J. Math. Sci. Univ. Tokyo
3 (1996), 15–44.

Notes on the integrability of exit times from unbounded domains of Brownian motion

By Atsushi Atsuji

Abstract. The integrability and the tail distribution of the first exit time from unbounded domain of Brownian motions will be considered. They are characterized by the growth order of the first eigen values of the intersection of domains and sphere with radius r and quasi-hyperbolic distance.

§1. Introduction

Let D be an unbounded domain with smooth boundary in a noncompact complete Riemannian manifold M and τ_D be the first exit time from D of Brownian motion on M or a diffusion with the generator $L = \frac{1}{2}\Delta_M + b$; b is a vector field. The purpose of this note is to give some conditions of some characteristics of D for the integrability of τ_D and to get some information on the tail of τ_D.

We consider some classes of diffusions such as Brownian motions on $\mathbf{R}^n$, spherically symmetric diffusions on $\mathbf{R}^n$ and Brownian motion on some Riemannian manifolds with a curvature conditions. In the case that $M = \mathbf{R}^n$ and $L = \Delta$ we will see the following.
Let $B(r) = \{x \in \mathbf{R}^n \ : \ |x| < r\}$ and $S(r) = \{x \in \mathbf{R}^n \ : \ |x| = r\}$. Suppose that $0 \in D$.
If for some $\nu \in (0, 1)$

$$\varliminf_{r\to\infty} \frac{1}{\log r} \int_0^{\nu r} \alpha_r dr > 2p,$$

1991 *Mathematics Subject Classification.* 58G32, 60J65.

then $E_0[\tau_D^p] < \infty$, where $\alpha_r = -\frac{n-2}{2r} + \sqrt{\lambda_r + \frac{(n-2)^2}{4r^2}}$ and λ_r is the first eigenvalue of the Laplacian with Dirichlet boundary condition on $D \cap S(r)$.

The key to the proof is an estimate on a hitting probability such as

$$P(B_{\tau_{D_r}} \in D \cap S(r)) \leq const. \exp(-\int_0^{\nu r} \alpha_r dr), \tag{1.1}$$

where D_r denotes $D \cap B(r)$.

This type estimate is known as Carleman-Tsuji inequality in classical complex function theory ([2,7,13,16]). We can easily extend this inequality for the other diffusions mentioned before. In combining Burkholder type inequality([1]) with such an estimate we have our result.
The converse of the above result depends on the following lower estimate.

$$P_x(B_{\tau_{D_r}} \in D \cap S(r)) \geq const. \exp(-\eta(x, D \cap S(r))), \tag{1.2}$$

where $\eta(x, D \cap S(r)) = \inf_{y \in D \cap S(r)} \eta(x, y)$ and $\eta(x, y)$ is quasi-hyperbolic distance in D from x to y ([17]). The definition of quasi-hyperbolic distance will be given in §3.

We first discuss these estimates in §2 and §3. Then we note how the above type estimates are joined to the Burkholder inequality in §4. Unfortunately we have different characteristics which are appeared in the upper and lower bounds in the right hand sides of (1.1) and (1.2) if D is general. But when D is a cone in $\mathbf{R}^n$, we can easily obtain equivalent upper and lower estimates with a little stochastic calculus in §5. These estimates will have several other applications to consider the tails of Brownian functionals. To give an example in §6 we obtain a condition for the finiteness of a special Feynman-Kac functional by using the method in the previous sections.

§2. A Carleman-Tsuji inequality

Let M be a complete Riemannian manifold. We first introduce geodesic polar coordinates which we often use from now on. In these coordinates the metric of M takes the form: $ds^2 = dr^2 + g_{ij} d\theta_i d\theta_j$. Let $G = det(g_{ij})^{1/2}$. The Laplacian takes the form:

$$\begin{aligned}\Delta_M &= \frac{\partial^2}{\partial r^2} + \frac{\frac{\partial G}{\partial r}}{G}\frac{\partial}{\partial r} + G^{-1}\frac{\partial}{\partial \theta_i}(G g^{ij}\frac{\partial}{\partial \theta_j}) \\ &= \frac{\partial^2}{\partial r^2} + \frac{\frac{\partial G}{\partial r}}{G}\frac{\partial}{\partial r} + \Delta_\theta,\end{aligned}$$

where (g^{ij}) is the inverse matrix of (g_{ij}) and Δ_θ denotes the Laplacian on $S(r)$ with respect to the induced metric.

We here discuss a reason why we consider a Riemannian structure even in $\mathbf{R}^n$. Let $n \geq 3$.
Let a second order differential elliptic operator on $\mathbf{R}^n$: $A = \sum \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j})$ $(a_{ij} = a_{ji})$ be given. Assume that A satisfies the uniform elliptic condition, that is, there exists constant $\lambda > 0$ such that for any $\xi \in \mathbf{R}^n$

$$\lambda^{-1}|\xi|^2 \leq \sum a_{ij}\xi_i\xi_j \leq \lambda|\xi|^2.$$

We can find a Riemannian metric on $\mathbf{R}^n$ from (a_{ij}) by $a_{ij} = Gg^{ij}$, $G = det(a_{ij})^{\frac{1}{n-2}}$. Hence we turn $\mathbf{R}^n$ into a Riemannian manifold M with a Riemannian metric (g_{ij}) in a global coordinate. We have that $\Delta_M = G^{-1}A$. Then Δ_M-diffusion is a time-changed process of $A-$diffusion. It should be remarked that the uniform ellipticity of A implies that there exist constants $a_1, a_2 > 0$ such that $a_1 \leq G = det\,(a_{ij})^{\frac{1}{n-2}} \leq a_2$. Let Δ_M-diffusion and $A-$diffusion be denoted by B_t and X_t respectively, and τ^i, $i = 1, 2$ be hitting times to a domain in $\mathbf{R}^n$ and σ^i, $i = 1, 2$ to another one of B_t and X_t respectively. Then

$$\begin{aligned} P(\tau^1 < \sigma^1) &= P(\int_0^{\tau^1} G(B_t)dt < \int_0^{\sigma^1} G(B_t)dt) \\ &= P(\tau^2 < \sigma^2). \end{aligned}$$

And since $P(\tau^1 > t) = P(\int_0^{\tau^1} G(B_t)dt > \int_0^t G(B_t)dt)$, hence there exist constants $c_1, c_2 > 0$ such that

$$P(\tau^1 > c_1 t) \leq P(\tau^2 > t) \leq P(\tau^1 > c_2 t).$$

Thus, as for our problem, the problem on $A-$diffusion is equivalent to one on Δ_M-diffusion.

Remark. It is usual to set that $g^{ij} = a_{ij}$. But the setting bears a drift term $\nabla \log G$. Since it is easier to treat a time-changed object than to do one with a drift, we prefer our setting to the other.

The diffusion process we consider in this paper has a generator with the form in this coordinates as

$$L = \frac{1}{2}\Delta_M + b_1(r)\frac{\partial}{\partial r} + b_2 \cdot \nabla_\theta,$$

where ∇_θ is the component to $TS(r)$ of ∇.

In this section we assume the following conditions on L.

$$\frac{G'(r,\theta)}{G(r,\theta)} \text{ is a radial function.}$$

Let us denote this function by $\psi(r)$.

Remark. The above condition is satisfied in the cases that M is $\mathbf{R}^n$ and the diffusion is a spherically symmetric diffusion (i.e. the distribution of the diffusion is invariant under the actions of rotations. Of course this class includes Brownian motion.) and that a Riemannian maifold with constant curvature.

We note that this assumption can be removed in the following discussions with some conditions. But this yields the complicated forms of quantities in the estimates, for example $\gamma(r)$ in Proposition 2.1. For brevity we impose this condition.

In this section we also assume that $B(\delta) \subset D$ for some δ.
Let $\theta_r = D \cap S(r)$ and $b_2(r) = \sup_{\theta\in\theta_r} |b_2(r,\theta)|$.

Let λ_r be the first eigenvalue for the Laplacian on $S(r)$ of the Dirichlet problem on θ_r, that is,

$$\lambda_r = \inf_{\substack{u\in C^\infty(\theta_r)\\ u=0 \text{ on } \partial\theta_r}} \frac{\int_{\theta_r} |grad_\theta u(x)|^2 ds_r}{\int_{\theta_r} |u(x)|^2 ds_r} \qquad \text{for } r > 0,$$

where ds_r is the volume form on $S(r)$ induced from the Riemannian metric and $\lambda_0 = 0$.

