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A Hamiltonian Path Integral for a Degenerate Parabolic Pseudo-Differential Operator

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Dedicated to Professor Hikosaburo Komatsu on his sixtieth birthday

Abstract. The symbol of the fundamental solution for a degenerate parabolic pseudo-differential operator of order $m\,(>0)$ can be described in terms of a Hamiltonian path integral. This Hamiltonian path integral converges in the topology of the symbol class $\boldsymbol{S}^{2m}_{\lambda,\rho,\delta}$ and in the weak topology of the symbol class $\boldsymbol{S}^{0}_{\lambda,\rho,\delta}$.

0. Introduction

In this paper, we construct the fundamental solution for a degenerate parabolic pseudo-differential operator of order $m\,(>0)$ in a different way from that in C.Tsutsumi [10]. In [10], she constructed the fundamental solution by Levi-Mizohata method. On the other hand, in this paper, we construct the fundamental solution by a Hamiltonian path integral. If we use a Hamiltonian path integral, we can actually give an expression of the symbol of the fundamental solution. Furthermore, this Hamiltonian path integral converges in the topology of the symbol class $\boldsymbol{S}^{2m}_{\lambda,\rho,\delta}$ and in the weak topology of the symbol class $\boldsymbol{S}^{0}_{\lambda,\rho,\delta}$.

In Section 1, we introduce some basic properties of pseudo-differential operators, which we use in Section 2. For the details, see Chapter 7 § 1 and § 2 in H.Kumano-go [6]. In Section 2, we construct the fundamental solution for a degenerate parabolic pseudo-differential operator by a Hamiltonian path integral. Theorem 2.1 is the main theorem in this paper.

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1. Pseudo-Differential Operators

For $x=(x_1,\dots,x_n)\in \boldsymbol{R}^n_x$, $\xi=(\xi_1,\dots,\xi_n)\in \boldsymbol{R}^n_\xi$ and multi-indices of non-negative integers $\alpha=(\alpha_1,\dots,\alpha_n)$, $\beta=(\beta_1,\dots,\beta_n)$, we employ the usual notation:

$$|\alpha|=\alpha_1+\cdots+\alpha_n,\quad |\beta|=\beta_1+\cdots+\beta_n,$$

$$\alpha!=\alpha_1!\cdots\alpha_n!,\quad \beta!=\beta_1!\cdots\beta_n!,$$

$$x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n,\quad \langle x\rangle=(1+|x|^2)^{1/2},\quad \langle\xi\rangle=(1+|\xi|^2)^{1/2},$$

$$\partial_{\xi_j}=\frac{\partial}{\partial\xi_j},\quad D_{x_j}=-i\frac{\partial}{\partial x_j},\quad \partial_\xi^\alpha=\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_n}^{\alpha_n},\quad D_x^\beta=D_{x_1}^{\beta_1}\cdots D_{x_n}^{\beta_n}.$$

$\mathcal{S}$ denotes the Schwartz space of rapidly decreasing C^∞-functions on $\boldsymbol{R}^n$. For $u\in\mathcal{S}$, we define semi-norms $|u|_{l,\mathcal{S}}(l=0,1,2,\dots)$ by

$$|u|_{l,\mathcal{S}}\equiv\max_{k+|\alpha|\le l}\sup_x|\langle x\rangle^k\partial_x^\alpha u(x)|\quad(l=0,1,2,\dots).$$

Then, $\mathcal{S}$ is a Fréchet space with these semi-norms.
For simplicity, we set $đ\eta\equiv(2\pi)^{-n}d\eta$ and $đ\xi\equiv(2\pi)^{-n}d\xi$.
Oscillatory integral of a function $a(\eta,y)$, is defined by the equality

$$\mathrm{O_s}\text{—}\iint e^{-iy\cdot\eta}a(\eta,y)dyđ\eta\equiv\lim_{\epsilon\to0}\iint e^{-iy\cdot\eta}\chi(\epsilon\eta,\epsilon y)a(\eta,y)dyđ\eta\,,$$

where $\chi(\eta,y)\in\mathcal{S}$ in $\boldsymbol{R}^{2n}_{\eta,y}$ and $\chi(0,0)=1$. For the details, see Chapter 1 § 6 in H.Kumano-go [6].

DEFINITION 1.1 (A weight function $\lambda(\xi)$).
We say that a real-valued C^∞-function $\lambda(\xi)$ on $\boldsymbol{R}^n_\xi$ is a weight function, if there exist constants $A_0, A_\alpha>0$ such that

$$1\le\lambda(\xi)\le A_0\langle\xi\rangle\,,\tag{1.1}$$

$$|\partial_\xi^\alpha\lambda(\xi)|\le A_\alpha\lambda(\xi)^{1-|\alpha|}\,.\tag{1.2}$$

Examples.
1° $\lambda(\xi)=\langle\xi\rangle$.
2° $\lambda(\xi)=\left\{1+\sum_{j=1}^n|\xi_j|^{2m_j}\right\}^{1/(2m)},\quad(m_j\in\mathbb{N},\quad m\equiv\max_{1\le j\le n}\{m_j\})$.

Definition 1.2 (Pseudo-differential operators).

