, in $\pi^{-1}(V^{(0)}) = \bigcup \overline{P_0(n) \backslash \mathcal{O}}$, the number of degenerating coordinates is ≤ 3. Also $D^{[2]}$ is a disjoint union of $\mathbf{P}^1$'s whose image in $\widetilde{V}_g(n)$ intersect with each other at $i_3(D^{[3]}) = \{\text{points}\}$.

Example 2. $g = 3$.

$V_3(n)^* = V_3(n) \amalg V^{(2)} \amalg V^{(1)} \amalg V^{(0)}$. Each cone of the D-V decomposition of Σ_3 is transformed into one of the followings by the action of $GL(3, \mathbf{Z})$.

$$\begin{aligned} \sigma_0 &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \sigma_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \ \lambda \geq 0 \right\}, \\ \sigma_2 &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}; \ \lambda_1, \lambda_2 \geq 0 \right\}, \\ \sigma_3 &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 + \lambda_2 & -\lambda_2 \\ 0 & -\lambda_2 & \lambda_2 + \lambda_3 \end{pmatrix}; \ \lambda_1, \lambda_2, \lambda_3 \geq 0 \right\}, \\ \sigma_4 &= \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}; \ \lambda_1, \lambda_2, \lambda_3 \geq 0 \right\}, \\ \sigma_5 &= \left\{ \begin{pmatrix} \lambda_4 & 0 & 0 \\ 0 & \lambda_1 + \lambda_2 & -\lambda_2 \\ 0 & -\lambda_2 & \lambda_2 + \lambda_3 \end{pmatrix}; \ \begin{matrix} \lambda_1, \lambda_2 \\ \lambda_3, \lambda_4 \end{matrix} \geq 0 \right\}, \\ \sigma_6 &= \left\{ \begin{pmatrix} \lambda_1 + \lambda_4 & -\lambda_1 & 0 \\ -\lambda_1 & \lambda_1 + \lambda_2 & -\lambda_2 \\ 0 & -\lambda_2 & \lambda_2 + \lambda_3 \end{pmatrix}; \ \begin{matrix} \lambda_1, \lambda_2 \\ \lambda_3, \lambda_4 \end{matrix} \geq 0 \right\}, \end{aligned}$$

$$\sigma_7 = \left\{ \begin{pmatrix} \lambda_4+\lambda_5 & -\lambda_5 & 0 \\ -\lambda_5 & \lambda_1+\lambda_2+\lambda_5 & -\lambda_2 \\ 0 & -\lambda_2 & \lambda_2+\lambda_3 \end{pmatrix} ; \begin{array}{c} \lambda_1, \lambda_2, \lambda_3 \\ \lambda_4, \lambda_5 \end{array} \geq 0 \right\},$$

$$\sigma_8 = \left\{ \begin{pmatrix} \lambda_4+\lambda_5+\lambda_6 & -\lambda_5 & -\lambda_6 \\ -\lambda_5 & \lambda_1+\lambda_2+\lambda_5 & -\lambda_2 \\ -\lambda_6 & -\lambda_2 & \lambda_2+\lambda_3+\lambda_6 \end{pmatrix} ; \begin{array}{c} \lambda_1, \lambda_2, \lambda_3 \\ \lambda_4, \lambda_5, \lambda_6 \end{array} \geq 0 \right\}.$$

As in the case of $g = 2$, we consider subdecomposition $\Sigma_0 = \{\sigma_0\}$, $\Sigma_1 = \{\sigma_0, \sigma_1\}$, $\Sigma_2 = \{GL(2, \mathbf{Z})$-transformations of $\sigma_0, \sigma_1, \sigma_2\}$.

For each generator of $\sigma_8 \cap \overline{\mathcal{Q}_3^+}$ over $\mathbf{Z}$,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

the dual bases as in case $g = 2$ are

$$\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1/2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix}.$$

Then we set as coordinates:

$$q_1 = \mathbf{e}\left(\frac{z_1+z_2+z_3}{n}\right),\ q_2 = \mathbf{e}\left(\frac{-z_2}{n}\right),\ q_3 = \mathbf{e}\left(\frac{-z_3}{n}\right),$$

$$q_4 = \mathbf{e}\left(\frac{z_2+z_4+z_5}{n}\right),\ q_5 = \mathbf{e}\left(\frac{-z_5}{n}\right),\ q_6 = \mathbf{e}\left(\frac{z_3+z_5+z_6}{n}\right).$$

Now we get an affine torus embedding:

$$\mathcal{T}_{\sigma_8} = Spec\ \mathbf{C}[q_1, q_2, q_3, q_4, q_5, q_6] \simeq \{t = (t_1, t_2, t_3, t_4, t_5, t_6)\}.$$

