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The Heat Flows of Harmonic Maps from S^2 to S^2

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Abstract. In this paper we show that the heat flow $u(x,t)$ with a rotationally symmetric initial map $u_0(x,t)$ converges to a harmonic map as $t \to \infty$ if $n = 1$ and $0 < h(\theta) < \pi$ $(\theta \in (0,\pi))$. Here we call $u_0 : S^2 \to S^2$ rotationally symmetric if there exists a function $h : [0,\pi] \to R$ such that $h(0) = 0$, $h(\pi) = n\pi$, and $u(\cos\tau \sin\theta, \sin\tau \sin\theta, \cos\theta) = (\cos\tau \sin h(\theta), \sin\tau \sin h(\theta), \cos h(\theta))$.

§0. Introduction

The heat flows of harmonic maps between compact Riemannian manifolds made their first appearance in the paper by Eells and Sampson [E-S]. In general the heat flows of harmonic maps are given by nonlinear parabolic partial differential systems. They showed that there exist global solutions of the heat flows of harmonic maps if the sectional curvatures of target manifolds are non-positive.

In 1981 Sacks and Uhlenbeck [S-U] proved that weakly harmonic maps from a compact Riemann surface to a compact Riemannian manifold with positive sectional curvatures are not always regular, using Ljusternik-Schnileman theory for a suitable sequence of functionals. In 1985, Struwe ([S1][S2]) established the existence of unique global solutions of the heat flows of harmonic maps from Riemannian surfaces. This unique solution is regular with the exception of at most finitely many singular points where non-constant harmonic maps of S^2 into the target manifolds.

It will be an interesting problem to obtain the maximal existence time of the regular solutions. Is it finite or infinite ? Chang, Ding, and Ye [C-D] [C-D-Y] showed in 1990 that either case occurs. They considered rotationally

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symmetric (see the definition in §1) initial maps from D^2 to S^2. (Their terminology is "spherically symmetric" instead of "rotationally symmetric".) In this case the partial differential system of the heat flow is reduced to a one-dimensional parabolic equation, which satisfies the Maximum Principle and the comparison theorem so that the behaviour of the solution is well controlled.

In this paper we shall consider the heat flows of harmonic maps from S^2 to S^2 ((1.4)) with rotationally symmetric initial maps. In §1, we recall the result of Struwe (Theorem[S2]) and state ours (Theorem). In §2, we study the rotationally symmetric solutions. In this case our partial differential system of the heat flow is reduced to a one-dimensional parabolic equation which is singular on the boundary. We shall show the Hopf boundary lemma does hold even in this case so that the comparison theorem can be applied under very milder conditions on initial maps. We prove our Theorem in §3.

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§1. The Heat Flows of the Harmonic Maps

Let S^2 be the unit sphere in the Euclidean space R^3. In order to formulate the problem, we introduce some notation: we denote by $C^{1,\alpha}(S^2, S^2)$ the sets of $u : S^2 \to S^2$ whose first differentials are α-Hölder continuous, where $0 < \alpha < 1$. For $u \in C^{1,\alpha}(S^2, S^2)$, the Hilbert-Schmidt norm $|\nabla u|$ is defined by

$$|\nabla u|^2 = \sum_{i=1}^{2} \gamma_{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta}, \tag{1.1}$$

where $x = (x^1, x^2)$ is a local coordinate system on S^2 with the metric tensor $\gamma = (\gamma_{\alpha\beta})_{1 \le \alpha,\beta \le 2}$, where $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$ and $u = (u^1, u^2, u^3) \in S^2 \subset R^3$.

The energy of $u \in C^{1,\alpha}(S^2, S^2)$ is defined by

$$E(u) = \frac{1}{2} \int_{S^2} |\nabla u|^2 dS^2. \tag{1.2}$$

We call a critical points of E a harmonic map. A harmonic map $u : S^2 \to S^2$ satisfies the Euler-Lagrange Equation

$$\Delta u + |\nabla u|^2 u = 0, \tag{1.3}$$

where

$$\Delta = \frac{1}{\sqrt{\det \gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\det \gamma}\, \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right).$$

The solutions to the evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + |\nabla u|^2 u \\ u(0,t) = u_0 \in C^{1,\alpha}(S^2, S^2) \end{cases} \tag{1.4}$$

are called the heat flows of Harmonic maps.

