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On the growth of meromorphic functions on the unit disc and conformal martingales

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Abstract. The relations between divergence of integrals of meromorphic functions on the unit disc along Brownian paths and value distribution of the meromorphic functions will be discussed.

In this note we wish to discuss some aspects of the behaviour of conformal martingales related to Nevanlinna theory.

Let (Z_t, P_z) be a complex Brownian motion on $\mathbf{C}$ and $\sigma = \inf\{t > 0 : |Z_t| \geq 1\}$.
We first consider the behaviour of conformal martingales under P_z defined by such a form as

$$X_t = \int_0^{\sigma \wedge t} f(Z_t) dZ_t,$$

where f is a meromorphic function on the unit disc $\Delta(1)$ in $\mathbf{C}$.

We have

Theorem 1. *Under the above situation if there exists* $\alpha > 0$ *such that*

$$\int_0^{\sigma} |f|^{\alpha}(Z_t) dt = \infty \qquad P_z - \ a.s. \tag{1}$$

for any z *in* $\Delta(1)$, *then*

$$\alpha T_f(\rho) \geq 2 \log \frac{1}{1-\rho} \qquad as \quad \rho \to 1, \tag{2}$$

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where $T_f(\rho)$ *is the Ahlfors-Shimizu characteristic function of* f *defined by*

$$T_f(\rho) = \int_{\Delta(\rho)} \frac{|f'(z)|^2}{(|f(z)|^2+1)^2} g_\rho(0,z)dxdy$$

where $\Delta(\rho) = \{z \in \mathbf{C} : |z| < \rho\}$ *and* $g_\rho(w,z)$ *is a usual green function of* $\frac{1}{2}\Delta_{\mathbf{C}}$ *on* $\Delta(\rho)$.

We have the following as a corollary to the above theorem.

Corollary 1. *If* f *satisfies the assumption of Theorem 1, then* f *omits at most* $2 + \frac{\alpha}{2}$ *values.*

It is a well-known fact that a conformal martingale X_t converges in $\mathbf{C}$ if and only if

$$< X >_\infty = \int_0^\sigma |f|^2(Z_t)dt < \infty \qquad \text{a.s.}$$

where $< X >_t$ is a usual quadratic variation of X_t.
Then we can relate the behaviour of X_t to the value distribution of f.

Corollary 2. *If* X_t *does not converge as* $t \to \infty$ P_z- *a.s. for all* z *in* $\Delta(1)$, *then* f *can omit at most three points of* $\mathbf{C} \cup \{\infty\}$.

Since the poles of f are polar, then $f'(Z_t)$ can be defined a.s for $0 < t < \sigma$. We also have the similar result for drivatives of meromorphic functions.

Corollary 3. *If* $f(Z_t)$ *does not converge as* $t \to \infty$ P_z- *a.s. for all* z *in* $\Delta(1)$, *then*

$$T_{f'}(\rho) \geq 2\log\frac{1}{1-\rho} \qquad \textit{as} \quad \rho \to 1.$$

Hence f' *can omit at most three points of* $\mathbf{C} \cup \{\infty\}$.

We can show a defect relation for such functions satisfying (1) and it leads the above corollaries. We define functions appeared in classical Nevanlinna theory as in [5].

$$m(\rho,a) = \int_0^{2\pi} \log^+ \frac{1}{|f(\rho e^{i\theta})-a|}\frac{d\theta}{2\pi}, \quad m(\rho,\infty) = \int_0^{2\pi} \log^+ |f|(\rho e^{i\theta})\frac{d\theta}{2\pi}$$

$$N(a,\rho) = \sum_{f(\zeta)=a, \zeta\in\Delta(\rho)} \log\frac{\rho}{|\zeta|} \quad (\text{ the sum counting with multiplicity})$$

Nevanlinna's second main theorem is as follows.