We say that a C^∞-function $p(x,\xi)$ on $\boldsymbol{R}^{2n}_{x,\xi}$ is a symbol of class $\boldsymbol{S}^m_{\lambda,\rho,\delta}$ ($m \in \boldsymbol{R}$, $0 \le \delta \le \rho \le 1$, $\delta < 1$), if for any α, β, there exists a constant $C_{\alpha,\beta}$ such that

$$|p^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha,\beta}\lambda(\xi)^{m+\delta|\beta|-\rho|\alpha|}\,, \tag{1.3}$$

where $p^{(\alpha)}_{(\beta)}(x,\xi) \equiv \partial^\alpha_\xi D^\beta_x p(x,\xi)$.
The pseudo-differential operator $p(X,D_x)$ with the symbol $p(x,\xi)$ is defined by

$$p(X,D_x)u(x) \equiv \iint e^{i(x-x')\cdot\xi} p(x,\xi)u(x')dx'\text{đ}\xi \quad (u \in \mathcal{S})\,, \tag{1.4}$$

where $\text{đ}\xi \equiv (2\pi)^{-n}d\xi$.

Remark.

1° For simplicity, we set $p^{(\alpha)}_{(\beta)}(x,\xi) \equiv \partial^\alpha_\xi D^\beta_x p(x,\xi)$, $p^{(\alpha)}(x,\xi) \equiv \partial^\alpha_\xi p(x,\xi)$ and $p_{(\beta)}(x,\xi) \equiv D^\beta_x p(x,\xi)$ for any α, β.

2° The symbol class $\boldsymbol{S}^m_{\lambda,\rho,\delta}$ is a Fréchet space with the semi-norms

$$|p|^{(m)}_l \equiv \max_{|\alpha+\beta|\le l} \sup_{(x,\xi)} \{|p^{(\alpha)}_{(\beta)}(x,\xi)|\lambda(\xi)^{-(m+\delta|\beta|-\rho|\alpha|)}\} \quad (l = 0,1,2,\dots)\,. \tag{1.5}$$

3° The continuity of $p(X,D_x) : \mathcal{S} \longrightarrow \mathcal{S}$ is clear. Furthermore, we can extend $p(X,D_x) : \mathcal{S} \longrightarrow \mathcal{S}$ to $p(X,D_x) : \mathcal{S}' \longrightarrow \mathcal{S}'$ by means of

$$(p(X,D_x)u, v) \equiv (u, p(X,D_x)^* v) \quad \text{for} \quad u \in \mathcal{S}'\,, v \in \mathcal{S}\,. \tag{1.6}$$

Theorem 1.3 (Multi-products).

Let M be a positive constant and let $\{m_j\}_{j=1}^\infty$ be a sequence of real numbers satisfying

$$\sum_{j=1}^\infty |m_j| \le M < \infty\,. \tag{1.7}$$

For any $\nu = 1, 2, \dots$ *and* $p_j(x,\xi) \in \boldsymbol{S}^{m_j}_{\lambda,\rho,\delta} (j = 1, 2, \dots, \nu+1)$, *there exists* $q_{\nu+1}(x,\xi) \in \boldsymbol{S}^{\bar{m}_{\nu+1}}_{\lambda,\rho,\delta} (\bar{m}_{\nu+1} \equiv m_1 + m_2 + \cdots + m_{\nu+1})$ *such that*

$$q_{\nu+1}(X, D_x) = p_1(X, D_x) p_2(X, D_x) \cdots p_{\nu+1}(X, D_x)\,. \tag{1.8}$$

Furthermore, for any l, *there exist a constant* A_l *and an integer* l' *such that*

$$|q_{\nu+1}|_l^{(\bar{m}_{\nu+1})} \le (A_l)^\nu \prod_{j=1}^{\nu+1} |p_j|_{l'}^{(m_j)}\,, \tag{1.9}$$

where A_l *and* l' *depend only on* M *and* l, *but are independent of* ν.

Proof. See Theorem 2.4 in Chapter 7 § 2 of H.Kumano-go [6]. □

Theorem 1.4.
Let $p_j(x,\xi) \in \boldsymbol{S}^{m_j}_{\lambda,\rho,\delta} (j = 1, 2)$. *Define* $q_\theta(x,\xi)$ $(|\theta| \le 1)$ *by*

$$q_\theta(x,\xi) \equiv \mathrm{O_s}\text{—}\iint e^{-iy\cdot\eta} p_1(x, \xi + \theta\eta) p_2(x+y, \xi) dy đ\eta\,. \tag{1.10}$$

Then $\{q_\theta(x,\xi)\}_{|\theta|\le 1}$ *is a bounded set of* $\boldsymbol{S}^{m_1+m_2}_{\lambda,\rho,\delta}$. *Furthermore, for any* l, *there exist a constant* A_l *and an integer* l' *independent of* θ *such that*

$$|q_\theta|_l^{(m_1+m_2)} \le A_l |p_1|_{l'}^{(m_1)} |p_2|_{l'}^{(m_2)}\,. \tag{1.11}$$

Proof. See Lemma 2.4 in Chapter 2 § 2 or Lemma 2.2 in Chapter 7 §2 of H.Kumano-go [6]. □

2. The Main Theorem

Theorem 2.1 (The main theorem).
Let $K(t,x,\xi) \in \mathcal{C}^0([0,T]; \boldsymbol{S}^m_{\lambda,\rho,\delta})$ $(m > 0\,, 0 \le \delta < \rho \le 1)$. *Assume that* $K(t,x,\xi)$ *satisfies the following conditions* (a1), (a2) *:*

