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Perturbation of the Navier-Stokes flow in an annular domain with the non-vanishing outflow condition

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Abstract. The boundary value problem of the Navier-Stokes equations has been studied so far only under the vanishing outflow condition due to Leray. We consider this problem in an annular domain $D = \{\boldsymbol{x} \in \mathbf{R}^2; R_1 < |\boldsymbol{x}| < R_2\}$, under the boundary condition with non-vanishing outflow. In a previous paper of the first author, an exact solution is obtained for a simple boundary condition of non-vanishing outflow type: $\boldsymbol{u} = \frac{\mu}{R_i}\boldsymbol{e}_r + b_i\boldsymbol{e}_\theta$ on Γ_i, $i = 1, 2$, where μ, b_1, b_2 are arbitrary constants. In this paper, we show the existence of solutions satisfying the boundary condition: $\boldsymbol{u} = \{\frac{\mu}{R_i} + \varphi_i(\theta)\}\boldsymbol{e}_r + \{b_i + \psi_i(\theta)\}\boldsymbol{e}_\theta$ on Γ_i, $i = 1, 2$, where $\varphi_i(\theta), \psi_i(\theta)$ are 2π-periodic smooth function of θ, under some additional condition.

Let D be an annular domain in $\mathbf{R}^2$:

$$D = \{\boldsymbol{x} \in \mathbf{R}^2; R_1 < |\boldsymbol{x}| < R_2\},$$

where $0 < R_1 < R_2$, and Γ_i its boundary:

$$\Gamma_i = \{\boldsymbol{x} \in \mathbf{R}^2; |\boldsymbol{x}| = R_i\}, \quad i = 1, 2.$$

We consider the boundary value problem of the Navier-Stokes equations:

$$(1) \qquad \begin{cases} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}\nabla p & = \mathbf{0} \quad \text{in} \quad D, \\ \operatorname{div}\boldsymbol{u} & = 0 \quad \text{in} \quad D, \\ \boldsymbol{u} & = \boldsymbol{b} \quad \text{on} \quad \partial D, \end{cases}$$

where $\boldsymbol{u}$ is the fluid velocity, p is the pressure, ν(the kinematic viscosity) and ρ (the density) are given positive constants, and the vector $\boldsymbol{b}$ is a given boundary value of the velocity.

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For the general bounded domain $D \subset \mathbf{R}^n, n \geq 2$, J. Leray [6] showed the existence of the solution to this problem under the following condition:

$$(H) \quad \int_{\Gamma_i} \boldsymbol{b} \cdot \boldsymbol{n}\, ds = 0, \quad 1 \leq i \leq k,$$

where $\partial D = \bigcup_{i=1}^k \Gamma_i$, Γ_i is the connected component of ∂D and $\boldsymbol{n}$ is the unit outward normal to the boundary ∂D. The condition (H) is stronger than the condition

$$(H)_0 \quad \int_{\partial D} \boldsymbol{b} \cdot \boldsymbol{n}\, ds = \sum_{i=1}^k \int_{\Gamma_i} \boldsymbol{b} \cdot \boldsymbol{n}\, ds = 0,$$

which is to be satisfied by the boundary value $\boldsymbol{b}$ of a solenoidal vector $\boldsymbol{u}$.

We are concerned with the problem whether does exist a solution to (1) under the non-vanishing outflow condition $(H)_0$, even if the boundary value does not satisfy the vanishing outflow condition (H) ([2], [6]). In the previous paper [7], the first author showed an exact solution to this equation in an annular domain under the boundary condition with non-vanishing outflow given by,

$$\boldsymbol{b} = \frac{\mu}{R_i} \boldsymbol{e}_r + b_i \boldsymbol{e}_\theta \;\text{ on } \Gamma_i, \; i = 1, 2,$$

where μ, b_1, b_2 are given constants and $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ are the unit vectors in the polar coordinates representation $\{r, \theta\}$.

