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L^2-theory of singular perturbation of hyperbolic equations III Asymptotic expansions of dispersive type

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Dedicated to Professor Hikosaburo Komatsu on his 60th birthday

Abstract. We consider Cauchy problems for linear strictly hyperbolic equations of order l with a small parameter $\epsilon \in (0, \epsilon_0]$:

$$\begin{aligned}(0.1) \qquad & \{(i\epsilon)^{l-m} L(t, x, D_t, D_x; \epsilon) + M(t, x, D_t, D_x; \epsilon)\} u(t, x; \epsilon) \\ & = f(t, x; \epsilon) \\ & \text{for} \quad (t, x) \in (0, T) \times \boldsymbol{R}^d_x,\end{aligned}$$

$$(0.2) \qquad D_t^j u(0, x; \epsilon) = g_j(x; \epsilon) \quad j = 0, 1, 2, \ldots, l-1$$

where L and M are linear strictly hyperbolic operators of order l and m ($l = m+1$ or $m+2$) with C^∞ bounded derivatives with respect to $(t, x, \epsilon) \in [0, \infty) \times \boldsymbol{R}^d \times [0, \epsilon_0]$. The aim of this paper is to give C^∞ asymptotic expansions of solutions to singularly perturbed Cauchy problems of this type, when the characteristic roots of L and M satisfy the separation conditions. The points are to construct formal solutions (Proposition 5.3, 5.4), consisting of the regular part and the singular one (correction part of dispersive type) expressed by Maslov's canonical operators, and to give the error estimates in order to obtain asymptotic expansions with respect to ϵ in the sense of arbitrarily higher order differentiability norms (Theorem 6.1, 6.2), when the supports of f and g_j's are contained in fixed compact sets.

1. Introduction

We consider Cauchy problems for a linear strictly hyperbolic equation of order l with a small parameter $\epsilon \in (0, \epsilon_0]$:

$$(1.1) \quad P(t, x, D_t, D_x; \epsilon) u(t, x; \epsilon) =$$

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$$\Big((i\epsilon)^{l-m}L(t,x,D_t,D_x;\epsilon)+M(t,x,D_t,D_x;\epsilon)\Big)\,u(t,x;\epsilon)= f(t,x;\epsilon)$$

for $(t,x)\in(0,T)\times \boldsymbol{R}^{\,d}_{\,x}$,

$$D_t^j u(0,x;\epsilon)=g_j(x;\epsilon) \qquad j=0,1,2,\ldots,l-1 \tag{1.2}$$

where L and M are linear strictly hyperbolic operators of order l and m ($l=m+1$ or $m+2$ cf. Ashino [2]) with C^∞ bounded derivatives with respect to $(t,x,\epsilon)\in[0,\infty)\times\boldsymbol{R}^{\,d}\times[0,\epsilon_0]$. The aim of this paper is to give C^∞ asymptotic expansions of solutions to singularly perturbed Cauchy problems of this type. This is a revisit of problems treated in [12].

We assume the data $f(t,x;\epsilon)\in C_0^\infty([0,\infty)\times\boldsymbol{R}^d\times[0,\epsilon_0])$ and $g_j(x;\epsilon)\in C_0^\infty(\boldsymbol{R}^d\times[0,\epsilon_0])$. They have asymptotic expansions with respect to ϵ:

$$f(t,x;\epsilon)=\sum_{n=0}^{N}\epsilon^n f_n(t,x)+R_{N+1}(f;\epsilon), \tag{1.3}$$

$$g_j(x;\epsilon)=\sum_{n=0}^{N}\epsilon^n g_{j,n}(x)+R_{N+1}(g_j;\epsilon). \tag{1.4}$$

We postulate that the solution has an expansion

$$u(t,x;\epsilon)\sim v(t,x;\epsilon)+w(t,x;\epsilon), \tag{1.5}$$

where v and w mean formal sums such that

$$v(t,x;\epsilon) = \sum_{n=0}^{\infty}\epsilon^n v_n(t,x) \quad \text{(regular part)}, \tag{1.6}$$

$$w(t,x;\epsilon) = \sum_{n=m}^{\infty}\epsilon^n w_n(t,x;\epsilon) \quad \text{(singular part)}, \tag{1.7}$$

$$Pv \sim f, \tag{1.8}$$

$$Pw \sim 0 \tag{1.9}$$

$$D_t^j(v+w)\Big|_{t=0} \sim g_j, \quad j=0,1,2,\ldots,l-1. \tag{1.10}$$

We investigated in Part I ([13]) a priori L^2 and higher order Sobolev norm estimates of the solution to (1.1) and (1.2) under various separation conditions of characteristic roots of L and M. In Part II ([14]), we dealt with the case where the singular part, that is, the correction terms (1.7) associated with (1.6) are of dissipative type (exponential decay as ϵ tends to 0). In this paper, we treat the case where the the correction terms are dispersive (highly oscillating as ϵ tends to 0). They are described by oscillating functions locally and by Maslov's canonical operators K_Λ globally. The estimates of the remainder terms of asymptotic expansions are given by a priori estimates in Part I ([13]).

We put

$$
\begin{aligned}
(1.11)\quad & u_N(t,x;\epsilon) \\
&= \begin{cases} \sum_{n=0}^{N} \epsilon^n v_n(t,x) + \sum_{n=m}^{N+m} \epsilon^n K_\Lambda h_n(t,x;\epsilon), & \text{when } l = m+1, \\ \sum_{n=0}^{N} \epsilon^n v_n(t,x) + \sum_{\substack{n=m \\ *=\pm}}^{N+m} \epsilon^n K_{\Lambda^*} h_n^*(t,x;\epsilon), & \text{when } l = m+2 \end{cases}
\end{aligned}
$$

and its remainder term by

$$R_{N+1}(u;\epsilon) = u(t,x;\epsilon) - u_N(t,x;\epsilon).$$

We have (in Propositions 6.1 and 6.2)

$$
\begin{aligned}
(1.12)\quad ((i\epsilon)^{(l-m)}L + M)R_{N+1}(u;\epsilon) &= R_{N+1}(f;\epsilon) \\
&\quad + \epsilon^{N+1}\rho(t,x;\epsilon) + \epsilon^{N+1}\chi(t,x;\epsilon), \\
(1.13)\quad D_t^j R_{N+1}(u;\epsilon)(0,x) &= R_{N+1}(g_j;\epsilon) + \epsilon^{N+1}\eta_j(x;\epsilon), \\
&\quad 0 \le j \le l-1.
\end{aligned}
$$

We apply a priori estimates (Theorem 2.1 or Theorem 2.2) to (1.12) and (1.13) in order to obtain estimates of $R_{N+1}(u;\epsilon)$. Thus, we have our main result Theorem 6.1 and Theorem 6.2. For an arbitrarily higher order Sobolev norm and $\nu \in \boldsymbol{N}$, there exist large number N, such that

$$
\begin{cases} ((i\epsilon)^{(l-m)}L + M)u_N = f + O(\epsilon^\nu) \\ D_t^j u_N = g_j + O(\epsilon^\nu) \end{cases}
$$

and

$$u = u_N(t,x;\epsilon) + O(\epsilon^\nu),$$

where $O(\epsilon^\nu)$'s are measured by the given Sobolev norm.

In §2, we state assumptions and a priori estimates quoted from Part I ([13]). In §3, singular characteristic roots (cf. Frank[5]) are introduced through principal symbol of ϵ-differential operators. They give nonhomogeneous Lagrangian manifolds.

In view point of propagation of waves, the regular part of the solution is governed by the principal part of M (the subcharacteristic wave in Whitham[15]). The singular part is governed by ϵ -principal part of $(i\epsilon)^{l-m}L + M$. In contrast with the propagation of singularity of the solution u, the principal part of L is not *principal* to determine the quantitative propagation of the singularly perturbed wave.

In §4, we give a brief review of canonical operators of Maslov. In §5, we construct each term of formal asymptotic expansions of solutions. In §6, we estimate the error terms of truncated expansion of the solutions, using a priori estimates quoted in §2. Our conclusion is Theorem 6.1 and Theorem 6.2. We have an asymptotic expansion of the solution with respect to ϵ in the sense of arbitrarily higher order local Sobolev norm.

It seems there are not many works on singular perturbation of hyperbolic-hyperbolic type with dispersive correction terms. In Gao [7] similar problems are studied under more restricted conditions than ours.

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2. A priori estimates

We consider two differential operators L and M, whose coefficients have smooth and bounded derivatives in (t, x, ϵ) :

$$(2.1) \quad L(t, x, D_t, D_x; \epsilon) = D_t^l + \sum_{j=1}^{l} L_j(t, x, D_x; \epsilon) D_t^{l-j}$$

$$(2.2) \quad M(t, x, D_t, D_x; \epsilon) = m_0(t, x; \epsilon) D_t^m + \sum_{j=1}^{m} M_j(t, x, D_x; \epsilon) D_t^{m-j}$$

Their homogeneous principal symbols are defined by

$$(2.3)\qquad l(t,x,\tau,\xi;\epsilon) = \tau^l + \sum_{j=1}^{l} l_j(t,x,\xi;\epsilon)\tau^{l-j},$$

$$(2.4)\qquad m(t,x,\tau,\xi;\epsilon) = m_0(t,x;\epsilon)\tau^m + \sum_{j=1}^{m} m_j(t,x,\xi;\epsilon)\tau^{m-j}.$$

We assume the following assumptions:
(H0) Regular hyperbolicity of L: $l(t,x,\tau,\xi;\epsilon)$ has the decomposition

$$(2.5)\qquad l(t,x,\tau,\xi;\epsilon) = \prod_{j=1}^{l}(\tau - \varphi_j(t,x,\xi;\epsilon))$$

where $\varphi_j(t,x,\xi;\epsilon)$ are real distinct elements such that

$$(2.6)\qquad \varphi_1(t,x,\xi;\epsilon) < \varphi_2(t,x,\xi;\epsilon) < \cdots < \varphi_l(t,x,\xi;\epsilon) \quad \text{uniformly :}$$

in $(t,x,\xi,\epsilon) \in [0,\infty) \times \boldsymbol{R}_x^d \times \{|\xi| = 1\} \times [0,\epsilon_0]$, that is, $\varphi_{j+1}(t,x,\xi;\epsilon) - \varphi_j(t,x,\xi;\epsilon)$ is uniformly positive for $j = 1, \cdots, l-1$.
(H1) Regular hyperbolicity of M: $m(t,x,\tau,\xi;\epsilon)$ has the decomposition

$$(2.7)\qquad m(t,x,\tau,\xi;\epsilon) = m_0(t,x;\epsilon)\prod_{j=1}^{m}(\tau - \psi_j(t,x,\xi;\epsilon))$$

where $\psi_j(t,x,\xi;\epsilon)$ are real distinct elements such that

$$(2.8)\qquad \psi_1(t,x,\xi;\epsilon) < \psi_2(t,x,\xi;\epsilon) < \ldots < \psi_m(t,x,\xi;\epsilon) \quad \text{uniformly.}$$

We assume
(D1): $l = m+1$,
and the following assumption
(H2): $m_0(t,x;\epsilon)$ is pure-imaginary and uniformly away from 0, that is,
(HP): $\Re m_0(t,x;\epsilon) \equiv 0$ and there exists a positive constant δ such that

$$\Im m_0(t,x;\epsilon) \geq \delta > 0,$$

or
(HN): $\Re m_0(t,x;\epsilon) \equiv 0$ and there exists a positive constant δ such that

$$\Im m_0(t,x;\epsilon) \leq -\delta < 0.$$

We assume also
(S0): $\{\psi_i\}$ separates $\{\varphi_j\}$ uniformly, that is,

$$\varphi_1 < \psi_1 < \varphi_2 < \cdots < \psi_m < \varphi_{m+1} \quad \text{uniformly.}$$

Remark 1. When L and M are pseudo-differetial operators, we introduced in Part I ([13]) the assumptions

(SP): $$\varphi_1 < \{\psi_1, \varphi_2\} < \cdots < \{\psi_{m-1}, \varphi_m\} < \{\psi_m, \varphi_{m+1}\}$$

with (HP) and

(SN): $$\{\psi_1, \varphi_1\} < \{\psi_2, \varphi_2\} < \{\psi_3, \varphi_3\} < \cdots < \{\psi_m, \varphi_m\} < \varphi_{m+1},$$

with (HN), where $* < \{a, b\}$ means $* < \min\{a, b\}$ and $\{c, d\} < *$ means $\max\{c, d\} < *$. (They are $(\mathrm{WS}^{\pm})$ and $(\mathrm{S}^{\pm})$ in [13].)

When L and M are differential operators, any one of the conditions (SP) and (SN) is equivalent to (S0).

Remark 2. In Part II ([14]), we assumed (D1), (H0), (H1), (S0) and (E1): uniformly strong ellipticity of m_0, that is,

$$\Re m_0(t, x; \epsilon) \geq \delta > 0.$$

We quote from Part I ([13])

Theorem 2.1. *We assume* (D1), (H0), (H1), (H2) *and* (S0). *For any natural number p, there exist $C > 0$ and γ_0 such that for any positive $\epsilon \leq \epsilon_0$, for any $\gamma \geq \gamma_0$ and for any $u(t) \in C^\infty\left([0, T]; C_0^\infty(\boldsymbol{R}_x^d)\right)$ we have*

$$\begin{aligned}(2.9)\quad C\Bigg\{ & \frac{1}{\gamma} \int_0^T e^{-2\gamma t} \frac{1}{\epsilon} \sum_{j=0}^{p} \left(\epsilon^2 \gamma\right)^j \parallel D^j f(t) \parallel^2 dt + \parallel D^{m-1} u(0) \parallel_{1/2}^2 \\ & + \gamma^p \left(\epsilon \sum_{j=0}^{p} \epsilon^{2j} \parallel D^m u(0) \parallel_j^2 + \sum_{j=1}^{p} \epsilon^{2j} \parallel D^m u(0) \parallel_{j-1/2}^2 \right. \\ & \left. + \ \epsilon \sum_{j=0}^{p-1} \epsilon^{2j} \parallel D^j f(0) \parallel^2 + \sum_{j=1}^{p-1} \epsilon^{2j} \parallel D^{j-1} f(0) \parallel_{1/2}^2 \right) \Bigg\}\end{aligned}$$

$$\geq \gamma \int_0^T e^{-2\gamma t} \sum_{j=0}^{p} \left(\epsilon^2\gamma\right)^j \left(\epsilon \parallel D^{m+j}u(t) \parallel^2 + \parallel D^{m+j-1}u(t) \parallel_{1/2}^2\right) dt$$

$$+e^{-2\gamma T} \sum_{j=0}^{p} \left(\epsilon^2\gamma\right)^j \left(\epsilon \parallel D^{m+j}u(T) \parallel^2 + \parallel D^{m+j-1}u(T) \parallel_{1/2}^2\right),$$

where $f(t) = (i\epsilon)Lu(t) + Mu(t)$.

