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Branching of singularities for some second or third order microhyperbolic operators

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Abstract. We study microhyperbolic operators of the form $(D_1 - \alpha_1 x_1 D_n)(D_1 - \alpha_2 x_1 D_n)$+lower, $(D_1 - x_1 D_n)D_1(D_1 + x_1 D_n)$+lower. When the lower order terms take a certain form, we can obtain a very detailed information about the singularity of a solution. We look at this problem from the viewpoint of boundary value problems. General arguments for m-th order operators are given.

Introduction

For (micro)differential operators whose characteristic variety has a non-involutory intersection, the propagation and branching of singularities is the most interesting problem. [H], [O], [Al], [A-N], [Tak], [N] and [T-T] are well-known results.

In this paper, we study the branching of singularities for second and third order (micro)hyperbolic operators. The principal symbols are $(\xi_1 - \alpha_1 x_1 \xi_n)(\xi_1 - \alpha_2 x_1 D_n)$, $(\xi_1 - x_1 \xi_n)\xi_1(\xi_1 + x_1 \xi_n)$. General arguments (heuristic in some parts) about m-th order operators are given.

Our approach is based on the study of boundary value problems. Although this viewpoint was already taken in [Al], it is more apparent in the present paper.

Another feature of our approach is that we use ODEs of Fuchs type, while in [Al], [A-N] and [T-T], ODEs with irregular singularities were used.

The plan of this paper is as follows. PART 0 gives a general background about an operator of arbitrary order. PARTS 1 and 2 are about the second

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and the third order cases respectively. The main theorems are found in PART 1 §1 and PART 2 §1.

This paper is an application of [Kat 2].

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Part 0 General background

Let $P(x, D) = D_1^m + P_1(x, D')D_1^{m-1} + \cdots + P_m(x, D')$ be an m-th order microhyperbolic microdiferential operator defined in a neighborhood of $p \in \{(x; i\xi) \in iT^*M; x_1 = \xi_1 = 0, \xi_n > 0\}$, $M = \mathbb{R}^n$. Here we write $x = (x_1, \ldots, x_n) = (x_1, x') \in \mathbb{R}^n = M$.

Assume that its principal symbol is of the form

$$\sigma(P)(x, \xi) = (\xi_1 - x_1^\lambda \alpha_1(x, \xi')) \cdots (\xi_1 - x_1^\lambda \alpha_m(x, \xi')) \qquad \lambda = 1, 2, \cdots ,$$

where $\alpha_j\,(1 \leq j \leq m)$ is real analytic in (x, ξ'), $\alpha_j \neq \alpha_k\,(j \neq k)$, and $\alpha_j(x, \xi') \in \mathbb{R}$, if x and ξ' are real. In addition, we impose the Levi condition on the lower order terms:

$$\mathrm{ord} \frac{\partial^q P_l}{\partial x_1^q}(0, x', D') \leq \frac{q + l}{\lambda + 1}(< l)$$

$$1 \leq l \leq m,\ 0 \leq q < \lambda l.$$

Branching of singularities for this kind of operator has been studied by Alinhac, Taniguchi-Tozaki, Amano and others. A typical method is to apply the partial Fourier transform and reduce the problem to that of an ordinary differential operator with an irregular singular point. Here in the present paper, we choose a different approach. Instead of the partial Fourier transform, we use a singular coordinate change and the quantized Legendre transform. Then the problem is reduced to that of an ordinary differential operator of Fuchs type. In PARTS 1 and 2, we treat the cases of $\lambda = 1$, $m = 2, 3$. In these cases, we encounter Gauss and Jordan-Pochhammer hypergeometric equations respectively.

The main idea is due to the theory of a coordinate change of fractional order in [Kat 2]. Here we introduce a rough sketch of his theory. Let

$b_j^{\pm} \subset \{\pm x_1 > 0\}$ be the half-bicharacteristic strip of $\xi_1 = x_1^{\lambda}\alpha_j$, issuing from p. We say that $u \in (\Gamma_{\{\pm x_1 > 0\}}\mathcal{C}_M^P)_p$ is a *j-null* solution if $u = 0$ on $b_j^{\pm}$, where $\mathcal{C}_M^P$ is the kernel sheaf of $P : \mathcal{C}_M \to \mathcal{C}_M$. Let *Null*$(j, \pm)$ be the totality of j-null solutions, that is,

$$Null(j, \pm) = \{u \in (\Gamma_{\{\pm x_1 > 0\}}\mathcal{C}_M^P)_p; u = 0 \,\text{on}\; b_j^{\pm}\}.$$