In this paper, we study the case where the boundary value depends on θ variable, more precisely, the vector $\boldsymbol{b}$ is given as follows:

$$(2) \qquad \boldsymbol{b} = \{a_i + \varphi_i(\theta)\} \boldsymbol{e}_r + \{b_i + \psi_i(\theta)\} \boldsymbol{e}_\theta \;\text{ on } \; \Gamma_i, \; i = 1, 2.$$

REMARK 1. Since the condition $(H)_0$ has to be satisfied,

$$(A1) \quad a_1 R_1 = a_2 R_2$$

should hold. We denote this common value by μ. If $\mu \neq 0$, the condition (H) does not hold.

On the other hand, without loss of generality, we can suppose the following:

$(A2)$ $\varphi_i(\theta), \psi_i(\theta)$ *be* 2π*-periodic smooth function of* θ, *satisfying*

$$\int_0^{2\pi} \varphi_i(\theta)d\theta = 0, \quad \int_0^{2\pi} \psi_i(\theta)d\theta = 0, \quad i = 1,2.$$

Finally we put

$$\omega_i = \frac{b_i}{R_i}, \quad i = 1,2.$$

If the absolute value $|\mu|$ of μ is small, we can show the existence of a solution to (1) (2) by the usual method (c.f. [5], [9]). We show, in the following, the existence of a solution to (1) (2) even for large $|\mu|$.

THEOREM 1. *Suppose* $(A1), (A2)$ *and the inequality*

$$|\omega_1 - \omega_2| \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \left(\log \frac{R_2}{R_1}\right)^2 < 2\nu$$

hold. Then there exists at most discrete countable set M *such that for each* $\mu \in \mathbf{R} \setminus M$ *the boundary value problem (1) (2) has a solution for sufficiently small* $\varphi_i(\theta)$, $\psi_i(\theta)\,(i = 1,2)$.

REMARK 2. Let μ, ω_1, ω_2 be constants and $\varphi_i(\theta) = \psi_i(\theta) = 0\,(i = 1,2)$. Then, we have the following exact solution $\boldsymbol{u}_0$ to (1) (2) of the form:

$$\boldsymbol{u}_0 = \frac{\mu}{r}\boldsymbol{e}_r + b(\mu, r)\boldsymbol{e}_\theta.$$

(i) For $\mu \neq -2\nu$,

$$b(\mu, r) = \frac{c_1}{r} + c_2 r^{1+\frac{\mu}{\nu}},$$

where

$$c_1 = \frac{\omega_1 R_1^2 R_2^{2+\frac{\mu}{\nu}} - \omega_2 R_2^2 R_1^{2+\frac{\mu}{\nu}}}{R_2^{2+\frac{\mu}{\nu}} - R_1^{2+\frac{\mu}{\nu}}}, \quad c_2 = \frac{\omega_2 R_2^2 - \omega_1 R_1^2}{R_2^{2+\frac{\mu}{\nu}} - R_1^{2+\frac{\mu}{\nu}}}.$$

(ii) For $\mu = -2\nu$,

$$b(\mu, r) = \frac{1}{r}(c_1 + c_2 \log r),$$

where

$$c_1 = \frac{\omega_1 R_1^2 \log R_2 - \omega_2 R_2^2 \log R_1}{\log R_2 - \log R_1}, \quad c_2 = \frac{\omega_2 R_2^2 - \omega_1 R_1^2}{\log R_2 - \log R_1}.$$

The pressure p_0 can be obtained from the equation. This solution is unique if $|\mu|$ and $|\omega_1-\omega_2|$ $(case(i))$ $(|\omega_1|, |\omega_2|$ $(case(ii)))$ are sufficiently small $(c.f.[7],[8]$).