When
(D2): $l = m + 2$,
we assume (H0), (H1) and the following assumptions
(WS): $\{\psi_i\}$ weakly separates $\{\varphi_j\}$ uniformly, that is,

$$\varphi_1 < \{\psi_1, \varphi_2\} < \cdots < \{\psi_m, \varphi_{m+1}\} < \varphi_{m+2} \quad \text{uniformly.}$$

and
(P): $m_0(t, x; \epsilon)$ is real and uniformly positive, that is,

$$\Im m_0(t, x; \epsilon) \equiv 0, \quad \text{and} \quad m_0(t, x; \epsilon) \geq \delta > 0.$$

We quote from Part I ([13]),

Theorem 2.2. *We assume* (D2), (H0), (H1), (P) *and* (WS). *For any natural number* p, *there exist positive constant* C *and* γ_0 *such that for any positive* $\epsilon \leq \epsilon_0$, *for any* $\gamma \geq \gamma_0$ *and for any* $u(t) \in C^\infty\left([0, T]; C_0^\infty\left(\boldsymbol{R}_x^d\right)\right)$ *we have*

$$(2.10)\ C\left\{\frac{1}{\gamma} \int_0^T e^{-2\gamma t} \frac{1}{\epsilon^2} \sum_{j=0}^{p} \left(\epsilon^2\gamma\right)^j \parallel D^j f(t) \parallel^2 dt + \gamma^p \|D^m u(0)\|^2\right.$$

$$\left. + \gamma^p \left(\sum_{j=0}^{p} \epsilon^{2j+2} \parallel D^{m+1}u(0) \parallel_j^2 + \sum_{j=0}^{p-1} \epsilon^{2j} \parallel D^j f(0) \parallel^2\right)\right\}$$

$$\geq \gamma \int_0^T e^{-2\gamma t} \sum_{j=0}^{p} \left(\epsilon^2\gamma\right)^j \left(\epsilon^2 \parallel D^{m+j+1}u(t) \parallel^2 + \parallel D^{m+j}u(t) \parallel^2\right) dt$$

$$+e^{-2\gamma T} \sum_{j=0}^{p} \left(\epsilon^2\gamma\right)^j \left(\epsilon^2 \parallel D^{m+j+1}u(T) \parallel^2 + \parallel D^{m+j}u(T) \parallel^2\right),$$

where $f(t) = (i\epsilon)^2 Lu(t) + Mu(t)$.

3. Singular characteristic roots

3.1. Degeneration of order 1

Let $l=m+1$. We define ϵ-principal symbol

$$ip(t,x,\tau,\xi)=il(t,x,\tau,\xi;0)+m(t,x,\tau,\xi;0).$$

We denote the roots of $p(\tau)=0$ by $\tau_j(t,x,\xi)$'s. In order to show the argument is microlocal, we state the assumptions (SP) and (SN) separately.

Proposition 3.1. *We assume* (**S2**): (D1), (H0), (H1), (HP), (SP) *or* (**S3**): (D1), (H0), (H1), (HN), (SN). *Then, τ_j's are real and uniformly distinct, that is, there exists a positive constant c such that*

$$\tau_{j+1}(t,x,\xi)-\tau_j(t,x,\xi)\geq c|\xi| \quad \textit{for} \quad j=1,2,\cdots,m. \tag{3.1}$$

Moreover, in case (**S2**), *the least root $\tau_1(t,x,\xi)$ satisfies $\tau_1(t,x,0)=-\Im m_0(t,x;0)$,*

$$\tau_j(t,x,\xi)-\tau_1(t,x,\xi)\geq c(1+|\xi|) \quad \textit{for} \quad j=2,\cdots,m+1, \tag{3.2}$$

and belongs to the nonhomogeneous smooth symbol class S^1. And in case (**S3**), *the greatest root $\tau_{m+1}(t,x,\xi)$ satisfies $\tau_{m+1}(t,x,0)=-\Im m_0(t,x;0)$,*

$$\tau_{m+1}(t,x,\xi)-\tau_j(t,x,\xi)\geq c(1+|\xi|) \quad \textit{for} \quad j=1,\cdots,m, \tag{3.3}$$

and belongs to the nonhomogeneous smooth symbol class S^1.

Proof. We prove the statements under the assumption (**S2**). We introduce notations $a\vee b=\max\{a,b\}$ and $a\wedge b=\min\{a,b\}$. We have

$$\begin{aligned}\operatorname{sgn}[p(\varphi_{m+1})] &= \operatorname{sgn}[\Im m_0]\operatorname{sgn}\left[\prod_{j=1}^{m}(\varphi_{m+1}-\psi_j)\right]=\operatorname{sgn}(\varphi_{m+1}-\psi_m)\\ \operatorname{sgn}[p(\psi_m)] &= \operatorname{sgn}\prod_{j=1}^{m+1}(\psi_m-\varphi_j)=\operatorname{sgn}(\psi_m-\varphi_{m+1})\end{aligned}$$

Therefore, since $p(\psi_m\vee\varphi_{m+1})\geq 0$ and $p(\psi_m\wedge\varphi_{m+1})\leq 0$, we have a root τ_{m+1} between $\psi_m\wedge\varphi_{m+1}$ and $\psi_m\vee\varphi_{m+1}$. In the same way, $p(\tau)=0$ has a root τ_{j+1} between $\psi_j\wedge\varphi_{j+1}$ and $\psi_j\vee\varphi_{j+1}$ for $j=1,2,\cdots,m$.

Since $\mathrm{sgn}[p(\varphi_1)] = (-1)^m$ and $\mathrm{sgn}[p(\tau)] = (-1)^{m+1}$ for $\tau < \varphi_1$ with sufficiently large $|\tau|$, there exists the $(m+1)$-th root τ_1 less than φ_1. By the assumption (SP), there exists a positive constant independent of (t, x, ξ) such that the roots $\{\tau_j\}$ satisfy $|\tau_{j+1} - \tau_j| \geq c|\xi|$ for $j = 1, 2, \cdots, m$.

Since the coefficients of $p(\tau)$ are uniformly bounded with respect to (t, x, ξ), there exists a positive constant C independent of $(t, x, \xi; \epsilon)$ such that

$$|\tau_i(t, x, \xi)| \leq C(1 + |\xi|), \quad (i = 1, 2, \cdots, m+1).$$

The estimates of the derivatives of τ_1 follows from the implicit function theorem. In fact, we need an estimate from below of $\partial p/\partial\tau(\tau_1) = \prod_{j=2}^{m+1}(\tau_1 - \tau_j)$. By the separation condition (SP),

$$|\tau_1 - \tau_j| \geq c|\xi| \quad \text{for} \quad j = 2, 3, \cdots, m+1,$$

for τ_j is between $\psi_{j-1} \wedge \varphi_j$ and $\psi_{j-1} \vee \varphi_j$ when $j \geq 2$. On the other hand,

$$p(\tau) = \tau^{m+1} + (\Im m_0(t, x; 0))\,\tau^m + \sum_{j=1}^{d} \xi_j p_j(\tau),$$

where p_j's are polynomials in τ of order at most m. When $\xi = 0$, $\tau_1 = -\Im m_0(t, x; 0)$ and $\tau_j = 0 \quad (j \geq 2)$. By Rouché's theorem, $|\tau_1 - \tau_j| \geq c$ for sufficiently small $|\xi|$. Hence, we have (3.2) and $|\partial p/\partial\tau(\tau_1)| \geq c(1 + |\xi|)^m$. $\square$

Remark. When the condition (**S2**) holds, we have for $j = 2, 3, \cdots, m+1$,

$$\tau_1 < \varphi_1 < \min\{\varphi_j, \psi_{j-1}\} \leq \tau_j \leq \max\{\varphi_j, \psi_{j-1}\}.$$

We call τ_1 the singular root, since $\tau_1(t, x, \epsilon\xi)/\,\epsilon$ is a root of $(i\epsilon)\; l(t, x, \tau, \xi; 0)$ $+m(t, x, \tau, \xi; 0) = 0$, which is singular when ϵ tends to 0. Aternatively, τ_{m+1} is the singular one, when the condition (**S3**) holds. Cf. Frank [5] Chap.3.9.

We assume (**S2**). We denote for simplicity, $p(t, x, \tau, \xi)$ by p, $\tau_1(t, x, \xi)$ by τ_1 and so on. We consider a Hamiltonian system for $(t(\sigma, y), x(\sigma, y), \tau(\sigma, y), \xi(\sigma, y))$:

$$(3.4) \qquad \begin{cases} \dfrac{dt}{d\sigma} = \dfrac{\partial p}{\partial\tau}, & \dfrac{dx_j}{d\sigma} = \dfrac{\partial p}{\partial\xi_j}, \qquad j = 1, 2, \cdots, d, \\[2ex] \dfrac{d\tau}{d\sigma} = -\dfrac{\partial p}{\partial t}, & \dfrac{d\xi_j}{d\sigma} = -\dfrac{\partial p}{\partial x_j}, \qquad j = 1, 2, \cdots, d, \end{cases}$$

with Cauchy data

$$\left\{\begin{array}{lll} t(0,y)=0, & x_j(0,y)=y_j, & j=1,2,\cdots,d, \\ \tau(0,y)=\tau_1(0,y,0), & \xi_j(0,y)=0, & j=1,2,\cdots,d. \end{array}\right. \tag{3.5}$$

Suppose $x=x(\sigma,y), \xi=\xi(\sigma,y), t=t(\sigma,y), \tau=\tau_1(t(\sigma,y),x(\sigma,y),\xi(\sigma,y))$ be a system of solutions to (3.4) and (3.5). Then,

$$\frac{dt}{d\sigma}=\frac{\partial p}{\partial\tau}\Big|_{\tau=\tau_1}=\sum_{k=1}^{m+1}\prod_{\substack{j=1\\ j\neq k}}^{m+1}(\tau-\tau_j)\Bigg|_{\tau=\tau_1}=\prod_{j=2}^{m+1}(\tau_1-\tau_j).$$

Therefore, $\mathrm{sgn}\frac{dt}{d\sigma}=(-1)^m$ and

$$\left|\frac{dt}{d\sigma}\right|\geq c(1+|\xi|)^m.$$

Hence, we have $\sigma=\sigma(t,y)$, the inverse function of $t=t(\sigma,y)$ with respect to σ.

We also consider the system for $(\tilde{x}(t,y),\tilde{\xi}(t,y))$

$$\left\{\begin{array}{ll} \dfrac{d\tilde{x}_j}{dt}=-\dfrac{\partial\tau_1}{\partial\xi_j}(t,\tilde{x},\tilde{\xi}), & j=1,2,\cdots,d, \\ \\ \dfrac{d\tilde{\xi}_j}{dt}=\dfrac{\partial\tau_1}{\partial x_j}(t,\tilde{x},\tilde{\xi}), & j=1,2,\cdots,d, \end{array}\right. \tag{3.6}$$

and Cauchy data

$$\tilde{x}(0,y)=y, \quad \tilde{\xi}(0,y)=0. \tag{3.7}$$

Put

$$\begin{aligned} \pi(t,y) &= \frac{\partial p}{\partial\tau}(t,\tilde{x}(t,y),\tau_1(t,\tilde{x}(t,y),\tilde{\xi}(t,y)),\tilde{\xi}(t,y)) \\ &= \prod_{j=2}^{m+1}(\tau_1(t,\tilde{x}(t,y),\tilde{\xi}(t,y))-\tau_j(t,\tilde{x}(t,y),\tilde{\xi}(t,y))). \end{aligned}$$

Then, we consider the equation

$$\left\{\begin{array}{lll} \dfrac{dt}{d\sigma} &=& \pi(t,y), \\ t(0,y) &=& 0. \end{array}\right.$$

A unique solution $t(\sigma, y)$ is the inverse function of $\sigma(t,y) = \int_0^t \frac{ds}{\pi(s,y)}$.

PROPOSITION 3.2 (Fedoriuk[4]). *We assume* (**S2**). *If the family of* $x = x(\sigma, y)$, $\xi = \xi(\sigma, y)$, $t = t(\sigma, y)$, $\tau = \tau_1(t(\sigma, y), x(\sigma, y), \xi(\sigma, y))$ *is a unique solution to* (3.4) *and* (3.5), *then* $\tilde{x}(t, y) = x(\sigma(t, y), y)$ *and* $\tilde{\xi}(t, y) = \xi(\sigma(t, y), y)$ *satisfy* (3.6) *and* (3.7).
Conversely, if $\tilde{x}(t, y)$ *and* $\tilde{\xi}(t, y)$ *make a system of solutions to* (3.6) *and* (3.7),

$$\begin{array}{rcl} x & = & x(\sigma,y) = \tilde{x}(t(\sigma,y),y), \quad \xi = \xi(\sigma,y) = \tilde{\xi}(t(\sigma,y),y) \\ t & = & t(\sigma,y), \quad \tau = \tau_1(t(\sigma,y),x(\sigma,y),\xi(\sigma,y)) \end{array}$$

consist of a solution to (3.4) *and* (3.5).

PROOF. We have

$$\begin{array}{rcl} \dfrac{dx_j}{d\sigma} & = & \left.\dfrac{\partial p}{\partial \xi_j}\right|_{\tau=\tau_1} = -\dfrac{\partial \tau_1}{\partial \xi_j}\displaystyle\prod_{j=2}^{m+1}(\tau_1 - \tau_j) \\ & = & -\dfrac{\partial \tau_1}{\partial \xi_j}\dfrac{dt}{d\sigma}. \end{array}$$

Hence,

$$\frac{d\sigma}{dt}\frac{dx_j}{d\sigma} = -\frac{\partial \tau_1}{\partial \xi_j}, \quad j = 1, 2, \cdots, d.$$

In the same way, we have

$$\begin{array}{rcl} \dfrac{d\sigma}{dt}\dfrac{d\xi_j}{d\sigma} & = & \dfrac{\partial \tau_1}{\partial x_j}, \quad j = 1, 2, \cdots, d. \\ \dfrac{d\sigma}{dt}\dfrac{d\tau}{d\sigma} & = & \dfrac{\partial \tau_1}{\partial t}, \quad j = 1, 2, \cdots, d. \end{array}$$

We define for $j = 1, 2, \cdots, d$,

$$\left\{ \begin{array}{rcl} \tilde{x}_j(t,y) & = & x_j(\sigma(t,y),y) \\ \tilde{\xi}_j(t,y) & = & \xi_j(\sigma(t,y),y). \end{array} \right.$$

We have

$$\left\{ \begin{array}{rcll} \dfrac{d\tilde{x}_j}{dt} & = & -\dfrac{\partial \tau_1}{\partial \xi_j}, & j = 1, 2, \cdots, d, \\ \dfrac{d\tilde{\xi}_j}{dt} & = & \dfrac{\partial \tau_1}{\partial x_j}, & j = 1, 2, \cdots, d, \end{array} \right. \tag{3.8}$$

and

$$\left\{\begin{array}{lclcl} \tilde{x}_j(0,y) & = & x_j(0,y) & = & y_j \\ \tilde{\xi}_j(0,y) & = & \xi_j(0,y) & = & 0. \end{array}\right. \tag{3.9}$$

The converse is proved in a similar way. □

REMARK. The bicharacteristic curves in $\boldsymbol{R}^{2d+2}$ with parameters (σ, y)

$$t = t(\sigma, y),\ x = x(\sigma, y),\ \tau = \tau(\sigma, y),\ \xi = \xi(\sigma, y)$$

have another expression

$$\left\{\begin{array}{lcl} x & = & \tilde{x}(t,y),\ \tau = \tau_1(t, \tilde{x}(t,y), \tilde{\xi}(t,y)) \\ \xi & = & \tilde{\xi}(t,y) \quad \text{with parameters } (t,y). \end{array}\right.$$

PROPOSITION 3.3. *We assume* (**S2**).
(i) *We have a unique system of* C^∞ *solutions* $\{\tilde{x}_i(t,y)\}$ *and* $\{\tilde{\xi}_i(t,y)\}$ *to* (3.6) *and* (3.7) *for all non-negtive* t. *There exists a positive constant* M *such that for any nonnegative* t

$$\sup_y \left|\tilde{x}_i(t,y) - y_i\right| \le Mt \qquad i = 1, 2, \cdots, d,$$

$$\sup_y \left|\tilde{\xi}_i(t,y)\right| \le e^{Mt} - 1, \qquad i = 1, 2, \cdots, d.$$

(ii) *There exist a nonnegative continuous function* $m(t)$ *with* $m(0) = 0$ *such that for any* i, a

$$\left|\frac{\partial \tilde{x}_i}{\partial y_a}(t,y) - \delta_{ia}\right| \le m(t).$$

Hence, there exist positive constants T_0 *and* δ *such that*

$$\left|\det\left(\frac{\partial \tilde{x}_i}{\partial y_a}(t,y)\right)\right| \ge \delta > 0 \quad (t,y) \in [0, T_0] \times \boldsymbol{R}^d.$$