Also the orbits associated with cones are

$$\begin{aligned}
\mathcal{O}(\sigma_0) &= T_3 \\
\mathcal{O}(\sigma_1) &= \{(t_1, t_2, t_3, t_4, t_5, 0\ );\ t_i \neq 0,\ i = 1,2,3,4,5\} \\
\mathcal{O}(\sigma_2) &= \{(t_1, t_2, t_3, 0\ , t_5, 0\ );\ t_i \neq 0,\ i = 1,2,3,5\} \\
\mathcal{O}(\sigma_3) &= \{(t_1, t_2, t_3, 0\ , 0\ , 0\ );\ t_i \neq 0,\ i = 1,2,3\} \\
\mathcal{O}(\sigma_4) &= \{(0\ , t_2, t_3, 0\ , t_5, 0\ );\ t_i \neq 0,\ i = 2,3,5\} \\
\mathcal{O}(\sigma_5) &= \{(0\ , t_2, t_3, 0\ , 0\ , 0\ );\ t_2 \neq 0,\ t_3 \neq 0\} \\
\mathcal{O}(\sigma_6) &= \{(0\ , 0\ , t_3, t_4, 0\ , 0\ );\ t_3 \neq 0,\ t_4 \neq 0\} \\
\mathcal{O}(\sigma_7) &= \{(0\ , 0\ , t_3, 0\ , 0\ , 0\ );\ t_3 \neq 0\} \\
\mathcal{O}(\sigma_8) &= \{(0\ , 0\ , 0\ , 0\ , 0\ , 0\ )\}.
\end{aligned}$$

(1) $\pi^{-1}(V_3(n)) = V_3(n)$.
(2) $\pi^{-1}(V^{(2)})$ $(g_0 = 2,\ g_1 = 1)$,

$$\begin{aligned}
\mathcal{O}_{\Sigma_1} &= \bigcup_{\sigma \cap \mathcal{Q}_1^+, \sigma \in \Sigma_1} \mathrm{H} \times V_{2,1} \times \mathcal{O}(\sigma) \\
&= \mathrm{H}_2 \times V_{2,1} \times \mathcal{O}(\sigma_1) \\
&= \mathrm{H}_2 \times V_{2,1} \times \{t_6 = 0\}.
\end{aligned}$$

Hence in $\pi^{-1}(V^{(2)}) = \bigcup \overline{P_2(n)} \backslash \mathcal{O}_{\Sigma_1}$, the number of degenerating coordinates is ≤ 1.
(3) $\pi^{-1}(V^{(1)})$ $(g_0 = 1, g_2 = 2)$,

$$\begin{aligned}
\mathcal{O}_{\Sigma_2} &= \bigcup_{\sigma \cap \mathcal{Q}_2^+, \sigma \in \Sigma_2} \mathrm{H}_1 \times V_{1,2} \times \mathcal{O}(\sigma) \\
&= \bigcup_{\{\sigma_2\}} \mathrm{H}_1 \times V_{1,2} \times \mathcal{O}(\sigma_2) \cup \bigcup_{\{\sigma_3\}} \mathrm{H}_1 \times V_{1,2} \times \mathcal{O}(\sigma_3) \\
&= \bigcup \mathrm{H}_1 \times V_{1,2} \times \{t_4 = t_6 = 0\} \cup \bigcup \mathrm{H}_1 \times V_{1,2} \times \{t_4 = t_5 = t_6 = 0\}.
\end{aligned}$$

Hence in $\pi^{-1}(V^{(1)}) = \bigcup \overline{P_1(n)} \backslash \mathcal{O}_{\Sigma_2}$, the number of degenerating coordinates is ≤ 3.
(4) $\pi^{-1}(V^{(0)})$ $(g_0 = 0, g_1 = 3)$,

$$\mathcal{O}_{\Sigma_3} = \bigcup_{\sigma \cap \mathcal{Q}_3^+, \sigma \in \Sigma_3} \mathcal{O}(\sigma)$$

$$
\begin{aligned}
&= \bigcup_{\{\sigma_4\}} \mathcal{O}(\sigma_4) \cup \bigcup_{\{\sigma_5\}} \mathcal{O}(\sigma_5) \cup \bigcup_{\{\sigma_6\}} \mathcal{O}(\sigma_6) \cup \bigcup_{\{\sigma_7\}} \mathcal{O}(\sigma_7) \cup \bigcup_{\{\sigma_8\}} \mathcal{O}(\sigma_8) \\
&= \bigcup \{t_1 = t_4 = t_6 = 0\} \cup \bigcup \{t_1 = t_4 = t_5 = t_6 = 0\} \\
&\cup \bigcup \{t_1 = t_2 = t_5 = t_6 = 0\} \cup \bigcup \{t_1 = t_2 = t_4 = t_5 = t_6 = 0\} \\
&\cup \bigcup \{t_1 = t_2 = t_3 = t_4 = t_5 = t_6 = 0\}.
\end{aligned}
$$

Hence in $\pi^{-1}(V^{(0)}) = \overline{P_0(n)} \backslash \mathcal{O}_{\Sigma_3}$, the number of degenerating coordinates is ≤ 6.

REMARK 6.2. We can obtain that $D^{[N-1]}$ is a union of $\mathbf{P}^1_{\mathbf{C}}$. Indeed the components of $D^{[N-1]}$ correspond to those $N-1$ dimensional cones in Σ_g. By the construction of the partial compactification each components of $D^{[N-1]}$ contains an affine line $\mathbf{A}^1$ as Zariski dense subset. Then we obtain our assertion.