(a1) *There exist constants* $c > 0$ *and* $m' (0 \le m' \le m)$ *such that*

$$Re\ K(t,x,\xi) \le -c\lambda(\xi)^{m'} \text{ on } [0,T] \times \boldsymbol{R}^{2n}_{x,\xi}\,. \tag{2.1}$$

(a2) *For any* α, β, *there exists a constant* $C_{\alpha,\beta}$ *such that*

$$|K^{(\alpha)}_{(\beta)}(t,x,\xi)/Re\ K(t,x,\xi)| \le C_{\alpha,\beta} \lambda(\xi)^{\delta|\beta| - \rho|\alpha|} \text{ on } [0,T] \times \boldsymbol{R}^{2n}_{x,\xi}\,. \tag{2.2}$$

Then we have the following $(1)-(5)$ *:*

(1) *Let* $\mathit{\Delta}_{t,s} : (T \geq)t \equiv t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} \equiv s(\geq 0)$ *be an arbitrary division of interval* $[s,t]$ *into subintervals, and let* $e^{(t_j-t_{j+1})K(t_{j+1})}(X,D_x)$ *be an operator defined by*

$$e^{(t_j-t_{j+1})K(t_{j+1})}(X,D_x)u(x) \equiv \iint e^{i(x-x')\cdot\xi} e^{(t_j-t_{j+1})K(t_{j+1},x,\xi)}u(x')dx'đ\xi\,. \tag{2.3}$$

Then there exists $p(\mathit{\Delta}_{t,s};x,\xi) \in \boldsymbol{S}^0_{\lambda,\rho,\delta}$ *such that*

$$p(\mathit{\Delta}_{t,s};X,D_x) = e^{(t-t_1)K(t_1)}(X,D_x)e^{(t_1-t_2)K(t_2)}(X,D_x) \cdots e^{(t_\nu-s)K(s)}(X,D_x)\,. \tag{2.4}$$

(2) *For any* l*, there exist constants* C_l, C'_l *and an integer* l' *such that*

$$|p(\mathit{\Delta}_{t,s})|_l^{(0)} \leq C_l\,, \tag{2.5}$$

and

$$|p(\mathit{\Delta}_{t,s}) - p(\mathit{\Delta}'_{t,s})|_l^{(2m)} \leq C'_l(t-s)\Big(|\mathit{\Delta}_{t,s}| + \sup_{|t'-t''|\leq|\mathit{\Delta}_{t,s}|} |K(t')-K(t'')|_{l'}^{(m)}\Big)\,. \tag{2.6}$$

Here, $\mathit{\Delta}_{t,s} : (T \geq)t \equiv t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} \equiv s(\geq 0)$ *is an arbitrary division of interval* $[s,t]$ *into subintervals,* $\mathit{\Delta}'_{t,s}$ *is an arbitrary refinement of* $\mathit{\Delta}_{t,s}$*,* $|\mathit{\Delta}_{t,s}|$ *denotes the size of division defined by* $|\mathit{\Delta}_{t,s}| \equiv \max_{0\leq j\leq\nu} |t_j - t_{j+1}|$*, and the constants* C_l, C'_l *and the integer* l' *are independent of* ν*,* $\mathit{\Delta}_{t,s}$ *and* $\mathit{\Delta}'_{t,s}$*.*

(3) *There exists* $p^\star(t,s;x,\xi) \in \boldsymbol{S}^0_{\lambda,\rho,\delta}$ *such that* $p(\mathit{\Delta}_{t,s};x,\xi)$ $(\in \boldsymbol{S}^0_{\lambda,\rho,\delta})$ *converges to* $p^\star(t,s;x,\xi)$ $(\in \boldsymbol{S}^0_{\lambda,\rho,\delta})$ *in* $\boldsymbol{S}^{2m}_{\lambda,\rho,\delta}$ *as* $|\mathit{\Delta}_{t,s}|$ *tends to* 0*. Furthermore,* $p^\star(t,s;x,\xi)$ *has the following expression:*

$$p^\star(t,s;x,\xi) = \lim_{|\mathit{\Delta}_{t,s}|\to 0} \mathrm{O_s}\text{—}\iint \cdots \iint e^{-i\sum_{j=1}^{\nu} y^j\cdot\eta^j} \times \exp\Big(\sum_{j=0}^{\nu}(t_j-t_{j+1})K(t_{j+1},x+\bar{y}^j,\xi+\eta^{j+1})\Big) \times dy^1đ\eta^1\cdots dy^\nu đ\eta^\nu\,, \tag{2.7}$$

where $\bar{y}^0 \equiv 0$, $\bar{y}^j \equiv y^1 + y^2 + \cdots + y^j$, *and* $\eta^{\nu+1} \equiv 0$.

(4) *For* $u \in \boldsymbol{L}^2$, *the pseudo-differential operator* $U(t,s) \equiv p^\star(t,s;X,D_x)$ *satisfies the following relation:*

(2.8)
$$\begin{aligned} &U(t,s)u(x) \\ &= \lim_{|\Delta_{t,s}|\to 0} e^{(t-t_1)K(t_1)}(X,D_x)e^{(t_1-t_2)K(t_2)}(X,D_x) \\ &\qquad\qquad \cdots e^{(t_\nu-s)K(s)}(X,D_x)u(x) \\ &= \lim_{|\Delta_{t,s}|\to 0} \iint \cdots \iint \exp\Bigg(\sum_{j=0}^{\nu} i(x^j - x^{j+1})\cdot \xi^{j+1} \\ &\qquad\qquad\qquad\qquad + (t_j - t_{j+1})K(t_{j+1}, x^j, \xi^{j+1}) \Bigg) \\ &\qquad\qquad\qquad \times u(x^{\nu+1}) dx^{\nu+1} đ\xi^{\nu+1} \cdots dx^1 đ\xi^1 , \end{aligned}$$

in $\boldsymbol{L}^2$ *where* $x^0 \equiv x$.