PROPOSITION 1. *Let f be a meromorphic function on $\Delta(1)$. Then we have for $\beta > 0$*

$$\sum_{i=1}^{q} m(\rho, a_i) + N_1(\rho) \leq 2T_f(\rho) + O(\log T_f(\rho)) + (1+\beta)\log\frac{1}{1-\rho}$$

holds for $\rho \notin E_\beta \subset [0,1)$ with $\int_{E_\beta} \frac{1}{1-\rho} d\rho < \infty$, where $N_1(\rho)$ is the counting function for critical points of f defined by

$$N_1(\rho) = \sum_{f'(\zeta)=0, \zeta\in\Delta(\rho)} \log\frac{\rho}{|\zeta|} \quad (\textit{ the sum counting with multiplicity}).$$

We define the defect for f by

$$\delta(a) = \liminf_{\rho\to 1}\left(1 - \frac{N(a,\rho)}{T_f(\rho)}\right).$$

Combining the above Nevanlinna's theorem with (2) we immediately have that under the assumption in the theorem 1

$$\sum_{i=1}^{q} \delta(a_i) \leq 2 + \frac{\alpha}{2}. \tag{3}$$

If f omits a, then $\delta(a) = 1$. Hence we have the Corollary 1.

(2) follows from the following.

LEMMA 1. *Let $u(z)$ be a nonnegative function on $\Delta(1)$ satisfying that $u(Z_t)$ is a local submartingale. If*

$$\int_0^\sigma e^{u(Z_t)} dt = \infty \qquad P_z - \text{ a.s.} \tag{4}$$

for Brownian motion Z_t starting from any points z inside $\Delta(1)$, then

$$E[u(Z_{\sigma_\rho})] \geq 2\log\frac{1}{1-\rho} \qquad \textit{as} \quad \rho \to 1,$$

where $\sigma_\rho = \inf\{t > 0 : |Z_t| \geq \rho\}$.

Since the poles of f are polar for Brownian motion, then $u = \log^+ |f|^\alpha(Z_t)$ is a local submartingale. Then we can apply the above lemma to this case.

The same method as Theorem 1 leads us to have the similar result for the holomorphic cuves from the unit disc to $\mathbf{P}^n(\mathbf{C})$. Let f be a holomorphic map from the unit disc to $\mathbf{P}^n(\mathbf{C})$. It can be expressed as $(f_0, \dots, f_n)$ in a homogeneous coordinate. Set $|f|^2 = (\frac{f_1}{f_0})^2 + \cdots + (\frac{f_n}{f_0})^2$ and $\|f\|^2 = |f_0|^2 + \cdots + |f_n|^2$. Let's define a characteristic function, counting function and defect of f. Let D be a hyperplane in $\mathbf{P}^n(\mathbf{C})$.

$$
\begin{aligned}
T(\rho) &= \int_0^{2\pi} \log \|f\|(\rho e^{i\theta}) \frac{d\theta}{2\pi} \\
N(\rho, D) &= \sum_{\zeta \in f^{-1}(D), \zeta \in \Delta(\rho)} \log \frac{\rho}{|\zeta|} \quad \text{(the sum counting with multiplicity)} \\
\delta(D) &= \liminf_{\rho \to 1} (1 - \frac{N(\rho, D)}{T(\rho)}).
\end{aligned}
$$

THEOREM 2. *Let f be a nondegenerate holomorphic curve from $\Delta(1)$ to $\mathbf{P}^n(\mathbf{C})$. If there exists $\alpha > 0$ such that*

$$\int_0^\sigma |f|^\alpha(Z_t) dt = \infty \qquad a.s.$$

for Brownian motion Z_t starting from any points inside $\Delta(1)$, then for any q hyperplanes $\{D_j\}$ in general position

$$\sum_{i=1}^q \delta(D_i) \leq n + 1 + \frac{n(n+1)}{4} \alpha.$$

As for the proof of this theorem the same procedure can be carried as the theorem 1 where we have only to be careful to the remainder term of the second main theorem. In this case it is just $\frac{n(n+1)}{2} \log \frac{1}{1-\rho}$([6]) as mentioned in §2.