7. Eisenstein series

In this section we review the some basic facts on holomorphic Eisenstein series. These are used in the proof of main results in §8 and §9.

7.1. Siegel Eisenstein series of higher weights

Let $\{e_0, \cdots, e_r\}$ be the set of all 0-dimensional cusps in $V_g(n)^*$, and choose e_0 to be the standard one. Fix an element $\gamma_i \in \Gamma_g$ such that $e_i = \gamma_i e_0$. Then we have only to construct holomorphic Siegel modular forms of weight $g+1$ for $\Gamma_g(n)$ on H_g such that the constant term of its Fourier expansion at some e_i does not vanish. Equivalently we construct holomorphic Siegel modular forms of weight $g+1$ such that $\Phi_i^g f = a_0(e_i) \neq 0$, here Φ_i is the Siegel operator at e_i.

We consider Siegel Eisenstein series of weight k for $\Gamma_g(n)$ on H_g associated with each e_i. It is defined for $k > g+1,\ Z \in \mathrm{H}_g$ as

$$E_{e_i}(Z; k) = \sum_{\sigma \in \Gamma_g(n) \cap P_i \backslash \Gamma_g(n)} j(\gamma_i^{-1}\sigma, Z)^{-k},$$

where $j(\sigma, Z) = \det(CZ + D)$, $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n)$, $P_i = \gamma_i P_0 \gamma_i^{-1}$, and P_0 is the stabilizer of e_0 in $Sp(g, \mathbf{R})$, that is the standard maximal

$\mathbf{Q}$-parabolic subgroup:

$$P_0 = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp(g, \mathbf{R}),\ A, D \in GL(g, \mathbf{R}) \right\}.$$

PROPOSITION 7.1. (i) *For $k > g+1$, the above infinite series is absolutely convergent, and it defines a holomorphic modular form of weight k for $\Gamma_g(n)$. We call this series Siegel Eisenstein series of weight k.*
(ii) *The constant term of the Fourier expansion of the series at the cusp e_i of $E_{e_i}(Z;k)$ does not vanish, and the constant terms at the other cusps e_j not $\Gamma_g(n)$-equivalent to e_i are equal to zero.*

PROOF. Statements for the absolutely convergence and the constant term of Fourier expansion at e_i are well known (Maass [13, §14]). We prove the last statement. Now we define a relation for elements of Γ_g.

$$\text{For } N_1,\ N_2 \in \Gamma_g,\ N_1 \overset{n}{\sim} N_2 \quad \Leftrightarrow \quad \exists M \in \Gamma_g(n) \text{ such that } N = N_1^{-1} M N_2 \in P_0 \cap \Gamma_g.$$

Then we claim that:
For $\gamma_i,\ \gamma_j \in \Gamma_g$ which are not equivalent to each other and $\gamma_i e_0 = e_i,\ \gamma_j e_0 = e_j$, the constant term of the Fourier expansion at e_j of $E_{e_i}(Z,k)$ is equal to zero. That is,

$$\Phi^g\Big(\sum_{\sigma \in \Gamma_g(n) \cap P_i \backslash \Gamma_g(n)} j(\gamma_i^{-1}\sigma, Z)^{-k} | \gamma_j\Big) = 0.$$

Proof of the claim.

$$\Phi^g\Big(\sum_{\sigma \in \Gamma_g(n) \cap P_i \backslash \Gamma_g(n)} j(\gamma_i^{-1}\sigma, Z)^{-k} | \gamma_j\Big) = \Phi^g\Big(j(\gamma_j, Z)^{-k} \sum j(\gamma_i^{-1}\sigma, \gamma_j\langle Z\rangle)^{-k}\Big)$$

$$= \lim_{\lambda \to \infty} \sum j(\gamma_i^{-1}\sigma\gamma_j, \sqrt{-1}\lambda \mathbf{1}_g)^{-k} = \sum \lim_{\lambda \to \infty} j(\gamma_i^{-1}\sigma\gamma_j, \sqrt{-1}\lambda \mathbf{1}_g)^{-k}.$$

We put $\gamma_i^{-1}\sigma\gamma_j = N_\sigma = \begin{pmatrix} A_\sigma & B_\sigma \\ C_\sigma & D_\sigma \end{pmatrix} \in \Gamma_g$ (N_σ depends on σ under fixed γ_i, γ_j). Then $j(\gamma_i^{-1}\sigma\gamma_j, \sqrt{-1}\lambda\mathbf{1}_g) = \det(\sqrt{-1}\lambda C_\sigma + D_\sigma)$ is a polynomial in λ (including the case of degree 0). And if the degree of this polynomial is greater than 0, then we have $\lim_{\lambda\to\infty} |\det(\sqrt{-1}\lambda C_\sigma + D_\sigma)| \to \infty$ as $\lambda \to \infty$. Hence in the above infinite series the term associated to this N_σ is equal to zero. Therefore we want to know in which case $\det(\sqrt{-1}\lambda C_\sigma + D_\sigma)$ is a constant independent of λ.