We explain how to construct j-null solutions. For convenience, we only consider the case of $+$. If u is a solution in $x_1 > 0$, it is mild and its canonical extention $\tilde{u}$ is defined. It satisfies

$$x_1^m P\tilde{u} = 0.$$

Put $t = \frac{1}{\lambda+1}x_1^{\lambda+1}$. We denote by τ the dual variable of t. Then $\xi_1 - x_1^{\lambda}\alpha_j = x_1^{\lambda}(\tau - \alpha_j)$. Since α_j's are mutually distinct (while $x_1^{\lambda}\alpha_j$'s are not), the problem has become easier at least from the geometric point of view. Moreover

$$\begin{aligned}
&\frac{1}{(\lambda+1)^m}x_1^m P(x, D) \\
&= \prod_{j=0}^{m-1}(tD_t - \frac{j}{\lambda+1}) \\
&+ \sum_{l=1}^{m}\sum_{q=0}^{\infty}(\lambda+1)^{\frac{q-\lambda l}{\lambda+1}}\frac{1}{q!}t^{\frac{q+l}{\lambda+1}}\frac{\partial^q P_l}{\partial x_1^q}(0, x', D')\prod_{j=0}^{m-l-1}(tD_t - \frac{j}{\lambda+1}).
\end{aligned}$$

Here assume that there is no contribution by the terms corresponding to $\frac{q+l}{\lambda+1} \neq 0, 1, 2, 3, \ldots$. Now apply the quantized Legendre transform β_n^+ with respect to (t, x'). (See [Kat 1] for the definition.) Then we obtain the following operator:

$$\begin{aligned}
\tilde{P} = &\prod_{j=0}^{m-1}(-D_\zeta\zeta - \frac{j}{\lambda+1}) \\
&+ \sum_{l=1}^{m}\sum_{q=0}^{\infty}\frac{(\lambda+1)^{\frac{q-\lambda l}{\lambda+1}}}{q!}\frac{\partial^q P_l}{\partial x_1^q}(0, x'', x_n + D_\zeta\zeta D_n^{-1}, D') \\
&\qquad \times (-iD_\zeta D_n^{-1})^{\frac{q+l}{\lambda+1}}\prod_{j=0}^{m-l-1}(-D_\zeta\zeta - \frac{j}{\lambda+1}),
\end{aligned}$$

$$x'' = (x_2, \dots, x_{n-1}).$$

We will see that $\tilde{P}$ is an ordinary differential operator of Fuchs type modulo perturbation. Set

$$\begin{aligned} Q = &\prod_{j=0}^{m-1}(-D_\zeta\zeta - \frac{j}{\lambda+1}) \\ &+ \sum_{l=1}^{m}\sum_{q=0}^{\lambda l}\frac{(\lambda+1)^{\frac{q-\lambda l}{\lambda+1}}}{q!}\sigma_0\left(\frac{\partial^q P_l}{\partial x_1^q}(0,x',\xi')\xi_n^{-\frac{q+l}{\lambda+1}}\right) \\ &\qquad \times (-iD_\zeta)^{\frac{q+l}{\lambda+1}}\prod_{j=0}^{m-l-1}(-D_\zeta\zeta - \frac{j}{\lambda+1}). \end{aligned}$$

PROPOSITION 1.

The operator Q is of Fuchs type in ζ if we freeze the parameters (x',ξ'). Its regular singular points are ∞ and $i\alpha_j(0,x',\xi')\xi_n^{-1}$ $(1 \le j \le m)$. The characteristic exponents at ∞ are $1, 1+\frac{1}{\lambda+1}, \dots, 1+\frac{m-1}{\lambda+1}$ and ∞ is a non-logarithmic singularity. The non-negative integers $0,1,2,3,\dots,m-2$ are characteristic exponents at $i\alpha_j(0,x',\xi')\xi_n^{-1}$. If the remaining characteristic exponent is a non-integer, then this is a non-logarithmic singularity.

PROOF.

The coefficient of D_ζ^m is

$$(-\zeta)^m + \sum_{l=1}^{m}\frac{1}{(\lambda l)!}\sigma_0\left(\frac{\partial^{\lambda l}P_l}{\partial x_1^{\lambda l}}(0,x',\xi')\xi_n^{-l}\right)(-i)^l(-\zeta)^{m-l}$$

$$= (-1)^m\left\{\zeta^m + \sum_{l=1}^{m}\frac{1}{(\lambda l)!}\sigma_l\left(\frac{\partial^{\lambda l}P_l}{\partial x_1^{\lambda l}}(0,x',\xi')\right)(\frac{i}{\xi_n})^l\zeta^{m-l}\right\}.$$