(5) $U(t,s) \equiv p^\star(t,s;X,D_x)$ *is the fundamental solution for the operator* $L \equiv \partial_t - K(t,X,D_x)$ *such that*

$$\begin{cases} LU(t,s) = 0 & \text{on } (s,T] \\ U(s,s) = I & (0 \le s \le T)\,. \end{cases} \tag{2.9}$$

Remark.

1° It is sufficient to satisfy the conditions $(a1)$ and $(a2)$ for $|\xi| \ge M$, with a constant $M \ge 0$. In fact, in this case, there exists a sufficiently large $R > 0$ such that the symbol $K_R(t,x,\xi) \equiv K(t,x,\xi) - R$ satisfies $(a1)$ and $(a2)$ for any ξ. Let $U_R(t,s)$ be the fundamental solution of $L_R \equiv \partial_t - K_R(t,X,D_x)$. Then $U(t,s) \equiv e^{(t-s)R}U_R(t,s)$ is the fundamental solution of L.

2° We can replace $(t_j - t_{j+1})K(t_{j+1},\cdot,\cdot)$ with $\int_{t_{j+1}}^{t_j} K(\tau,\cdot,\cdot)d\tau$. Furthermore, in this case, we can replace (2.6) with

$$|p(\Delta_{t,s}) - p(\Delta'_{t,s})|_l^{(2m)} \le C'_l(t-s)|\Delta_{t,s}|\,, \tag{2.6'}$$

and the proof of Theorem 2.1 becomes a little easier.

Example.
Consider

$$L \equiv \partial_t + a(t)|x|^{2l}(-\triangle)^m + (-\triangle)^{m'} \quad (0 \le a(t) \in \mathcal{C}[0,T]\,,\ m - m' < l)\,.$$

If we set $\rho = 1$, $\delta = (m - m')/l$, $m \to 2m$ and $m' \to 2m'$, then the symbol $a(t)|x|^{2l}|\xi|^{2m} + |\xi|^{2m'}$ satisfies the conditions $(a1)$ and $(a2)$. Therefore, we see that these conditions are satisfied not only by the usual parabolic operators, but also by parabolic operators of a degenerate type.

Before we prove Theorem 2.1, we prepare some lemmas:
To begin with, for $T \ge t \ge s \ge 0$, we define $p(t,s;x,\xi)$ by

$$p(t,s;x,\xi) \equiv \exp\Big((t-s)K(s,x,\xi)\Big)\,. \tag{2.10}$$

The next lemma is a generalization of asymptotic expansion formulas, and an essential part in this paper. Especially, it is important that all constants are independent of $\varDelta_{t_0,t_{\nu+1}}$ and ν.

Lemma 2.2 (A key lemma).
Let $\varDelta_{t_0,t_{\nu+1}}\,:\,(T \ge) t_0 \ge t_1 \ge \cdots \ge t_\nu \ge t_{\nu+1} (\ge 0)$, $\nu = 1,2,\ldots,$ and let N_0 be a fixed positive integer such that $(\rho - \delta)N_0 \ge 2m$. Define $q(\varDelta_{t_0,t_1};x,\xi)$, $q(\varDelta_{t_0,t_{\nu+1}};x,\xi)$, and $r(\varDelta_{t_0,t_{\nu+1}};x,\xi)$ respectively by

$$q(\varDelta_{t_0,t_1};x,\xi) \equiv p(t_0,t_1;x,\xi)\,, \tag{2.11}$$

$$\begin{aligned} q(\varDelta_{t_0,t_{\nu+1}};x,\xi) \equiv & \sum_{|\alpha^1|+|\alpha^2|+\cdots+|\alpha^\nu|<N_0} \frac{1}{\alpha^1!\alpha^2!\cdots\alpha^\nu!} \\ & \times p_{(\alpha^\nu)}(t_\nu,t_{\nu+1};x,\xi)\partial_\xi^{\alpha^\nu}\Big(p_{(\alpha_{\nu-1})}(t_{\nu-1},t_\nu;x,\xi) \\ & \times \partial_\xi^{\alpha^{\nu-1}}\Big(\cdots p_{(\alpha^2)}(t_2,t_3;x,\xi)\partial_\xi^{\alpha^2}\Big(p_{(\alpha^1)}(t_1,t_2;x,\xi) \\ & \times \partial_\xi^{\alpha^1}\Big(p(t_0,t_1;x,\xi)\Big)\Big)\cdots\Big)\Big)\,. \end{aligned} \tag{2.12}$$

and

$$r(\Delta_{t_0,t_{\nu+1}};x,\xi) \equiv \sum_{|\alpha^1|+|\alpha^2|+\cdots+|\alpha^\nu|=N_0,\,|\alpha^\nu|\neq 0} \frac{|\alpha^\nu|}{\alpha^1!\alpha^2!\cdots\alpha^\nu!} \tag{2.13}$$
$$\times \int_0^1 (1-\theta)^{|\alpha^\nu|-1}\mathrm{O_s}\text{—}\iint e^{-iy\cdot\eta}p_{(\alpha^\nu)}(t_\nu,t_{\nu+1};x+y,\xi)$$
$$\times \partial_\xi^{\alpha^\nu}\Big(p_{(\alpha_{\nu-1})}(t_{\nu-1},t_\nu;x,\xi+\theta\eta)$$
$$\times \partial_\xi^{\alpha^{\nu-1}}\Big(\cdots p_{(\alpha^2)}(t_2,t_3;x,\xi+\theta\eta)$$
$$\times \partial_\xi^{\alpha^2}\Big(p_{(\alpha^1)}(t_1,t_2;x,\xi+\theta\eta)$$
$$\times \partial_\xi^{\alpha^1}\Big(p(t_0,t_1;x,\xi+\theta\eta)\Big)\Big)\cdots\Big)\Big)dy\,{d\!\!\bar{}}\eta\,d\theta\,.$$