Moreover, for any multi-index α, *there exists a nonnegative continuous function* $m_\alpha(t)$ *with* $m_\alpha(0) = 0$ *such that*

$$\left|\frac{\partial^{|\alpha|}\tilde{x}_i}{\partial y^\alpha}(t,y)\right| \le m_\alpha(t), \ \text{when } |\alpha| > 1,$$

and that

$$\left|\frac{\partial^{|\alpha|}\tilde{\xi}_i}{\partial y^\alpha}(t,y)\right| \le m_\alpha(t).$$

PROOF. We omit the parameter y in the solutions.
(i) We will show the global existence and uniqueness of solutions to the system of integral equations:

$$\tilde{x}_i(t) = y_i - \int_0^t \frac{\partial \tau_1}{\partial \xi_i}(s, \tilde{x}(s), \tilde{\xi}(s))ds \tag{3.10}$$

$$\tilde{\xi}_i(t) = \int_0^t \frac{\partial \tau_1}{\partial x_i}(s, \tilde{x}(s), \tilde{\xi}(s))ds \tag{3.11}$$

for $j = 1, 2, \cdots, d$. We fix $T > 0$ arbitrarily. For $t \in [0, T]$, we define successively

$$\begin{cases} \tilde{x}_i^{(0)}(t) = y_i, \\ \tilde{x}_i^{(n)}(t) = y_i - \displaystyle\int_0^t \frac{\partial \tau_1}{\partial \xi_i}(s, \tilde{x}^{(n-1)}(s), \tilde{\xi}^{(n-1)}(s))ds \\ \text{for} \quad n \ge 1, \end{cases} \tag{3.12}$$

and

$$\begin{cases} \tilde{\xi}_i^{(0)}(t) = 0, \\ \tilde{\xi}_i^{(n)}(t) = \displaystyle\int_0^t \frac{\partial \tau_1}{\partial x_i}(s, \tilde{x}^{(n-1)}(s), \tilde{\xi}^{(n-1)}(s))ds \\ \text{for} \quad n \ge 1. \end{cases} \tag{3.13}$$

We will give a priori estimates of approximate sequences. By Proposition 3.1, there exists a constant $M \ge 1$ such that

$$\begin{aligned} \left|\frac{\partial^{|\alpha|}\tau_1}{\partial x^\alpha}\right| &\le M(1+|\xi|) \quad \text{for } |\alpha| \le 2, \\ \left|\frac{\partial^{|\alpha|+1}\tau_1}{\partial x^\alpha \partial \xi_i}\right| &\le M \quad \text{for } |\alpha| \le 1 \quad \text{and} \\ \left|\frac{\partial^2 \tau_1}{\partial \xi_i \partial \xi_j}\right| &\le M(1+|\xi|)^{-1}. \end{aligned}$$

Then,

$$|\tilde{x}_i^{(1)}(t) - y_i| = |-\int_0^t \frac{\partial \tau_1}{\partial \xi_i}(s,y,0)ds| \leq Mt,$$

and

$$|\tilde{\xi}_i^{(1)}(t)| = |\int_0^t \frac{\partial \tau_1}{\partial x_i}(s,y,0)ds| \leq Mt.$$

By induction, we have

$$\begin{aligned} |\tilde{x}_i^{(n)}(t) - y_i| &\leq Mt, \\ |\tilde{\xi}_i^{(n)}(t)| &\leq \sum_{q=1}^{n} \frac{M^q t^q}{q!} \leq e^{Mt} - 1. \end{aligned}$$

We will show the global convergence of the approximate sequences. We put

$$\|\tilde{x}(t)\| = \sum_{j=1}^{d} |\tilde{x}_i(t)| \quad \text{and} \quad \|\tilde{\xi}(t)\| = \sum_{j=1}^{d} |\tilde{\xi}_i(t)|.$$

We claim that there exists a positive constant C_T such that

$$\|\tilde{x}^{(k)}(t) - \tilde{x}^{(k-1)}(t)\| + \|\tilde{\xi}^{(k)}(t) - \tilde{\xi}^{(k-1)}(t)\| \leq \frac{{C_T}^k}{k!}$$

for any $k \in \boldsymbol{N}$ and any $t \in [0,T]$. In fact, if we put $C_T = Md(1+\sqrt{d}e^{MT})T$, this is derived by induction. Hence, $\lim_{k\to\infty} \tilde{x}_j^{(k)}$ and $\lim_{k\to\infty} \tilde{\xi}_j^{(k)}$ exist and they are the desired solutions.
(ii) Differentiating the equations (3.8) succesively with respect to y, we have a sequence of linear equations satisfied by $\{\frac{\partial^{|\alpha|}\tilde{x}_i}{\partial y^\alpha}, \frac{\partial^{|\alpha|}\tilde{\xi}_j}{\partial y^\alpha}; 1 \leq i,j \leq d, |\alpha| \geq 1\}$. The desired estimates follow from it by Gronwall's inequality and by induction. □

We consider $\boldsymbol{R}^{d+1}_{t,x} \oplus \boldsymbol{R}^{d+1}_{\tau,\xi}$ as symplectic space with the fundamental 1-form $\tau dt + \sum_{j=1}^d \xi_j dx_j$. Let Λ^{d+1} be the flow-out of $\boldsymbol{R}^d_x \times \{0\} \subset \boldsymbol{R}^d_x \oplus \boldsymbol{R}^d_\xi$ by the trajectory defined by (3.6) and (3.7) for $t \in [0,\infty)$. That is,

$$\begin{aligned} \Lambda^{d+1} = \Big\{ &(t,x,\tau,\xi) \in \boldsymbol{R}^{d+1}_{t,x} \oplus \boldsymbol{R}^{d+1}_{\tau,\xi}; 0 \leq t < \infty, \\ &x = \tilde{x}(t,y), \tau = \tau_1(t,\tilde{x}(t,y),\tilde{\xi}(t,y)), \\ &\xi = \tilde{\xi}(t,y), y \in \boldsymbol{R}^d \Big\}. \end{aligned} \tag{3.14}$$

We put $\Lambda^d_s = \Lambda^{d+1}\Big|_{t=s}$ and $\Lambda^{d+1}_{[0,T]} = \{(t,x,\tau,\xi) \in \Lambda^{d+1}; 0 \le t \le T\}$.

PROPOSITION 3.4 (Fedoriuk[4]). (i) Λ^{d+1} *is a* $(d+1)$*-dimensional simply connected Lagrangian* C^∞ *manifold with boundary:*

$$\begin{aligned} \Lambda^d_0 &= \{(0,y,\tau_1(0,y,0),0); y \in \boldsymbol{R}^d\}, \\ &\cong \boldsymbol{R}^d_x. \end{aligned}$$

(ii) *The variable* t *can be used as a member of local coordinates of every chart of* Λ^{d+1}.
(iii) *There exists a positive* T_0 *such that the projection of* $\Lambda^{d+1}_{[0,T_0]}$ *onto* $\boldsymbol{R}^{d+1}_{t,x}\Big|_{[0,T_0]}$ *is a diffeomorphism.*

Let $I = \{i_1, i_2, \cdots, i_k\}$ be an empty or nonempty subset of $\{1,2,\cdots,d\}$ and $\overline{I} = \{i_{k+1},\cdots, i_d\}$ be its complement. $\boldsymbol{R}^{|I|}_x$ and $\boldsymbol{R}^{|\overline{I}|}_\xi$ are the spaces of coordinates $x_I = (x_{i_1}, x_{i_2}, \cdots, x_{i_k})$ and $\xi_{\overline{I}} = (\xi_{i_{k+1}}, \cdots, \xi_{i_d})$ respectively. We use a fixed canonical atlas $\{\Lambda_I, \pi_I; I = I(k), k \in \boldsymbol{N}\}$ where Λ_I is an open domain and π_I is a projection

$$\pi_I : \boldsymbol{R}^{d+1}_{t,x} \oplus \boldsymbol{R}^{d+1}_{\tau,\xi} \to \boldsymbol{R}_t \oplus \boldsymbol{R}^{|I|}_x \oplus \boldsymbol{R}^{|\overline{I}|}_\xi,$$

which is a diffeomorphism from Λ_I onto a domain

$$\begin{aligned} \tilde{U}_I \;=\; &\{(t, \tilde{x}_I(t,y), \tilde{\xi}_{\overline{I}}(t,y)); \\ &(t,y) \in \text{a rectangular set } U_I \text{ of } [0,+\infty) \times \boldsymbol{R}^d_y\}. \end{aligned} \tag{3.15}$$

The domain Λ_I is expressed by a graph of mapping

$$x_{\overline{I}} = X_{\overline{I}}(t, x_I, \xi_{\overline{I}}), \quad \xi_I = \Xi_I(t, x_I, \xi_{\overline{I}}). \tag{3.16}$$

Abuse of notation. I of Λ_I means the label of a local chart and also the multi-index $\{i_1, i_2, \cdots, i_k\}$ specifying the canonical coordinates of Λ_I.

The case (**S3**) is treated in the same way as (**S2**).

3.2. Degeneration of order 2

Let $l = m + 2$. We define ϵ-principal symbol

$$-p(t,x,\tau,\xi) = -l(t,x,\tau,\xi;0) + m(t,x,\tau,\xi;0).$$

We denote the roots of $p(\tau) = 0$ by $\tau_j(t,x,\xi)$'s.

PROPOSITION 3.5. *We assume* (H0),(H1),(P) *and* (WS). *Then,* τ_j*'s are real and uniformly distinct, that is, there exists a positive constant* c *such that*

$$\tau_{j+1}(t,x,\xi) - \tau_j(t,x,\xi) \geq c|\xi|.$$

Moreover, the least root $\tau_1(t,x,\xi)$ *and the greatest root* $\tau_{m+2}(t,x,\xi)$ *satisfy* $\tau_1(t,x,0) = -\sqrt{m_0(t,x;0)}$, $\tau_{m+2}(t,x,0) = \sqrt{m_0(t,x;0)}$ *and*

$$(3.17) \begin{cases} \tau_j(t,x,\xi) - \tau_1(t,x,\xi) & \geq & c(1+|\xi|), \quad j = 2,\cdots,m+2, \\ \tau_{m+2}(t,x,\xi) - \tau_j(t,x,\xi) & \geq & c(1+|\xi|), \quad j = 1,\cdots,m+1. \end{cases}$$

They belong to the nonhomogeneous smooth symbol class S^1.

PROOF. Since $p(\tau) > 0$ for sufficiently big τ and $p(\varphi_{m+2}) < 0$, we have a root τ_{m+2} bigger than φ_{m+2}. Then, we have for $j = 1,2,\cdots,m$,

$$\mathrm{sgn}[p(\varphi_{j+1})] = (-1)^{m-j+1}\mathrm{sgn}[\varphi_{j+1} - \psi_j]$$

and

$$\mathrm{sgn}[p(\psi_j)] = (-1)^{m-j}\mathrm{sgn}[\varphi_{j+1} - \psi_j].$$

Therefore, we have a root in the interval $[\varphi_{j+1} \wedge \psi_j, \varphi_{j+1} \vee \psi_j]$. In fact, it is trivial, if $\varphi_{j+1} = \psi_j$. It follows from

$$\begin{array}{rcl} \mathrm{sgn}[p(\varphi_{j+1} \wedge \psi_j)] & = & (-1)^{m-j} \\ \mathrm{sgn}[p(\varphi_{j+1} \vee \psi_j)] & = & (-1)^{m-j+1} \end{array}$$

when $\varphi_{j+1} \neq \psi_j$.

Especially,

$$\mathrm{sgn}[p(\varphi_2 \wedge \psi_1)] = (-1)^{m-1} \quad \mathrm{sgn} p(\varphi_2 \vee \psi_1) = (-1)^m.$$

Combining the facts

$$\begin{array}{rcl} \mathrm{sgn}[p(\varphi_1)] & = & (-1)^{m+1}, \\ \mathrm{sgn}[p(\tau)] & = & (-1)^{m+2} \quad \text{for sufficiently negative } \tau, \end{array}$$

we know the existence of the roots $\{\tau_j\}$ such that

$$\tau_1 < \varphi_1 < \varphi_j \wedge \psi_{j-1} \leq \tau_j \leq \varphi_j \vee \psi_{j-1} < \varphi_{m+2} < \tau_{m+2},$$

where $j = 2, 3, \cdots, m+1$.

The rest of proof follows as in the proof of Proposition 3.1. $\square$

REMARK. As we have seen, we have for $j = 2, 3, \cdots, m+1$,

$$\tau_1 < \varphi_1 < \min\{\varphi_j, \psi_{j-1}\} \leq \tau_j \leq \max\{\varphi_j, \psi_{j-1}\} < \varphi_{m+2} < \tau_{m+2}.$$

We call τ_1 and τ_{m+2} singular roots.

We consider the Hamiltonian systems of the same type as in the previous subsection, except one condition in the Cauchy data,

$$\begin{aligned} \tau|_{\sigma=0} &= \tau_i(0, y, 0) \quad \text{for} \quad i = 1 \quad \text{or} \quad m+2 \\ &= \pm\sqrt{m_0(0, x; 0)}, \quad \text{for} \quad i = 1 \quad \text{or} \quad m+2. \end{aligned} \tag{3.18}$$

We obtain the solutions $(t^*(\sigma), x^*(\sigma), \xi^*(\sigma))$ and $\left(\tilde{x}^*(t, y), \tilde{\xi}^*(t, y)\right)$, where $* = \pm$ according to the signature of the Cauchy data (3.18). They define the Lagrangian manifolds Λ^* as before. We introduce in the same way their canonical atlas $\{\Lambda_I^*, \pi_I^*\}$ etc.

4. Review of canonical operators of Maslov

We summarize basic definitions and results in the theory of canonical operators ([9]). We refer details to [10], [4]; [11], [6].

4.1. Preliminaries

stationary phase method. We quote a version of the stationary phase method (see [3], [1], [8]).

We assume the following three conditions.

(C-I) $\phi(x, \eta)$ is a real valued C^∞ function on a neighborhood of a compact set K in $\boldsymbol{R}^m \times \boldsymbol{R}^n$.

(C-II) There exists a positive constant C_0 such that

$$\left|\det \frac{\partial^2 \phi(x, \eta)}{\partial \eta_j \partial \eta_k}\right| \geq C_0 \text{ for any } (x, \eta) \in K.$$

(C-III) $a(x, \eta) \in C_0^\infty(\boldsymbol{R}^m \times \boldsymbol{R}^n)$ with support in K.

Then, we assume for $x \in K$, the system of equations

$$\frac{\partial}{\partial \eta_j} \phi(x, \eta) = 0, \quad j = 1, \ldots, n$$

has a unique solution $\eta = \eta(x)$. We put

$$h(x,\eta) = \phi(x,\eta) - \phi(x,\eta(x)) - \frac{1}{2} < H(x)w, w >$$

where

$$H(x) = \left(\frac{\partial^2 \phi}{\partial \eta_j \partial \eta_k}(x,\eta(x))\right)_{1\le j,k\le n}$$

and

$$w = \eta - \eta(x).$$

LEMMA 4.1.

$$\begin{aligned}
(4.1)\quad & \int_{\boldsymbol{R}^n} a(x,\eta) \exp\left[i\frac{\phi(x,\eta)}{\epsilon}\right] d\eta \\
&= (2\pi\epsilon)^{n/2} |\det H(x)|^{-1/2} \exp\frac{\pi i}{4}(n - 2\mathrm{Ind}H(x)) \\
&\quad \times \exp\left[\frac{i}{\epsilon}\phi(x,\eta(x))\right] \left\{\sum_{k=0}^{N} \frac{1}{k!}\left(-\frac{i\epsilon}{2} < H^{-1}(x)D_\eta, D_\eta >\right)^k a(x,\eta)\right. \\
&\quad \left.\times\ \exp\left[\frac{i}{\epsilon}h(x,\eta)\right]\Big|_{\eta=\eta(x)}\right\} + \tilde{r}_{N+1}(x,\epsilon) \\
&= (2\pi\epsilon)^{n/2} |\det H(x)|^{-1/2} \exp\frac{\pi i}{4}(\mathrm{sgn}H(x)) \exp\left[\frac{i}{\epsilon}\phi(x,\eta(x))\right] \\
&\quad \times \left\{\sum_{k=0}^{N}\left(-\frac{i\epsilon}{2}\right)^k \sum_{p=0}^{2k} \frac{2^{-p}}{(k+p)!p!} < H^{-1}(x)D_\eta, D_\eta >^{k+p}\right. \\
&\quad \left.(h(x,\eta))^p\, a(x,\eta)\big|_{\eta=\eta(x)}\right\} + r_{N+1}(x,\epsilon)
\end{aligned}$$

Here, $\mathrm{Ind}H(x)$ *is the dimension of the eigenspace with negative eigenvalues of* $H(x)$*. The remainder term* $r_{N+1}(x,\epsilon)$ *(and also* $\tilde{r}_{N+1}(x,\epsilon)$*) have the following estimate:*
for any multi-index α*, there exist a positive integer* $l = l(\alpha, N)$ *and a positive constant* C *which are independent of* ϵ

$$\left|\left(\epsilon\frac{\partial}{\partial x}\right)^\alpha r_{N+1}(x,\epsilon)\right| \le C \sup_{\substack{x,\eta\\ \beta\le\alpha\\ |\gamma|\le l}} \left|\partial_x^\beta \partial_\eta^\gamma a(x,\eta)\right| \epsilon^{N+1}.$$

REMARK. $l(\alpha, N)$ is a linear function of $|\alpha|$, N.