We want to determine its zeroes. Recall that

$$\begin{aligned} &\sigma_m\left(\xi_1^m + P_1(x,\xi')\xi_1^{m-1} + \dots + P_m(x,\xi')\right) \\ &= (\xi_1 - x_1^\lambda\alpha_1(x,\xi'))\dots(\xi_1 - x_1^\lambda\alpha_m(x,\xi')). \end{aligned}$$

Hence

$$\begin{aligned}&(x_1^\lambda\tau)^m+\sigma_1(P_1)(x,\xi')(x_1^\lambda\tau)^{m-1}+\cdots+\sigma_m(P_m)(x,\xi')\\ &=(x_1^\lambda\tau-x_1^\lambda\alpha_1)\dots(x_1^\lambda\tau-x_1^\lambda\alpha_m).\end{aligned}$$

and $\sigma_l(P_l)(x,\xi')$ is divisible by $(x_1^\lambda)^l$. Comparing the coefficients of $(x_1^\lambda)^m$ in the above equality, we obtain

$$\begin{aligned}&\tau^m+\sum_{l=1}^m\frac{1}{(\lambda l)!}\frac{\partial^{\lambda l}}{\partial x_1^{\lambda l}}\sigma_l(P_l)(0,x',\xi')\tau^{m-l}\\ &=(\tau-\alpha_1(0,x',\xi'))\dots(\tau-\alpha_m(0,x',\xi')).\end{aligned}$$

Set $\tau=\frac{\xi_n}{i}\zeta$, then

$$\begin{aligned}&(\frac{\xi_n}{i}\zeta)^m+\sum_{l=1}^m\frac{1}{(\lambda l)!}\frac{\partial^{\lambda l}}{\partial x_1^{\lambda l}}\sigma_l(P_l)(0,x',\xi')(\frac{\xi_n}{i}\zeta)^{m-l}\\ &=(\frac{\xi_n}{i}\zeta-\alpha_1(0,x',\xi'))\dots(\frac{\xi_n}{i}\zeta-\alpha_m(0,x',\xi')).\end{aligned}$$

Multiplication by $(\frac{i}{\xi_n})^m$ yields

$$\begin{aligned}&\zeta^m+\sum_{l=1}^m\frac{1}{(\lambda l)!}\frac{\partial^{\lambda l}}{\partial x_1^{\lambda l}}\sigma_l(P_l)(0,x',\xi')(\frac{i}{\xi_n})^l\zeta^{m-l}\\ &=(\zeta-\frac{i}{\xi_n}\alpha_1(0,x',\xi'))\dots(\zeta-\frac{i}{\xi_n}\alpha_m(0,x',\xi')).\end{aligned}$$

Hence we know the location of the singularities of Q. At $i\alpha_j\xi_n^{-1}$, Q has the form

$$Q=(\text{nonzero function})\times\left((\zeta-i\alpha_j\xi_n^{-1})D_\zeta^m+\cdots\right).$$

So the assertion about $i\alpha_j\xi_n^{-1}$ is obvious.

To study Q at ∞, we need the following lemma. □

Lemma 2.

Set

$$L=\prod_{j=0}^{m-1}(-\zeta D_\zeta-\frac{j+\lambda+1}{\lambda+1})+\sum_{l=1}^{m}\sum_{q}a_{lq}D_\zeta^{\frac{q+l}{\lambda+1}}\prod_{j=0}^{m-l-1}(-\zeta D_\zeta-\frac{j+\lambda+1}{\lambda+1}).$$

Here $\sum_q$ *is taken with respect to* $q=0,1,\ldots,\lambda l$, $q\equiv\lambda l \mod \lambda+1$. *Then* ∞ *is a regular singular point of* Q *and its characteristic exponents are* $1, 1+\frac{1}{\lambda+1},\ldots,1+\frac{m-1}{\lambda+1}$.

PROOF. Set $\zeta^{-\frac{1}{\lambda+1}}=x$, $\theta=xD_x$. Then $\zeta^{-1}=x^{\lambda+1}$, $\zeta D_\zeta=-\frac{1}{\lambda+1}\theta$ and $D_\zeta=-\frac{1}{\lambda+1}x^{\lambda+1}\theta$. We have

$$\begin{aligned}L=&\prod_{j=0}^{m-1}(\frac{1}{\lambda+1}\theta-\frac{j+\lambda+1}{\lambda+1})\\&+\sum_{l=1}^{m}\sum_{q}a_{lq}(-\frac{1}{\lambda+1}x^{\lambda+1}\theta)^{\frac{q+l}{\lambda+1}}\prod_{j=0}^{m-l-1}(\frac{1}{\lambda+1}\theta-\frac{j+\lambda+1}{\lambda+1}).\end{aligned}$$