Let us write "$f'(r)$" for "$\frac{\partial f}{\partial r}$ ".

PROPOSITION 2.1. *Let M have injectivity radius $i(o)$ for a point o and $\psi(r) = \frac{G'(r,\theta)}{G(r,\theta)}$ independent of θ.*
Assume that

$$\frac{2(n-2)}{(n-1)^2}\psi^2 + \frac{2(n-2)}{n-1}\psi' \geq 0 \quad \textit{for } r < i(o)$$

and

$$b_2(r)\lambda_r^{-1/2} \leq 1.$$

(i) If $b_1 \equiv 0$, then there exists a positive constant c_1 such that for $0 < \nu < 1$ and for $i(o) < r$

$$P_o(X_{\tau_{D_r}} \in \theta_r) \leq c_1\, exp(-\int_\delta^{\nu r} \alpha_r dr),$$

where

$$\alpha_r = -\gamma(r) + \sqrt{(1 - b_2(r)\lambda_r^{-1/2})\lambda_r + \gamma(r)^2},$$
$$\gamma(r) = \sqrt{\frac{1}{4}\{(n-2)^2(\frac{g'(r)}{g(r)})^2 + 2(n-2)\frac{g''(r)}{g(r)}\}},$$

and $g(r)$ is defined by

$$\psi(r) = (n-1)g'(r)/g(r) \ \textit{with } g(\delta) = 1.$$

(ii) If $b_1 \not\equiv 0$ and for some $0 < p < 1$

$$\sup_{x\in M}\int_M \int_{d(x,y)}^\infty e^{-\int_1^r \psi(t)dt} b_1(r)^2 dr dV(y) \leq \sqrt{2e^{-1}p^{-1} + 1} - 1,$$

then there exists a positive constant c_2 such that for $i(o) < r$

$$P_o(X_{\tau_{D_r}} \in \theta_r) \leq c_2\, exp(-(1 - \frac{1}{p})\int_\delta^{\nu r} \alpha_r dr),$$

where α_r is same as (i).

Proof of Proposition 2.1. (i) This proof is a slight modification of [16].
Set $u(x) = P_x(X_{\tau_{D_R}} \in \theta_R)$ and $m(r)^2 = \int_{\theta_r} u(x)^2 ds_r$, for $r < R$ where ds_r is the volume form on $S(r)$ induced by the Riemannian metric on M.
Since u vanishes on $\partial D_r \setminus \theta_r$,by Green formula

$$\begin{aligned}\int_{\theta_r} u^2 ds_r &= \int_{\theta_r} u^2 \frac{\partial r}{\partial r} ds_r \\ &= \int_{D_r} \left\{ \Delta r u^2 + < grad\, r, grad\; u^2 > \right\} dV \\ &= \int_{D_r} \left\{ \frac{G'}{G} u^2 + 2uu' \right\} dV.\end{aligned}$$

Differentiating $m(r)^2$ in r, we have

$$2m(r)m'(r) = 2\int_{\theta_r} uu' ds_r + \int_{\theta_r} u^2 \frac{G'}{G} ds_r \tag{2.1}$$

Using Green formula again on (2.1), we have

the right hand side of (2.1)

$$\begin{aligned}&= \int_{\theta_r} \frac{\partial}{\partial r} \log G(r,\theta) u^2 ds_r + 2\int_{\theta_r} uu' ds_r \\ &= \int_{D_r} \{ \Delta \log G(r,\theta) u^2 + < grad\, u^2, grad\, \log G > \\ &\qquad +2|grad\; u|^2 + 2u\Delta u \} \, dV \\ &= \int_{D_r} u^2 (\frac{\partial^2}{\partial r^2} + \frac{G'}{G}\frac{\partial}{\partial r}) \log G dV + \int_{D_r} \Delta_\theta \log G(r,\theta) u^2 dV \\ &\quad + \int_{D_r} 2uu' \frac{G'}{G} dV \\ &\quad + \int_{D_r} \{ < grad_\theta\, u^2, grad_\theta\; \log G > +2|grad\; u|^2 + 2u\Delta u \} dV,\end{aligned}$$

where $grad_\theta$ denotes gradient on $S(r)$.
Using Green formula on θ_u for $\delta < u < r$ again, we have that

$$\text{the right hand side } = \int_{D_r} \left\{ \frac{G''}{G} u^2 + 2uu' \frac{G'}{G} + 2|grad\; u|^2 + 2u\Delta u \right\} dV.$$

Then differentiating the both sides of (3.1) in r again, since u is L-harmonic,

$$
\begin{aligned}
(2.2)\qquad (2mm')' &= \int_{\theta_r} \left\{ \frac{G''}{G}u^2 + 2uu'\frac{G'}{G} + 2|grad\ u|^2 + 2u\Delta u \right\} ds_r \\
&= \int_{\theta_r} \left\{ \frac{G''}{G}u^2 + 2uu'\frac{G'}{G} + 2|grad\ u|^2 - 2(bu)u \right\} ds_r \\
&= \int_{\theta_r} \left\{ \frac{G''}{G}u^2 + 2uu'\frac{G'}{G} + 2(u')^2 \right. \\
&\qquad\qquad \left. + 2|grad_\theta u|^2 - 2b_2uu \right\} ds_r.
\end{aligned}
$$

By Schwarz inequality

$$
\begin{aligned}
(2.3)\qquad -\int_{\theta_r} b_2uuds_r &\geq -b_2(r)\left(\int_{\theta_r}|grad_\theta u|^2ds_r\right)^{1/2}\left(\int_{\theta_r}u^2ds_r\right)^{1/2} \\
&\geq -b_2(r)\lambda_r^{-1/2}\int_{\theta_r}|grad_\theta u|^2ds_r.
\end{aligned}
$$

Using (2.1) and Schwarz inequality

$$
\begin{aligned}
(2.4)\qquad & \int_{\theta_r}(u')^2ds_r \\
&\geq \frac{1}{m^2}\left(\int_{\theta_r}uu'ds_r\right)^2 \\
&= \frac{1}{m^2}\left(mm' - \frac{1}{2}\int_{\theta_r}u^2\frac{G'}{G}ds_r\right)^2 \\
&= \frac{1}{m^2}\left\{m^2(m')^2 - mm'\int_{\theta_r}u^2\frac{G'}{G}ds_r + \frac{1}{4}(\int_{\theta_r}u^2\frac{G'}{G}ds_r)^2\right\} \\
&\geq (m')^2 - \psi mm' + \frac{1}{4}\psi^2m^2.
\end{aligned}
$$

From (2.1),(2.3),(2.4) the right hand side of (2.2) is bounded from below by

$$\{2(1 - b_2(r)\lambda_r^{-1/2})\lambda_r + \frac{1}{2}\psi^2 + \psi'\}m(r)^2 + 2(m')^2.$$

Hence, from this and (2.3), we obtain

$$m''(r) \geq \frac{1}{2}\{2(1-b_2(r)\lambda_r^{-1/2})\lambda_r + \frac{1}{2}\psi^2 + \psi'\}m(r). \tag{2.5}$$

Let $M(r)^2 = \frac{1}{A(r)}m(r)^2$ where $A(r) = \int_{S(r)} ds_r$.
From (2.5) we have

$$\frac{M''}{M} + \psi\frac{M'}{M} \geq \frac{1}{2}(1-b_2(r)\lambda_r^{-1/2})\lambda_r. \tag{2.6}$$

We define $g(r)$ by

$$\psi(r) = (n-1)\frac{g'(r)}{g(r)} \text{ with } g(\delta) = 1.$$

We set $f(t) = \log M(r)^2 + (n-2)\log g(r)$ and change the variable by $t = \int_\delta^r \frac{dr}{g(r)}$. From (2.6), using $\dot{r} = g(r)$ and $\ddot{r} = gg'$, we have

$$\begin{aligned}(\dot{r})^{-2}(2\ddot{f}(t) + (\dot{f}(t))^2) \geq & 4(1-b_2(r)\lambda_r^{-1/2})\lambda_r \\ & + (\dot{r})^{-2}\{(n-2)^2(g'(r))^2 + 2(n-2)g''(r)g(r)\}.\end{aligned}$$