Let us prove Theorem 1 in several steps. Let $C^{\infty}_{0,\sigma}(D)$ be all smooth solenoidal functions with compact support in the domain D, H_σ the closure of $C^{\infty}_{0,\sigma}(D)$ in $L^2(D)$, and V the closure of $C^{\infty}_{0,\sigma}(D)$ in the Sobolev space $H^1(D)$.

Let $\boldsymbol{u}_0, p_0$ be the solution as above. Let $\boldsymbol{v}_0$ satisfy the condition

$$\text{div}\boldsymbol{v}_0 = 0 \;\; in \;\; D, \; and \; \boldsymbol{v}_0 = \varphi_i(\theta)\boldsymbol{e}_r + \psi_i(\theta)\boldsymbol{e}_\theta \;\; on \;\; \Gamma_i, \;\; i=1,2. \tag{3}$$

The existence of such function is known(c.f. [1]) but for our convenience, we choose:

$$\begin{aligned}\boldsymbol{v}_0 \;=\; & \Big[R_1 r^{-1}\int_r^{R_2}\alpha(t)dt\,\varphi_1(\theta) + R_2 r^{-1}\int_{R_1}^{r}\alpha(t)dt\,\varphi_2(\theta) \\ & \quad -r^{-1}\int_{R_1}^{r}\beta_1(t)dt\,\psi_1'(\theta) - r^{-1}\int_{R_1}^{r}\beta_2(t)dt\,\psi_2'(\theta)\Big]\boldsymbol{e}_r \\ & + \Big[\int_0^\theta \{R_1\varphi_1(t) - R_2\varphi_2(t)\}dt\,\alpha(r) + \beta_1(r)\psi_1(\theta) + \beta_2(r)\psi_2(\theta)\Big]\boldsymbol{e}_\theta,\end{aligned} \tag{4}$$

where, $\alpha(r),\ \beta_i(\theta)$ $(i=1,2)$ are smooth functions such that

$$\alpha(R_1) = \alpha(R_2) = 0, \quad \int_{R_1}^{R_2}\alpha(t)dt = 1,$$

$$\beta_i(R_j) = \delta_{ij} \ (i,j=1,2), \quad \int_{R_1}^{R_2}\beta_i(t)dt = 0 \ (i=1,2).$$

Then, we have the following estimate.

LEMMA 1. *There exists a positive constant c_0 such that*

$$||\boldsymbol{v}_0||_{C^2(D)} \le c_0\sum_{i=1}^{2}(||\varphi_i||_{C^2(I)} + ||\psi_i||_{C^3(I)})$$

holds, where I is the closed interval $[0,2\pi]$.

Suppose $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{u}_0 + \boldsymbol{v}_0$ satisfy (1) with $\boldsymbol{b}$ given in (2). Then, the equation for $\boldsymbol{w}$ is as follows:

$$(5) \quad \begin{cases} -\nu\Delta\boldsymbol{w} & + \quad (\boldsymbol{w}\cdot\nabla)\boldsymbol{u}_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + \dfrac{1}{\rho}\nabla q \\ & \quad +(\boldsymbol{w}\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{w} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad D, \\ \operatorname{div}\boldsymbol{w} & = \quad 0 \quad \text{in} \quad D, \\ \boldsymbol{w} & = \quad \boldsymbol{0} \quad \text{on} \quad \partial D, \end{cases}$$

where $\boldsymbol{f}_0 = -\nu\Delta\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{u}_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{v}_0$.