DEFINITION. We introduce the Fourier transformation with ϵ, which is $\lambda - Fourier$ transform in [10].

For $u(x_I) \in C_0^\infty\left(\boldsymbol{R}\,{}_x^{|I|}\right)$,

$$\left(F_{\epsilon, x_I \to \xi_I} u\right)(\xi_I) = \frac{e^{-|I|\pi i/4}}{(2\pi\epsilon)^{|I|/2}} \int_{\boldsymbol{R}_x^{|I|}} \exp\left[-\frac{i}{\epsilon} x_I \cdot \xi_I\right] u\left(x_I\right) dx_I.$$

The inverse transformation is defined by

$$\left(F^{-1}_{\epsilon, \xi_I \to x_I} v\right)(x_I) = \frac{e^{|I|\pi i/4}}{(2\pi\epsilon)^{|I|/2}} \int_{\boldsymbol{R}_\xi^{|I|}} \exp\left[\frac{i}{\epsilon} \xi_I \cdot x_I\right] v\left(\xi_I\right) d\xi_I$$

for $v \in C_0^\infty\left(\boldsymbol{R}_\xi^{|I|}\right)$.

phase function. Let Λ be the Lagrangian manifold defined by (3.14), denoted in the sequel by $x(t,y)$ and $\xi(t,y)$ without the tildes. We designate the origin in $\boldsymbol{R}^{2d+2}_{t,x,\tau,\xi}$ by $\lambda_0 \in \Lambda$. For $\lambda \in \Lambda$, we integrate the form $\tau dt + \xi dx$ along a curve connecting λ_0 and λ on Λ:

$$S(\lambda) = \int_{\lambda_0}^{\lambda} \tau dt + \xi dx.$$

This is well-defined, since $\tau dt + \xi dx$ is a closed form on the simply connected Λ. When Λ_I is a local chart with coordinates $(t, x_I, \xi_{\overline{I}})$, we define by (3.16)

$$S_I\left(t, x_I, \xi_{\overline{I}}\right) = S\left(\lambda(t, x_I, \xi_{\overline{I}})\right) - < \xi_{\overline{I}}, X_{\overline{I}}(t, x_I, \xi_{\overline{I}}) > .$$

By construction we have

$$\frac{\partial S_I}{\partial x_i} = \Xi_i(t, x_I, \xi_{\overline{I}}) \quad \text{for} \quad i \in I$$

and

$$\frac{\partial S_I}{\partial \xi_j} = -X_j(t, x_I, \xi_{\overline{I}}) \quad \text{for} \quad j \in \overline{I}.$$

invariant density. We fix an invariant measure $d\mu = dt \wedge dy_1 \wedge \ldots \wedge dy_d$ with respect to the hamiltonian flow. When D is a compact set contained in a single chart Λ_I,

$$\mu(D) = \int_0^\infty dt \int_{\pi_I(D)} \left| \det \frac{\partial(t,y)}{\partial(t,x_I,\xi_{\overline{I}})} \right| dx_I d\xi_{\overline{I}}$$

where $\pi_I(D)$ is the projection of $D|_t$ to $\boldsymbol{R}_x^{|I|} \times \boldsymbol{R}_\xi^{|\overline{I}|}$. Since $\det \frac{\partial(t,y)}{\partial(t,x_I,\xi_{\overline{I}})} = \det \frac{\partial(y)}{\partial(x_I,\xi_{\overline{I}})}$, the density of μ is denoted by

$$\mu_I(t,x_I,\xi_{\overline{I}}) = \left| \det \frac{\partial y}{\partial(x_I,\xi_{\overline{I}})}(t,x_I,\xi_{\overline{I}}) \right|.$$

index δ_I. We assume always from now on, Λ^{d+1} is of general position, that is,

$$\dim \left\{ y; \det \frac{\partial x(t,y)}{\partial y} = 0 \right\} \leq d-1.$$

Let Λ_I and Λ_J have local coordinates $(t,x_I,\xi_{\overline{I}})$ and $(t,x_J,\xi_{\overline{J}})$. Suppose $\lambda \in \Lambda_I \cap \Lambda_J$ is a nonsingular point, at which, by definition, $\det \frac{\partial x(t,y)}{\partial y} \neq 0$.

The index in $\mathbf{Z}_4$ of an ordered pair of nondisjoint charts is defined by

$$\gamma(\Lambda_I \cap \Lambda_J) = \text{ Ind } \left(\frac{\partial x_{\overline{I}}}{\partial \xi_{\overline{I}}}(\lambda) \right) - \text{ Ind } \left(\frac{\partial x_{\overline{J}}}{\partial \xi_{\overline{J}}}(\lambda) \right),$$

where Ind (A) denotes the dimension of the eigenspace with the negative eigenvalues of A.

REMARK. $\gamma(\Lambda_I \cap \Lambda_J)$ in $\mathbf{Z}_4$ is independent from choice of regular points λ in $\Lambda_I \cap \Lambda_J$ (Lemma 6.4 in [10]).

DEFINITION. Let Λ_I be a local chart of a fixed atlas $\left\{ \Lambda_{I(i)};\ i \in \mathbf{N} \right\}$. We choose a chain $\left\{ \Lambda_{I(i_k)} \right\}_{0 \leq k \leq s}$ such that

$$\Lambda_{I(i_0)} = \Lambda^{d+1}_{[0,T_0]}, \quad \Lambda_{I(i_s)} = \Lambda_I; \quad \Lambda_{I(i_k)} \cap \Lambda_{I(i_{k+1})} \neq \emptyset \quad \text{(connected)}.$$

We define δ_I in $\mathbf{Z}_4$ by

$$\delta_I = \sum_{k=0}^{s-1} \gamma(\Lambda_{I(i_k)} \cap \Lambda_{I(i_{k+1})}).$$

δ_I is independent of choice of chains, since Λ is simply connected. This follows from the fact that the difference of the two values

$$\sum_{k=0}^{s-1} \gamma\left(\Lambda_{I(i_k)} \cap \Lambda_{I(i_{k+1})}\right) - \sum_{l=0}^{t-1} \gamma\left(\Lambda_{I(j_l)} \cap \Lambda_{I(j_{l+1})}\right)$$

is considered as the closed path index ([10]).

4.2. Canonical operators and commutator relation

Maslov's canonical operators. The precanonical operators K_I are defined as follows.

1. In case Λ_I is a nonsingular chart where $\det \frac{\partial x(t,y)}{\partial y}$ never vanishes by definition:
$\lambda \in \Lambda_I$ is represented by $\lambda = \lambda_I(t,x)$. Let $h \in C_0^\infty(\Lambda_I)$.

$$K_I(h)(t,x) = \sqrt{\mu_I(t,x)}\, h(\lambda_I(t,x))\, e^{\frac{i}{\epsilon} S(\lambda_I(t,x))}. \tag{4.2}$$

2. In case Λ_I is a singular chart where the set of zeros of $\det \frac{\partial x(t,y)}{\partial y}$ is not empty by definition:
Suppose $\lambda \in \Lambda_I$ is represented by $\lambda = \lambda_I(t, x_I, \xi_{\overline{I}})$. Let $h \in C_0^\infty(\Lambda_I)$.

$$\begin{aligned} K_I(h)(t,x) &= e^{\frac{\pi i}{2}\delta_I} F^{-1}_{\epsilon,\xi_{\overline{I}} \to x_{\overline{I}}} \Big[e^{\frac{i}{\epsilon} S_I(t,x_I,\xi_{\overline{I}})} \\ &\qquad \times h(\lambda_I(t,x_I,\xi_{\overline{I}})) \sqrt{\mu_I(t,x_I,\xi_{\overline{I}})} \Big]. \end{aligned} \tag{4.3}$$

We fix a set of canonical charts $\{\Lambda_I\}$ on Λ and a partition of unity $\{e_I\}$ subordinate to this covering. Then, the canonical operator for $h \in C_0^\infty(\Lambda)$ is defined by

$$(K_\Lambda h)(t,x) = \sum_I K_I(e_I h)(t,x). \tag{4.4}$$

3. Let T be a fixed positive constant and K be a fixed compact set in Λ. For any nonnegative integer j, there exists a constant C such that for any $h \in C_0^\infty(\Lambda)$ with $\operatorname{supp} h \subset K$

$$\int_0^T \epsilon^{2j} \|D^j_{t,x} K_\Lambda h(t)\|^2 dt \le C \int_0^T \|D^j_{t,y} h(t)\|^2 dt,$$

where h in the right hand side is identified with an element in $C^\infty([0,T];$ $C_0^\infty(\boldsymbol{R}^d_y))$.

asymptotic transition operator.

LEMMA 4.2 (Lemma 9.1 in [10]). *Let Λ_I and Λ_J be non-disjoint local charts. Then, for the precanonical operators K_I and K_J there exists an infinite set of differential operators $\left\{V_{IJ}^{(k)}, \quad k=0,1,2,\cdots\right\}$ on $\Lambda_I \cap \Lambda_J$ and integral operators $\{R_N(V_{IJ};\epsilon), \quad N=1,2,\cdots\}$ such that , for any $h \in C_0^\infty(\Lambda_I \cap \Lambda_J)$ with $\operatorname{supp} h$ in $\operatorname{supp} e_I \cap \operatorname{supp} e_J$, and for any natural number N, we have*

$$(K_J h)(t,x) = K_I \sum_{k=0}^{N} \epsilon^k V_{IJ}^{(k)} h(t,x) + R_{N+1}(V_{IJ};\epsilon) h(t,x).$$

Here, $V_{IJ}^{(k)}$ is of degree $2k$, and the remainder satisfies the following estimate: for any fixed $T>0$ and for any nonnegative integer j, there exists a constant C and an integer l such that

$$\int_0^T \epsilon^{2j} \|D^j R_{N+1}(V_{IJ};\epsilon) h(t)\|^2 dt \leq C \int_0^T \epsilon^{2(N+1)} \|D^l h(t)\|^2 dt. \tag{4.5}$$

h is identified with an element in $C^\infty([0,T]; C_0^\infty(\boldsymbol{R}_y^d))$ and $l = l(j,N)$.

PROOF. We put

$$I_1 = I \cap J, \quad I_2 = I \cap \overline{J}, \quad I_3 = \overline{I} \cap J, \quad I_4 = \overline{I} \cap \overline{J}.$$

Then, we have

$$I = I_1 \cup I_2, \quad J = I_1 \cup I_3, \quad \overline{I} = I_3 \cup I_4, \quad \overline{J} = I_2 \cup I_4.$$

Let $h \in C_0^\infty(\Lambda_I \cap \Lambda_J)$. By the definition of the precanonical operator, we have

$$\begin{aligned}
&(K_J h)(t, x_J, x_{\overline{J}}) \\
&\quad = e^{\frac{\pi i}{2}\delta_I} F^{-1}_{\epsilon, \xi_{\overline{I}} \to x_{\overline{I}}} \left[e^{-\frac{\pi i}{2}\delta_I} F_{\epsilon, x_{\overline{I}} \to \xi_{\overline{I}}} (K_J h)(t,x) \right] \\
&\quad = e^{\frac{\pi i}{2}\delta_I} F^{-1}_{\epsilon, \xi_{\overline{I}} \to x_{\overline{I}}} \left[e^{\frac{\pi i}{2}(\delta_J - \delta_I)} F_{\epsilon, x_{I_3} \to \xi_{I_3}} F^{-1}_{\epsilon, \xi_{I_2} \to x_{I_2}} \sqrt{\left| \frac{\partial \mu_J}{\partial(\xi_{\overline{J}}, x_J)} \right|} \right. \\
&\quad \left. \times \exp\left(\frac{i}{\epsilon} S_J(t, x_J, \xi_{\overline{J}}) \right) h(\lambda) \right] \\
&\quad = e^{\frac{\pi i}{2}\delta_I} F^{-1}_{\epsilon, \xi_{\overline{I}} \to x_{\overline{I}}} [b_I],
\end{aligned}$$

where

$$\begin{aligned} b_I &= e^{\frac{\pi i}{2}(\delta_J-\delta_I)-\frac{\pi i}{4}|I_3|+\frac{\pi i}{4}|I_2|}(2\pi\epsilon)^{(-|I_3|-|I_2|)/2} \\ &\times \int \exp\left[\frac{i}{\epsilon}\phi(x_{I_1}, x_{I_2}, \xi_{I_3}, \xi_{I_4}; x_{I_3}, \xi_{I_2})\right] \\ &\times \sqrt{\left|\frac{\partial \mu_J}{\partial \xi_{\bar{J}} \partial x_J}\right|} h(\lambda) d\xi_{I_2} dx_{I_3}, \end{aligned}$$

with

$$\phi = -x_{I_3} \cdot \xi_{I_3} + x_{I_2} \cdot \xi_{I_2} + S_J(t, x_J, \xi_{\tilde{J}}).$$

Notice that ϕ restricted on the stationary points is equal to S_I on $\Lambda_I \cap \Lambda_J$. We have the expansion of b_I by the stationary phase method. □

Corollary. *There exist differential operators $W_{IJ}^{(k)}$ of degree $2k$ ($k \geq 0$) on $\Lambda_I \cap \Lambda_J$ such that $\sum_{k=0}^{\infty} \epsilon^k \sum_J W_{IJ}^{(k)} e_J$ is the formal inverse of $\sum_{k=0}^{\infty} \epsilon^k \sum_J V_{IJ}^{(k)} e_J$. The remainder, defined by*

$$R_{I,N+1} g = g - \sum_{k=0}^{N} \epsilon^k \sum_J V_{IJ}^{(k)} e_J \left(\sum_{l=0}^{N} \epsilon^l \sum_K W_{IK}^{(l)} e_K g\right)$$

for $g \in C^\infty(\Lambda_I)$ is a differential operator of degree $4N$ on Λ_I ($N \geq 1$). Its coefficients are of order $N+1$ with respect to ϵ.

commutator relation. Let

$$p(t, x, \tau, \xi; \epsilon) = \sum_{j=0}^{m} p_j(t, x, \xi; \epsilon) \tau^{m-j}$$

be a symbol, where p_j belongs to the usual nonhomogeneous symbol class S^j with smooth parameters t and ϵ. An ϵ-pseudodifferential operator

$$P(t, x, \epsilon D_t, \epsilon D_x; \epsilon) = \sum_{j=0}^{m} P_j(t, x, \epsilon D_x; \epsilon)(\epsilon D_t)^{m-j}$$

is defined by

$$P_j(t, x, \epsilon D_x; \epsilon) u(t, x) = F_{\epsilon,\xi\to x}^{-1}\left[p_j(t, x, \xi; \epsilon)\left(F_{\epsilon,x\to\xi} u\right)\right].$$