Theorem (Struwe)[S2, p.197]. *Let $u_0 \in C^{1,\alpha}(S^2, S^2)$. Then there exists a solution of (1.4), $E(u(\cdot,t)) \leq E(u_0)$, which is regular with the exception of at most finitely many points, and which is unique in this class. Let $S = \{(x_i, t_i) \in S^2 \times R^+; 1 \leq i \leq K\}$ be the set consisting of all the singular points.*

For $(\bar{x}, \bar{t}) \in S$, there exist sequences $R_m \searrow 0, t_m \nearrow \bar{t}, x_m \to \bar{x}$ such that

$$u_m(x) \equiv u(\exp_{x_m}(R_m x), t_m) : R^2 \to S^2 \tag{1.5}$$

converges to $\bar{u} : S^2 \backsimeq R^2 \cup +\infty \to S^2$ uniformly on each compact subsets in R^2.

Finally $u(\cdot, t)$ converges to a harmonic map $u_\infty : S^2 \to S^2$ on any compact subsets in $S^2 \backslash \{x_i; t_i = +\infty\}$ uniformly.

Remark. Struwe proved the existence of the solution u with initial map $u_0 \in H^{1,2}(\Sigma, N)$, Σ being a Riemannian surface and N a Riemannian manifold. He shows that for a singular point $(\bar{x}, \bar{t} < +\infty) \in \Sigma \times [0, \infty]$ and for any $R \in]0, i_\Sigma[$ (i_Σ is the injective radius of the domain manifold Σ), we have the following:
(i) if $\bar{t} < +\infty$,

$$\limsup_{t_m \nearrow \bar{t}} E(u(t_m); B_R(\bar{x})) \geq \varepsilon_1,$$

(ii) if $\bar{t} = +\infty$,

$$\liminf_{t_m \nearrow \bar{t}} E(u(t_m); B_R(\bar{x})) \geq \frac{\varepsilon_1}{2},$$

where ε_1 is a positive constant which depends on only the geometry of Σ and N. See Chapter 5 of [S2] for the detail.

We call the points of S the blow up points for u and the minimum of $\{t_i\}$ the maximal existence time (denoted by T) of the regular solution for u_0. We are concerned with T. Chang and Ding showed that a solution with an initial map from D^2 to S^2 does not always converge to a harmonic map even if it is regular for all time. On the other hand we shall show that a solution with a particular initial map from S^2 to S^2 is regular for all time and converges to a harmonic map.

We call $u : S^2 \to S^2$ rotationally symmetric if there exists a function $h : [0, \pi] \to R$ such that $h(0) = 0$, $h(\pi) = n\pi$, and

$$u(\varphi(\tau, \theta)) = \varphi(\tau, h(\theta)), \tag{1.6}$$

tag1.6 where $\varphi(\tau, \theta) = (\cos\tau \sin\theta, \sin\tau \sin\theta, \cos\theta)$.

The main result of this paper is as follows.

Theorem. *Let $u_0 \in C^{1,\alpha}(S^2, S^2)$ be a rotationally symmetric map which is given by continuous $h_0 : [0, \pi] \to R$ in form (1.6). If h_0 satisfies that $0 \leq h_0(\theta) \leq \pi$, $h_0 = (0)$, and $h_0(\pi) = \pi$, then there exists uniquely the smooth solution $u(x, t)$ of (1.4) for all time t and $u(x, t)$ converges to a harmonic map $u_\infty : S^2 \to S^2$ uniformly as $t \to \infty$.*

§2. Solutions with Rotationally Symmetric Initial Maps

At first we give some lemmas for rotationally symmetric maps. Now let R_η be the rotation around the x_3-axis by an angle η and R_{13} the reflection with respect to the x_1x_3 plane.

Lemma 2.1. *$u_0 \in C^{1,\alpha}(S^2, S^2)$ is rotationally symmetric if and only if u_0 is R_η-invariant for all η and R_{13}-invariant.*

Proof. We remark that u is R_η-invariant if and only if

$$u \circ R_\eta = R_\eta \circ u, \tag{2.1}$$

and that u is R_{13}-invariant if and only if

$$u \circ R_{13} = R_{13} \circ u. \tag{2.2}$$

If u is rotationally symmetric, it is easy to see u satisfies (2.1) (2.2).