Then it follows that

$$\begin{aligned} &q(\Delta_{t_0,t_\nu};X,D_x)p(t_\nu,t_{\nu+1};X,D_x) \\ &= q(\Delta_{t_0,t_{\nu+1}};X,D_x)+r(\Delta_{t_0,t_{\nu+1}};X,D_x)\,. \end{aligned} \tag{2.14}$$

Furthermore, there exist constants $C_{1,l},C_{2,l},C_{3,l}$ *such that*

$$|q(\Delta_{t_0,t_\nu})|_l^{(0)} \le C_{1,l}\,, \tag{2.15}$$

$$\begin{aligned} &|q(\Delta_{t_0,t_{\nu+1}})-p(t_0,t_{\nu+1})|_l^{(2m)} \\ &\le C_{2,l}(t_0-t_{\nu+1}) \\ &\quad\times\Big((t_0-t_{\nu+1})+\sup_{t_0\ge t'\ge t''\ge t_{\nu+1}}|K(t')-K(t'')|_l^{(m)}\Big)\,, \end{aligned} \tag{2.16}$$

and

$$|r(\Delta_{t_0,t_{\nu+1}})|_l^{(0)} \le C_{3,l}(t_0-t_\nu)(t_\nu-t_{\nu+1})\,, \tag{2.17}$$

for any $\Delta_{t_0,t_{\nu+1}}:(T\ge)t_0\ge t_1\ge\cdots\ge t_\nu\ge t_{\nu+1}(\ge 0)$ *and* $\nu=1,2,\ldots$.

PROOF.

1° For $T \geq t \geq s \geq 0$, we set

$$\eta(t,s;x,\xi) \equiv -(t-s)\mathrm{Re}\ K(s,x,\xi)\,(\geq 0)\,. \tag{2.18}$$

Furthermore, for $\Delta_{t_0,t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ and $\nu = 1,2,\ldots$, we define $d(\Delta_{t_0,t_\nu};x,\xi)$ by

$$d(\Delta_{t_0,t_\nu};x,\xi) \equiv \prod_{j=0}^{\nu-1} p(t_j,t_{j+1};x,\xi)\,, \tag{2.19}$$

and we set

$$\eta(\Delta_{t_0,t_\nu};x,\xi) \equiv \sum_{j=0}^{\nu-1} \eta(t_j,t_{j+1};x,\xi)\,. \tag{2.20}$$

Clearly, we have

$$|d(\Delta_{t_0,t_\nu};x,\xi)| = \exp\Big(-\eta(\Delta_{t_0,t_\nu};x,\xi)\Big)\,. \tag{2.21}$$

2° Define $d_{\alpha,\beta}(\Delta_{t_0,t_\nu};x,\xi)$ by

$$d^{(\alpha)}_{(\beta)}(\Delta_{t_0,t_\nu};x,\xi) \equiv d_{\alpha,\beta}(\Delta_{t_0,t_\nu};x,\xi)d(\Delta_{t_0,t_\nu};x,\xi)\,. \tag{2.22}$$

Then, by induction, for any α,β ($|\alpha+\beta| \geq 1$) and α',β', there exists a constant $C_{\alpha,\beta,\alpha',\beta'}$ such that

$$\begin{aligned} &|d_{\alpha,\beta}{}^{(\alpha')}_{(\beta')}(\Delta_{t_0,t_\nu};x,\xi)| \\ &\quad \leq C_{\alpha,\beta,\alpha',\beta'}\eta(\Delta_{t_0,t_\nu};x,\xi)\Big(\eta(\Delta_{t_0,t_\nu};x,\xi)+1\Big)^{|\alpha+\beta|-1} \\ &\qquad \times \lambda(\xi)^{\delta|\beta+\beta'|-\rho|\alpha+\alpha'|}\,, \end{aligned} \tag{2.23}$$

for any $\Delta_{t_0,t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ and $\nu = 1,2,\ldots$.