Let P be the orthogonal projection from $L^2(D)$ onto H_σ and $A = -P\Delta$ be the Stokes operator. Applying the orthogonal projection P to the first equation in (5), we get:

$$(6) \quad \begin{cases} A\boldsymbol{w} & + \quad \dfrac{1}{\nu}P\{(\boldsymbol{w}\cdot\nabla)\boldsymbol{u}_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{w}\} \\ & \quad +\dfrac{1}{\nu}P\{(\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{w} + \boldsymbol{f}_0\} = \boldsymbol{0}. \end{cases}$$

As is well known, A is a self-adjoint positive operator in H_σ and the inverse A^{-1} is a compact operator on H_σ (e.g., [5], [9]). Applying A^{-1} to the equation (6), we obtain:

$$(7) \quad \boldsymbol{w} - T(\mu)\boldsymbol{w} + \frac{1}{\nu}A^{-1}P\{(\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{w} + \boldsymbol{f}_0\} = \boldsymbol{0},$$

where

$$T(\mu)\boldsymbol{w} \equiv -\frac{1}{\nu}A^{-1}P\{(\boldsymbol{w}\cdot\nabla)\boldsymbol{u}_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{w}\}.$$

Since A^{-1} is compact operator in H_σ, and its range is the domain of the operator A which is compactly imbedded in V, we obtain the following:

LEMMA 2. *The operator $T(\mu)$ is a compact linear operator on V.*

Let

$$\mu_n = -2\nu + \frac{2n\pi\nu}{\log R_2 - \log R_1}i, \; n \in \mathbf{Z}, \; i = \sqrt{-1}.$$

We define $b(\mu, r)$ for all $\mu \in \mathbf{C}$ letting $b(\mu, r) = b(-2\nu, r)$ for $\mu = \mu_n$. Then, $b(\mu, r)$ is continuous and holomorphic in $\mu \in \mathbf{C}$ even at the points $\mu = \mu_n$. Therefore, we have:

LEMMA 3. *(c.f.Kato[4]) The compact operator $T(\mu)$ is an entire function of μ.*

LEMMA 4. *If $|\omega_1 - \omega_2|$ is sufficiently small, then 1 is not the eigenvalue of the operator $T(0)$.*

PROOF. Suppose that 1 is an eigenvalue of $T(0)$, i.e., there exists a nonzero $\boldsymbol{w} \in V$ such that $T(0)\boldsymbol{w} = \boldsymbol{w}$. Then,

$$-\frac{1}{\nu}A^{-1}P\{(\boldsymbol{w}\cdot\nabla)\tilde{\boldsymbol{u}}_0 + (\tilde{\boldsymbol{u}}_0\cdot\nabla)\boldsymbol{w}\} = \boldsymbol{w}$$

holds, that is,

$$\nu A\boldsymbol{w} = -P\{(\boldsymbol{w}\cdot\nabla)\tilde{\boldsymbol{u}}_0 + (\tilde{\boldsymbol{u}}_0\cdot\nabla)\boldsymbol{w}\}$$

holds, where $\tilde{\boldsymbol{u}}_0 = b(0,r)\boldsymbol{e}_\theta$ (See Remark 2). Without loss of generality, we may assume $\boldsymbol{w}$ is real since $\tilde{\boldsymbol{u}}_0$ is real. Taking the inner product with $\boldsymbol{w}$, we have

$$\nu||\nabla\boldsymbol{w}||^2 = -((\boldsymbol{w}\cdot\nabla)\tilde{\boldsymbol{u}}_0, \boldsymbol{w}).$$

The right hand side is equal to:

$$2c_1(0)\int_D \frac{w_r w_\theta}{r} dr d\theta, \qquad \text{where} \qquad c_1(0) = \frac{\omega_1 - \omega_2}{R_2^2 - R_1^2}R_1^2 R_2^2.$$

Let $f \in H_0^1(D)$. Then the inequality:

$$\int\int_D \left(\frac{f(r,\theta)}{r}\right)^2 r dr d\theta \leq \frac{1}{2}\left(\log\frac{R_2}{R_1}\right)^2 \int\int_D \left(\frac{\partial f}{\partial r}\right)^2 r dr d\theta$$

holds. Therefore, we have :

$$|((\boldsymbol{w}\cdot\nabla)\tilde{\boldsymbol{u}}_0, \boldsymbol{w})| \leq \frac{|\omega_1 - \omega_2|}{2}\frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left(\log\frac{R_2}{R_1}\right)^2 ||\nabla\boldsymbol{w}||^2.$$