Because $N_\sigma \in \Gamma_g$, we have $C_\sigma {}^t D_\sigma = D_\sigma {}^t C_\sigma$ for C_σ, D_σ (It means that C_σ and D_σ make a symmetric pair). Suppose rank $C_\sigma = r$, then we can take $U_1, U_2 \in GL(g, \mathbf{Z})$ such that $U_1 C_\sigma = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} {}^t U_2$, where C_1 is a $r \times r$ matrix and its determinant is not zero. Write formally as $U_1 D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} U_2^{-1}$, then we get $U_1 C_\sigma {}^t(U_1 D_\sigma) = U_1 C_\sigma {}^t D_\sigma {}^t U_1 = U_1 D_\sigma {}^t(U_1 C_\sigma)$. This means that $U_1 C_\sigma$ and $U_1 D_\sigma$ is also a symmetric pair. In particular, since $\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {}^t D_1 & {}^t D_2 \\ {}^t D_3 & {}^t D_4 \end{pmatrix} = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \begin{pmatrix} {}^t C_1 & 0 \\ 0 & 0 \end{pmatrix}$, we get $C_1 {}^t D_1 = D_1 {}^t C_1$ and $D_3 = 0$. Under this consideration, we investigate $\det(\sqrt{-1}\lambda C_\sigma + D_\sigma)$ in the followings. Our consideration is separated into three cases.

(*Case* 1) rank $C_\sigma = g$.

In this case, since $\det(\sqrt{-1}\lambda C_\sigma + D_\sigma) = \det C_\sigma \det(\sqrt{-1}\lambda \mathbf{1}_g + C_\sigma^{-1} D_\sigma)$ and $\det(\sqrt{-1}\lambda \mathbf{1}_g + C_\sigma^{-1} D_\sigma)$ has non-zero degree g-term, we obtain that $|\det(\sqrt{-1}\mathbf{1}_g C_\sigma + D_\sigma)| \to \infty$ as $\lambda \to \infty$.

(*Case* 2) $0 < \text{rank } C_\sigma = k \le g - 1$.

First we have

$$\begin{aligned} 0 &\neq \det((\sqrt{-1}\lambda C_\sigma + D_\sigma) \\ &= \det U_1^{-1} \det(\sqrt{-1}\lambda \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} {}^t U_2 + \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix} U_2^{-1}). \end{aligned}$$

where $C_1 \in M_k(\mathbf{Z})$ is of rank k, $D_2 \in M_{g-k,k}(\mathbf{Z})$, and $\det D_4 \neq 0$. Moreover we have that

$$\begin{aligned} &\det(\sqrt{-1}\lambda \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} {}^t U_2 + \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix} U_2^{-1}) \\ = &\det(\sqrt{-1}\lambda \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} {}^t U_2 U_2 + \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix}) \det U_2^{-1}. \qquad (*) \end{aligned}$$

Here we considerthe matrix ${}^tU_2U_2 = \begin{pmatrix} V & * \\ * & * \end{pmatrix}$, with ${}^tV = V \in M_k(\mathbf{Z})$, for $U_2 \in GL(g, \mathbf{Z})$. Then the $k \times k$ symmetric matrix V is of full rank. Indeed if V is not invertible then take a nonzero vector v in Ker$V$, and make $w = (v, 0) \in \mathbf{R}^g$. Then ${}^tw{}^tU_2U_2w = |U_2w|^2 = 0$, for $w \neq 0, \in \mathbf{R}^g$, which contradicts to the condition that $\det U_2 \neq 0$. Therefore we obtain that

$$\begin{aligned} (*) &= \det(\sqrt{-1}\lambda \begin{pmatrix} C_1V & * \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix}) \times (\text{nonzero constant}) \\ &= \det(\sqrt{-1}\lambda C_1 V + D_1) \times (\text{nonzero constant}) \\ &= \det(\sqrt{-1}\lambda \mathbf{1}_k + *) \times (\text{nonzero constant}). \end{aligned}$$

(We remark that $\det C_1V \neq 0$ in the last equality.)

Thus, since $\det(\sqrt{-1}\lambda\mathbf{1}_k + *)$ has a nontrivial term of degree k of λ, the term $|j(N_\sigma, \sqrt{-1}\lambda\mathbf{1}_g)^{-k}|$ associated with this N_σ vanishes, when $\lambda \to \infty$.

(*Case* 3) rank $C_\sigma = 0$.

This means $\gamma_i^{-1}\sigma\gamma_j \in P_0 \cap \Gamma_g$. Since $\sigma \in \Gamma_g(n)$, it contradicts to that γ_i and γ_j is not equivalent to each other.