Its ϵ-principal symbol is by definition

$$p(t,x,\tau,\xi)=\sum_{j=0}^{m}p_j(t,x,\xi;0)\tau^{m-j}.$$

PROPOSITION 4.1. *Let $P(t,x,\epsilon D_t,\epsilon D_x;\epsilon)$ be an $\epsilon-$pseudodifferential operator. Let $p(t,x,\tau,\ \xi)$ be its ϵ-principal symbol. For K_I, there exist a set of differential opertators on Λ_I $\{T_I^{(k)};\ k=0,1,2,\cdots\}$ independent of ϵ and a set of integral opertators $\{R_N(K_I,P;\ \epsilon);\ N=1,2,\cdots\}$ dependent on ϵ such that for $h\in C_0^\infty(\Lambda_I)$ with $\operatorname{supp} h$ in $\operatorname{supp} e_I$,*

$$P(t,x,\epsilon D_t,\epsilon D_x;\epsilon)K_I(h)=K_I\sum_{k=0}^{N}\epsilon^k T_I^{(k)}h+R_{N+1}\left(K_I,P;\epsilon\right)h,$$

and that for a fixed T and for any nonnegative integer j, there exists a constant C_I and an integer l such that

$$\int_0^T\epsilon^{2j}\|D_{t,x}^jR_{N+1}h(t)\|^2dt\le C_I\int_0^T\epsilon^{2(N+1)}\|D_{t,y}^lh(t)\|^2dt.$$

More precisely,

$$T_I^{(0)} = p\left(t,x_I,-\frac{\partial S_I}{\partial\xi_{\overline{I}}},\frac{\partial S_I}{\partial t},\frac{\partial S_I}{\partial x_I},\xi_{\overline{I}}\right) \tag{4.6}$$

$$\begin{aligned}
T_I^{(1)} &= \frac{1}{i\sqrt{\mu_I}}\left(\frac{\partial p}{\partial\tau}\frac{\partial}{\partial t}+\sum_{i\in I}\frac{\partial p}{\partial\xi_i}\frac{\partial}{\partial x_i}-\sum_{i\in\overline{I}}\frac{\partial p}{\partial x_i}\frac{\partial}{\partial\xi_i}\right)\sqrt{\mu_I}\\
&\quad+\frac{1}{i}\left[\frac{1}{2}\left(\frac{\partial^2S_I}{\partial t^2}\frac{\partial^2p}{\partial\tau^2}+2\sum_{j\in I}\frac{\partial^2S_I}{\partial t\partial x_j}\frac{\partial^2p}{\partial\tau\partial\xi_j}\right.\right.\\
&\quad+\sum_{i,j\in I}\frac{\partial^2S_I}{\partial x_i\partial x_j}\frac{\partial^2p}{\partial\xi_i\partial\xi_j}+\sum_{i,j\in\overline{I}}\frac{\partial^2S_I}{\partial\xi_i\partial\xi_j}\frac{\partial^2p}{\partial x_i\partial x_j}\\
&\quad\left.-2\sum_{j\in\overline{I}}\frac{\partial^2S_I}{\partial t\partial\xi_j}\frac{\partial^2p}{\partial\tau\partial x_j}-2\sum_{i\in I,j\in\overline{I}}\frac{\partial^2S_I}{\partial x_i\partial\xi_j}\frac{\partial^2p}{\partial\xi_i\partial x_j}\right)\\
&\quad\left.-\sum_{i\in\overline{I}}\frac{\partial^2p}{\partial x_i\partial\xi_i}+i\frac{\partial P}{\partial\epsilon}\bigg|_{\epsilon=0}\right]
\end{aligned} \tag{4.7}$$

and $T_I^{(k)}$ *are linear differential operators of order* k *with the coefficients in* $C^\infty(\Lambda_I)$.

This expansion follows from the stationary phase method. In our case where P is an ϵ-differential operator, this is only differentiation of oscillatory functions under the integral.

The global canonical operator is defined by

$$K_\Lambda h = \sum_I K_I(e_I h).$$

We fix a positive T. We put

$$\begin{aligned} T_I^N &= \sum_{k=0}^N \epsilon^k T_I^{(k)}, \quad V_{IJ}^N = \sum_{k=0}^N \epsilon^k V_{IJ}^{(k)}, \\ W_{IJ}^N &= \sum_{k=0}^N \epsilon^k W_{IJ}^{(k)}. \end{aligned}$$

Let f_I be a function in $C_0^\infty(\Lambda_I)$, such that $f_I \equiv 1$ on $\mathrm{supp} e_I$. The global commutation relation is given as follows ([10]).

$$\begin{aligned} P(K_\Lambda h) &= P \sum_I K_I(e_I h) \\ &= \sum_I \left\{ K_I T_I^N(e_I h) + R_{N+1}(K_I, P; \epsilon)(e_I h) \right\} \\ &= \sum_I \Bigg\{ K_I \left(\sum_J V_{IJ}^N e_J \right) f_I \left(\sum_K W_{IK}^N e_K \right) \\ &\qquad\qquad + K_I R_{I,N+1} \Bigg\} T_I^N(e_I h) \\ &\quad + \sum_I R_{N+1}(K_I, P; \epsilon)(e_I h) \\ &= \sum_{I,J,K} K_I V_{IJ}^N e_J f_I W_{IK}^N e_K T_I^N(e_I h) \\ &\quad + \sum_I K_I R_{I,N+1} T_I^N(e_I h) + \sum_I R_{N+1}(K_I, P; \epsilon)(e_I h) \\ &= \sum_{I,J,K} (K_J - R_{N+1}(V_{IJ}; \epsilon)) e_J f_I W_{IK}^N e_K T_I^N(e_I h) \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_I K_I R_{I,N+1} T_I^N (e_I h) + \sum_I R_{N+1}(K_I, P; \epsilon)(e_I h) \\
&= \sum_J K_J e_J \sum_{I,K} f_I W_{IK}^N e_K T_I^N (e_I h) \\
&\quad - \sum_{I,J,K} R_{N+1}(V_{IJ}; \epsilon) e_J f_I W_{IK}^N e_K T_I^N (e_I h) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(transition remainder)} \\
&\quad + \sum_I K_I R_{I,N+1} T_I^N (e_I h) \quad \text{(inverse remainder)} \\
&\quad + \sum_I R_{N+1}(K_I, P; \epsilon)(e_I h) \quad \text{(commutation remainder)} \\
&= \sum_J K_J e_J T^N h + R_{N+1}(K_\Lambda, P; \epsilon) h. \quad \text{(by definition)}
\end{aligned}
$$

The remainder term $R_{N+1}(K_\Lambda, P; \epsilon)h$ has an estimate of the same type as in Proposition 4.1. T^N has an expansion $T^N = \sum_{k=0}^N \epsilon^k T^{(k)}$. If $T_I^{(0)} = 0$ for all I, which is the case in §§5,6, we have $T^{(1)} = \sum_I T_I^{(1)} e_I$.

5. Formal construction of asymptotic solutions

For any $n \in \boldsymbol{N}$, we have the Taylor expansion of L:

$$
L(t, x, D_t, D_x; \epsilon) = \sum_{n=0}^N \epsilon^n L^{(n)}(t, x, D_t, D_x) + R_{N+1}(L; \epsilon),
$$

where $L(t, x, D_t, D_x; \epsilon)$ and $R_{N+1}(L; \epsilon)$ are differential operators of order l. We have also

$$
M(t, x, D_t, D_x; \epsilon) = \sum_{n=0}^N \epsilon^n M^{(n)}(t, x, D_t, D_x) + R_{N+1}(M; \epsilon),
$$

where $M(t, x, D_t, D_x; \epsilon)$ and $R_{N+1}(M; \epsilon)$ are differential operators of order m.

We recall the notation for the Taylor expansions with respect of ϵ of the inhomogeneous data $f(t, x; \epsilon) \in C_0^\infty([0, \infty) \times \boldsymbol{R}^d \times [0, \epsilon_0])$ in (1.1) and $g_j(x; \epsilon) \in C_0^\infty(\boldsymbol{R}^d \times [0, \epsilon_0])$ in (1.2) of the Introduction:

$$
f(t, x; \epsilon) = \sum_{n=0}^N \epsilon^n f_n(t, x) + R_{N+1}(f; \epsilon), \tag{5.1}
$$

$$g_j(x;\epsilon) = \sum_{n=0}^{N} \epsilon^n g_{j,n}(x) + R_{N+1}(g_j;\epsilon). \tag{5.2}$$

We introduce for simplicity of the statements the following

Definition. Let T be fixed. Firstly, if there is a correspondence

$$\begin{aligned}\prod_{j=1}^{n} C^\infty([0,T];C_0^\infty(\boldsymbol{R}_x^d)) \times \prod_{k=1}^{m} C_0^\infty(\boldsymbol{R}_x^d) \ni (f_j(t,x), g_k(x))\\ \to u(t,x) \in C^\infty([0,T];C_0^\infty(\boldsymbol{R}_x^d))\end{aligned}$$

equipped with the following estimate (5.3), we call u is *well determined* by $\{f_j, g_k\}$:

for any given natural number p, there exist a constant C, natural numbers q_j, r_j, real numbers μ_j, ν_j, and σ_k such that

$$\begin{aligned}&\int_0^T \|D^p u(t)\|^2 dt \\ &\quad \le C\left\{\sum_{j=1}^{n}\int_0^T \|D^{q_j} f_j(t)\|_{\mu_j}^2 dt + \sum_{j=1}^{n} \|D^{r_j} f_j(0)\|_{\nu_j}^2 + \sum_{k=1}^{m} \|g_k\|_{\sigma_k}^2\right\}.\end{aligned} \tag{5.3}$$

Secondly, if there is a correspondence

$$\prod_{j=1}^{n} C_0^\infty(\boldsymbol{R}_x^d) \ni \{g_j(x)\} \to h(t,x) \in C^\infty([0,T];C_0^\infty(\boldsymbol{R}_x^d))$$

which satisfies the following estimate (5.4), we call h is *well determined* by $\{g_j\}$:

for any given natural number p, there exist a constant C and real numbers σ_k, such that

$$\int_0^T \|D^p h(t)\|^2 dt \le C \sum_{k=1}^{n} \|g_k\|_{\sigma_k}^2. \tag{5.4}$$

Lastly, if there is a correspondense

$$\prod_{j=1}^{n} C_0^\infty(\boldsymbol{R}_x^d) \ni \{g_j(x)\} \to v(x) \in C_0^\infty(\boldsymbol{R}_x^d)$$

which satisfies the following estimate, we call v is *well determined* by $\{g_k\}$:

for any given natural number p, there is a constant C and real numbers σ_k such that

$$\|v\|_p^2 \le C \sum_{k=1}^{n} \|g_k\|_{\sigma_k}^2. \tag{5.5}$$

5.1. Degeneration of order 1

The problem is

$$\begin{cases} (i\epsilon L + M)u(t,x;\epsilon) = f(t,x;\epsilon), \\ D_t^j u(0,x;\epsilon) = g_j(x;\epsilon), \qquad 0 \le j \le m. \end{cases} \tag{5.6}$$

We construct a formal expansion of the solution u along the outline in the introduction §1. We define $P = \epsilon L + i^{-1}M$ and introduce

$$\tilde{P}(t,x,\epsilon D_t,\epsilon D_x;\epsilon) = \epsilon^m P(t,x,D_t,D_x;\epsilon)$$

and its ϵ-principal symbol

$$p(t,x,\tau,\xi) = l(t,x,\tau,\xi;0) + i^{-1}m(t,x,\tau,\xi;0).$$

The singular characteristic root τ_1 or τ_{m+1} defined in §3.1 gives the Lagrangian manifold Λ and the global canonical operator of Maslov K_Λ. We seek for the singular part in the form of

$$w \sim \sum_{n=m}^{\infty} \epsilon^n w_n = \sum_{n=m}^{\infty} \epsilon^n K_\Lambda h_n,$$

where $h_n(\lambda)$'s are functions on Λ. $\tilde{P}$ has the Taylor expansion with respect to ϵ:

$$\begin{aligned} \tilde{P}(t,x,\epsilon D_t,\epsilon D_x;\epsilon) &= \sum_{n=0}^{N} \epsilon^n \tilde{P}^{(n)}(t,x,\epsilon D_t,\epsilon D_x) \\ &+ R_{N+1}(\tilde{P};\epsilon). \end{aligned}$$

$\tilde{P}^{(n)}(t,x,\tau,\xi)$'s are polynomial symbols of order at most $m+1$. $R_{N+1}(\tilde{P};\epsilon)$ is a differential operator of order at most $m+1$ and its coefficients $\tilde{a}(t,x;\epsilon)$ satisfy $\sup_{t,x}\left|D_t^j D_x^\alpha \tilde{a}(t,x;\epsilon)\right| \le C\epsilon^{N+1}$.

We have a sequence of equations for the regular part

$$v(t,x;\epsilon)=\sum_{n=0}^{\infty}\epsilon^n v_n(t,x)$$

satisfying

$$M^{(0)}v_0(t,x)=f_0(t,x), \tag{5.7}$$

and

$$M^{(0)}v_n(t,x) = f_n(t,x)-\sum_{p=0}^{n-1}(L^{(p)}+M^{(p+1)})v_{n-1-p}(t,x) \quad \text{for} \quad n\geq 1. \tag{5.8}$$

We set

$$h_0=\cdots=h_{m-1}=0. \tag{5.9}$$

Using the global commutation relation with $T^{(0)}=0$ on Λ, we have formally

$$Pw=\tilde{P}\sum_{n=0}^{\infty}\epsilon^n K_\Lambda h_{m+n}=\sum_{n=0}^{\infty}\epsilon^n K_\Lambda\sum_{k=1}^{\infty}\epsilon^k T^{(k)}h_{m+n}.$$

We have equations on Λ with global coordinates (t,y):

$$T^{(1)}h_m(t,y) = 0, \tag{5.10}$$

$$T^{(1)}h_{m+n}(t,y) = -\sum_{k=2}^{n+1}T^{(k)}h_{m+n+1-k}(t,y) \quad \text{for} \quad n\geq 1. \tag{5.11}$$

$T^{(1)}=\sum T_I^{(1)}e_I$ is a hyperbolic operator of 1st order, since $\frac{\partial p}{\partial\tau}\geq c(1+|\xi|)^m$ in (4.7). From the initial conditions (5.6), we have

$$(\epsilon D_t)^j v(0,x;\epsilon)+(\epsilon D_t)^j K_\Lambda\sum_{n=m}^{\infty}\epsilon^n\, h_n|_{t=0} \sim \epsilon^j g_j(x;\epsilon) \quad \text{for} \quad 0\leq j\leq m. \tag{5.12}$$

Since $[0,T_0]\times\boldsymbol{R}_x^d$ is the canonical chart for small $T_0>0$, we denote $h(\lambda(t,x))$ simply by $h(t,x)$, when $0\leq t<T_0$.

We assume (H0), (H1), (HP) and (S0). The argument is similar, when (HN) is assumed instead of (HP).

Lemma 5.1. *Let $S(t,x)$ be the solution to the eikonal equation $p(t,x,S_t,S_x)=0$ with Cauchy data $S(0,x)=0$ and $S_t(0,x)=\tau_1(0,x,0)$. Then,*

$$\begin{aligned} D_t^j & \left(\exp\left[\frac{i}{\epsilon}S(t,x)\right]\sqrt{\mu(t,x)}h(t,x)\right)\Big|_{t=0} \\ &= \epsilon^{-j}\left\{W_j^{(0)}(h)+\epsilon W_j^{(1)}(h)+\cdots+\epsilon^j W_j^{(j)}(h)\right\}, \end{aligned}$$

where $W_j^{(k)}$'s are linear combinations of trace operators of order at most k on $t=0$. The coefficients of $W_j^{(k)}$ have bounded derivatives on $\boldsymbol{R}_x^d$.