Our assumption implies that $(n-2)^2(g'(r))^2 + 2(n-2)g''(r)g(r) \geq 0$.
We let

$$\tilde{\gamma}(t) = \sqrt{\frac{1}{4}\{(n-1)^2(g'(r))^2 + 2(n-2)g''(r)g(r)\}}$$

(regarding as a function of $t = \int_\delta^r \frac{dr}{g(r)}$),

$$\tilde{\beta}_t = -\tilde{\gamma}(t) + \sqrt{(1-b_2(r)\lambda_r^{-1/2})\lambda_r g(r)^2 + \gamma(t)^2}$$

and $\tilde{\gamma}(t) = g(r)^2\gamma(r), \tilde{\beta}_t = g(r)^2\beta_r$. We have

$$\left(\dot{f}(t) + \frac{\ddot{f}(t)}{\dot{f}(t)}\right)^2 \geq (2\tilde{\beta}_t + 2\tilde{\gamma}(t))^2.$$

From (2.1) and Green formula we have

$$\begin{aligned}\frac{M'}{M} &= 2\frac{m'}{m} - \psi \\ &= 2\int_{\theta_r} uu' ds_r \\ &= \int_{D_r} \{u\Delta u + |grad\ u|^2\} dV \\ &\geq \int_{D_r} \{(u')^2 + |grad_\theta\ u|^2 - (b_2\, u)u\} dV.\end{aligned}$$

Then from Schwarz inequarity, (2.4) and the assumptions on b we have that $\frac{M'}{M} > 0$.

Since $\frac{M'}{M} > 0$ implies that $\dot{f}$ is positive, so is $\dot{f}(t) + \frac{\ddot{f}(t)}{\dot{f}(t)}$, then

$$\dot{f}(t) + \frac{\ddot{f}(t)}{\dot{f}(t)} \geq 2\tilde{\beta}_t + 2\tilde{\gamma}(t).$$

From this we have for $t_2 > t_1 \geq t(\delta)$

$$\int_{t_1}^{t_2} \exp f(s)\dot{f}(s) ds \geq \exp f(t_1)\dot{f}(t_1) \int_{t_1}^{t_2} \exp\left(\int_{t_1}^{s} (2\tilde{\beta}_u + 2\tilde{\gamma}(u)) du\right) ds.$$

Rewriting this in the variable of r again

$$\begin{aligned}&M(r_2)^2 g(r_2)^{n-2} - M(r_1)^2 g(r_1)^{n-2} \\ &\geq M(r_1)^2 g(r_1)^{n-2} \dot{f}(t_1) \int_{r_1}^{r_2} \exp\,(\int_{r_1}^{s} 2\beta_t + 2\gamma(t) dt) \frac{ds}{g(s)} \\ &\geq M(r_1)^2 g(r_1)^{n-2} \dot{f}(t_1) \int_{\nu r_2}^{r_2} \exp\,(\int_{r_1}^{s} 2\gamma(t) dt) \frac{ds}{g(s)} \cdot \exp \int_{r_1}^{\nu r_2} \beta_t dt.\end{aligned}$$

Our assumption on the metric yields

$$2\gamma(r) \geq (n-2) g'(r)/g(r).$$

This implies that

$$\frac{\int_{\nu r_2}^{r_2} \exp\,(\int_{r_1}^{s} 2\gamma(t) dt) ds}{g(r_2)^{n-2}} \geq const. > 0.$$

Hence

$$1 \geq M(r)^2 \geq const.M(\delta)^2 \exp \int_\delta^{\nu r} \beta_t dt.$$

In combining this with sub-mean property of u^2 or Harnack inequality on u we obtain i). (We can show Harnack inequality in this situation ([3])).

(ii) Let $X_t = (r_t, \theta_t)$ be the diffusion treated in (i) with $b_1 \equiv 0$ and

$$dr_t = dB_t + \frac{1}{2}\psi(r_t)dt,$$

where B_t is an 1-dimensional standard Brownian motion.
Set

$$M_t = \exp(\int_0^t b_1(r_s)dB_s - \frac{1}{2}\int_0^t b_1(r_s)^2 ds).$$

Let $\hat{X}_t$ be the diffusion in this case with $b_1 \not\equiv 0$. The formula of transformation of drift ([8,15]) implies that

$$\hat{P}_{x_0}(\hat{X}_{\tau_{D_r}} \in \theta_r) = E[M_{\tau_{D_r}} \ ; \ X_{\tau_{D_r}} \in \theta_r].$$

In rather vulgar way, using Hölder inequality the above right hand side is bounded by

$$E[M^p_{\tau_{D_r}}]^{1/p} P(X_{\tau_{D_r}} \in \theta_r)^{1-1/p}.$$

Thus we estimate $E[M^p_{\tau_{D_r}}]$.
Set $Y_t = \int_0^t b_1(r_s)dB_s - \frac{1}{2}\int_0^t b_1(r_s)^2 ds$ and $c_r = \sup_{x \in D_r} E_x[|Y_{\tau_{D_r}}|]$.
From the proof of John-Nirenberg inequality for BMO-martingale([6,15]) we have

$$E[M^p_{\tau_{D_r}}] \leq \frac{c_r}{1 - epc_r}.$$

On the other hand

$$\begin{aligned} E_x[|Y_{\tau_{D_r}}|] &\leq E_x[|\int_0^{\tau_r} b_1(r_s)dB_s|] + \frac{1}{2}E[\int_0^{\tau_r} b_1(r_s)^2 ds] \\ &\leq E[\int_0^{\tau_r} b_1(r_s)^2 ds]^{1/2} + \frac{1}{2}E[\int_0^{\tau_r} b_1(r_s)^2 ds] \\ &\leq E[\int_0^{\infty} b_1(r_s)^2 ds]^{1/2} + \frac{1}{2}E[\int_0^{\infty} b_1(r_s)^2 ds]. \end{aligned}$$

Unless the global green function for X_t exists, then the right hand side of the above inequality is divergent. Thus we may assume that the global green function exists.
It takes the form as

$$g(x,y) = \int_{d(x,y)}^{\infty} e^{-\int_1^r \psi(t)dt} dr.$$

Then

$$c_r \leq \sup_{x\in M}\{(\int_M \int_{d(x,y)}^{\infty} e^{-\int_1^r \psi(t)dt} b_1(r)^2 dr dV(y))^{1/2} \\ + \frac{1}{2}\int_M \int_{d(x,y)}^{\infty} e^{-\int_1^r \psi(t)dt} b_1(r)^2 dr dV(y)\}.$$

Combining these estimates complete the proof. □

The above proposition gives us a bit information on the decay of the tail distribution of τ_D.

Theorem 2.2. *Suppose the assumption of Proposition 2.1 and that* $i(o) = \infty$.
(i) Assume that $\psi(r) \leq \frac{n-1}{r}$.
If $\underline{\lim}_{r\to\infty} \frac{1}{\log r}\int_0^{\nu r} \alpha_r dr < \infty$, *then*

$$\underline{\lim}_{t\to\infty} -\frac{1}{\log t}\log P(\tau_D > t) \geq \underline{\lim}_{r\to\infty} \frac{1}{2}\frac{1}{\log r}\int_0^{\nu r} \alpha_r dr.$$

(ii) Suppose that $0 < a \leq \psi(r)$.
If $\underline{\lim}_{r\to\infty} \frac{1}{\log r}\int_0^{\nu r} \alpha_r dr < \infty$, *then*

$$\underline{\lim}_{t\to\infty} -\frac{1}{\log t}\log P(\tau_D > t) \geq \underline{\lim}_{r\to\infty} \frac{1}{\log r}\int_0^{\nu r} \alpha_r dr.$$

(iii) Suppose that $0 < a \leq \psi(r)$. *If* $0 < \sigma = \underline{\lim}_{r\to\infty} \frac{1}{r}\int_0^{\nu r} \alpha_r dr < \infty$, *then*

$$\underline{\lim}_{t\to\infty} -\frac{1}{t}\log P(\tau_D > t) \geq \frac{a^2\sigma}{8\sigma + 4a}.$$

By the discussion in the beginning of this section and Corollary 2.5 below we have the following.