These prove the above claim. □

PROOF OF THE PROPOSITION 7.1. Now we take two cusps $e_i = \gamma_i e_0$ and $e_j = \gamma_j e_0$ which are not $\Gamma_g(n)$-equivalent with each other. Suppose $\gamma_i \overset{n}{\sim} \gamma_j$, thus there exists an element $M \in \Gamma_g(n)$ such that $N = \gamma_i^{-1}M\gamma_j \in P_0 \cap \Gamma_g$. Then we have the following equality for those maximal $\mathbf{Q}$-parabolic subgroup corresponding to each of $e_i,\ e_j$

$$P_j = \gamma_j P_0 \gamma_j^{-1} = M^{-1}\gamma_i N P_0 N^{-1}\gamma_i^{-1}M = M^{-1}P_iM,$$

which contradicts to the choice of e_i and e_j. Therefore, together with above claim, the proposition is proved. □

7.2. Eisenstein series of low weight

As far as $Gr^W_{2N}\mathrm{H}^N(V_g(n), \mathbf{C})$ is concerned, what we want to obtain is a holomorphic modular form of weight $g+1$. But for $k = g+1$, above infinite series which defines $E_{e_i}(Z, k)$ does not converge. Therefore some modification is needed for above series to make sense. For this purpose we consider a series $E_{e_i}(Z,s;g+1)$ as below. This infinite series of $Z \in \mathrm{H}_g$, $s \in$ **C** is defined as

$$E_{e_i}(Z,s;g+1) = \sum_{\sigma \in \Gamma_g(n)\cap P_i \backslash \Gamma_g(n)} j(\gamma_i^{-1}\sigma, Z)^{-(g+1)} |j(\gamma_i^{-1}\sigma, Z)|^{-s}.$$

This series absolutely converges for Re $s > 0$, and is known to be meromorphically continued to all s-plane (after some modification [22, p.426], then by a theory of Langlands [12], Arthur [2]). Moreover, we recall some theorems of Shimura ([22, Theorem (7.1)]).

Theorem 7.2. (Shimura)
(i) *If* $g \geq 2$, $E_{e_i}(Z,s;g+1)$ *is holomorphic in* s *at* $s=0$, *and moreover* $E_{e_i}(Z,s;g+1)|_{s=0}$ *defines a holomorphic function of* $Z \in \mathrm{H}_g$.
(ii) *The same result as in* (ii) *of Proposition (7.1) is valid for the constant terms of the Fourier expansions of* $E_{e_i}(Z,s;g+1)|_{s=0}$ *at* 0*-dimensional cusps.*

Shimura proved this theorem by explicit calculations of the Fourier coefficients of some related Eisenstein series at some cusp (image of an intertwining operator) and showing its analytic continuation. He also get some other statement on Eisenstein series (see remarks below). We remark, in the case of $g=1$, Eisenstein series of weight 2 of above type (without non-trivial character) does not define a holomorphic function of $Z \in \mathrm{H}_g$ at $s=0$. (This is, in other terms, the residue theorem for a Riemann surface.) The holomorphy of above Eisenstein series at $s=0$ is a property of Siegel ($g \geq 2$) modular forms.

8. Weight filtration of $\mathrm{H}^N(V,\mathbf{C})$

Finally we state some results about the graded quotients $Gr_\bullet^W \mathrm{H}^N(V_g(n), \mathbf{C})$. $N = \frac{1}{2}g(g+1)$ is the dimension of $V_g(n)$. For some related other results, in particular in the case of $g=2$, see also Oda-Schwermer [19]. By the definition of $W_\bullet$ on $\Omega^\bullet_{\widetilde{V}}(logD)$, we obtain that $W_m\mathrm{H}^N(V_g(n),\mathbf{C}) = 0$ for $m \leq N$, and $W_{2N}\mathrm{H}^N(V_g(n),\mathbf{C}) = \mathrm{H}^N(V_g(n),\mathbf{C})$. By Deligne [5] Cor(3.2.17), we also know that $W_N\mathrm{H}^N(V_g(n),\mathbf{C})$ is the image of the natural map: $\mathrm{H}^N(\widetilde{V}_g(n),\mathbf{C}) \to \mathrm{H}^N(V_g(n),\mathbf{C})$.

Theorem 8.1. *Assume that* $g \geq 2$.

(i) *The dimension of the space* $Gr^W_{2N}\mathrm{H}^N(V_g(n), \mathbf{C})$ *is equal to the number of 0-dimensional rational cusps in* $V_g(n)^*$. *The corresponding classes to this space are constructed by the global automorphic Siegel Eisenstein series of weight* $g+1$ *for* $\Gamma_g(n)$ *on* H_g.
(ii) $Gr^W_{2N-1}\mathrm{H}^N(V_g(n), \mathbf{C}) = \{0\}$.

PROOF. First we prove (i). We consider induced residue map:

$$Res_{[N]} : \mathbf{H}^N(\widetilde{V}_g(n), Gr^W_N \Omega^{\bullet}_{\widetilde{V}}(logD)) \simeq \mathrm{H}^0(D^{[N]}, \mathbf{C})(-N).$$

In the right hand side of above isomorphism, $D^{[N]} = \{\text{points}\}$ is a finite union of points. Hence,

$$\mathrm{H}^0(D^{[N]}, \mathbf{C})(-N) = \coprod_{p \in D^{[N]}} \mathbf{C}(-N) \text{ a direct sum of } \mathbf{C}(-N).$$

Therefore the Hodge type of this space is only of (N, N), and the whole space is itself an edge component. Then from Lemma (3.1) we can consider a restriction of $Res_{[N]}$ to this subspace:

$$Res_{[N]} : Gr^W_{2N}\mathrm{H}^N(V_g(n), \mathbf{C}) \to \mathrm{H}^0(D^{[N]}, \mathbf{C})(-N),$$

which defines an isomorphism into its image. Also as in §3, we consider the map $Res_{[n]}$ on $\Gamma(\widetilde{V}_g(n), \Omega^N_{\widetilde{V}}(logD))$. By §6, $i_N(D^{[N]})$ must be all contained within the fiber of π over 0-dimensional cusps of $V_g(n)^*$.