Especially,

$$W_j^{(0)}(h)=\left(\frac{\partial S}{\partial t}(0,x)\right)^j h(0,x).$$

Applying Lemma 5.1 to (5.12), we have

$$(\epsilon D_t)^j\, v(0,x;\epsilon)+\sum_{k=0}^{j}\epsilon^k W_j^{(k)}\left(\sum_{n=m}^{\infty}\epsilon^n h_n\right)\sim \epsilon^j g_j(x;\epsilon) \quad \text{for} \quad j=0,1,\cdots,m. \tag{5.13}$$

Hence,

$$D_t^j v_0(0,x)=g_{j,0}(x) \quad \text{for} \quad j=0,1,\cdots,m-1. \tag{5.14}$$

We will verify that $\{v_n\}$ and $\{h_n\}$ are well determined successively by the coefficients of asymtotic expansions of f and g_j's, when the supports of f and g_j's are contained in fixed compact sets.

Proposition 5.1. *Under the assumption* (H1), *$v_0(t,x)\in C^\infty([0,\infty);C_0^\infty(\boldsymbol{R}^d))$ is determined by* (5.7) *and* (5.14). *Moreover, $v_0(t,x)$ is well determined by $f_0(t,x)$ and $\{g_{j,0}(x);\,0\le j\le m-1\}$. $D_t^k v_0(0,x)$ is well determined by $\{D_t^l f_0(0);\,0\le l\le k\}$ and $\{g_{j,0};\,0\le j\le m-1\}$.*

Proof. $T^{(1)}$ is a first order ordinary smooth differential operator along the Hamilton flow. The supports of the data are contained in the fixed compact sets. Hence, the estimate easily follows. □

PROPOSITION 5.2. *The Cauchy problem of 1st order equation on* Λ *with coordinates* (t, y)

$$\begin{cases} T^{(1)}h_m(t) &= 0, \qquad (0 < t < T), \\ W_m^{(0)}h_m &= g_{m,0} - D_t^m v_0(0) \end{cases}$$

has a unique solution.

Moreover, $h_m(t)$ *and its traces* $D_t^k h_m(0)$ *are well determined by* $f_0(0, x)$ *and* $\{g_{j,0}(x);\ \ 0 \le j \le m\}$*, when the supports of the data are contained in fixed compact sets.*

PROPOSITION 5.3. *We assume all supports of data are contained fixed compact sets.*
(i) *Under the assumptions* (H1) *and* (5.9)*, there exist uniquely* $\{v_n(t, x);\ n \ge 1\}$ *and* $\{h_{m+n}(\lambda); n \ge 1\}$ *such that* $v_n(t, x)$ *satisfies*

$$(5.15) \qquad \begin{cases} M^{(0)}v_n(t) &= f_n(t) - \displaystyle\sum_{k=0}^{n-1}\left(L^{(k)} + M^{(k+1)}\right)v_{n-1-k}(t) \\ D_t^j v_n(0) &= g_{j,n} - \displaystyle\sum_{k=0}^{j} W_j^{(k)}\left(h_{j+n-k}\right) \\ & \quad j = 0, 1, \cdots, m-1. \end{cases}$$

and that $h_{m+n}(\lambda)$ *satisfies*

$$(5.16) \qquad \begin{cases} T^{(1)}h_{m+n}(t) &= -\displaystyle\sum_{p=2}^{n+1} T^{(p)}h_{m+n+1-p}(t) \\ W_m^{(0)}h_{m+n} &= g_{m,n} - D_t^m v_n(0) - \displaystyle\sum_{k=1}^{m} W_m^{(k)}\left(h_{m+n-k}\right). \end{cases}$$

(ii) *Moreover,* $v_n(t, x) \in C^\infty\left([0, T]; C^\infty(\boldsymbol{R}^d)\right)$ *is well determined by* $\{f_k(t, x); 0 \le k \le n\}$*,* $\{g_{j,k}(x); 0 \le j \le m,\ 0 \le k \le n-1\}$ *and* $\{g_{j,n}(x);\ 0 \le j \le m-1\}$*.* $D_t^{m+k}v_n(0, x)$ *is well determined by* $\{D_t^l f_q(0, x);\ 0 \le l+q \le k+n,\ 0 \le q \le n\}$*.*
(iii) $h_{m+n}(t)$ *and its traces* $D_t^k h_{m+n}(0)$ *are well determined by* $\left\{D_t^l f_q(0, x); 0 \le l+q \le n\right\}$ *and* $\{g_{j,k}(x);\ 0 \le j \le m, \quad 0 \le k \le n\}$.

PROOF. Let $n=1$. From (5.8) and (5.13), $v_1(t,x)$ satisfies:

$$\begin{cases} M^{(0)}v_1(t) &= f_1(t)-(L^{(0)}+M^{(1)})v_0(t) \\ D_t^j v_1(0) &= g_{j,1}, \qquad 0\le j\le m-2 \\ D_t^{m-1}v_1(0) &= g_{m-1,1}-W_{m-1}^{(0)}(h_m). \end{cases}$$

$v_1(t)$ is thus well determined by $\{f_0(t), f_1(t)\}$, $\{g_{j,0}; 0\le j\le m\}$ and $\{g_{j,1}; 0\le j\le m-1\}$. $D_t^{m+k}v_1(0)$ is well determined by $\{D_t^l f_q(0); 0\le l+q\le k+1, q=0,1\}$ and the same $\{g_{j,0},\ g_{j,1}\}$ as above.
From (5.11) and (5.13), $h_{m+1}(\lambda(t,y))$ satisfies:

$$\begin{cases} T^{(1)}h_{m+1}(t) &= -T^{(2)}h_m(t), \\ W_m^{(0)}(h_{m+1}) &= g_{m,1}-D_t^m v_1(0)-W_m^{(1)}(h_m). \end{cases}$$

$h_{m+1}(t)$ and $D_t^k h_{m+1}(0)$ are thus well determined by $\{f_0(0), f_1(0), D_t f_0(0)\}$ and $\{g_{j,k}; 0\le j\le m,\ k=0,1\}$.

We assume the proposition for $\{v_0,\cdots,v_{n-1}\}$ and $\{h_m, h_{m+1},\cdots, h_{m+n-1}\}$. Then, $v_n(t)$ is given by (5.15). h_{m+n} is given by (5.16). $v_n(t)$ is well determined by $f_n(t)$, $\{v_j(t); 0\le j\le n-1\}$, $\{g_{j,n}; 0\le j\le m-1\}$ and $\{D_t^k h_{n+l}(0); 0\le k+l\le m-1\}$. $h_{m+n}(t)$ is well determined by $\{h_m(t),\cdots,h_{m+n-1}(t)\}$, $g_{m,n}$, $D_t^m v_n(0)$ and $\{D_t^k h_{n+l}(0); 0\le k+l\le m,\ l\le m-1\}$. By induction, the assertion (ii) and then (iii) follow. □

5.2. Degeneration of order 2

The problem is

$$\begin{cases} (-\epsilon^2 L+M)u(t,x;\epsilon)=f(t,x;\epsilon), \\ D_t^j u(0,x;\epsilon)=g_j(x;\epsilon), \qquad 0\le j\le m+1. \end{cases}$$

We define $P=(\epsilon)^2L-M$ and introduce

$$\tilde{P}(t,x,\epsilon D_t,\epsilon D_x;\epsilon)=\epsilon^m P(t,x,D_t,D_x;\epsilon)$$

and its ϵ-principal symbol

$$p(t,x,\tau,\xi)=l(t,x,\tau,\xi;0)-m(t,x,\tau,\xi;0).$$

The singular roots $\tau_1(=\tau_-)$ and $\tau_{m+2}(=\tau_+)$ defined in §3.2 give the Lagrangian manifolds $\Lambda^*(*=+,-)$ and the global canonical operators of

Maslov K_{Λ^*}. We assume for the singular part

$$w \sim \sum_{n=m}^{\infty} \epsilon^n w_n = \sum_{n=m}^{\infty} \epsilon^n \sum_{*=\pm} w_n^* = \sum_{\substack{n=m \\ *=\pm}}^{\infty} \epsilon^n K_{\Lambda^*} h_n^*.$$

We have a sequence of equations for the regular part

$$v(t,x;\epsilon) = \sum_{n=0}^{\infty} \epsilon^n v_n(t,x)$$

satisfying

$$M^{(0)} v_0(t,x) = f_0(t,x), \tag{5.17}$$

$$M^{(0)} v_1(t,x) = f_1(t,x) - M^{(1)} v_0(t,x), \tag{5.18}$$

and

$$\begin{aligned} M^{(0)} v_n(t,x) &= f_n(t,x) - M^{(1)} v_{n-1}(t,x) \\ &\quad + \sum_{p=0}^{n-2} (L^{(p)} - M^{(p+2)}) v_{n-2-p}(t,x) \quad \text{for} \quad n \geq 2. \end{aligned} \tag{5.19}$$

We seek solutions under the assumption

$$h_0^* = \cdots = h_{m-1}^* = 0. \tag{5.20}$$

Using the global commutation relation with $T^{*(0)} = 0$ on Λ^*, we have formally

$$Pw = \tilde{P} \sum_{\substack{n=0 \\ *=\pm}}^{\infty} \epsilon^n K_{\Lambda^*} h_{m+n}^* = \sum_{\substack{n=0 \\ *=\pm}}^{\infty} \epsilon^n K_{\Lambda^*} \sum_{k=1}^{\infty} \epsilon^k T^{*(k)} h_{m+n}^*.$$

$\{h_{m+n}^*\}_{n\geq 0}$ should satisfy equations on Λ^* with global coordinates (t,y):

$$T^{*(1)} h_m^*(t) = 0 \tag{5.21}$$

and

$$T^{*(1)} h_{m+n}^*(t) = -\sum_{k=2}^{n+1} T^{*(k)} h_{m+n+1-k}(t) \quad \text{for} \quad n = 1, 2, \cdots. \tag{5.22}$$

The initial conditions give

$$(5.23)\qquad \begin{aligned} D_t^j u(0,x;\epsilon) &\sim \sum_{n=0}^{\infty} \epsilon^n D_t^j v_n(0,x) \\ &\quad + \sum_{n=m}^{\infty} \epsilon^n \{D_t^j w_n^+(0,x;\epsilon) + D_t^j w_n^-(0,x;\epsilon)\} \\ &\sim \sum_{n=0}^{\infty} \epsilon^n g_{j,n}(x). \end{aligned}$$

Near the initial plane, we have

$$w_n^*(t,x;\epsilon) = \sqrt{\mu_*(t,x)} \exp\left[\frac{iS^*(t,x)}{\epsilon}\right] h_n^*(t,x).$$

Here, $\mu_*(t,x) = |J_*(t,x)|^{-1}$ and $J_*(t,x^*(t,y)) = \det(\partial x_i^*(t,y)/\partial y_j)$ with $J_*(0,y) = 1$. $S^*(t,x)$ is the solution to

$$\begin{aligned} \frac{\partial S^-}{\partial t} - \tau_1\left(t,x,\frac{\partial S^-}{\partial x}\right) &= 0, \\ \frac{\partial S^+}{\partial t} - \tau_{m+2}\left(t,x,\frac{\partial S^+}{\partial x}\right) &= 0 \end{aligned}$$

with initial data

$$S^-(0,x) = 0, \qquad S^+(0,x) = 0$$

$$\frac{\partial S^-}{\partial t}(0,x) = \tau_1(0,x,0), \qquad \frac{\partial S^+}{\partial t}(0,x) = \tau_{m+2}(0,x,0).$$

Then, by the Lemma 5.1,

$$(5.24)\qquad \begin{aligned} &\sum_{p=m}^{\infty} \epsilon^p \left(D_t^j w_p^*\right)(0,x) \\ &= \sum_{p=m}^{\infty} \epsilon^{p-j} \sum_{k=0}^{j} \epsilon^k W_j^{*(k)}(h_p^*)(x) \\ &= \epsilon^{m-j} \sum_{l=0}^{\infty} \epsilon^l \sum_{q=\max\{l-j,0\}}^{l} W_j^{*(l-q)}\left(h_{q+m}^*(x)\right). \\ &\qquad\qquad \text{for} \quad j = 0,1,\cdots,m+1. \end{aligned}$$

From (5.24) with $j = m+1$, we have

$$(5.25) \qquad \left(S_t^+\right)^{m+1} h_m^+(0,x) + \left(S_t^-\right)^{m+1} h_m^-(0,x) = 0.$$

When (j,n) satisfies the inequality $0 \le n+j \le m-1$, (5.23) implies

$$(5.26) \qquad D_t^j v_n(0,x) = g_{j,n}(x),$$

since $m-j \ge n+1$. For the rest of (j,n), we have

$$(5.27) \qquad \begin{aligned} D_t^j v_n(0) \quad &+ \sum_{q=0}^{\min\{j,n-m+j\}} W_j^{+(q)} \left(h_{n+j-q}^+\right) \\ &+ \sum_{q=0}^{\min\{j,n-m+j\}} W_j^{-(q)} \left(h_{n+j-q}^-\right) = g_{j,n}, \end{aligned}$$

that is,

$$(5.28) \qquad \begin{aligned} & D_t^j v_n(0) + W_j^{+(0)}\left(h_{n+j}^+\right) + W_j^{-(0)}\left(h_{n+j}^-\right) \\ = \quad & g_{j,n} - \sum_{q=1}^{\min\{j,n-m+j\}} \left\{ W_j^{+(q)}\left(h_{n+j-q}^+\right) + W_j^{-(q)}\left(h_{n+j-q}^-\right) \right\}. \end{aligned}$$

Here, $n+j \ge m$ and the sum in the right hand side should read 0, if $n-m+j=0$.

Later, we need the initial conditions for the transport equations of $h_{m+n}^{\pm}$. They will be given by (5.23) and (5.24).

Proposition 5.4. *We assume all supports of data are contained fixed compact sets.*
(i) *Under the assumptions* (D2), (H0), (H1), (P) *and* (WS), *there exist uniquely* $\{v_n(t,x);\, n \ge 1\}$ *and* $\{h_{m+n}^*(\lambda); n \ge 1\}$ *such that* $v_n(t,x)$ *satisfies*

$$(5.29)\, M^{(0)} v_n(t) = \begin{cases} f_1(t) - M^{(1)} v_0(t), \quad n=1 \\ f_n(t) - M^{(1)} v_{n-1}(t) \\ \qquad + \displaystyle\sum_{p=0}^{n-2} \left(L^{(p)} - M^{(p+2)}\right) v_{n-2-p}(t), \quad n \ge 2 \end{cases}$$

and

$$
(5.30)\quad \begin{cases} D_t^j v_n(0,x) &= g_{j,n}(x) \\ & \quad \textit{for} \quad j = 0,1,\cdots,m-1-n, \quad \textit{if} \quad n < m, \\ D_t^j v_n(0,x) &= g_{j,n}(x) - \displaystyle\sum_{q=0}^{\min\{j,n-m+j\}} \Big\{ W_j^{+(q)}\left(h^+_{n+j-q}\right) \\ & \qquad + W_j^{-(q)}\left(h^-_{n+j-q}\right)\Big\} \\ & \quad \textit{for} \quad j = \max\{0,m-n\},\cdots,m-1. \end{cases}
$$

and that $h^*_{m+n}(\lambda)$ *satisfies*

$$
(5.31)\qquad T^{*(1)}h^*_{m+n}(t) = -\sum_{p=2}^{n+1} T^{*(p)}h^*_{m+n+1-p}(t)
$$

and

$$
(5.32)\quad \begin{cases} \displaystyle\sum_{*=\pm} W_m^{*(0)}\left(h^*_{m+n}\right) \\ \quad = g_{m,n} - \displaystyle\sum_{\substack{q=1 \\ *=\pm}}^{\min\{m,n\}} W_m^{*(q)}\left(h^*_{n+m-q}\right) - D_t^m v_n(0) \\ \displaystyle\sum_{*=\pm} W_{m+1}^{*(0)}\left(h^*_{m+n}\right) \\ \quad = g_{m+1,n-1} - \displaystyle\sum_{\substack{q=1 \\ *=\pm}}^{\min\{m+1,n\}} W_{m+1}^{*(q)}\left(h^*_{n+m-q}\right) \\ \qquad - D_t^{m+1} v_{n-1}(0). \end{cases}
$$

(ii) *Moreover,* $v_n(t,x) \in C^\infty\left([0,T]; C^\infty(\boldsymbol{R}^d)\right)$ *is well determined by* $\{f_k(t,x); 0 \le k \le n\}$, $\{g_{j,n}; 0 \le j \le m-1\}$, $\{g_{j,n-1}; 0 \le j \le m\}$ *and* $\{g_{j,k}; 0 \le j \le m+1,\ 0 \le k \le n-2\}$.