By choosing suitable constants a'_{lq},

$$\begin{aligned}M&:=(\lambda+1)^mL\\&=\prod_{j=0}^{m-1}\{\theta-(j+\lambda+1)\}+\sum_{l=1}^{m}\sum_{q}a'_{lq}(x^{\lambda+1}\theta)^{\frac{q+l}{\lambda+1}}\prod_{j=0}^{m-l-1}\{\theta-(j+\lambda+1)\}.\end{aligned}$$

Then this lemma follows from the one below. □

LEMMA 3.

$$\begin{aligned}&x^{-(\lambda+1)}M(x,D)x^{\lambda+1}\\&=x^mf(x)\{D_x^m+M_1(x)D_x^{m-1}+\cdots+M_m(x)\}\end{aligned}$$

where $f, M_1,\ldots,M_m$ *are holomorphic functions defined in a neighborhood of* $x=0$ *and* $f(0)\neq 0$.

Proof. By using $x^{-1}\theta x = \theta + 1$ repeatedly, we obtain

$$x^{-(\lambda+1)}\theta x^{\lambda+1} = \theta + \lambda + 1.$$

So we have

$$\begin{aligned}
&x^{-(\lambda+1)}M(x,D)x^{\lambda+1} \\
&= \prod_{j=0}^{m-1}(\theta - j) + \sum_{l=1}^{m}\sum_{q} a'_{lq}\{x^{\lambda+1}(\theta+\lambda+1)\}^{\frac{q+l}{\lambda+1}} \prod_{j=0}^{m-l-1}(\theta - j) \\
&= x^m D_x^m + \sum_{l=1}^{m}\sum_{q} a'_{lq}\{x^{\lambda+1}(\theta+\lambda+1)\}^{\frac{q+l}{\lambda+1}} x^{m-l} D_x^{m-l}.
\end{aligned}$$

Obviously, the coefficient of D_x^m has the form $x^m + \mathcal{O}(x^{m+1})$. The proof is finished as soon as we prove

$$x^{-(\lambda+1)} M x^{\lambda+1} \in x^m \mathcal{D}_{x=0}.$$

We have only to prove that

$$\{x^{\lambda+1}(\theta+\lambda+1)\}^{\frac{q+l}{\lambda+1}} x^{m-l} D_x^{m-l} \in x^m \mathcal{D}_{x=0}.$$

This inclusion follows from the sublemma below. □

Sublemma 4. $\{x^{\lambda+1}(\theta+\lambda+1)\}^a x^b \in x^{(\lambda+1)a+b}\mathcal{D}_{x=0}$.

Proof. Induction on a. The case $a = 0$ is obvious. We have

$$\begin{aligned}
&\{x^{\lambda+1}(\theta+\lambda+1)\}x^{(\lambda+1)a+b} \\
&= x^{\lambda+1}[x^{(\lambda+1)a+b}\theta + \{(\lambda+1)a + b + \lambda + 1\}x^{(\lambda+1)a+b}] \\
&\in x^{(\lambda+1)(a+1)+b}\mathcal{D}_{x=0}
\end{aligned}$$

and induction proceeds. We have finally proved Proposition 1. □

In PARTS 1 and 2, the following lemma will be convenient.

LEMMA 5.

$$L:\zeta^{-1}\mathbb{C}\{\zeta^{-\frac{1}{\lambda+1}}\}\to\zeta^{-(\frac{m}{\lambda+1}+1)}\mathbb{C}\{\zeta^{-\frac{1}{\lambda+1}}\}$$

is surjective. Here $\mathbb{C}\{\cdot\}$ *is the set of convergent power series.*

PROOF. We have the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{O}_{x=0} & \xrightarrow{x^{-(\lambda+1)}Mx^{\lambda+1}} & x^m\mathcal{O}_{x=0} & \longrightarrow & 0(\text{exact}) \\
{\scriptstyle x^{\lambda+1}\wr}\downarrow & & \downarrow{\scriptstyle \wr x^{\lambda+1}} & & \\
x^{\lambda+1}\mathcal{O}_{x=0} & \xrightarrow{M} & x^{m+\lambda+1}\mathcal{O}_{x=0} & \longrightarrow & 0(\text{exact})
\end{array}$$