Now we apply Theorem (7.2) to $E_{e_i}(Z, s=0; g+1)$. Then for $\tilde{e} \in i(D^{[n]})$ we obtain that

$$\begin{aligned} &\tilde{e}\text{-component of } Res_{[N]}\, E_{e_i}(Z, 0\,; g+1) \\ &= \begin{cases} a_0(E_{e_i}(Z, 0; g+1); e_i) \neq 0, & \text{if } \pi(\tilde{e}) = e_i \\ 0, & \text{if } \pi(\tilde{e}) = e_j \neq e_i \end{cases} \end{aligned}$$

For $f \in M_{g+1}(\Gamma_g(n))$, we have shown that $Res_{[N]}f \in \coprod_{\tilde{e} \in i(D^{[N]})} \mathbf{C}(-N)$ is determined by the constant terms of Fourier expansions at $\pi(\tilde{e})$'s. On the other hand, in the above consideration we have attached to each of cusps in $V_g(n)^*$ one Eisenstein series with nonzero residue. Thus we get (i) of the theorem.

We want to prove (ii). Recall that $D^{[N]}$ is a union of boundary components of codimension $N-1$ of a toric variety of dimension $N = \frac{1}{2}g(g+1)$, and is a union of $\mathbf{P}^1$'s (see Remark 6.2. For $g = 2, 3$, see Igusa [11], Namikawa [16]). Then in the residue map:

$$Res_{[N-1]} : \mathbf{H}^N(\widetilde{V}_g(n), Gr^W_{N-1}\Omega^{\bullet}_{\widetilde{V}}(logD)) \simeq \mathrm{H}^1(D^{[N-1]}, \mathbf{C})(-N+1),$$

we have that the target space:

$$\mathrm{H}^1(D^{[N-1]}, \mathbf{C})(-N+1) \simeq \coprod \mathrm{H}^1(\mathbf{P}^1, \mathbf{C})(-N+1) = 0.$$

Therefore we obtain (ii). □

Remarks.

(1) For lower weight quotient it might be need to consult with properties of something like Klingen Eisenstein series as nearly holomorphic forms studied by Shimura.

(2) We know that Fourier coefficients of above Siegel Eisenstein series are all in $\mathbf{Q}^{ab}$ by the results of Shimura [22, Theorem (7.1)]. This arithmetic structure of Fourier coefficients should be compatible with the rational structure of the de Rham realization of the mixed Hodge structure.

(3) The following is suggested by T.Oda. Start with the exact sequence of relative cohomology:

$$\cdots \to \mathrm{H}^N(V_g(n), \mathbf{Q}) \to \mathrm{H}^N(\partial\overline{V}_g(n), \mathbf{Q}) \to \mathrm{H}^{N+1}_c(V_g(n), \mathbf{Q}) \to \cdots,$$

where $\overline{V}_g(n)$ is the Borel-Serre compactification of the Siegel modular variety, $\partial\overline{V}_g(n)$ its boundary, and $\mathrm{H}^*_c(*)$ means the cohomology with compact support. Then we can consider the above sequence as an exact sequence of mixed Hodge structure thanks to the theory of mixed Hodge structure on the cohomology of links (A.H.Durfee and M.Saito [8]). Then we have that the term $\mathrm{H}^N(V_g(n), \mathbf{Q})$ has weight $N, \cdots, 2N$. On the other hand $\mathrm{H}^{N+1}_c(V_g(n), \mathbf{Q})$ has weight $2, \cdots, N+1$. (This is a dual of $\mathrm{H}^{N-1}(V_g(n), \mathbf{Q})$ with weight $N-1, \cdots, 2N-2$.) Then the term $\mathrm{H}^N(\partial\overline{V}_g(n), \mathbf{Q})$ has possible weight $2, \cdots, 2N$. Hence, for the weight compatibility of above exact sequence, we conclude that $\mathrm{H}^N(V_g(n), \mathbf{Q})$ maps surjectively to those spaces with weights $N, \cdots, 2N$ derived from $\mathrm{H}^N(\partial\overline{V}_g(n), \mathbf{Q})$.