THEOREM 2.3. *Let τ_D be an exit time from D of a spherical symmetric diffusion on $\mathbf{R}^n$ whose generator A is uniformly elliptic.*
If $\varliminf_{r\to\infty} \frac{1}{\log r}\int_0^{\nu r}\alpha_r dr < \infty$, then there exists a constant $c > 0$ depending only on the elliptic constant such that

$$\varliminf_{t\to\infty} -\frac{1}{\log t}\log P(\tau_D > t) \geq \varliminf_{r\to\infty} \frac{c}{2}\frac{1}{\log r}\int_0^{\nu r}\alpha_r dr.$$

In particular we can take $c = 1$ in the case that A is a usual Laplacian on $\mathbf{R}^n$.

To prove these we prepare an elementary estimate.

LEMMA 2.4. *Let D be a bounded domain in M and τ_D be the first exit time from D of a diffusion corresponding to Δ_M. Let $Q_D(t,x,y)$ be a heat kernel with Dirichlet boundary condition on D. Assume that $Q_D(t,x,y) \leq p(t)$. We have for $t > 2$*

$$P(\tau_D > t) \leq e^{-\frac{\lambda_1}{2}t} vol(D)^{3/2} p(t/2-1)p(2)^{1/2},$$

where λ_1 is the first eigenvalue of Dirichlet problem of Δ_M on D.

PROOF. By eigenfunction expansion of it $Q_D(t,x,y) = \sum e^{-\lambda_m t}\phi_m(x)\phi_m(y)$. Since

$$\sum e^{-\lambda_m t}\phi_m(x)^2 = Q_D(t,x,x) \leq p(t),$$

then

$$\sum e^{-\lambda_m t} = \int_D Q_D(t,x,x)dx \leq vol(D)p(t) \tag{2.9}$$

And $e^{-\frac{\lambda_m}{2}t}|\phi_m(x)| \leq p(t)^{1/2}$, so

$$e^{-\lambda_m}|\phi_m(x)| \leq p(2)^{1/2}. \tag{2.10}$$

From (2.9) and (2.10), for $t > 2$

$$\begin{aligned}
P(\tau_D > t) &= \int_D Q_D(t,x,y)dy \\
&\le \int_D \sum e^{-\lambda_m t}|\phi_m(x)||\phi_m(y)|dy \\
&\le vol(D)^{1/2}p(2)^{1/2}\sum e^{-\lambda_m(t-1)} \\
&\le vol(D)^{1/2}p(2)^{1/2}e^{-\lambda_1 t/2}\sum e^{-\lambda_m(t-1)+\lambda_m t/2} \\
&\le vol(D)^{3/2}p(2)^{1/2}e^{-\lambda_1 t/2}p(t/2-1). \ \square
\end{aligned}$$

COROLLARY 2.5.
(i) If the assumption of Theorem 2.2 (i) is satisfied, or $\tau_{B(r)}$ *is the first exit time from* $B(r)$ *of a diffusion on* $\mathbf{R}^n$ *whose generator* A *is uniformly elliptic, then there exist constants* $c_1, c_2 > 0$ *in each case such that for* $t > 2$

$$P(\tau_{B(r)} > t) \le c_1 e^{-c_2\frac{t}{r^2}} vol(B(r))^{3/2}(t/2-1)^{-n/2}.$$

(ii) Suppose that $i(o) = \infty$*. If* $\psi(r)$ *is away from 0, then there exist constants* $c_1, c_2, c_3 > 0$ *for* $t > 2$

$$P(\tau_{B(r)} > t) \le c_1 e^{-c_2(\frac{t}{r^2}+t)} vol(B(r))^{3/2}(t/2-1)^{-n/2}.$$

PROOF. (i) We have only to compare τ with one of radial motion of Eucledian Brownian motion.
(ii) We have only to note that uniform ellipticity implies that $p(t) = const.t^{-n/2}$ ([4]) and $\lambda_{B(r)} = const.\frac{1}{r^2}$. $\square$

LEMMA 2.6. *Suppose that* $a \le \psi(r)$*. Then*

$$P(\tau_r > t) \le e^{\frac{a}{2}-\frac{a^2}{8}t}.$$

Proof. Let $r_t^1 - r_0 = w_t + \frac{a}{2}t$ where w_t is an one dimensional standard Brownian motion and $\tau_r^1 = \inf\{t > 0 \ : \ r_t^1 \geq r\}$. Then direct calculation says that

$$P(\tau_r^1 \in ds) = e^{\frac{a}{2} - \frac{a^2}{8}s} \frac{r}{\sqrt{2\pi s^3}} e^{-\frac{r^2}{2s}}.$$

Hence by comparison argument we have

$$\begin{aligned} P(\tau_r > t) &\leq P(\tau_r^1 > t) \\ &\leq e^{\frac{a}{2} - \frac{a^2}{8}t} \int_t^\infty \frac{r}{\sqrt{2\pi s^3}} e^{-\frac{r^2}{2s}} \\ &\leq e^{\frac{a}{2} - \frac{a^2}{8}t}. \ \square \end{aligned}$$

Proof of Theorem 2.2. (i) From lemma 2.4 and Corollary 2.5 we have

$$\begin{aligned} P(\tau_{B(r)} > t) &\leq const.e^{-c\frac{t}{r^2}} r^{3n/2} (t/2 - 1)^{-n/2} p(2)^{1/2} \\ &= const. \exp(-\{c\frac{t}{r^2} - \log r^{3n/2} + \log{(t/2 - 1)^{n/2}}\}). \end{aligned}$$

Set $t = r^2 \log(r^{3n/2}(t/2-1)^{-n/2} r^p)$ for $p > \underline{\lim}_{r\to\infty} \frac{1}{\log r} \int_0^{\nu r} \alpha_r dr$. We note that $\log t = 2\log r + o(\log t)$. For such t by Proposition 2.1 we have as $r \to \infty$

$$\begin{aligned} P(\tau_D > t) &\leq P(\tau_D > \tau_r) + P(\tau_r > t) \\ &\leq const.e^{-\int_\delta^{\nu r} \alpha_r dr} + const.\, r^{-p} \\ &\leq const.e^{-\int_\delta^{\nu r} \alpha_r dr}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{\log t} \log P(\tau_D > t) &\leq -\frac{\log r}{\log t} \frac{1}{\log r} \int_0^{\nu r} \alpha_r dr + o(1) \\ &= -\frac{1}{2} \frac{1}{\log r} \int_0^{\nu r} \alpha_r dr + o(1). \end{aligned}$$

(ii) and (iii) By lemma 2.6 we can carry the similar argument to (i). $\square$

§3. Lower estimate

In this section we treat diffusions with generator $L = \Delta_M$ without a bector field.
Before we mention the lower bound $P(X_{\tau_{D_r}} \in \theta_r)$ we need some notations. For a simple curve $\phi(t)$ $(0 \leq t < \infty)$ in D with $\phi(0) = x$ (x is the origin of our coordinate) let $\rho(t)$ denote the distance from $\phi(t)$ to ∂D.
Let $\Gamma_r = \{\phi : [0, r) \to D_r,$ simple rectifiable curve $, \phi(r) \in \theta_r$ and $\phi(0) = x\}$.
We first have the following.

Proposition 3.1. *Assume $Ric_M \geq k$.*
We define $g(r)$ by

$$g''(r) + kg(r) = 0 \text{ with } g(0) = 0 \text{ and } g'(0) = 1.$$

Let $v(x)$ be a positive subharmonic function on D_r. Let $I_t = \inf\{v(x) : d(x, \phi(t)) \leq \kappa\rho(t)\}$ for $0 < \kappa < 1$. Then we have

$$I_0 \geq I_r \exp\left(- \int_0^r (\kappa + 1) \frac{g(\kappa\rho(t))^{-n+1}}{\int_{\kappa\rho(t)}^{\rho(t)} g(s)^{-n+1} ds} |\dot{\phi}| dt \right).$$

In particular if $M = \mathbf{R}^n$ and $\Delta_M = \Delta_{\mathbf{R}^n}$,

$$I_0 \geq I_r \exp(-c \int_0^r |\dot{\phi}| \frac{1}{\rho(t)} dt),$$

where c satisfies that $c(\log c - 1) = 1$ if $n = 2$ and that $c = (n-1)^{\frac{n-1}{n-2}}$ if $n \geq 3$.