3° Let $\widetilde{\alpha}^{\nu} \equiv (\alpha^1, \dots, \alpha^{\nu})$ denote a multi-index of $\boldsymbol{R}^{\nu n}$. Define $f_{\widetilde{\alpha}^{\nu}}(\varDelta_{t_0,t_{\nu+1}}; x, \xi)$ by

$$\begin{aligned}(2.24)\quad & f_{\widetilde{\alpha}^{\nu}}(\varDelta_{t_0,t_{\nu+1}}; x, \xi) d(\varDelta_{t_0,t_{\nu+1}}; x, \xi) \\ & \equiv p_{(\alpha^{\nu})}(t_{\nu}, t_{\nu+1}; x, \xi) \partial_{\xi}^{\alpha^{\nu}} \Big(p_{(\alpha_{\nu-1})}(t_{\nu-1}, t_{\nu}; x, \xi) \partial_{\xi}^{\alpha^{\nu-1}} \Big(\\ & \cdots p_{(\alpha^2)}(t_2, t_3; x, \xi) \partial_{\xi}^{\alpha^2} \Big(p_{(\alpha^1)}(t_1, t_2; x, \xi) \partial_{\xi}^{\alpha^1} \Big(p(t_0, t_1; x, \xi) \Big) \Big) \cdots \Big) \Big) .\end{aligned}$$

Then, by induction, for any $N = 1, 2, \dots$ and α, β, there exists a constant $C_{N,\alpha,\beta}$ such that

$$\begin{aligned}(2.25)\quad & |f_{\widetilde{\alpha}^{\nu}}{}^{(\alpha)}_{(\beta)}(\varDelta_{t_0,t_{\nu+1}}; x, \xi)| \\ & \le C_{N,\alpha,\beta} \Big(\prod_{k=1}^{J} \eta(t_{j_k}, t_{j_k+1}; x, \xi) \Big) \eta(\varDelta_{t_0,t_{\nu+1}}; x, \xi) \\ & \quad \times \Big(\eta(\varDelta_{t_0,t_{\nu+1}}; x, \xi) + 1 \Big)^{2(N-1)} \lambda(\xi)^{-(\rho-\delta)N + \delta|\beta| - \rho|\alpha|} ,\end{aligned}$$

where

$$1 \le j_1 < j_2 < \cdots < j_J \le \nu , \quad |\alpha^{j_k}| \ne 0 \, (k = 1, 2, \dots, J) ,$$

and

$$\sum_{j=1}^{\nu} |\alpha^j| = \sum_{k=1}^{J} |\alpha^{j_k}| = N ,$$

for any $\varDelta_{t_0,t_{\nu+1}} : (T \ge) t_0 \ge t_1 \ge \cdots \ge t_{\nu} \ge t_{\nu+1} (\ge 0)$ and $\nu = 1, 2, \dots$.

4° For $N = 1, 2, \dots$, define $g_N(\varDelta_{t_0,t_{\nu+1}}; x, \xi)$ by

$$\begin{aligned}(2.26)\quad & g_N(\varDelta_{t_0,t_{\nu+1}}; x, \xi) \\ & \equiv \sum_{|\alpha^1| + |\alpha^2| + \cdots + |\alpha^{\nu}| = N} \frac{1}{\alpha^1! \alpha^2! \cdots \alpha^{\nu}!} f_{\widetilde{\alpha}^{\nu}}(\varDelta_{t_0,t_{\nu+1}}; x, \xi) .\end{aligned}$$

By (2.25), we have

$$
\begin{aligned}
(2.27)\quad & |g_N{}^{(\alpha)}_{(\beta)}(\Delta_{t_0,t_{\nu+1}};x,\xi)| \\
&\le \sum_{J=1}^{N} \sum_{1\le j_1<j_2<\cdots<j_J\le\nu} \sum_{\sum_{k=1}^{J}|\alpha^{j_k}|=N,\,|\alpha^{j_k}|\ne 0} \frac{1}{\alpha^{j_1}!\alpha^{j_2}!\cdots\alpha^{j_J}!} \\
&\quad \times C_{N,\alpha,\beta}\Big(\prod_{k=1}^{J}\eta(t_{j_k},t_{j_k+1};x,\xi)\Big)\eta(\Delta_{t_0,t_{\nu+1}};x,\xi) \\
&\quad \times \Big(\eta(\Delta_{t_0,t_{\nu+1}};x,\xi)+1\Big)^{2(N-1)}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\
&\le (nN)^N C_{N,\alpha,\beta}\eta(\Delta_{t_0,t_{\nu+1}};x,\xi) \\
&\quad \times \Big(\eta(\Delta_{t_0,t_{\nu+1}};x,\xi)+1\Big)^{2(N-1)}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\
&\quad \times \left(\sum_{J=1}^{N}\sum_{1\le j_1<j_2<\cdots<j_J\le\nu}\prod_{k=1}^{J}\eta(t_{j_k},t_{j_k+1};x,\xi)\right).
\end{aligned}
$$

Hence, for any $N=1,2,\dots$ and α,β, there exists a constant $C'_{N,\alpha,\beta}$ such that

$$
\begin{aligned}
(2.28)\quad & |g_N{}^{(\alpha)}_{(\beta)}(\Delta_{t_0,t_{\nu+1}};x,\xi)| \\
&\le C'_{N,\alpha,\beta}\Big(\eta(\Delta_{t_0,t_{\nu+1}};x,\xi)\Big)^2 \\
&\quad \times \Big(\eta(\Delta_{t_0,t_{\nu+1}};x,\xi)+1\Big)^{3(N-1)}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|},
\end{aligned}
$$

for any $\Delta_{t_0,t_{\nu+1}} : (T\ge)t_0\ge t_1\ge\cdots\ge t_\nu\ge t_{\nu+1}(\ge 0)$ and $\nu=1,2,\dots$.