If $|\omega_1 - \omega_2|$ is sufficiently small, then the inequality

$$\frac{|\omega_1 - \omega_2|}{2}\frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left(\log\frac{R_2}{R_1}\right)^2 < \nu$$

holds, and contradiction. □

Let K be any compact subset of $\mathbf{C}$, containing $\{0\}$. According to Theorem 1.9 in Chapter VII §1 of [4], there exist a finite set $\{\mu_1^*, \mu_2^*, \ldots, \mu_k^*\}$ such that for any μ in $K \setminus \{\mu_1^*, \mu_2^*, \ldots, \mu_k^*\}$, $(T(\mu) - 1)^{-1}$ exists and is bounded on V. Let $\mu \in K \setminus \{\mu_1^*, \mu_2^*, \ldots, \mu_k^*\}$. From (7), we obtain:

$$(8)\quad \boldsymbol{w} = \frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1}P\{(\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{w} + \boldsymbol{f}_0\}.$$

Let us denote the right hand side of (8) by $N(\mu)\boldsymbol{w}$:

$$(9)\qquad N(\mu)\boldsymbol{w} = \frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1} \times P\{(\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{w} + \boldsymbol{f}_0\},$$

and

$$\sigma = \sum_{i=1}^{2}(||\varphi_i||_{C^2(I)} + ||\psi_i||_{C^3(I)}).$$

According to Lemma 1, we have:

$$(10)\quad ||\frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1}P\{(\boldsymbol{w}\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{w}\}||_V \le \frac{C\sigma}{\nu}||\boldsymbol{w}||_V,$$

$$(11)\quad ||\frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1}P\boldsymbol{f}_0||_V \le \frac{C\sigma}{\nu}\{(\nu + c_0\sigma)|D|^{1/2} + ||\boldsymbol{u}_0||_V\},$$

where c_0 is the constant in Lemma 1, $C = c_0||(1 - T(\mu))^{-1}A^{-1/2}||$ and $|D|$ is the measure of D.

LEMMA 5. *There exists a positive constant c_D such that the estimate*

$$||A^{-1}P(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}||_V \le c_D||\boldsymbol{v}||_V||\boldsymbol{w}||_V, \quad \forall \boldsymbol{v}, \boldsymbol{w} \in V$$

holds.

It is known that there exists an absolute constant c such that

$$||A^{-1/4}P(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}|| \le c||A^{1/2}\boldsymbol{v}||||A^{1/2}\boldsymbol{w}||, \qquad \forall \boldsymbol{v}, \boldsymbol{w} \in C_{0,\sigma}^{\infty}(D)$$

holds. See, e.g., Fujita-Kato [3]. Using this inequality, we obtain Lemma 5 easily.

Put

(12) $\rho_0 = \max\left[\ \frac{C}{\nu},\ \frac{c_D}{\nu}||(T(\mu)-1)^{-1}||,\ \frac{C}{\nu}\left\{(\nu+c_0)|D|^{1/2}+||\boldsymbol{u}_0||_V\right\}\ \right],$

where c_0, c_D, C are constants given in Lemma 1, in Lemma 5, and in (10), respectively. Now we have the following estimate for the nonlinear operator $N(\mu)$:

(13) $$||N(\mu)\boldsymbol{w}||_V \leq \rho_0(||\boldsymbol{w}||_V^2 + \sigma||\boldsymbol{w}||_V + \sigma).$$

Let $\sigma_0 = \dfrac{1}{\rho_0}\{1+2\rho_0-\sqrt{(1+2\rho_0)^2-1}\}$.