Proof. We have only to modify the proposition in [13] only a little. Let $C(t) = \{x | \kappa\rho(t) < d(\phi(t), x) < \rho(t)\}$. We define $u(x)$ by

$$u(x) = \frac{\int_{d(\phi(t),x)}^{\rho(t)} g(s)^{-n+1} ds}{\int_{\kappa\rho(t)}^{\rho(t)} g(s)^{-n+1} ds} I_t,$$

where $I_t = \inf\{v(x) \ : \ d(x,\phi(t)) \le \kappa\rho(t)\}$. Take a polar coordinate around $\phi(t)$.

We can write the radial part of X_t such as in [6] :

$$r_t = r_0 + w_t + \frac{1}{2}\int_0^t \frac{\partial G}{\partial r}/G(X_s)ds - L_t,$$

where L_t is an increasing process which increases only on the cut locus of $\phi(t)$. Set $\tilde{u}(d(x,\phi(t))) = u(x)$. Then comparison argument leads us $\tilde{u}(r_t)$ is a submartingale, that is, $u(x)$ is subharmonic on $C(t)$. u satisfies that

$$u(x) = 0 \text{ on } d(\phi(t),x) = \kappa\rho(t) \quad u(x) = I_t \text{ on } d(\phi(t),x) = \rho(t).$$

Thus maximum principle implies that $u(x) \le v(x)$ on $C(t)$.
We can calculate the left differential $(I_t)'_-$ of I_t in t as in [13]. By maximum principle again

$$I_{t-\triangle t} \ge \tilde{u}(d(\phi(t),\phi(t-\triangle t)) + \kappa\rho(t-\triangle t)).$$

It is obvious that $\rho(t-\triangle t) \le d(\phi(t),\phi(t-\triangle t)) + \rho(t)$. Then

$$\begin{aligned} I_{t-\triangle t} &\ge \frac{\int_{(1+\kappa)d(\phi(t),\phi(t-\triangle t))+\kappa\rho(t)}^{\rho(t)} g(s)^{-n+1}ds}{\int_{\kappa\rho(t)}^{\rho(t)} g(s)^{-n+1}ds} I_t \\ &= I_t - \frac{\int_{\kappa\rho(t)}^{(1+\kappa)d(\phi(t),\phi(t-\triangle t))+\kappa\rho(t)} g(s)^{-n+1}ds}{\int_{\kappa\rho(t)}^{\rho(t)} g(s)^{-n+1}ds} I_t. \end{aligned}$$

We have

$$\frac{(I_t)'_-}{I_t} \le (\kappa+1)\frac{g(\kappa\rho(t))^{-n+1}}{\int_{\kappa\rho(t)}^{\rho(t)} g(s)^{-n+1}ds}|\dot{\phi}|.$$

Hence integrating the both sides fom 0 to r

$$I_0 \ge I_r \exp\left(-\int_0^r (\kappa+1)\frac{g(\kappa\rho(t))^{-n+1}}{\int_{\kappa\rho(t)}^{\rho(t)} g(s)^{-n+1}ds}|\dot{\phi}|dt\right). \ \Box$$

We now return to our diffusion treated in §2, namely, its generator has the form

$$L = \frac{1}{2}\Delta_M = \frac{1}{2}(\frac{\partial^2}{\partial r^2} + \psi(r)\frac{\partial}{\partial r} + \Delta_\theta).$$

Let $\tilde{\Gamma_r}$ denote all of simple smooth curves belonging to Γ_r and satisfying the following condition. Let $B(x, l)$ be a ball with center x and radius l. *The condition* ;

$$\int_{\kappa\rho(r)}^{\nu\rho(r)} \alpha_t dt \geq const. > 0, \text{ uniformly in } r,$$

where α_t is defined in Proposition 2.1 with $L = \Delta_M$ and $D = B(\phi(r), \rho(r)) \cap B(r)$.

Here we introduce quasi-hyperbolic distance $\eta_D(x, y)$ on $D \subset \mathbf{R}^n$.

$$\eta_D(x, y) = \inf_{\phi \in \Gamma} \int_\phi \frac{1}{d(x, \partial D)}|dx|,$$

where $d(x, \partial D)$ is Euclidean distance from x to ∂D and

$$\Gamma = \{\phi : \text{ a rectifiable curve in } D \text{ from } x \text{ to } y\}.$$

Cororally 3.2.
i) Let $M = \mathbf{R}^n$ *and*

$$\eta_r = c \inf_{y \in \theta_r} \eta_D(x, y)$$

where c is a sonstant satisfying that $c(\log c - 1) = 1$ *and* $c > 1$ *if* $n = 2$ *and that* $c = (n-1)^{\frac{n-1}{n-2}}$ *if* $n \geq 3$. *Then there exists a constant* $C > 0$ *such that*

$$P_x(X_{\tau_{D_r}} \in \theta_r) \geq C\, exp(-\eta_r).$$

ii) Assume that $Ric_M \geq k$*.(i.e.* $\psi(r)$ *is bounded for all* r *and in any local coordinates.) We define* $g(r)$ *as in the Proposition 3.1. We have*

$$P_x(X_{\tau_{D_r}} \in \theta_r) \geq c\, exp(-\eta_r) \text{ for } i(x) > 2r,$$

where

$$\eta_r = \inf_{\phi\in\tilde{\Gamma}_r}\int_0^r (1+\kappa)|\dot{\phi}|\frac{g(\kappa\rho(t))^{-n+1}}{\int_{\kappa\rho(t)}^{\rho(t)} g(s)^{-n+1}ds}dt$$

and $0<\kappa<1$.

PROOF. i) We can evaluate I_r without any constraint on Γ_r. We consider the region $\tilde{B}=B(\phi(r),\rho(r))\cap H_-$, where H_- is a half space separated by the hyperplane H tangential to ∂D_r at $\phi(r)$ and including D_r. We define $\tilde{p}(x)$ by

$$\begin{aligned}\Delta\tilde{p}(x)&=0 \quad x\in\tilde{B}\\ \tilde{p}(x)&=0 \quad \text{on } (\partial B(\phi(r),\rho(r)))\cap H_- \text{ and}\\ \tilde{p}(x)&=1 \quad \text{on } B(\phi(r),\rho(r))\cap H.\end{aligned}$$

Then maximum principle yields $\tilde{p}(x)\leq P_x(X_{\tau_{D_r}}\in\theta_r)$ on $C(\phi(r))$. Since $\tilde{B}$ is a cone with a vertex $\phi(r)$, we can estimate $\tilde{p}(x)$ from below by Proposition 2.1 or direct calculation in §5. Then

$$\begin{aligned}I_r&\geq\inf_{x\in B(\phi(r),\kappa\rho(r))}\tilde{p}(x)\\ &\geq 1-\sup Px(X_{\tau_{\tilde{B}}}\in\partial B(\phi(r),\rho(r)))\\ &\geq 1-\exp(-const.\int_{\kappa\rho(r)}^{\rho(r)}\frac{dt}{t})\\ &\geq const.>0.\end{aligned}$$

Then we have

$$I_0\geq const.\exp(-\inf_{\phi\in\Gamma_r}c_\kappa\int_0^r|\dot{\phi}|\frac{1}{\rho(t)}dt),$$

where $c_\kappa=(1+1/\kappa)\log(1/\kappa)$ if $n=2$ and $=\frac{n-2}{\kappa(1-\kappa^{n-2})}$ if $n\geq 3$.

We choose κ such that $1/\kappa(\log(1/\kappa)-1)=1$ if $n=2$, $\kappa=(n-1)^{-\frac{1}{n-2}}$ which are minimizing c_κ.

ii) The condition of $\tilde{\Gamma}_r$ implies that $I_r\geq const.>0$, uniformly.

$$I_0\geq const.\exp(-\eta_r).$$

This completes the proof. □

If the first eigenvalue of Dirichlet problem on $\tilde{B}$ could be estimated, we could know whether $\tilde{\Gamma}_r$ is empty or not. But I don't know such estimates on the first eigenvalues in general. In the following two cases we are not bothered with this problem.