$D_t^{m+k}v_n(0)$ *is well determined by* $\{D_t^l f_q(0); 0 \le l \le 2\left[\frac{n-q}{2}\right] + k, 0 \le q \le n\}$ *and the same* $\{g_{j,k}\}$ *as above. Here,* $[r]$ *is the greatest integer less than or equal to* r.

(iii) $h^*_{m+n}(\lambda)$ *and* $D_t^k h_{m+n}(0)$ *are well determined by* $\{D_t^l f_q(0,x); 0 \le l+q \le n\}$, $\{g_{j,n}(x);\ 0 \le j \le m\}$ *and* $\{g_{j,k}(x); 0 \le j \le m+1,\ 0 \le k \le n-1\}$.

PROOF. At first, from (5.23),

$$\left\{\begin{array}{lcl} M^{(0)}v_0(t) & = & f_0(t) \\ D_t^j v_0(0) & = & g_{j,0}, \quad j=0,1,\cdots,m-1. \end{array}\right. \tag{5.33}$$

Since $M^{(0)}$ is regularly hyperbolic, $v_0(t,x) \in C^\infty\left([0,\infty);C_0^\infty(\boldsymbol{R}^d)\right)$ is well determined by $\{f_0(t)\}$ and $\{g_{j,0}; 0 \le j \le m-1\}$. $D_t^{m+k}v_0(0)$ is well determined by $\{D_t^l f_0(0); 0 \le l \le k\}$ and $\{g_{j,0}; 0 \le j \le m-1\}$.

The transportation equations for $h_m^\pm$ are

$$\left\{\begin{array}{lcl} T^{(1)}h_m^+(t) & = & 0 \\ T^{(1)}h_m^-(t) & = & 0 \end{array}\right. \tag{5.34}$$

with the initial condition

$$\left\{\begin{array}{lcl} W_m^{+(0)}(h_m^+) + W_m^{-(0)}(h_m^-) & = & g_{m,0} - D_t^m v_0(0) \\ W_{m+1}^{+(0)}(h_m^+) + W_{m+1}^{-(0)}(h_m^-) & = & 0. \end{array}\right. \tag{5.35}$$

(5.35) comes from (5.25) and (5.28) with $n=0$. (5.35) is rewritten by

$$\left\{\begin{array}{lcl} \left(S_t^+(0)\right)^m h_m^+(0) + \left(S_t^-(0)\right)^m h_m^-(0) & = & g_{m,0} - D_t^m v_0(0) \\ \left(S_t^+(0)\right)^{m+1} h_m^+(0) + \left(S_t^-(0)\right)^{m+1} h_m^-(0) & = & 0. \end{array}\right. \tag{5.36}$$

This gives $h_m^\pm(0,x) \in C_0^\infty\left(\boldsymbol{R}^d\right)$. From (5.34), we have $h_m^\pm \in C^\infty(\Lambda^\pm)$ and $D_t^k h_m^\pm(0)$, well determined by $\{f_0(0)\}$ and $\{g_{j,0}; 0 \le j \le m\}$.

Then, we see similarly that $v_1(t)$ and $D_t^{m+k}v_1(0)$, $h_{m+1}(t)$ and $D_t^k h_{m+1}(0)$, $v_2(t)$ and $D_t^{m+k}v_2(0)$ are well determined successively.

We will construct $\{v_n; n=0,1,\cdots\}$, $\{h_{m+n}^\pm; n=0,1,\cdots\}$ by induction. We assume the proposition for $v_0,\cdots,v_{n-1}$ and $h_m^\pm,\cdots,h_{m+n-1}^\pm$, from which we derive v_n and $h_{m+n}^\pm$. In fact, we note that the traces of $h_{n+j-q}^\pm$ in (5.30) are known, since $m \le n+j-q \le n+j \le n+m-1$. With the initial data (5.30), the equation (5.29) gives $v_n(t,x) \in C^\infty\left([0,\infty);C_0^\infty(\boldsymbol{R}^d)\right)$ well determined by $f_n(t)$, $\{v_j(t); 0 \le j \le n-1\}$, $\{g_{j,n};\ 0 \le j \le m-1\}$, and traces of $\{h_{m+j}^\pm(t); 0 \le j \le n-1\}$. By induction, we have (ii). Then, from (5.28), we have (5.32), of which the right hand sides are all known. The initial values $h_{m+n}^\pm(0)$ are thus determined. Hence, $h_{m+n}^\pm(t) \in C^\infty(\Lambda^\pm)$ are well determined by $\{h_j^\pm(t); m \le j \le n+m-1\}$, $g_{m,n}$, $g_{m+1,n-1}$ and by $D_t^{m+1}v_{n-1}(0)$, $D_t^m v_n(0)$. By induction, we have (iii). □

6. Remainder estimates of asymptotic solutions

6.1. Degeneration of order 1

We define the partial sum by

$$u_N(t,x;\epsilon)=\sum_{n=0}^{N}\epsilon^n v_n(t,x)+\sum_{n=m}^{N+m}\epsilon^n K_\Lambda h_n(t,x;\epsilon)$$

and its remainder term by

$$R_{N+1}(u;\epsilon)=u(t,x;\epsilon)-u_N(t,x;\epsilon).$$

Our main result is

Theorem 6.1. *Let T be a fixed positive number. Let $f\in C_0^\infty([0,T]\times \boldsymbol{R}^d\times[0,\epsilon_0])$ and $g_j\in C_0^\infty(\boldsymbol{R}^d\times[0,\epsilon_0])$ with their supports containd in fixed compact sets independent of ϵ. For any $p,N\in\boldsymbol{N}$, there exists a positive constant C independent of ϵ such that for any $\epsilon\in(0,\epsilon_0]$,*

$$\begin{aligned}
&C\,\epsilon^{2(N+1)-1}\\
\geq\ &\int_0^T\sum_{j=0}^{p}\epsilon^{2j}\Big(\epsilon\parallel D^{m+j}R_{N+1}(u;\epsilon)(t)\parallel^2\\
&\qquad\qquad+\parallel D^{m+j-1}R_{N+1}(u;\epsilon)(t)\parallel_{1/2}^2\Big)\,dt\\
&+\sum_{j=0}^{p}\epsilon^{2j}\Big(\epsilon\parallel D^{m+j}R_{N+1}(u;\epsilon)(T)\parallel^2\\
&\qquad\qquad+\parallel D^{m+j-1}R_{N+1}(u;\epsilon)(T)\parallel_{1/2}^2\Big)\,.
\end{aligned}$$

Corollary. *For any $k,N_0\in\boldsymbol{N}$ and positive T, there exist $N_1\in\boldsymbol{N}$ such that for any $N\geq N_1$ there exists a positive constant C_{N,N_0} independent of ϵ such that*

$$\sup_{\substack{0\leq t\leq T\\ x\in R^d}}\sum_{j+|\alpha|\leq k}|D_t^jD_x^\alpha R_{N+1}(u;\epsilon)(t,x)|\leq C_{N,N_0}\epsilon^{N_0}.$$

In order to estimate $R_{N+1}(u;\epsilon)$ by Theorem 2.1, we need

PROPOSITION 6.1. *The remainder term $R_{N+1}(u;\epsilon)$ satisfies*

$$(6.1)\quad \begin{aligned} (i\epsilon L+M)R_{N+1}(u;\epsilon) &= R_{N+1}(f;\epsilon) \\ &\quad +\epsilon^{N+1}\rho(t,x;\epsilon)+\epsilon^{N+1}\chi(t,x;\epsilon), \\ D_t^j R_{N+1}(u;\epsilon)(0,x) &= R_{N+1}(g_j;\epsilon)+\epsilon^{N+1}\eta_j(x;\epsilon), \\ &\quad 0\le j\le m, \end{aligned}$$

where

$$(6.2)\quad \begin{aligned} \rho(t,x;\epsilon) &= \sum_{\substack{p+q\ge N\\ 0\le p\le N-1\\ 1\le q\le N}} \epsilon^{p+q-N}L^{(p)}v_q+\sum_{\substack{p+q\ge N+1\\ 1\le p\le N\\ 1\le q\le N}} \epsilon^{p+q-N-1}M^{(p)}v_q \\ &\quad -\sum_{q=0}^{N} \epsilon^q\left(\epsilon^{-N}R_N(L;\epsilon)+\epsilon^{-N-1}R_{N+1}(M;\epsilon)\right)v_q, \end{aligned}$$

$$(6.3)\quad \begin{aligned} \chi(t,x;\epsilon) &= K_\Lambda \sum_{\substack{p+q\ge N+1\\ 1\le p,q\le N}} \epsilon^{p+q-N-1}T^{(p)}h_{m+q} \\ &\quad +\epsilon^{-N-1}R_{N+1}(K_\Lambda,\tilde{P};\epsilon)\left(\sum_{q=0}^{N} \epsilon^q h_{m+q}\right), \end{aligned}$$

and where

$$(6.4)\quad \begin{cases} \eta_j(x;\epsilon) = -\displaystyle\sum_{\substack{m-j+p+q\ge N+1\\ \max\{N-m+1,0\}\le p\le N\\ 0\le q\le j}} \epsilon^{m-j+p+q-N-1}W_j^{(q)}(h_{p+m}) \\ \eta_m(x;\epsilon) = -\displaystyle\sum_{\substack{p+q\ge N+1\\ \max\{N-m+1,0\}\le p\le N\\ 1\le q\le m}} \epsilon^{p+q-N-1}W_m^{(q)}(h_{p+m}). \end{cases}$$

PROOF. It is long but straightforward computation from construction of $\{v_n\}$ and $\{h_{m+n}\}$. (See the proof of Proposition 6.2.) □

PROOF OT THE MAIN THEOREM 6.1.
(i) By the assumption on $f(t,x;\epsilon)$, there exists a constant $C_{j,N}(f)$ such that

$$\int_0^T e^{-2\gamma t}\|D^j R_{N+1}(f;\epsilon)(t)\|^2 dt \le \frac{C_{j,N}(f)}{\gamma}\epsilon^{2(N+1)},$$

and

$$\|D^j R_{N+1}(f;\epsilon)(0)\|^2 \leq C_{j,N}(f)\epsilon^{2(N+1)}.$$

$C_{j,N}(f)$ depends on the norms of $\left(\frac{\partial}{\partial\epsilon}\right)^{N+1} f$, but it is bounded when ϵ tends to 0.

(ii)

$$\begin{aligned}\int_0^T e^{-2\gamma t}\|D^j\rho(t;\epsilon)\|^2 dt &\leq C_N \int_0^T e^{-2\gamma t}\sum_{q=0}^{N}\|D^{m+1+j}v_q(t)\|^2 dt \\ &= \frac{C'_{N,j}}{\gamma}.\end{aligned}$$

Here, by Proposition 5.3, $C'_{j,N}$ depends on the norms of $\{f_j(t); 0 \leq j \leq N\}$, $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N-1\}$, $\{g_{j,N}; 0 \leq j \leq m-1\}$ and on their supports, but it is bounded when ϵ tends to 0. We have

$$\begin{aligned}\|D^j\rho(0;\epsilon)\|^2 &\leq C_N \sum_{q=0}^{N} \|D^{m+1+j}v_q(0)\|^2 \\ &\leq C''_{j,N}.\end{aligned}$$

The dependence of $C''_{j,N}$ is just like that of $C'_{j,N}$. In fact, it depends on the norms of $\{D_t^l f_q(0);\ 0 \leq l+q \leq j+1+N, 0 \leq q \leq N\}$ and the same $\{g_{j,k}\}$ as above.

(iii)

$$\epsilon^{2j}\int_0^T e^{-2\gamma t}\|D^j\chi(t)\|^2 dt \leq \frac{C_{j,N}}{\gamma},$$

$C_{j,N}$ depends on the norms of $\{D_t^l f_q(0);\ 0 \leq l+q \leq N\}$ and $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N\}$ and on their supports by Propositions 4.1, 5.3, but it is bounded when ϵ tends to 0. We have also

$$\epsilon^{2j}\|D^j\chi(0)\|^2 \leq C'_{j,N}.$$

(iv)

$$\sum_{k=0}^{m}\|R_{N+1}(g_k;\epsilon)\|^2_{m-k+j} \leq C'_{j,N}\epsilon^{2(N+1)},$$

$$\sum_{k=0}^{m}\|\eta_k(\epsilon)\|^2_{m-k+j} \leq C''_{j,N}.$$

$C'_{j,N}$ depends on the norms of $\{\frac{\partial^{N+1} g_k}{\partial \epsilon^{N+1}}; k = 0, \cdots, m\}$, but stays bounded for ϵ. By Proposition 5.3, $C''_{j,N}$ depends on the norms of $\{D_t^l f_q(0); 0 \leq l + q \leq N\}$ and $\{g_{j,k}; 0 \leq j \leq m, 0 \leq k \leq N\}$ and on their supports, but it is bounded when ϵ tends to 0. We have the conclusion by Theorem 2.1 applicable to (6.1). □

6.2. Degeneration of order 2

We define the partial sum by

$$u_N(t,x;\epsilon) = \sum_{n=0}^{N} \epsilon^n v_n(t,x) + \sum_{\substack{n=m \\ *=\pm}}^{N+m} \epsilon^n K_{\Lambda^*} h_n^*(t,x;\epsilon)$$

and its remainder term by

$$R_{N+1}(u;\epsilon) = u(t,x;\epsilon) - u_N(t,x;\epsilon).$$

Our main result is

THEOREM 6.2. *Let T be a fixed positive number. Let $f \in C_0^\infty([0,T] \times \boldsymbol{R}^d \times [0,\epsilon_0])$ and $g_j \in C_0^\infty(\boldsymbol{R}^d \times [0,\epsilon_0])$ with their supports contained in fixed compact sets independent of ϵ. For any $p, N \in \boldsymbol{N}$, there exists a positive constant C independent of ϵ such that for any $\epsilon \in (0,\epsilon_0]$,*

$$\begin{aligned} & C\,\epsilon^{2(N+1)-2} \\ \geq\ & \int_0^T \sum_{j=0}^{p} \epsilon^{2j} \Big(\epsilon^2 \parallel D^{m+j+1} R_{N+1}(u;\epsilon)(t) \parallel^2 \\ & \qquad\qquad + \parallel D^{m+j} R_{N+1}(u;\epsilon)(t) \parallel^2\Big)\, dt \\ & + \sum_{j=0}^{p} \epsilon^{2j} \Big(\epsilon^2 \parallel D^{m+j+1} R_{N+1}(u;\epsilon)(T) \parallel^2 \\ & \qquad\qquad + \parallel D^{m+j} R_{N+1}(u;\epsilon)(T) \parallel^2\Big). \end{aligned}$$

COROLLARY. *For any $k, N_0 \in \boldsymbol{N}$ and positive T, there exist $N_1 \in \boldsymbol{N}$ such that for any $N \geq N_1$ there exists a positive constant C_{N,N_0} independent of ϵ such that*

$$\sup_{\substack{0 \leq t \leq T \\ x \in R^d}} \sum_{j+|\alpha| \leq k} |D_t^j D_x^\alpha R_{N+1}(u;\epsilon)(t,x)| \leq C_{N,N_0} \epsilon^{N_0}.$$