Conversely suppose u satisfies (2.1) (2.2). Let

$$C_\tau = \{(\varphi(\tau, \theta)); 0 \le \theta \le \pi\} \subset S^2$$

and

$$\tilde{C_\tau} = C_\tau \cup C_{\tau+\pi} = C_\tau \cup R_\pi(C_\tau).$$

C_0 is in the x_1x_3 plane, then $R_{13}(C_0) = C_0$. By (2.2) $R_{13} \circ u(C_0) = u \circ R_{13}(C_0) = u(C_0)$. Since $R_{13}(C_0) = C_0$, $u(C_0) \subset \tilde{C_0}$.

The behaviour of u from $u(C_0)$ to $\tilde{C_0}$ exactly decides $u : S^2 \to S^2$. So u is rotationally symmetric. □

We can easily prove lemmas 2.2 and 2.3 by lemma 2.1 and the uniqueness of the solution (Theorem [S2]).

Lemma 2.2. *Let $u : S^2 \to S^2$ be rotationally symmetric and represented by h in form* (1.6). *$u \in C^{1,\alpha}$ if and only if $h \in \Phi^{1,\alpha} = \{h \in C^{1,\alpha}([0, \pi]); h(0) = 0, h(\pi) = n\pi, n = 0, \pm 1, \pm 2, \dots\}$.*

Lemma 2.3. *If $u_0 \in C^{1,\alpha}(S^2, S^2)$ is rotationally symmetric, then so is the solution $u(x,t)$ of* (1.4) *for all $t \in [0,T]$, where T is the maximal existence time of the regular solution for u_0.*

We suppose u_0 is rotationally symmetric. Without loss of generality, we may assume $u((0,0,1)) = (0,0,1)$. Then there exists a function $h_0 : [0, \pi] \to R$ such that $h_0(0) = 0, h_0(\pi) = n\pi$, and $u_0(\varphi(\tau, \theta)) = \varphi(\tau, h_0(\theta))$. By lemma 2.3 the solution of (1.4) is also rotationally symmetric and there exists a function $h : [0, \pi] \times R \to R$ such that $h(0,t) = 0, h(\pi, t) = n\pi$, and $u(\varphi(\tau, \theta)) = \varphi(\tau, h(\theta))$. Then (1.4) is reduced to the one-dimensional parabolic equation,

$$h_t = h_{\theta\theta} + \frac{\cos\theta}{\sin\theta} h_\theta - \frac{\sin 2h}{2\sin^2\theta}, \tag{2.3}$$

$$h(\theta, 0) = h_0(\theta), \theta \in [0, \pi] \tag{2.4}$$

$$h(0,t) = 0, \qquad h(\pi, t) = n\pi.$$

For our theorem, we need only to consider the case $n = 1$, that is

$$h(0,t) = 0, \qquad h(\pi,t) = \pi. \tag{2.5}$$

LEMMA 2.4. *Let $h(\theta,t)$ be the solution of* (2.3) (2.4) (2.5), *where $h_0 \in \Phi^{1,\alpha}$. Suppose that $0 \leq h_0(\theta) \leq \pi$ for $\theta \in (0,\pi)$. Then $0 < h(\theta,t) < \pi$ on $(0,\pi)\times(0,T)$, where T is the maximal existence time of the regular solution for h_0.*

PROOF. At first we shall prove $h(\theta,t) > 0$.

We write the equation in the form of

$$h_t = h_{\theta\theta} + \frac{\cos\theta}{\sin\theta} h_\theta + q(\theta,t)h,$$

where

$$q(\theta,t) = \begin{cases} -\dfrac{\sin 2h}{2h\sin^2\theta} & (h(\theta,t) \neq 0) \\ -\dfrac{1}{\sin^2\theta} & (h(\theta,t) = 0). \end{cases} \tag{2.6}$$

We shall check whether $q(\theta,t)$ is bounded from above on the neighborhood of $\theta = 0$ and π. Since $h(0,t) = 0$, for each $s \in (0,T)$, there exists $\delta = \delta_s$ such that $q(\theta,t) < 0$ on $(0,\delta_s) \times (0,s)$.