LEMMA 3.3. $\inf_r \inf_{\phi\in\Gamma_r} I_r > 0$ *holds in the following cases.*
i) X_t *is a spherically symmetric diffusion on* $\mathbf{R}^n$*.*
ii) Δ_θ *takes the form as*

$$\Delta_\theta = div(\mathcal{A}\nabla)$$

with satisfying that there exists a constant $a > 0$ *such that*

$$a^{-1}|\xi|^2 \le < \mathcal{A}_x\xi, \xi > \le a|\xi|^2$$

for all $(x,\xi) \in TS^{n-1}(r)$ *where* $S^{n-1}(r)$ *is a sphere with centere* o *and radius* r *in* $\mathbf{R}^n$*.*

PROOF. Immediate from the proof of Corollary 3.2. □

THEOREM 3.4. *Assume the assumption of Corollary 3.2 with* $k = 0$ *and that* $i(o) = \infty$*.*
i) If

$$\overline{\lim_{r\to\infty}} \frac{1}{\log r}\eta_r < \infty,$$

then

$$\overline{\lim_{t\to\infty}} -\frac{1}{\log t}\log P(\tau_D > t) \le \overline{\lim_{r\to\infty}} \frac{1}{2}\frac{1}{\log r}\eta_r.$$

ii) If

$$\overline{\lim_{r\to\infty}} \frac{1}{r}\eta_r < \infty,$$

then

$$\overline{\lim_{t\to\infty}} -\frac{1}{t}\log P(\tau_D > t) \le 4(\overline{\lim_{r\to\infty}} \frac{1}{r}\eta_r)^2.$$

PROOF. The proof of ii) is quite similar to i)'s. Then we give only one in the case of i). It is well-known that $P(\tau_r < t) \le const.e^{-\frac{r^2}{4t}}$ for Eucledian

Brownian motion. On the other hand let r_t be the distance on M from o to X_t. Set $r_t^{(0)} - r_0^{(0)} = w_t + \int_0^t \frac{n-1}{r_s^0} ds$ be the radial motion of a Brownian motion on $\mathbf{R}^n$. Then the curvature assumption and comparison theorem imply that

$$r_t \leq r_t^{(0)}.$$

Then we have

$$P(t > \tau_r) \leq P(t > \tau_r^{(0)}) \leq const.e^{-\frac{r^2}{4t}}.$$

Hence

$$P(\tau_D > \tau_r) \leq P(\tau_D > t) + P(t > \tau_r) \leq P(\tau_D > t) + const.e^{-\frac{r^2}{4t}}.$$

We set $t = \frac{r^2}{4p \log r}$ with $p > \overline{\lim}_{r \to \infty} \frac{1}{\log r} \eta_r$ so that $\log r / \log t \to 1/2$ as $t \to \infty$. Then as Theorem 2.2 we have the desired result. □

§4. Burkholder type inequalities and a basic argument

Theorem 4.1. *Assume the assumptions of Proposition 2.1 with* $i(o) = \infty$ *and that* $b_1(r) \geq 0$. *If a moderately increasing function* $\phi(r)$ *satisfies that*

$$\underline{\lim}_{r \to \infty} \frac{1}{\log \phi(r)} \int_0^{\nu r} \alpha_r dr > 1$$

and $\phi(d(o,x))$ *is* $L-$*subharmonic, then*

$$\begin{cases} E[\phi(\tau_D)] < \infty & \text{if } \psi(r) \geq c > 0, \\ E[\phi(\tau_D^{1/2})] < \infty & \text{if } \psi(r) \geq 0. \end{cases}$$

We also have a necessary condition for the integrability of τ_D.

Theorem 4.2. *Suppose the assumption of Theorem 3.4 and use the notation there. Let* $\phi(r)$ *be a positive moderately increasing function.*
If $E[\phi(\tau_D^{1/2})] < \infty$, *then*

$$\underline{\lim}_{r \to \infty} \frac{1}{\log \phi(r)} \eta_r \geq 1.$$

Since $\phi(r) = r^p$ is a moderately increasing and $\phi(d(o,x))$ is L-subharmonic in each cases in the above theorems, in particular we recover the following.

Corollary 4.3. *i) Assume the assumption of Theorem 4.1 and that* $\psi(r) \geq 0$.
If

$$\varliminf_{r\to\infty} \frac{1}{\log r} \int_0^{\nu r} \alpha_r dr > 2p,$$

then $E[\tau_D^p] < \infty$.
ii) Assume the assumption of Theorem 4.1 and that $\psi(r) \geq c > 0$.
If

$$\varliminf_{r\to\infty} \frac{1}{\log r} \int_0^{\nu r} \alpha_r dr > p,$$

then $E[\tau_D^p] < \infty$.
iii) Assume the assumption of Theorem 4.2.
If $E[\tau_D^p] < \infty$, *then*

$$\varliminf_{r\to\infty} \frac{1}{\log r} \eta_r \geq 2p.$$

We use the method in [1], so we need the following Burkholder type inequalities. We define $h(B_t)^*$ by $\sup_{0<s<t} h(B_s)$.

Lemma 4.4. *Let* $o \in M$ *be fixed and* $\phi(r)$ *be a moderately increasing function.*
i)[1] *Let* B_t *denote a Brownian motion on* $\mathbf{R}^n$. *There exist constants c and* C *such that for any stopping time* τ

$$cE[\phi(\tau^{1/2})] \leq E[\phi(|B_\tau|^*)] \leq CE[\phi(\tau^{1/2})].$$

ii) Assume the assumption of Theorem 4.1.
If $\psi(r) > c > 0$, *then there exist constants* C, $c_p > 0$ *such that*

$$cE[\phi(\tau)] \leq E[\phi(d(o,X_\tau)^*)] + C.$$

If $\psi(r) \geq 0$, *then*

$$cE[\phi(\tau^{1/2})] \leq E[\phi(d(o, X_\tau)^*)].$$

iii) Assume the assumption of Theorem 4.1. There exists a constant C *such that*

$$E[\phi(d(o, X_t)^*)] \leq CE[\phi(\tau^{1/2})]$$

Proof of ii) and iii). We write the radial part of B_t such as in [10] again :

$$r_t = r_0 + w_t + \frac{1}{2}\int_0^t \frac{\partial G}{\partial r}/G(B_s)ds - L_t,$$

where L_t is an increasing process which increases only on the cut locus of B_0. In the case of ii) $L_t \equiv 0$ and $\phi(r) = \frac{\partial G}{\partial r}/G$. Then we can compare r_t with

$$r_t^{(1)} = r_0^{(1)} + w_t + const.t$$

and

$$r_t^{(2)} = r_0^{(2)} + w_t.$$

It is easy to see that "good λ inequalities" for r_t and any stopping time are verified.
Set

$$r_t^{(0)} = r_0^{(0)} + w_t + \frac{1}{2}\int_0^t \frac{n-1}{r_s^{(0)}}ds.$$

Comparing r_t with $r_t^{(0)}$, iii) is a direct consequence of i). $\square$

The following argument is essentially due to Tsuji([16]).

Proposition 4.5.
i)Assume the assumption of Theorem 4.1. If $\underline{\lim}_{r\to\infty} -\frac{\log P(X_{\tau_{D_r}} \in \theta_r)}{\log \phi(r)} > 1$ *and* $\phi(d(o,x))$ *is L-subharmonic for a moderately increasing function* $\phi(r)$, *then* $E[\phi(\tau_D)] < \infty$.
ii) Conversely if the assumption of Theorem 4.2 is satisfied and $E[\phi(\tau_D^{1/2})] < \infty$, *then* $\underline{\lim}_{r\to\infty} -\frac{\log P(X_{\tau_r} \in \theta_r)}{\log \phi(r)} \geq 1$.