5° Set

$$
(2.29)\qquad h_N(\Delta_{t_0,t_{\nu+1}};x,\xi)\equiv g_N(\Delta_{t_0,t_{\nu+1}};x,\xi)d(\Delta_{t_0,t_{\nu+1}};x,\xi).
$$

Here we note that

$$
(2.30)\qquad \sup_{\eta>0}\eta^k e^{-\eta}<\infty \quad (k=0,1,2,\dots).
$$

By (2.21), (2.23) and (2.28), there exist constants $C'_{\alpha,\beta}$, $C''_{\alpha,\beta}$, $C''_{N,\alpha,\beta}$, $C'''_{N,\alpha,\beta}$, $C''''_{N,\alpha,\beta}$ such that

$$(2.31)\quad |d^{(\alpha)}_{(\beta)}(\varDelta_{t_0,t_\nu};x,\xi)| \le \begin{cases} C'_{\alpha,\beta}\lambda(\xi)^{\delta|\beta|-\rho|\alpha|} \\ C''_{\alpha,\beta}(t_0-t_\nu)\lambda(\xi)^{m+\delta|\beta|-\rho|\alpha|} & (|\alpha+\beta|\ge 1)\,, \end{cases}$$

and

$$(2.32)\qquad |h_N{}^{(\alpha)}_{(\beta)}(\varDelta_{t_0,t_{\nu+1}};x,\xi)| \le \begin{cases} C''_{N,\alpha,\beta}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\ C'''_{N,\alpha,\beta}(t_0-t_{\nu+1})\lambda(\xi)^{m-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\ C''''_{N,\alpha,\beta}(t_0-t_{\nu+1})^2\lambda(\xi)^{2m-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|}\,, \end{cases}$$

for any $\varDelta_{t_0,t_{\nu+1}} : (T\ge)t_0\ge t_1\ge\cdots\ge t_\nu\ge t_{\nu+1}(\ge 0)$ and $\nu=1,2,\ldots$.

6° Now we note that

$$(2.33)\qquad q(\varDelta_{t_0,t_{\nu+1}};x,\xi) = d(\varDelta_{t_0,t_{\nu+1}};x,\xi) + \sum_{N=1}^{N_0-1} h_N(\varDelta_{t_0,t_{\nu+1}};x,\xi)\,,$$

and

$$\begin{aligned}(2.34)\qquad & d(\varDelta_{t_0,t_{\nu+1}};x,\xi) - p(t_0,t_{\nu+1};x,\xi) \\ &= \sum_{j=0}^{\nu}(t_j-t_{j+1})\Big(K(t_{j+1},x,\xi)-K(t_{\nu+1},x,\xi)\Big) \\ &\times \int_0^1 \exp\Big(\theta\sum_{j=0}^{\nu}(t_j-t_{j+1})K(t_{j+1},x,\xi)\Big) \\ &\times \exp\Big((1-\theta)(t_0-t_{\nu+1})K(t_{\nu+1},x,\xi)\Big)d\theta\,.\end{aligned}$$

By (2.31) and (2.32), we get (2.15) and (2.16). Furthermore, we note that

$$
\begin{aligned}
(2.35)\quad & r(\varDelta_{t_0,t_{\nu+2}};x,\xi) \\
&= \sum_{0<|\alpha^{\nu+1}|<N_0} \frac{|\alpha^{\nu+1}|}{\alpha^{\nu+1}!}\int_0^1 (1-\theta)^{|\alpha^{\nu+1}|-1} \\
&\quad \times \mathrm{Os}\text{—}\iint e^{-iy\cdot\eta} h_{N_0-|\alpha^{\nu+1}|}{}^{(\alpha^{\nu+1})}(\varDelta_{t_0,t_{\nu+1}};x,\xi+\theta\eta) \\
&\quad \times d_{(\alpha^{\nu+1})}(\varDelta_{t_{\nu+1},t_{\nu+2}};x+y,\xi)dy\bar{d}\eta d\theta \\
&\quad + \sum_{|\alpha^{\nu+1}|=N_0} \frac{|\alpha^{\nu+1}|}{\alpha^{\nu+1}!}\int_0^1 (1-\theta)^{|\alpha^{\nu+1}|-1} \\
&\quad \times \mathrm{Os}\text{—}\iint e^{-iy\cdot\eta} d^{(\alpha^{\nu+1})}(\varDelta_{t_0,t_{\nu+1}};x,\xi+\theta\eta) \\
&\quad \times d_{(\alpha^{\nu+1})}(\varDelta_{t_{\nu+1},t_{\nu+2}};x+y,\xi)dy\bar{d}\eta d\theta\,.
\end{aligned}
$$

By (2.31), (2.32) and Theorem 1.4 , we get (2.17).

7° By induction, we get (2.14). □

The idea of the next lemma is found in Fujiwara [3].

Lemma 2.3 (Fujiwara's skip).
Define $\varUpsilon(\varDelta_{t_0,t_{\nu+1}};x,\xi) \in \boldsymbol{S}^0_{\lambda,\rho,\delta}$ *by*

$$
\begin{aligned}
(2.36)\qquad & p(t_0,t_1;X,D_x)p(t_1,t_2;X,D_x)\cdots p(t_\nu,t_{\nu+1};X,D_x) \\
&\equiv q(\varDelta_{t_0,t_{\nu+1}};X,D_x)+\varUpsilon(\varDelta_{t_0,t_{\nu+1}};X,D_x)\,.
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
(2.37)\qquad & \varUpsilon(\varDelta_{t_0,t_{\nu+1}};X,D_x) \\
&= \sum{}' r(\varDelta_{t_0,t_{j_1+1}};X,D_x)r(\varDelta_{t_{j_1+1},t_{j_2+1}};X,D_x) \\
&\qquad \cdots r(\varDelta_{t_{j_{J-1}+1},t_{j_J+1}};X,D_x)q(\varDelta_{t_{j_J+1},t_{\nu+1}};X,D_x)\,,
\end{aligned}
$$