The related results of Theorem 1 and Theorem 2 are exposed in [4].

We next consider an analogy of the above result for a more general conformal martingale under P_z, defined by such a form as

$$M_t^0 = \int_0^{t\wedge\sigma} g(Z_s)dZ_s$$
$$M_t^1 = \int_0^{t\wedge\sigma} M_s^0 dZ_s,$$

where g is a locally bounded complex valued function on $\Delta(1)$.

We have

THEOREM 3. *If M_t^1 does not converge as $t \to \infty$ P_z- a.s. for all z in $\Delta(1)$, then*

$$E[\ [M^0, M^0]_{\sigma_\rho}\ ] \geq \log \frac{1}{1-\rho},$$

where $[M^0, M^0]_t$ is the Riemannian quadratic variation (c.f.[3]) of M^0 on $\mathbf{P}^1$.

Since M^1 never hits any single points a.s, the phrase that 'M_t omits a point' is meaningless. But we can verify the phrase that 'M_t feels a point' in a sense as we will define later.

COROLLARY 4. *Assume that*

$$E[\log |g|^2(Z_{\sigma_\rho})] \leq o(E[\ [M^{(0)}, M^{(0)}]_{\sigma_\rho}\ ]) \qquad as\ \rho \uparrow \infty.$$

If M_t^1 does not converge as $t \to \infty$ P_z- a.s. for all z in $\Delta(1)$, then M_t^0 does not feel at most three points.

These topics will be treated in §3.

§1. Proof of Lemma 1

We note several facts on Brownian motion. Let Y_t be a Brownian motion on $\Delta(1)$ associated to the hyperbolic metric given by

$$ds^2 = \frac{4}{(1-|z|^2)^2}|dz|^2.$$

We can write

$$Y_{\phi_t} = Z_t \qquad \text{with} \quad \phi_t = \int_0^t \frac{4}{(1-|Z_s|^2)^2} ds. \tag{5}$$

Note that $\phi_\infty = \infty$a.s.

From now on we use hyperbolic length r for Euclidean length ρ, namely $r = \log \frac{1+\rho}{1-\rho}$. Let r_t be the hyperbolic length of Y_t from the origin. Define $\tau_r = \inf\{t > 0 : r_t \geq r\}$. Then our goal is to show

$$E[u(Y_{\tau_r})] \geq 2r \qquad \text{as} \qquad r \to \infty.$$

Remark that by time change argument the condition (4) in the lemma 1 says in the words of Y_t

$$\int_0^\infty e^{u(Y_t)} e^{-2r_t} dt = \infty. \tag{4'}$$

Assume that there exists a sequence $\{r_\nu \uparrow \infty \quad \nu = 1, 2, \dots\}$ and $0 < \delta < 1$ such that

$$\frac{E[u(Y_{\tau_{r_\nu}})]}{r_\nu} \leq 2(1-\delta) < 0.$$

Fix $\epsilon > 0$. Set

$$\kappa_\nu = \inf\{t \geq \tau_{r_\nu(1-\epsilon)} : \sigma(Y_t) e^{-2r_t} \geq \frac{1}{r_t^2 + 1}\}$$