9. An appendix: On a mixed Hodge structure of universal family over $V_2(n)$

Let $A \xrightarrow{f} V_2(n)$ be the universal family of principal polarized abelian varieties of dimension 2 with level n structure. The f is a proper smooth morphism and a fiber over Z mod $\Gamma_2(n)$ is the complex torus $\mathbf{C}^2/(1, Z)(\mathbf{Z})^4$. Denote by e the canonical 0-section of f. The family A is an open quasi-projective variety over $\mathbf{C}$ of dimension 5. We realize a compactification $\widetilde{A}$ of A as a smooth irreducible divisor in $\widetilde{V}_3(n) - V_3(n)$ (Namikawa [16]). In terms of §6 example (2), its local defining equation in $\widetilde{V}_3(n)$ is given by $\{t_6 = 0\}$. It is a smooth compactification of A. We can write $\widetilde{A} - A = \bigcup_i Y_i$ with each Y_i a smooth irreducible divisor of $\widetilde{A}$. The structure of $Y^{[m]}$ is following.

$m = 0$ $\quad Y^{[0]} = \widetilde{P}_2(n)\backslash \mathrm{H}_2 \times V_{2,1} \times \mathcal{O}(\sigma_1) = \widetilde{P}_2(n)\backslash \mathrm{H}_2 \times V_{2,1}.$

$m = 1$ $\quad Y^{[1]} = \bigcup \widetilde{P}_1(n)\backslash \mathrm{H}_1 \times V_{1,2} \times \mathcal{O}(\sigma_2).$

$$\mathcal{O}(\sigma_2) = \{t_4 = t_6 = 0\} \subset \mathbf{C}^3 = \{(t_4, t_5, t_6)\}.$$

$m = 2$ $\quad Y^{[2]} = \bigcup \widetilde{P}_1(n)\backslash \mathrm{H}_1 \times V_{1,2} \times \mathcal{O}(\sigma_3) \cup \bigcup \widetilde{P}_0(n)\backslash \mathcal{O}(\sigma_4).$

$$\mathcal{O}(\sigma_3) = \{t_4 = t_5 = t_6 = 0\} \subset \mathbf{C}^3 = \{(t_4, t_5, t_6)\},$$
$$\mathcal{O}(\sigma_4) = \{t_1 = t_4 = t_6 = 0\} \subset \mathbf{C}^6 = \{(t_1, t_2, t_3, t_4, t_5, t_6)\}.$$

$m = 3$ $\quad Y^{[3]} = \bigcup \widetilde{P}_0(n)\backslash \mathcal{O}(\sigma_5) \cup \bigcup \widetilde{P}_0(n)\backslash \mathcal{O}(\sigma_6),$

$$\mathcal{O}(\sigma_5) = \{t_1 = t_4 = t_5 = t_6 = 0\} \subset \mathbf{C}^6,$$
$$\mathcal{O}(\sigma_6) = \{t_1 = t_2 = t_5 = t_6 = 0\} \subset \mathbf{C}^6.$$

$m = 4$ $\quad Y^{[4]} = \bigcup \widetilde{P}_0(n)\backslash \mathcal{O}(\sigma_7).$

$$\mathcal{O}(\sigma_7) = \{t_1 = t_2 = t_4 = t_5 = t_6 = 0\} \subset \mathbf{C}^6.$$

$m = 5$ $\quad Y^{[5]} = \bigcup \widetilde{P}_0(n)\backslash \mathcal{O}(\sigma_8),$

$$\mathcal{O}(\sigma_8) = \{t_1 = t_2 = t_3 = t_4 = t_5 = t_6 = 0\}.$$

The $Y^{[1]}$ above is a fiber space over $V_1(n)$ whose fiber is an extension by $\mathbf{P}^1$'s of 2-copies of an elliptic curve $E \simeq \mathbf{C}^2/(1,z)(\mathbf{Z})^2, z \bmod \Gamma_1(n) \in V_1(n)$.

THEOREM 9.1. *In the above case, we have that*
(i) *the dimension of the space of* $Gr^W_{10}\mathrm{H}^5(A,\mathbf{C})$ *is equal to the number of* 0*-dimensional cusps in* $V_2(n)^*$. *And its cohomology classes are constructed by using the global Siegel Eisenstein series of degree two and weight four.*
(ii) $Gr^W_9\mathrm{H}^5(A,\mathbf{C}) = \{0\}$.

PROOF. (i). Since $\mathrm{H}^0(Y^{[5]},\mathbf{C})(-5)$ has only edge components, we can restrict residue map on $Gr^W_{10}\mathrm{H}^5(A,\mathbf{C})$ by Lemma (3.1).

$$Res_{[5]} : Gr^W_{10}\mathrm{H}^5(A,\mathbf{C}) \to \mathrm{H}^0(Y^{[5]},\mathbf{C})(-5).$$

We may consider $Res_{[5]}$ (the same symbol as above) the canonical map from $\Gamma(\widetilde{A},\ \Omega^5_A(logY))$ to $\mathrm{H}^0(Y^{[5]},\mathbf{C})(-5)$ as before (Lemma (3.2)).