where $\sum'$ *stands for the summation with respect to the sequences of integers* $(j_1,j_2,\ldots,j_J)$ *with the property*

$$
(2.38)\quad 0<j_1<j_1+1<j_2<j_2+1<\cdots<j_{J-1}<j_{J-1}+1<j_J\le\nu\,,
$$

and, in the special case of $j_J = \nu$*, we set* $q(\varDelta_{t_{j_J+1},t_{\nu+1}}; X, D_x) \equiv I$*.*
Furthermore, there exists a constant $C_{4,l}$ *such that*

$$|\varUpsilon(\varDelta_{t_0,t_{\nu+1}})|_l^{(0)} \leq C_{4,l}(t_0 - t_{\nu+1})^2\,, \tag{2.39}$$

for any $\varDelta_{t_0,t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ *and* $\nu = 1, 2, \ldots$*.*

Proof. Using (2.14) inductively, we get (2.37). Now let A_l, l' be the same constants in Theorem 1.3, and let $C_{1,l}, C_{3,l}$ be the same constants in Lemma 2.2. By (2.15), (2.17) and Theorem 1.3, we have

$$\begin{aligned}
&|\varUpsilon(\varDelta_{t_0,t_{\nu+1}})|_l^{(0)} \\
&\quad \leq {\sum}' (A_l)^J |r(\varDelta_{t_0,t_{j_1+1}})|_{l'}^{(0)} |r(\varDelta_{t_{j_1+1},t_{j_2+1}})|_{l'}^{(0)} \\
&\qquad \cdots |r(\varDelta_{t_{j_{J-1}+1},t_{j_J+1}})|_{l'}^{(0)} |q(\varDelta_{t_{j_J+1},t_{\nu+1}})|_{l'}^{(0)} \\
&\quad \leq {\sum}' (A_l)^J \Big(\prod_{k=1}^{J} C_{3,l'}(t_0 - t_{\nu+1})(t_{j_k} - t_{j_k+1}) \Big) C_{1,l'} \\
&\quad \leq C_{1,l'} \bigg(\prod_{j=0}^{\nu} \Big(1 + A_l C_{3,l'}(t_0 - t_{\nu+1})(t_j - t_{j+1}) \Big) - 1 \bigg) \\
&\quad \leq C_{4,l}(t_0 - t_{\nu+1})^2\,. \ \square
\end{aligned} \tag{2.40}$$

Now we prove Theorem 2.1:

Proof of Theorem 2.1.
1° Define $p(\varDelta_{t,s}; x, \xi)$ by

$$p(\varDelta_{t,s}; x, \xi) \equiv q(\varDelta_{t,s}; x, \xi) + \varUpsilon(\varDelta_{t,s}; x, \xi)\,. \tag{2.41}$$

Then (1) is clear.
2° By (2.14) and (2.39), we get (2.5). Next, we note that

$$\begin{aligned}
&p(\varDelta'_{t_j,t_{j+1}}; x, \xi) - p(t_j, t_{j+1}; x, \xi) \\
&\quad = \Big(q(\varDelta'_{t_j,t_{j+1}}; x, \xi) - p(t_j, t_{j+1}; x, \xi) \Big) + \varUpsilon(\varDelta'_{t_j,t_{j+1}}; x, \xi)\,,
\end{aligned} \tag{2.42}$$

Hence, by (2.16) and (2.39), there exists a constant $C_{5,l}$ such that

$$\begin{aligned}(2.43)\quad & |p(t_j,t_{j+1}) - p(\Delta'_{t_j,t_{j+1}})|_l^{(2m)} \\ & \le C_{5,l}(t_j - t_{j+1})\Big((t_j - t_{j+1}) + \sup_{t_j \ge t' \ge t'' \ge t_{j+1}} |K(t') - K(t'')|_l^{(m)}\Big)\,.\end{aligned}$$

Here we can write

$$\begin{aligned}(2.44)\quad p(\Delta_{t,s};X,D_x) - p(\Delta'_{t,s};X,D_x) = & \sum_{j=0}^{\nu} p(\Delta'_{t_0,t_j};X,D_x) \\ & \circ \Big(p(t_j,t_{j+1};X,D_x) - p(\Delta'_{t_j,t_{j+1}};X,D_x)\Big) \\ & \circ p(\Delta_{t_{j+1},t_{\nu+1}};X,D_x)\,.\end{aligned}$$

By (2.5), (2.43) and Theorem 1.3, we get (2.6).

3° By (2.6) and (2.5), there exists $p^\star(t,s;x,\xi) \in \boldsymbol{S}^0_{\lambda,\rho,\delta}$ such that

$$(2.45)\qquad |p^\star(t,s)|_l^{(0)} \le C_l\,,$$

and

$$\begin{aligned}(2.46)\quad & |p(\Delta_{t,s}) - p^\star(t,s)|_l^{(2m)} \\ & \le C'_l(t-s)\Big(|\Delta_{t,s}| + \sup_{|t'-t''| \le |\Delta_{t,s}|} |K(t') - K(t'')|_{l'}^{(m)}\Big)\,.\end{aligned}$$

Hence we get (3).

4° By the result of (3), we get (4). See Chapter 3 § 7 in H.Kumano-go [6].

5° Using the results of (2) and (3), it is easy to check (5). □

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