On the neighborhood of $\theta = \pi$, since $h(\pi,t) = \pi$, by choosing $\varepsilon = \varepsilon_s$ sufficiently small we may assume

$$h(\theta,t) > \frac{\pi}{2}(>0) \qquad \text{for} \quad \theta \in (\pi - \varepsilon_s, \pi). \tag{2.7}$$

$q(\theta,t)$ is a continuous function on $(0,\pi)\times(0,s)$, then $q(\theta,t)$ is upper bounded on $(0,\pi-\varepsilon_s)\times(0,s)$. Namely there exists a positive constant $c = c_s$ which is independent of θ and t such that $q(\theta,t) < c$.

We replace $h(\theta,t)$ by $\tilde{h}(\theta,t) = e^{-ct}h(\theta,t)$. Since $h(\theta,t)$ satisfies (2.3), $\tilde{h}$ satisfies

$$\tilde{h}_t = \tilde{h}_{\theta\theta} + \frac{\cos\theta}{\sin\theta}\tilde{h}_\theta + (q-c)\tilde{h}.$$

Using $q - c < 0$, we can apply the Strong Maximum Principle on each of the domain whose boundary is contained in $(0,\pi-\varepsilon_s)\times(0,s)$. Consequently

$\tilde{h}(\theta,t) > 0$ on $(0,\pi-\varepsilon_s)\times(0,s)$. By (2.7) we can conclude $h(\theta,t) > 0$ on $(0,\pi)\times(0,s)$. Because each s is in (0,T), we have $h(\theta,t) > 0$ on $(0,\pi)\times(0,T)$.

Next we shall prove that $h(\theta,t) < \pi$.

Let $f = \pi - h$ and $f_0 = \pi - h_0 \geq 0$. We can write the equation in the form of

$$f_t = f_{\theta\theta} + \frac{\cos\theta}{\sin\theta} f_\theta - \frac{\sin 2f}{2\sin^2\theta}.$$

By the same argument as above we can conclude $f(\theta,t) > 0$ on $(0,\pi)\times(0,T)$. Then $h(\theta,t) < \pi$ on $(0,\pi)\times(0,T)$. □

We can prove next lemma by applying the same argument in the proof of lemma 2.4 to $g = h^2 - h^1 > 0$. We omit details.

Lemma 2.5. *Let h^1, h^2 be the solutions of* (2.3)(2.4)(2.5) *with initial values h_0^1, $h_0^2 \in \Phi^{1,\alpha}$, and their maximal existence time T^1, T^2 for the regular solution for h_0^1, h_0^2 respectively.*

If

$$0 \leq h_0^1(\theta) < h_0^2(\theta) \leq \pi \qquad \textit{for all} \quad \theta \in (0,\pi),$$

then

$$h^1(\theta,t) < h^2(\theta,t) \qquad \textit{for all} \quad (\theta,t) \in (0,\pi)\times(0,T),$$

where $T = \min\{T^1, T^2\}$.

Let us study the stationary solution of (2.3).

The solution $\bar{h} : [0,\pi] \to [0,\pi]$ which satisfies

$$\begin{cases} \bar{h}_{\theta\theta} + \dfrac{\cos\theta}{\sin\theta}\bar{h}_\theta - \dfrac{\sin 2\bar{h}}{2\sin^2\theta} = 0 \\ \bar{h}(0) = 0 \\ \bar{h}(\pi) = \pi \end{cases} \tag{2.8}$$

is given by tag2.9

$$\tan\frac{\bar{h}}{2} = A\tan\frac{\theta}{2}, \tag{2.9}$$

where A is a positive number. In fact we replace θ by $s = \log\tan\frac{\theta}{2}$. Then $\tilde{h}(s) = \bar{h}(\theta)$ satisfies

$$\tilde{h}_{ss} = \frac{1}{2}\sin 2\tilde{h}.$$

We can solve this equation and obtain (2.9). By (2.9) $\bar{h}'(0) = A$ and $\bar{h}'(\pi) = \frac{1}{A}$. $\bar{h}(\theta)$ converges to $\bar{h}(\theta) \equiv 0$ uniformly as $A \to 0$ except $\theta = \pi$. Also $\bar{h}(\theta)$ converges to $\bar{h}(\theta) \equiv \pi$ uniformly as $A \to \infty$ except $\theta = 0$.