Proof. We fix a reference point $o \in D$.
(i) From the assumption there is a r_0 such that

$$P_x(X_{\tau_r} \in \theta_r) \leq const.\phi(r)^{-1-\epsilon} \text{ for } r \geq r_0,\ \epsilon > 0.$$

Define $v_r(x)$ as

$$\begin{aligned} Lv_r(x) = 0 \quad x \in D, \quad v_r(x) &= 1 \quad x \in \partial D \setminus \partial D_r \\ &= 0 \quad x \in \partial D_r \setminus \theta_r. \end{aligned}$$

Then $v_r(x) \leq P_x(X_{\tau_{D_r}} \in \theta_r)$ on D_r by the maximum principle.
Set $v(x) = \int_0^\infty \phi'(r)v_r(x)dr$. This is a bounded harmonic function on D. And $v(x) = \int_0^{r_1} \phi'(r)dr = \phi(r_1)$ if $x \in \partial D$ and $d(o,x) = r_1$. Since $\phi(d(o.x))$ is L-subharmonic, by maximum principle on D_r we have

$$\phi(d(o,x)) \leq \phi(r)P_x(X_{\tau_r} \in \theta_r) + v(x).$$

Since $b_1(r) \geq 0$, it is clear from lemma 4.4 that for any stopping time τ

$$const.E[\tau^p] \leq E[d(o,X_\tau)^{2p}].$$

It is easy to see by the routine argument that $E[\tau_D^p] < \infty$ ([1]).
iv) immediately follows from lemma 4.4. □

PROOF OF THEOREM 4.1 AND 4.2. Combine Proposition 4.5 with Proposition 2.1 and Corollary 3.2 respectively. □

Next we add a remark on the case $D = M \setminus \overline{U}$: U is an open set in M. Let $D = M \setminus B(1)$ and X_t be a Brownian motion on M. In view of our problem it is reasonable to consider only the case that $P_x(\tau_D < \infty)$ for any $x \in D$, namely, X_t is recurrent. We borrow the results from [11]. P.Li and L-F.Tam showed the following.

LEMMA 4.6. *Assume that the Ricci curvature of M is nonnegative on D and $\int_1^\infty \frac{dt}{A(t)} = \infty$ where $A(t)$ is $n-1$ dimensional volume of $S(t)$. Then there exists a harmonic function $g(x)$ on D satisfying that*

$$\begin{aligned} g(x) \text{ is harmonic on } D, \qquad & g(x) = 0 \quad \text{on} \quad S(1), \\ & g(x) \to \infty \text{ as } d(o,x) \to \infty, \end{aligned}$$

and there exists r_0 such that for $r \geq r_0$

$$c_1 \int_1^r \frac{dt}{A(t)} \leq g(x) \leq c_2 \int_1^r \frac{dt}{A(t)} \text{ on } S(r).$$

The above lemma and maximum principle immediately lead us to the following proposition.

PROPOSITION 4.7. *Under the condition of the Lemma 4.6 we have*

$$c_1\Big(\int_1^r \frac{dt}{A(t)}\Big)^{-1} \leq P(\tau_r < \tau_D) \leq c_2\Big(\int_1^r \frac{dt}{A(t)}\Big)^{-1}.$$

By Burkholder's argument and the method we have done by now, we get the following.

THEOREM 4.8.
i) Assume the same condition of the Lemma 4.6. Let $f(r)$ be a positive increasing function on $[0,\infty)$ satisfying that $\int_1^r \frac{dt}{A(t)} \leq f(r)$. Then $E[f(\tau_D)] = \infty$.
ii)Assume that $A(t) \geq \delta > 0$ for $t \geq 1$ together with the assumption of Lemma 4.6. We have for $\epsilon > 0$

$$P(\tau_D > t) \geq const.\Big(\int_1^{t^{\frac{1}{2-\epsilon}}} \frac{dt}{A(t)}\Big)^{-1}.$$

§5. Examples

1. *Cone.* Let $M = \mathbf{R}^n$, G be an open set on $S(1)$ and X_t be Brownian motion B_t. We define a cone C_G with respect to G by $C_G = \{x | x = a\xi, \quad \xi \in G, \quad 0 < a < \infty\}$. We can directly compute $P(B_{\tau_{D_r}} \in \theta_r)$ where $D = C_G$. It is well-known that B_t has skew-product representation [8] such as

$$B_t = (r_t, \Theta(\int_0^t \frac{ds}{r_s^2})).$$

Where r_t is a Bessel process : $r_t = r_0 + w_t + \int_0^t \frac{(n-1)}{r_s} ds$ and Θ_t is a Brownian motion on S^{n-1} independent of r_t. We first consider the distribution of $\int_0^t \frac{ds}{r_s^2}$. By Ito's formula

$$\log r_t = \log r_0 + W(\int_0^t \frac{ds}{r_s^2}) + \frac{n-2}{2}\int_0^t \frac{ds}{r_s^2}. \tag{5.1}$$

Where W_t is a one dimensional Brownian motion. We define one dimensional diffusion Y_t by

$$Y_t = r_0 + W_t + \frac{n-2}{2}t, \tag{5.2}$$

and $T_l = \inf\{t > 0 : Y_t = l\}$. Then it is easy to see that

$$E_{l_0}[e^{-\alpha T_l}] = \exp\{-(\frac{-(n-2) + \sqrt{(n-2)^2 + 8\alpha}}{2})(l - l_0)\}. \tag{5.3}$$

Let τ_r denotes $\tau_{B(r)}$. From (5.1) we have $\int_0^{\tau_r} \frac{ds}{r_s^2} = T_{\log r}$. On the other hand it is well known that $P(\sigma_G > t) \sim e^{-\lambda_G t/2} (t \uparrow \infty)$ ("a ~ b" means that there are constants c_1, c_2 such that $c_1 b \leq a \leq c_2 b$.), where $\sigma_G = \inf\{t > 0 : \Theta_t \notin G\}$. For simplicity we assume $r_0 = 1$. Therefore

$$\begin{aligned} P(B_{\tau_{D_r}} \in \theta_r) &= P(\sigma_G > \int_0^{\tau_r} \frac{ds}{r_s^2}) \\ &= \int_0^\infty P(\sigma_G > t) P(\int_0^{\tau_r} \frac{ds}{r_s^2} \in dt) \\ &= \int_0^\infty P(\sigma_G > t) P(T_{\log r} \in dt) \\ &\sim E[e^{-1/2 \lambda_G T_{\log r}}] \\ &= \exp\{-(\frac{-(n-2) + \sqrt{(n-2)^2 + 4\lambda_G}}{2}) \log r\} \end{aligned}$$

Making the same argument as Theorem 2.2 and Theorem 3.3, we have

$$\lim_{t \to \infty} \frac{1}{\log t} \log P(\tau_{C_G} > t) = -\frac{-(n-2) + \sqrt{(n-2)^2 + 4\lambda_G}}{4}.$$

Hence our upper estimste in Theorem 2.2 is sharp in this case.

This fact has already been known ([5]). We remark that recently this estimte was used for application to estimste occupation times at cone by Meyre and Werner[11]. They have this estimate using direct calculation without Dirichlet problem.

When M has a constant negative sectional curvature and D is a cone, $P(\tau_D = \infty) > 0$. Hence this case is not fit for our problem. From Theorem 4.1 we know $\lambda_r \sim const.e^{\sqrt{-k}r}/r^2$ is sufficient.

2. Let $M = \mathbf{R}^2$. Then $\lambda_r = \pi^2/l(r)^2 : l(r) =$ length of θ_r. Define D_d by

$$D_d = \{(x,y)|y > |x|^d\} \qquad d > 1.$$

We have

Proposition 5.1. *i) There exist positive constants* c_1, c_2, C_1, C_2 *such that*

$$C_1 e^{-c_1 r^{\frac{d-1}{d}}} \le P(X_{\tau_{D_r}} \in \theta_r) \le C_2 e^{-c_2 r^{\frac{d-1}{d}}}.$$

ii) There exist positive constants c_3, c_4, C_3, C_4 *such that*

$$C_3 e^{-c_3 t^{\frac{d-1}{d+1}}} \le P(\tau_{D_d} \ge t) \le C_4 e^{-c_4 t^{\frac{d-1}{3d-1}}}.$$

iii) $E[\tau_{D_d}^p] < \infty \quad for \quad 0 < p < \infty.$

Proof. We have $l(r) \sim 2r^{1/d}$ and $\rho(r) \sim r^{1/d}$. Proposition 2.1, Corollary 3.2 and Theorem 4.1 imply the desire results. □

As for the case of $d < 1$ we consider D_d^c with $d > 1$. Then we have

$$l_{D^c}(r) \sim 2\pi r - 2r^{1/d} \qquad \rho_{D^c} \sim r - r^{1/d}.$$

§6. Finiteness of a stopped Feynman-Kac functional

Let $q(x)$ be a measurable function on M. We call $E_x[\exp(\int_0^{\tau_D} q(X_s)ds)]$ the stopped Feynman-Kac functional on D. On the finiteness of stopped Feynman-Kac functional on a bounded domain many authors have considered. It is known via large deviation theory that the one is either finite or not according as $\sup Re(spec(\Delta + q))$ is negative or not.([14]) case D is unbounded, then we cannot apply large deviation theory directly. We obtain a sufficient condition on D for the finiteness of this functional for special potentials. Let $M = \mathbf{R}^n (n \ge 3)$, $L = \Delta$ and X_t be a Brownian motion on M throughout this section. We have the following result.