The assumption (4′) implies that $\kappa_\nu < \infty$ a.s. for $0 \leq \nu < \infty$ and $\kappa_\nu \uparrow \infty$ as $\nu \uparrow \infty$ a.s. Let θ_t be a shift on the Brownian path space Ω *i.e.* $(\omega \circ \theta_t)(s) = \omega(t+s)$ for $\omega \in \Omega$. We can write $\kappa_\nu = \kappa_0 \circ \theta_{\tau_{r_\nu(1-\epsilon)}} + \tau_{r_\nu(1-\epsilon)}$, where $\kappa_\nu = \inf\{t \geq 0 : \sigma(Y_t) e^{-2r_t} \geq \frac{1}{r_t^2+1}\}$. Using the positivity of u and the definition of κ_ν

$$\begin{aligned} E[u(Y_{\kappa_\nu \wedge \tau_{r_\nu}})] &= E[u(Y_{\tau_{r_\nu}}) : \kappa_\nu > \tau_{r_\nu}] + E[u(Y_{\kappa_\nu}) : \kappa_\nu \leq \tau_{r_\nu}] \\ &\geq E[2r_{\kappa_\nu} + \log \frac{1}{r_{\kappa_r}^2 + 1} : \kappa_\nu \leq \tau_{r_\nu}] \end{aligned}$$

Since $u(Y_t)$ is a submartingale, the left hand side of the first line is bouded by

$$E[u(Y_{\tau_{r_\nu}})] \leq 2(1-\delta) r_\nu.$$

On the other hand side if $P(\kappa_\nu \le \tau_{r_\nu}) \to 1$ as $\nu \to \infty$, it is easy to see that letting $r_\nu \to \infty$, we have

$$\frac{E[2r_{\kappa_\nu} + \log \frac{1}{r_{\kappa_\nu}^2+1} : \kappa_\nu \le \tau_{r_\nu}]}{r_\nu} \to 2(1-\epsilon).$$

We here used that $\frac{r_t}{t} \to 1$ a.s. By the following lemma 2 and taking $\epsilon < \delta$, this leads a contradiction.

LEMMA 2. *The notations are as above.*

$$P(\kappa_\nu \le \tau_{r_\nu}) \to 1 \qquad \textit{as } \nu \to \infty.$$

PROOF. It is obvious that

$$P(\kappa_\nu \le \tau_{r_\nu}) \ge P(\int_{\tau_{r_\nu(1-\epsilon)}}^{\tau_r} \sigma(Y_t)e^{-2r_t}dt \ge \int_{\tau_{r_\nu(1-\epsilon)}}^{\tau_{r_\nu}} \frac{1}{r_t^2+1}dt).$$

It is easy to see that for a hyperbolic Brownian motion

$$\frac{r_t}{t} \underset{t\uparrow\infty}{\longrightarrow} 1 \qquad \text{and} \qquad \frac{\tau_r}{r} \underset{r\uparrow\infty}{\longrightarrow} 1 \quad a.s.$$

Using this fact we have

$$P(\int_{\tau_{(1-\epsilon)r_\nu}}^{\tau_{r_\nu}} \frac{1}{r_t^2+1}dt \le \frac{8}{(1-\epsilon)r_\nu}) \to 1 \quad \text{ as } r_\nu \to \infty.$$

We write this event G_r. Set $q_r = \frac{8}{(1-\epsilon)r_\nu}$ and $C_t = \int_0^t \sigma(Y_s)e^{-2r_s}ds$. Hence the underlined probaility is bounded below by

$$P(C_{\tau_{r_\nu}} - C_{\tau_{(1-\epsilon)r_\nu}} \ge q_r, G_r).$$

We estimate

$$P(C_{\tau_{r_\nu}} \circ \theta_{\tau_{(1-\epsilon)r_\nu}} \ge q_r).$$

Let $S_l = \inf\{t > 0 : C_t \geq l\}$. By strong Markov property

$$\begin{aligned}
P(C_{\tau_{r_\nu}} \circ \theta_{\tau_{(1-\epsilon)r_\nu}} \geq q_r) &= E[P_{Y_{\tau_{(1-\epsilon)r_\nu}}}(C_{\tau_{r_\nu}} \geq q_r)] \\
&= E[P_{Y_{\tau_{(1-\epsilon)r_\nu}}}(C_{\tau_{r_\nu}} \geq C_{S_{q_r}})] \\
&= E[P_{Y_{\tau_{(1-\epsilon)r_\nu}}}(\tau_{r_\nu} \geq S_{q_r})] \\
&= P(\tau_{r_\nu} \circ \theta_{\tau_{(1-\epsilon)r_\nu}} \geq S_{q_r} \circ \theta_{\tau_{(1-\epsilon)r_\nu}}).
\end{aligned}$$