Now we show that Hopf boundary lemma (see [P-W] p.170 Theorem 3) holds in our case. We note again that the equation is singular itself at the boundary. Although the method of the proof is well-known, we perform the details because of their importance.

Lemma 2.6. *Let* $h_0 \in \Phi^{1,\alpha}$ *and* $0 \le h_0(\theta) \le \pi$ *for* $\theta \in (0,\pi)$. *Then the solution of* $h(\theta, t)$ *of* (2.3)(2.4)(2.5) *satisfies*

$$h_\theta(0,t) > 0 \qquad \textit{for} \quad 0 < t < T,$$

and

$$h_\theta(\pi,t) < 0 \qquad \textit{for} \quad 0 < t < T.$$

Proof. We show $h_\theta(0,t_0) > 0$, where $0 < t_0 < T$. On the (θ,t), let K be a disk plane whose center is (R, t_0) and radius is R, where $0 < R < \min\{t_0, T - t_0, \frac{\pi}{2}\}$. Let K' be a half disk whose center is $(0,t_0)$ and radius is $\bar{\theta}$, where $\bar{\theta}$ is a small positive number such that $h(\theta,t) < \frac{\pi}{2}$ if $(\theta,t) \in K'$. Let $C = \partial K \cap K'$ and $C' = K \cap \partial K'$, where C contains its end points and C' does not contain its end points. Let D be a domain which is enclosed with $C \cup C'$.

By lemma 2.5,

$$\begin{aligned} &h(0,t_0) = 0 \\ &h(\theta,t) > 0 \qquad (\theta,t) \in C \setminus \{(0,t_0)\} \\ &h(\theta,t) \ge \eta > 0 \qquad (\theta,t) \in C' \end{aligned}$$

where η is a positive constant. Let

$$v(\theta,t) = e^{-\alpha\{(\theta-R)^2+(t-t_0)^2\}} - e^{-\alpha R^2}.$$

It is easy to see

$$\begin{aligned} &v > 0 \qquad \text{in} \quad D \setminus \partial D \\ &v \equiv 0 \qquad \text{on} \quad C. \end{aligned}$$

Let us assume tag2.10

$$v_{\theta\theta}+\frac{\cos\theta}{\sin\theta}v_\theta+q(\theta,t)v-v_t>0 \qquad \text{in} \quad D\cup\partial D, \tag{2.10}$$

where $q(\theta,t)$ is defined by (2.6). Later we shall prove this assumption holds for sufficiently large α.

Set

$$w(\theta,t)=h(\theta,t)-\varepsilon v(\theta,t).$$

By choosing a small positive constant ε, we have

$$\begin{aligned} &w(0,t_0)=0\\ &w(\theta,t)=h(\theta,t)>0 \qquad (\theta,t)\in C\setminus\{(0,t_0)\}\\ &w(\theta,t)\ge\eta-\varepsilon v>0 \qquad (\theta,t)\in C'. \end{aligned}$$

Namely we have

$$w_{\theta\theta}+\frac{\cos\theta}{\sin\theta}w_\theta+q(\theta,t)w-w_t<0$$

on $D\cup\partial D\setminus\{(0,t_0)\}$ by (2.10). Since $\bar\theta$ is a small positive number such that $h(\theta,t)<\frac{\pi}{2}$, we can see $q(\theta,t)<0$ in $D\cup\partial D$. Applying the Strong Maximum Principle, the minimum of w attains on $(0,t_0)$ only. Then

$$\begin{aligned} &w_\theta(0,t_0)\ge 0,\\ i.e.\qquad &h_\theta(0,t_0)-\varepsilon v_\theta(0,t_0)\ge 0. \end{aligned}$$

Consequently we can conclude that $h_\theta(0,t_0)>0$ by $v_\theta(0,t_0)>0$.