We first consider $\Gamma(A,\Omega^5_A)$. For f is a smooth morphism, one has an exact sequence

$$0 \to f^*\Omega^1_{V_2(n)} \to \Omega^1_A \to \Omega^1_{A/V} \to 0.$$

Then $\Omega^5_A \simeq \bigwedge^3 f^*\Omega^1_V \otimes \bigwedge^2 \Omega^1_{A/V} \simeq f^*\Omega^3_V \otimes \bigwedge^2 \Omega^1_{A/V}$. Therefore $\Gamma(A,\Omega^5_A) \simeq \Gamma(V, f_*\Omega^5_A) \simeq \Gamma(V,\Omega^3_V \otimes f_* \bigwedge^2 \Omega^1_{A/V})$. Since each fiber of f is an abelian variety, its space of invariant differential is identified with the cotangent space at e. Hence we have $f^*e^*\Omega^1_{A/V} \simeq \Omega^1_{A/V}$. Moreover as f is proper and its (geometric) fiber is connected, $f_*\mathcal{O}_A \simeq \mathcal{O}_V$. Then we conclude the following isomorphisms:

$$f_*\Omega^1_{A/V} \simeq f_*(f^*e^*\Omega^1_{A/V} \otimes \mathcal{O}_A) \simeq e^*\Omega^1_{A/V}. \qquad (*)$$

Here $\omega_{A/V} := \bigwedge^2 e^*\Omega^1_{A/V} \simeq e^* \bigwedge^2 \Omega^1_{A/V}$ is an invertible sheaf which defines an automorphic factor (Chai-Faltings [4]). Since we know $\Omega^3_V \simeq \omega^{\otimes 3}_{A/V}$ by Kodaira-Spencer map, $\Gamma(V,\ \Omega^3_V \otimes \omega_{A/V})$ is isomorphic to the holomorphic Siegel modular forms of weight 4: $M_4(\Gamma_2(n))$. Because every element in $M_4(\Gamma_2(n))$ extends holomorphically to rational boundaries (Koecher principle), by above isomorphism $(*)$, one has $\Gamma(\widetilde{A},\Omega^5_{\widetilde{A}}(logY)) \simeq M_4(\Gamma_2(n))$. As in §5, take $\omega_0 = f(z_1, z_2, z_4)dz_1 \wedge dz_2 \wedge dz_4 \wedge d\zeta_1 \wedge d\zeta_2$ for $\omega \in \Gamma(\widetilde{A}, \Omega^5_{\widetilde{A}}(logY))$. Here $z_i,\ i = 1, 2, 4$ are coordinates in $V_2(n)$, $\zeta_1 = z_3$ and $\zeta_2 = z_5$ are coordinates in fiber variety (use the same symbol in §6 example 2). The function $f(Z)$ is in $M_4(\Gamma_2(n))$. In terms of the local coordinates for the smooth compactification given in §6 example 2, we have the following description of ω_0.

$$\omega_0 = C \cdot \{a_0(f) + \sum_{\{T\}} a_T(f)(q_1q_3)^{t_1} \cdot q_2^{t_1+t_4-2t_2} \cdot (q_4q_5)^{t_4}\} \bigwedge_{i=1}^{5} \frac{dq_i}{q_i},$$

where $q_1, \cdots, q_5$ are defined as in the example 2, §6 and $T = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_4 \end{pmatrix}$ are half integral semipositive matrices. Then $\tilde{e}_i$ $(\in Y^{[5]})$-component of $Res_{[5]}\omega$ is equal to $a_0(f)$: the constant term of a Fourier expansion of $f(Z)$ at a 0-dimensional Satake boundary.

We consider the Siegel Eisenstein series of weight 4:

$$E_{e_i}(Z; 4) = \sum_{\sigma \in P_i \cap \Gamma_2(n) \backslash \Gamma_2(n)} j(\gamma_i^{-1}\sigma, Z)^{-4}.$$

This series absolutely converges and satisfies the properties of Proposition (7.1). Thus (i) follows as before in the case of modular variety.

We can prove (ii) in the same way as in §8, Theorem (8.1). Indeed, $Y^{[4]}$ is a union of $\mathbf{P}^1$'s, hence $\mathbf{H}^5(\widetilde{A}, Gr^W_4 \Omega^\bullet_A(logY)) \simeq \mathrm{H}^1(Y^{[4]}, \mathbf{C})(-4) = 0$, which implies (ii) immediately.

We remark that the Leray spectral sequence for $A \xrightarrow{f} V_2(n)$,

$$E_2^{p,q} = \mathrm{H}^p(V_2(n), R^q f_* \mathbf{Q}) \Rightarrow \mathrm{H}^{p+q}(A, \mathbf{Q}),$$

degenerates at E_2-terms (Liebermann's trick). □

REMARK. To see the situation concretely we discussed only the case of $g = 2$ in this section. However after some more work we will also obtain similar results for a Hodge structure of the universal family A_g over $V_g(n),\ g \geq 2$. The results are the followings.
(i) $\dim Gr^W_{2M} \mathrm{H}^M(A_g, \mathbf{C}) =$ the number of 0-dimensional cusps in $V_g(n)^*$.
(ii) $Gr^W_{2M-1} \mathrm{H}^M(A_g, \mathbf{C}) = 0$.

Here $M = \frac{1}{2}g(g+1) + g$ is the dimension of A_g. The proof is completely similarly as the case of $g = 2$.

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