Since $\tau_{r_\nu} > \tau_{(1-\epsilon)r_\nu}$,

$$\tau_{r_\nu} \circ \theta_{\tau_{(1-\epsilon)r_\nu}} = \tau_{r_\nu} - \tau_{(1-\epsilon)r_\nu}.$$

By the property of hyperbolic Brownian motion mentioned a few lines before the right hand side diverges almost surely as r_ν tends to infinity. And the continuity of C_t in t concludes $S_\epsilon \to 0$ as $\epsilon \to 0$. This completes the proof.

§2. Proof of Theorem 2

As mentioned in the Introduction we have only to note the following second main theorem. We use the notations in the Introduction.

Proposition 2. *Let f be a nondegenerate holomorphic map from $\Delta(1)$ to $\mathbf{P}^n$ and $\{D_i\ ;\ i = 1, \ldots, q\}$ be hyperplanes of $\mathbf{P}^n$ in general position. Then we have for $\beta > 0$*

$$(q - n - 1)T(\rho) \leq \sum_{i=1}^{q} N(\rho, D_i) + O(\log T(\rho)) + (1+\beta)\frac{n(n+1)}{2} \log \frac{1}{1-\rho}$$

holds for $\rho \notin E_\beta \subset [0,1)$ with $\int_{E_\beta} \frac{1}{1-\rho} d\rho < \infty$.

In Shabat's book([6]) we can see the proof of the case that f is from $\mathbf{C}$ to $\mathbf{P}^n$. We understand that $\frac{n(n+1)}{2}$ comes from $1 + 2 + \cdots + n$ in the proof. (In the entire case the remainder term turns to $\beta \frac{n(n+1)}{2} \log \rho$.) The $\log \frac{1}{1-\rho}$ remainder term comes from the following estimate. We use the notations in §2.

LEMMA 3([1]). *Let Y_t be a hyperbolic Brownian motion on $\Delta(1)$ and $h(x)$ a function on $\Delta(1)$ satisfying that $E[h(Y_{\tau_r})] < \infty$ for $0 < r < \infty$. Then*

$$E[h(Y_{\tau_r})] \le e^{\beta r}(E[\int_0^{\tau_r} h(Y_s)ds])^{(\beta+1)^2}$$

holds for $r \notin E_\beta \subset [0, \infty)$ where the Lebesgue measure of E_β is finite.

When rewrite the above inequality in the word of a complex Brownian motion, the remainder term appears.

COROLLARY 5. *Let Z_t be a complex Brownian motion. Under the above situation*

$$E[\log h(Z_{\tau_r})] \le (\beta+2)^2 \log E[\int_0^{\sigma_\rho} h(Y_s)\frac{4}{(1-|Z_s|^2)^2}ds] + (2+\beta)\log\frac{1}{1-\rho} + O(1)$$

holds for $\rho \notin E_\beta \subset [0,1)$ with $\int_{E_\beta} \frac{1}{1-\rho} d\rho < \infty$.

PROOF. From Lemma 3 we have

$$E[h(Z_{\sigma_\rho}\frac{4}{(1-|Z_{\sigma_\rho}|^2)^2}] \le e^{\beta r}(E[\int_0^{\sigma_\rho} h(Z_s)\frac{4}{(1-|Z_s|^2)^2}ds])^{(\beta+1)^2},$$

with $r = \frac{\rho+1}{\rho-1}$. With Jensen's inequality we immediately have the desired inequality. □

REMARK. Though the remainder term of Corollary 5 is different from one in the second main theorem, because we apply Corollary 5 to the proof with deviding the both side of the corollary by 2, then in the final conclusion it turns from 2 to 1.