Finally we shall prove (2.10). Setting $r^2=r^2(\theta,t)=(\theta-R)^2+(t-t_0)^2$, we have

$$\begin{aligned} v_{\theta\theta}&+\frac{\cos\theta}{\sin\theta}v_\theta+q(\theta,t)v-v_t\\ =&\,2\alpha e^{-\alpha r^2}\{2\alpha(R-\theta)^2-1+(t-t_0)\}\\ &+\frac{e^{-\alpha r^2}}{\sin^2\theta}\{2\alpha(R-\theta)\cos\theta\sin\theta-\frac{\sin 2h}{2h}(1-e^{-\alpha(R^2-r^2)})\}\\ \ge&\,2\alpha e^{-\alpha r^2}\{2\alpha(R-\theta)^2-1+(t-t_0)\}+\frac{e^{-\alpha r^2}}{\sin^2\theta}I(\theta), \end{aligned}$$

where

$$I(\theta) = \alpha(R-\theta)\sin 2\theta - (1 - e^{-\alpha(R^2-r^2)}).$$

If $\theta = 0$, then $t = t_0$ on ∂D. So $I(0) = 0$. We have at $t = t_0$,

$$\begin{aligned} I'(\theta) &= 2\alpha(R-\theta)\{\cos 2\theta - e^{-\alpha(R^2-r^2)}\} - \alpha \sin 2\theta \\ I'(0) &= 0 \\ I''(\theta) &= 2\alpha(R-\theta)\{-2\sin 2\theta + 2\alpha(R-\theta)e^{-\alpha(R^2-r^2)}\} \\ &\qquad - 2\alpha\{\cos 2\theta - e^{-\alpha(R^2-r^2)}\} - 2\alpha\cos 2\theta \\ I''(0) &= 2\alpha(2\alpha R^2 - 1). \end{aligned}$$

Choosing sufficiently large α, we get $I''(0) > 0$ and

$$\lim_{\theta\to 0}\frac{I(\theta)}{\sin^2\theta} = \lim_{\theta\to 0}\frac{I'(\theta)}{\sin 2\theta} = \lim_{\theta\to 0}\frac{I''(\theta)}{2\cos 2\theta} = \frac{1}{2}I''(0) > 0.$$

Consequently by choosing α so large and $\bar{\theta}$ so small, we get (2.10). □

§3. The Proof of Theorem

In the proof of theorem [S2] in Ch.1, it is shown that the center of the radiation of u_m at (1.5) is x_m, which is chosen such that for some $R_0 > 0$ and $R_m \to 0$

$$E(u(t_m); B_{R_m}(x_m)) = \sup_{\substack{x\in 2R_0(\bar{x}) \\ t_m-\tau_m\le t\le t_m}} E(u(t); B_{R_m}(x)) = \frac{\varepsilon_1}{4},$$

where τ_m depends on ε_1, R_m, and $E(u_0)$. But we shall prove that we can choose $x_m \equiv \bar{x}$ (independent of m) if the fixed point by rotation is a singular point of the solution.

Lemma 3.1. *Let $u_0 \in C^{1,\alpha}(S^2, S^2)$ be rotationally symmetric. Suppose that the solution $u(x,t)$ of (1.4) is singular at $((0,0,1),T)$. Then $\bar{u} : S^2 \backsimeq R^2 \cup +\infty \to S^2$ in theorem (Struwe[S2]) is rotationally symmetric.*

Proof. If $\bar{x} = ((0,0,1),T)$ is a singular point, then there exist sequences $R_m \searrow 0, t_m \nearrow T, x_m \to \bar{x}$ such that

$$E(u(t_m); B_{R_m}(x_m)) = \frac{\varepsilon_1}{4}.$$

We compare the convergence of $R_m \searrow 0$ with that of $d_m := dist(x_m, \bar{x}) \searrow 0$. As $m \to +\infty$, we have either

$$\text{(i)} \quad \frac{R_m}{d_m} \to 0 \quad \text{or} \quad \text{(ii)} \quad \frac{R_m}{d_m} > 0.$$