REMARK 3. $\rho_0\sigma_0$ is the smallest positive root of the equation

$$X^2 - 2(1+2\rho_0)X + 1 = 0.$$

The inequality $0 < \sigma_0 < 1$ follows easily. If $0 < \sigma < \sigma_0$, then the equation

$$\rho_0(X^2 + \sigma X + \sigma) = X$$

has two positive roots. Let r_σ be the smaller one:

$$r_\sigma = \frac{1}{2\rho_0}\left\{1-\rho_0\sigma-\sqrt{(1-\rho_0\sigma)^2-4\rho_0^2\sigma}\right\}.$$

LEMMA 6. *If $0 < \sigma < \sigma_0$, then the operator $N(\mu)$ maps the ball*

$$B(r_\sigma) \equiv \{\boldsymbol{w} \in V\ ;\ ||\boldsymbol{w}||_V \leq r_\sigma\}$$

into itself.

PROOF. Let $\boldsymbol{w} \in B(r_\sigma)$. Then,

$$||N(\mu)\boldsymbol{w}||_V \leq \rho_0(||\boldsymbol{w}||_V^2 + \sigma||\boldsymbol{w}||_V + \sigma) \leq \rho_0(r_\sigma^2 + \sigma r_\sigma + \sigma) = r_\sigma. \ \square$$

LEMMA 7. *If $0 < \sigma < \sigma_0$, the operator $N(\mu)$ is a contraction operator on $B(r_\sigma)$.*

PROOF. Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ be arbitrary elements in $B(r_\sigma)$. Then, we have:

$$\begin{aligned}
&N(\mu)\boldsymbol{w}_1 - N(\mu)\boldsymbol{w}_2 \\
&\quad = \frac{1}{\nu}(T(\mu)-1)^{-1}A^{-1}P\{(\boldsymbol{w}_1\cdot\nabla)\boldsymbol{w}_1 - (\boldsymbol{w}_2\cdot\nabla)\boldsymbol{w}_2 \\
&\qquad\qquad + ((\boldsymbol{w}_1-\boldsymbol{w}_2)\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)(\boldsymbol{w}_1-\boldsymbol{w}_2)\}.
\end{aligned}$$

Since

$$(\boldsymbol{w}_1\cdot\nabla)\boldsymbol{w}_1 - (\boldsymbol{w}_2\cdot\nabla)\boldsymbol{w}_2 = ((\boldsymbol{w}_1-\boldsymbol{w}_2)\cdot\nabla)\boldsymbol{w}_1 + (\boldsymbol{w}_2\cdot\nabla)(\boldsymbol{w}_1-\boldsymbol{w}_2),$$

therefore,

$$\begin{aligned}
&||N(\mu)\boldsymbol{w}_1 - N(\mu)\boldsymbol{w}_2||_V \\
&\quad \leq \frac{c_D}{\nu}||(T(\mu)-1)^{-1}||\,||\boldsymbol{w}_1-\boldsymbol{w}_2||_V\,(||\boldsymbol{w}_1||_V + ||\boldsymbol{w}_2||_V) \\
&\qquad + \frac{C\sigma}{\nu}||\boldsymbol{w}_1-\boldsymbol{w}_2||_V \\
&\quad \leq \rho_0(||\boldsymbol{w}_1||_V + ||\boldsymbol{w}_2||_V + \sigma)||\boldsymbol{w}_1-\boldsymbol{w}_2||_V,
\end{aligned}$$

where we used (10) and Lemma 5. Since $\boldsymbol{w}_1, \boldsymbol{w}_2 \in B(r_\sigma)$ and $\sigma < \sigma_0$, we have:

$$\rho_0(||\boldsymbol{w}_1||_V + ||\boldsymbol{w}_2||_V + \sigma) \leq \rho_0(2r_\sigma + \sigma) = 1 - \sqrt{(1-\rho_0\sigma)^2 - 4\rho_0^2\sigma} < 1.$$

Therefore the operator $N(\mu)$ is a contraction and has a fixed point in the ball $B(r_\sigma)$. Theorem 1 is thus proved. □

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