§3. Related conformal martingales

We first see an analogy of Nevanlinna's theorems as the Proposition 1 for M^0 and M^1.

We regard M^0 and M^1 as conformal martingales on $\mathbf{P}^1$. We introduce a Riemannian qudratic variation of conformal martingale M on $\mathbf{P}^1$ by

$$[M,M]_t = \int_0^t \frac{d < M >_s}{(1+|M_s|^2)^2},$$

where $< M >_t$ is the usual quadratic variation of M.

We remark that $\frac{w \wedge \overline{w}}{(1+|w|^2)^2}$ is a cannonical Kähler form on $\mathbf{P}^1$, what is called, Fubini-Study metric.
We note that if $M_t = f(Z_t)$ for f is a meromorphic function and Z_t is a complex Brownian motion, then we have

$$E[\ [f(Z), f(Z)]_{\sigma_\rho}\ ] = T_f(\rho).$$

Let

$$\|x,a\| = \begin{cases} \frac{|x-a|}{\sqrt{1+|x|^2}\sqrt{1+|a|^2}} & \text{if } a < \infty, \\ \frac{1}{\sqrt{1+|x|^2}} & \text{if } a = \infty \quad (\text{chordal distance on } P^1)\ . \end{cases}$$

Set $u_a(w) = \log \|x,a\|^{-1}$.
We first have the following as in [2].

PROPOSITION 3. *Let M be a conformal martingale and T a stopping time satisfying that $E[\ [M,M]_T\ ] < \infty$. We have*

$$E[u_a(M_T)] - E[u_a(M_0)] + N(T,a) = E[\ [M,M]_T\ ],$$

where $N(T,a) = \lim_{\lambda \to \infty} \lambda P(u_a(M_T)^* > \lambda)$ with $u_a(M_t)^* = \sup_{0 \le s < t} u_a(M_t)$.

$N(T,a)$ is an analogy Nevanlinna's counting function of $a-$points.
Thus we say that a conformal martingale M does not feel $a \in \mathbf{P}^1$ if $N(T,a) = 0$ for any stopping time T.

We remarked that in [1] we can get Nevanlinna's second main theorem only using estimates on some functionals of complex Brownian motions and time-change argument. Since a conformal martingale is a time changed

Brownian motion, we can carry the same procedure as the proof of Nevanlinna's theorem in [1].
Define conformal martingales M^0, M^1 mentioned introduction by

$$M_t^0 = \int_0^{t\wedge\sigma} g(Z_s)dZ_s$$
$$M_t^1 = \int_0^{t\wedge\sigma} M_s^0 dZ_s,$$

where g is a locally bounded complex valued function on $\Delta(1)$ and Z_t and σ are the same as in the introduction.

Proposition 4. *Let $a_1, \ldots, a_q$ be distinct points on $\mathbf{P}^1$. Then we have for $\beta > 0$*

$$\sum_{i=1}^{q} E[u_a(M^j_{\sigma_\rho})] + N_1^j(\rho) \le 2E[\ [M^j, M^j]_{\sigma_\rho}\ ] + O(\log E[\ [M^j, M^j]_{\sigma_\rho}\ ]) + (1+\beta)\log\frac{1}{1-\rho} \qquad j = 0, 1$$

holds for $\rho \notin E_\beta \subset [0,1)$ with $\int_{E_\beta} \frac{1}{1-\rho} d\rho < \infty$, where

$$N_1^j(\rho) = \begin{cases} E[\log |g|^2(Z_{\sigma_\rho})] & j = 0 \\ E[\log |M^0_{\sigma_\rho}|^2] & j = 1 \end{cases}$$

We remark that $N_1^1(\rho)$ is lower bounded.

The quite same method as the previous section leads us to the Theorem 3 and its corollary.

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