In case (i) we consider $B_{R_m+d_m(\bar{x})}$ and $B_{R_m(x_m)}$. Let θ_m be defined by $\sin\theta_m = \frac{R_m}{d_m}$. By the symmetry there are $[\frac{2\pi}{2\theta_m}]$ balls, B_{R_m}, whose centers are on $\partial B_{R_m+d_m(\bar{x})}$ and which have no intersections. Then

$$E(u(t_m); B_{R_m+d_m(\bar{x})}) \geq [\frac{2\pi}{2\theta_m}]\frac{\varepsilon_1}{4} \to +\infty,$$

this is a contradiction. In case (ii) we have $B_{d_m}(\bar{x}) \subset B_{R_m}(x_m)$ for sufficiently large m and

$$E(u(t_m); B_{R_m+d_m}(\bar{x})) \geq \frac{\varepsilon_1}{4}.$$

By choosing $R'_m(< R_m + d_m) \to 0$ in the place of R_m, we have

$$E(u(t_m); B_{R'_m}(\bar{x})) = \sup_{\substack{x \in 2R_0(\bar{x}) \\ t_m - \tau_m \leq t \leq t_m}} E(u(t); B_{R'_m}(x)) = \frac{\varepsilon_1}{4}.$$

Then we can take $\bar{x}$ as the center of the radiation of u_m at (1.5). □

Now we shall prove Theorem.

Proof of Theorem. By lemma 2.3, if the initial map u_0 is rotationally symmetric, then so is the solutions of (1.4). The singular points are at most finite (Theorem [S2]). So if there exist singular points, they must be fixed points of R_η. Therefore they are included in $(0, 0, 1), (0, 0, -1)$. Namely $\theta \in 0, \pi$ in form (1.6).

We suppose the maximal existence time $T \leq +\infty$. If $(0, 0, 1)$ is a singular point, then

$$h_m(\theta) \equiv h(R_m\theta, t_m) \to h_\infty(\theta),$$

(We may choose $x_m \equiv (0, 0, 1)$ for all m in (1.5).) where $h_\infty : [0, \pi] \to R, h_\infty(0) = 0$ in the form of (1.6). Because u_∞ is non constant, there exists

$\theta \in [0,\pi]$ such that $h_\infty(\theta) = \pi$. Since $R_m\theta \to 0$ as $m \to +\infty$, there exist $\theta_m = R_m\theta$ and $t_m \to T$ such that

$$h(\theta_m, t_m) \to \pi, \tag{3.1}$$

Fix any $t_0 \in (0,T)$, for lemma 2.6 we have $h_\theta(\theta, t_0) > 0$. So choosing stationary solutions h^1, h^2 properly given in (2,10), we have

$$h^1(\theta) < h(\theta, t_0) < h^2(\theta).$$

Then by lemma 2.5 we can concluded

$$h^1(\theta) < h(\theta, t) < h^2(\theta).$$

This is a contradiction to (3.1).

Consequently we can conclude (0,0,1) is a regular point. Similarly, so is (0,0,-1). Hence there is no singular point for all $T \leq +\infty$. □

References

[C-D] Chang, K. C. and W. Y. Ding, A result on the global existence for heat flows of harmonic maps from D^2 into S^2, Nematics (J.M.Coron et al.(eds.)), 1990, pp. 37–47.

[C-D-Y] Chang, K. C., Ding, W. Y. and R. Ye, Finite-time blow up of the heat flow of harmonic maps from surfaces, J. Diff. Geom. **36** (1992), 507–515.

[E-S] Eells, J. and J. H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. **86** (1964), 109–169.

[J] Jost, J., Harmonic Mappings between Riemannian Manifolds, Proc. C.M.A., vol. 4, ANU-Press, 1983.

[P-W] Protter, M. H. and H. F. Weinberger, Maximum Principles in Differential Equations, Prentice Hall, 1967.

[S-U] Sacks, J. and K. Uhlenbeck, The existence of minimal immersions of two spheres, Annals of Math. **113** (1981), 1–24.

[S1] Struwe, M., On the evolution of harmonic maps of Riemannian surfaces, Comment. Math. Helv. (1985), 558–581.

[S2] Struwe, M., Variational methods:applications to nonlinear partial differential equations and Hamiltonian systems, Springer-Verlag, 1990.

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