PROPOSITION 6.2. *The remainder term $R_{N+1}(u;\epsilon)$ satisfies*

$$
\begin{aligned}
(6.5)\qquad & \{\,(i\epsilon)^2L+M\}R_{N+1}(u;\epsilon)\\
&= \;\; R_{N+1}(f;\epsilon)+\epsilon^{N+1}\rho(t,x;\epsilon)+\epsilon^{N+1}\chi(t,x;\epsilon),\\
& D_t^{\,j}R_{N+1}(u;\epsilon)(0,x)=R_{N+1}(g_j;\epsilon)+\epsilon^{N+1}\eta_j(x;\epsilon),\\
&\qquad 0\le j\le m,\\
& D_t^{\,m+1}R_{N+1}(u;\epsilon)(0,x)=R_{N+1}(g_{m+1};\epsilon)+\epsilon^{N}\eta_{m+1}(x;\epsilon),
\end{aligned}
$$

where

$$
\begin{aligned}
(6.6)\;\; \rho(t,x;\epsilon) \;\; = & \sum_{\substack{p+q\ge N-1\\ 0\le p\le N-2\\ 1\le q\le N}} \epsilon^{p+q+1-N}L^{(p)}v_q\\
& - \sum_{\substack{p+q\ge N+1\\ 1\le p\le N\\ 1\le q\le N}} \epsilon^{p+q-N-1}M^{(p)}v_q\\
& + \sum_{q=0}^{N} \epsilon^q\left(\epsilon^{1-N}R_{N-1}(L;\epsilon)-\epsilon^{-N-1}R_{N+1}(M;\epsilon)\right)v_q,\\
(6.7)\;\; \chi(t,x;\epsilon) \;\; = & \sum_{*=\pm}\Bigg\{K_{\Lambda^*}\sum_{\substack{p+q\ge N+1\\ 1\le p,q\le N}} \epsilon^{p+q-N-1}T^{*(p)}h^*_{m+q}\\
& \qquad + \epsilon^{-N-1}R_{N+1}(K^*,\tilde{P};\epsilon)\left(\sum_{q=0}^{N}\epsilon^q h^*_{m+q}\right)\Bigg\},
\end{aligned}
$$

and where

$$
(6.8)\quad \begin{cases}
\eta_j(x;\epsilon) = \displaystyle\sum_{\substack{m-j+p+q\geq N+1\\ \max\{N-m+1,0\}\leq p\leq N\\ 0\leq q\leq j,*=\pm}} \epsilon^{m-j+p+q-N-1} W_j^{*(q)}(h^*_{p+m})\\
\eta_m(x;\epsilon) = -\displaystyle\sum_{\substack{p+q\geq N+1\\ \max\{N-m+1,0\}\leq p\leq N\\ 1\leq q\leq m,*=\pm}} \epsilon^{p+q-N-1} W_m^{*(q)}(h^*_{p+m}),\\
\eta_{m+1}(x;\epsilon) = -\epsilon \displaystyle\sum_{\substack{p+q\geq N+2\\ \max\{N-m+1,0\}\leq p\leq N\\ 2\leq q\leq m+1,*=\pm}} \epsilon^{p+q-N-2} W_{m+1}^{*(q)}(h^*_{p+m})\\
\qquad + g_{m+1,N}\\
\qquad - \left\{ D_t^{m+1} v_N(0,x) + \displaystyle\sum_{\substack{1\leq q\leq \min\{m+1,N+1\}\\ *=\pm}} W_{m+1}^{*(q)}\left(h^*_{m+N+1-q}\right)\right\}.
\end{cases}
$$

Proof. Firstly,

$$
\begin{aligned}
&\{\ (i\epsilon)^2 L + M\} \left(u(t,x;\epsilon) - \sum_{q=0}^{N} \epsilon^q v_q(t,x)\right)\\
= \quad & f(t,x;\epsilon) + \left\{ \sum_{p=0}^{N-2}\sum_{q=0}^{N} \epsilon^{p+q+2} L^{(p)} - \sum_{p=0}^{N}\sum_{q=0}^{N} \epsilon^{p+q} M^{(p)} \right\} v_q\\
& + \sum_{q=0}^{N} \epsilon^q \left(\epsilon^2 R_{N-1}(L;\epsilon) - R_{N+1}(M;\epsilon)\right) v_q\\
= \quad & f(t,x;\epsilon) - M^{(0)} v_0 - \epsilon \left(M^{(0)} v_1 + M^{(1)} v_0 \right)\\
& + \sum_{n=2}^{N} \epsilon^n \left\{ \sum_{p=0}^{n-2} \left(L^{(p)} - M^{(p+2)}\right) v_{n-2-p} - M^{(0)} v_n - M^{(1)} v_{n-1}\right\}\\
& + \epsilon^{N+1} \left\{ \sum_{\substack{p+q\geq N-1\\ 0\leq p\leq N-2\\ 1\leq q\leq N}} \epsilon^{p+q+1-N} L^{(p)} v_q - \sum_{\substack{p+q\geq N+1\\ 1\leq p\leq N\\ 1\leq q\leq N}} \epsilon^{p+q-N-1} M^{(p)} v_q \right\}
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{q=0}^{N} \epsilon^q \left(\epsilon^2 R_{N-1}(L;\epsilon) - R_{N+1}(M;\epsilon)\right) v_q \\
= \quad & R_{N+1}(f;\epsilon) + \epsilon^{N+1}\rho(t,x;\epsilon) \quad \text{by definition.}
\end{aligned}
$$

Secondly,

$$
\begin{aligned}
& \{ (i\epsilon)^2 L + M\} \sum_{n=m}^{N+m} \epsilon^n w_n(t,x;\epsilon) \\
= \quad & \{-\epsilon^{m+2} L + \epsilon^m M\} \sum_{n=0}^{N} \epsilon^n w_{m+n}(t,x;\epsilon) \\
= \quad & \tilde{P}(t,x,\epsilon D_t,\epsilon D_x;\epsilon) \sum_{q=0}^{N} \epsilon^q \sum_{*=\pm} K_{\Lambda^*}\left(h^*_{m+q}\right) \\
= \quad & \sum_{q=0}^{N} \epsilon^q \sum_{*=\pm} \left\{ K_{\Lambda^*} \sum_{p=1}^{N} \epsilon^p T^{*(p)} h^*_{m+q} + R^*_{N+1}(K,\tilde{P};\epsilon) h^*_{m+q} \right\} \\
= \quad & \sum_{*=\pm} \left\{ K_{\Lambda^*} \sum_{n=1}^{N} \epsilon^n \sum_{p=1}^{n} T^{*(p)} h^*_{m+n-p} \right\} \\
& + \epsilon^{N+1} \sum_{*=\pm} K_{\Lambda^*} \sum_{\substack{p+q\geq N+1 \\ 1\leq p,q\leq N}} \epsilon^{p+q-N-1} T^{*(p)} h^*_{m+q} \\
& + \sum_{*=\pm} R^*_{N+1}(K,\tilde{P};\epsilon) \left(\sum_{q=0}^{N} \epsilon^q h^*_{m+q} \right) \\
= \quad & \epsilon^{N+1}\chi(t,x;\epsilon) \quad \text{by definition.}
\end{aligned}
$$

We compute the initial condition.

$$
\begin{aligned}
& D_t^j R_{N+1}(u;\epsilon)(0,x) = g_j(x;\epsilon) - D_t^j u_N(0,x;\epsilon) \\
= \quad & g_j(x;\epsilon) - \left\{ \sum_{n=0}^{N} \epsilon^n \left(D_t^j v_n\right)(0,x) + \sum_{p=m}^{N+m} \epsilon^p D_t^j \left[\sum_{*=\pm} K_{\Lambda^*} h^*_p \right]_{t=0} \right\} \\
= \quad & R_{N+1}(g_j;\epsilon) + \sum_{n=0}^{N} \epsilon^n \left\{ g_{j,n}(x) - D_t^j v_n(0,x) \right\} \\
& - \epsilon^{m-j} \sum_{n=0}^{N} \epsilon^n \sum_{\substack{0\leq q\leq \min\{j,n\} \\ *=\pm}} W_j^{*(q)} h^*_{m+n-q}
\end{aligned}
$$

$$-\epsilon^{N+1} \sum_{\substack{p+q\geq N+1 \\ \max\{0,N-j+1\}\leq p\leq N \\ 1\leq q\leq j \\ *=\pm}} \epsilon^{m+p+q-j-N-1} W_j^{*(q)}\left(h^*_{p+m}\right).$$

When $0 \leq j \leq m-1$, this turns out to be

$$\begin{aligned}
&D_t^j R_{N+1}(u;\epsilon)(0,x) \\
= \;& R_{N+1}(g_j;\epsilon) + \sum_{n=0}^{m-j-1} \epsilon^n \left\{ g_{j,n}(x) - D_t^j v_n(0,x) \right\} \\
&+ \sum_{n=m-j}^{N} \epsilon^n \left\{ g_{j,n}(x) - D_t^j v_n(0,x) - \sum_{\substack{q=0 \\ *=\pm}}^{\min\{j,n-m+j\}} W_j^{*(q)}\left(h^*_{n+j-q}\right) \right\} \\
&-\epsilon^{N+1} \sum_{\substack{m-j+p+q\geq N+1 \\ \max\{0,N-m+1\}\leq p\leq N \\ 0\leq q\leq j \\ *=\pm}} \epsilon^{m+p+q-j-N-1} W_j^{*(q)}\left(h^*_{p+m}\right) \\
= \;& R_{N+1}(g_j;\epsilon) + \epsilon^{N+1}\eta_j(x;\epsilon).
\end{aligned}$$

When $j = m$, we have

$$\begin{aligned}
&D_t^m R_{N+1}(u;\epsilon)(0,x) \\
= \;& R_{N+1}(g_m;\epsilon) + \sum_{n=0}^{N} \epsilon^n \left\{ g_{m,n}(x) - D_t^m v_n(0,x) \right\} \\
&- \sum_{p=0}^{N} \epsilon^p \sum_{\substack{q=0 \\ *=\pm}}^{m} \epsilon^q W_m^{*(q)}\left(h^*_{p+m}\right) \\
= \;& R_{N+1}(g_m;\epsilon) \\
&+ \sum_{n=0}^{N} \epsilon^n \left\{ g_{m,n}(x) - D_t^m v_n(0,x) - \sum_{\substack{q=0 \\ *=\pm}}^{\min\{m,n\}} W_m^{*(q)}\left(h^*_{m+n-q}\right) \right\} \\
&-\epsilon^{N+1} \sum_{\substack{p+q\geq N+1 \\ \max\{N-m+1,0\}\leq p\leq N \\ 1\leq q\leq m \\ *=\pm}} \epsilon^{p+q-N-1} W_m^{*(q)}\left(h^*_{p+m}\right) \\
= \;& R_{N+1}(g_m;\epsilon) + \epsilon^{N+1}\eta_m(x;\epsilon).
\end{aligned}$$

Finally, when $j = m+1$, we have

$$D_t^{m+1} R_{N+1}(u;\epsilon)(0,x)$$

$$
\begin{aligned}
&= R_{N+1}(g_{m+1};\epsilon) + \sum_{n=0}^{N} \epsilon^n \left\{ g_{m+1,n}(x) - D_t^{m+1} v_n(0,x) \right\} \\
&\quad - \epsilon^{-1} \sum_{p=0}^{N} \epsilon^p \sum_{\substack{q=0 \\ *=\pm}}^{m+1} \epsilon^q W_{m+1}^{*(q)} \left(h_{p+m}^* \right) \\
&= R_N(g_{m+1};\epsilon) - \epsilon^{-1} \sum_{*=\pm} W_{m+1}^{*(0)}(h_m^*) \\
&\quad + \sum_{n=0}^{N-1} \epsilon^n \left\{ g_{m+1,n} - D_t^{m+1} v_n(0,x) \right. \\
&\qquad \left. - \sum_{\substack{q=0 \\ *=\pm}}^{\min\{m+1,n+1\}} W_{m+1}^{*(q)} \left(h_{m+n+1-q}^* \right) \right\} \\
&\quad + \epsilon^N g_{m+1,N} \\
&\quad - \epsilon^N \left\{ D_t^{m+1} v_N(0,x) + \sum_{\substack{q=1 \\ *=\pm}}^{\min\{m+1,N+1\}} W_{m+1}^{*(q)} \left(h_{m+N+1-q}^* \right) \right\} \\
&\quad - \epsilon^{N+1} \sum_{\substack{p+q \geq N+2 \\ \max\{N-m+1,0\} \leq p \leq N \\ 2 \leq q \leq m+1 \\ *=\pm}} \epsilon^{p+q-N-2} W_{m+1}^{*(q)} \left(h_{p+m}^* \right) \\
&= R_{N+1}(g_{m+1};\epsilon) + \epsilon^N \eta_{m+1}(x;\epsilon). \ \square
\end{aligned}
$$

PROOF OF THE MAIN THEOREM 6.2.
(i) By the assumption on $f(t,x;\epsilon)$, we have

$$
\int_0^T e^{-2\gamma t} \|D^j R_{N+1}(f;\epsilon)(t)\|^2 dt \leq \frac{C_{j,N}(f)}{\gamma} \epsilon^{2(N+1)},
$$

and

$$
\|D^j R_{N+1}(f;\epsilon)(0)\|^2 \leq C_{j,N}(f) \epsilon^{2(N+1)}.
$$

(ii)

$$
\begin{aligned}
\int_0^T e^{-2\gamma t} \|D^j \rho(t;\epsilon)\|^2 dt &\leq C_N \int_0^T e^{-2\gamma t} \sum_{q=0}^{N} \|D^{m+2+j} v_q\|^2 dt \\
&\leq C'_N.
\end{aligned}
$$

Here, C'_N depends on the norms of $\{f_j; 0 \le j \le N\}$, $\{g_{j,k}; 0 \le j \le m+1,\ 0 \le k \le N-2\}$, $\{g_{j,N-1}; 0 \le j \le m\}$, $\{g_{j,N}; 0 \le j \le m-1\}$ and on their supports but it is bounded for ϵ by Proposition 5.4.

$$\begin{aligned}\|D^j \rho(0;\epsilon)\|^2 &\le C_N \sum_{q=0}^{N} \|D^{m+2+j} v_q(0)\|^2 \\ &\le C''_N.\end{aligned}$$

The dependence of C''_N is just like that of C'_N.
(iii)

$$\epsilon^{2j} \int_0^T e^{-2\gamma t} \|D^j \chi(t)\|^2 dt \le C'_N.$$

C'_N depends on the norms of $\{D_t^l f_q(0);\ 0 \le q \le N,\ 0 \le l+q \le N\}$, $\{g_{j,N};\ 0 \le j \le m\}$, $\{g_{j,k};\ 0 \le j \le m+1, 0 \le k \le N-1\}$, but it is bounded for ϵ by Proposition 5.4. We have also

$$\epsilon^{2j} \|D^j \chi(0)\|^2 \le C''_N.$$

(iv)

$$\sum_{k=0}^{m+1} \|R_{N+1}(g_k;\epsilon)\|^2_{m+1-k+j} \le C'_N \epsilon^{2(N+1)},$$

$$\sum_{j=0}^{p} \left\{ \epsilon^{2j+2} \sum_{k=0}^{m} \|\epsilon^{N+1} \eta_k(\epsilon)\|^2_{m+1-k+j} + \epsilon^{2j+2} \|\epsilon^N \eta_{m+1}\|^2_j \right\} \le C''_N\ \epsilon^{2(N+1)}.$$

C''_N depends on the norms of (a part of) $\{D_t^l f_q(0);\ 0 \le l+q \le N+1, 0 \le q \le N\}$, and $\{g_{j,k};\ 0 \le j \le m+1, 0 \le k \le N\}$. We have the conclusion from Theorem 2.2 applicable to (6.5). $\square$

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