THEOREM 6.1.
i)Let $0 \le q(x) \le c\frac{1}{|x|^2}$. *If*

$$\varlimsup_{r\to\infty} \frac{1}{\log r}\int_\delta^{\nu r} \frac{1}{r}\Big(-\frac{n-2}{2}+\sqrt{\lambda_r r^2+\frac{(n-2)^2}{4}}\Big)dr > \frac{n-2}{2}-\sqrt{\frac{(n-2)^2}{4}-4c},$$

then

$$E_x[\exp(\int_0^{\tau_D} q(X_s)ds)] < \infty \quad x \in D\setminus\{0\},$$

where λ_r *is defined in* §*2.*

ii)Let D *be a cone* C_G *defined in* §*5 and* $q(x) = c\frac{1}{|x|^2}$. *We have*

$$E_x[\exp(\int_0^{\tau_D} q(X_s)ds)] < \infty \quad x \in D\setminus\{0\}$$

$$\textit{if and only if} \qquad 2c < \lambda_G.$$

PROOF. ii) is obvious by skew product representation of X_t in §5. We are going to show i). Set $r_n = n^\gamma r_0,\ n = 1, 2, \ldots$, with $|X_0| = r_0$ and $\gamma > 0$.

$$\begin{aligned}(6.1)\, & E[\exp(\int_0^{\tau_D} q(X_s)ds)] \\ & \le E[\exp(\int_0^{\tau_D} c\frac{1}{|X_s|^2}ds)] \\ & \le \sum E[\exp(\int_0^{\tau_{r_{n+1}}} c\frac{1}{|X_s|^2}ds); \tau_{r_n} < \tau_D \le \tau_{r_{n+1}}] \\ & = \sum E[\exp(\int_0^{\tau_{r_n}} c\frac{1}{|X_s|^2}ds) \\ & \qquad \cdot E_{X_{\tau_{r_n}}}[\exp(\int_0^{\tau_{r_{n+1}}} c\frac{1}{|X_s|^2}ds); \tau_D \le \tau_{r_{n+1}}]; \tau_{r_n} < \tau_D \le \tau_{r_{n+1}}] \\ & \le \sum E[\exp(\int_0^{\tau_{r_n}} c\frac{1}{|X_s|^2}ds) \\ & \qquad \cdot E_{X_{\tau_{r_n}}}[\exp(\int_0^{\tau_{r_{n+1}}} c\frac{1}{|X_s|^2}ds)]; \tau_{r_n} < \tau_D \le \tau_{r_{n+1}}]\end{aligned}$$

By (5.1) and (5.2)

$$E_{X_{\tau_{r_n}}}[\exp(\int_0^{\tau_{r_{n+1}}} c\frac{1}{|X_s|^2}ds)] = E_0[\exp(cT_{\log r_{n+1}/r_n})] < \infty, \qquad \text{uniformly in } n \quad \text{if } c < (n-2)^2/8.$$

Then the last term in (6.1)

$$\begin{aligned} &= \sum E[\exp(\int_0^{\tau_{r_n}} c\frac{1}{|X_s|^2}ds); \tau_{r_n} < \tau_D] E_0[\exp(cT_{\log r_{n+1}/r_n})] \\ &\leq \mathit{const.} \sum E[\exp(\int_0^{\tau_{r_n}} 2c\frac{1}{|X_s|^2}ds)]^{1/2} P(\tau_{r_n} < \tau_D)^{1/2} \\ &\qquad\qquad \text{(by Schwarz inequality.)} \end{aligned}$$

Using the observation in §5 again

$$\text{(6.2)} \quad \text{the last term} = \mathit{const.} \sum E[\exp(2cT_{\log r_{n+1}/r_0})]^{1/2} P(\tau_{r_n} < \tau_D)^{1/2}$$

It is easy to see

$$E[\exp(2cT_{\log r_{n+1}/r_0})]^{1/2} = e^{\frac{1}{2}(\frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - 4c}) \log \frac{r_n}{r_0}}$$

On the other hand if

$$\infty > \varliminf_{r\to\infty} \frac{1}{\log r} \int_\delta^{\nu r} \frac{1}{r}(-\frac{n-2}{2} + \sqrt{\lambda_r r^2 + \frac{(n-2)^2}{4}})dr > p,$$

then there exists r_0 such that $P(\tau_r < \tau_D) \leq \mathit{const.} r^{-p}$ for $r > r_0$. Hence if $p > \frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - 4c}$ and set $\gamma = 2/(p - \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - 4c})$, then the right hand side of (6.2) is finite. This completes the proof.

Remark. In the case of $n = 2$ we can easily see that $E[\exp(\int_0^{\tau_D} c\frac{1}{|X_s|^2}ds)] = \infty$ for any $c > 0$ by the observation as in §4. Assume $0 \in D$. Then $E[\exp(\int_0^{\tau_D} c\frac{1}{|X_s|^2}ds)] = \infty$ for any $c > 0$ even if D is bounded, because $\int_0^{\tau_{B(1)}} \frac{1}{|X_s|^2}ds = \sigma_1$:the first exit time from $(-\infty, 1]$ of one dimensional Brownian motion, which has only $L^p(0 < p < 1/2)$ integrability.

References

[1] Burkholder, D. L., Exit times of Brownian motion,harmonic majorization and Hardy spaces, Adv. in Math. **26** (1977), 182–201.

[2] Carleman, T., Sur une inégarite différentielle dans la théorie des fonctions analytiques, C.R. Acad. Sci. Paris **196** (1933).

[3] Cranston, M., Gradient estimates on manifolds using coupling, J. Funct. Anal. **99** (1991), 110–124.

[4] Davies, E. B., *Heat kernels and spectral theory*, Cambridge Univ. Press, Cambridge, 1989.

[5] Dante DeBlassie, R., Exit times from cones in $\mathbf{R}^n$ of Brownian motion, Probab. Theory Rel. Fields **74** (1987), 1–29.

[6] Durret, R., *Brownian motion and martingales in analysis*, Wadsworth, Belmont, 1984.

[7] Hayman, W. K., *Subharmonic functions*, second vol. S, Academic Press, London, 1989.

[8] Ikeda, N. and S. Watanabe, *Stochastic differential equations and diffusion processes*, second ed., North-Holland : Kodansha, 1989.

[9] Ito, K. and H. Mckean, *Diffusion Processes and Their Sample Paths*, Springer, Berlin, 1965.

[10] Kendall, W. S., The radial part of Brownian motion on a manifolds :a semimartingale property, Ann. Probab. **15** (1987), 1491–1500.

[11] Meyre, T. and W. Werner, On the occupation times of cones by Brownian motion, Probab. Theory Rel. Fields **101** (1995), 409–419.

[12] Li, P. and L-F. Tam, Positive harmonic functions on complete manifolds with nonnegative curvature outside a compact set, Ann. Math. **125** (1987), 171–207.

[13] Nevanlinna, R., *Analytic functions*, Springer, Berlin, 1970.

[14] Pinsky, R., A spectral criterion for the finiteness or infiniteness of stopped Feynman-Kac functionals of diffusion processes, Ann. Prob. **14** (1986), 1180–1187.

[15] Revuz, D. and M. Yor, *Continuous martingales and Brownian Motion*, Springer, Berlin, 1991.

[16] Tsuji, M., *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.

[17] Vourinen, M., *Conformal Geometry and Quasiregular Mappings*, Lect. Note Math. Vol. 1319, Springer, Berlin, 1988.

(Received September 28, 1994)

Department of Mathematical Sciences
University of Tokyo
Komaba, Tokyo 153

Japan

Current address
Department of Mathematics
Graduated School of Science
Osaka University
Toyonaka, Osaka 560
Japan