□

We continue the explanation of how to construct a j-null solution. Assume that we are given a microdifferential operator $E_j(\zeta,x',D')$ satisfying the following conditions:
(i) $\tilde{P}(\zeta,x',\partial_\zeta,D')E_j(\zeta,x',D')=0$
(ii) E_j is defined in $(\{\mathrm{Re}\,\zeta>0\}\cup\{\zeta=i\alpha_j(0,x',\xi')\xi_n^{-1}\})\times$ (a conic neighborhood $\subset iT^*N$ of p') where $p'=\rho(p)$, $\rho:N\underset{M}{\times}iT^*M\to iT^*N$, $N=\{x_1=0\}\subset M$.
(iii) At $\zeta=\infty$, E_j has the form

$$E_j=\sum_{k=0}^{\infty}\zeta^{-1-\frac{k}{\lambda+1}}E_{jk}(x',D').$$

(iv) $E_j(\zeta,x',D')f(x')$ is an element of $\mathcal{CO}_+^\infty$ ($\{\mathrm{Re}\,\zeta\geq 0\}\times$ (a conic neighborhood of p')) for any $f(x')\in\mathcal{C}_{N,p'}$. Here $\mathcal{CO}_+^\infty$ is a sheaf introduced in [Kat 1]. We quote from [Kat 1] p.369.

Set $L,L_+,\tilde{L},\tilde{L}_+$ and π,λ as follows.
$L=\mathbb{P}^1\times N=\{(\zeta,x')=(\zeta,x_2,\dots,x_n)\in(\mathbb{C}\cup\{\infty\})\times N\}$,
$L_+=\frac{1}{2}\mathbb{P}^1\times N=\{(\zeta,x')\in L;\ \mathrm{Re}\,\zeta\geq 0 \text{ or } \zeta=\infty\}$,
$\tilde{L}=\mathbb{P}^1\times i\overset{\circ}{T}{}^*N=\{(\zeta,x';i\xi')\in(\mathbb{C}\cup\{\infty\})\times N\times(i\mathbb{R}^{n-1}\backslash\{0\})\}$,
$\tilde{L}_+=\frac{1}{2}\mathbb{P}^1\times i\overset{\circ}{T}{}^*N=\{(\zeta,x',i\xi')\in\tilde{L};\ \mathrm{Re}\,\zeta\geq 0 \text{ or } \zeta=\infty\}$,

$$\begin{array}{ccc}
\tilde{L}_+ & \xrightarrow{\text{inclusion}} & \tilde{L} \\
{\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\
L_+ & \xrightarrow{\text{inclusion}} & L
\end{array}$$

$$\begin{array}{ccc} \mathrm{int}(\tilde{L}_+) & \xrightarrow{\lambda} & \tilde{L}_+ \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ \mathrm{int}(L_+) & \xrightarrow{\lambda} & L_+ \end{array}$$

where $\mathrm{int}(\tilde{L}_+) = \tilde{L}_+ \cap \{\mathrm{Re}\,\zeta > 0,\, \zeta \neq \infty\}$, $\mathrm{int}(L_+) = L_+ \cap \{\mathrm{Re}\,\zeta > 0,\, \zeta \neq \infty\}$. We denote by $\mathcal{BO}$ the sheaf of germs of hyperfunctions on L depending holomorphically on ζ. Then the sheaf $\mathcal{CO}_+^\infty$ is defined on $\tilde{L}_+$ as follows.

$$\mathcal{CO}_+^\infty = \text{Image of } (\pi^{-1}\lambda_*(\mathcal{BO}|_{\mathrm{int}(L_+)}) \to \lambda_*(\mathcal{CO}|_{\mathrm{int}(\tilde{L}_+)})).$$

$\mathcal{CO}_+^\infty$ coincides with $\mathcal{CO}$ on $\mathrm{int}\,(\tilde{L}_+)$, but $\mathcal{CO}_+^\infty \underset{\neq}{\subset} \lambda_*(\mathcal{CO}|_{\mathrm{int}(\tilde{L}_+)})$ on $\partial\tilde{L}_+$. In fact sections of $\mathcal{CO}_+^\infty$ have boundary values. More precisely, the injective sheaf homomorphism is well-defined.

$$\mathcal{CO}_+^\infty|_{S^1\times i\mathring{T}^*N} \xrightarrow{b} \mathcal{H}_F^0(\mathcal{C}_{S^1\times N})|_{S^1\times i\mathring{T}^*N}$$

$$f(\zeta, x') \mapsto f(is+0, x')$$

where $S^1 \times i\mathring{T}^*N = \{(\zeta, x'; i\xi') \in \tilde{L};\, \mathrm{Re}\,\zeta = 0 \text{ or } \zeta = \infty\}$, $S^1 \times N = \{(\zeta, x') \in L = \mathbb{P}^1 \times N;\, \mathrm{Re}\,\zeta = 0 \text{ or } \zeta = \infty\}$, and $S^1 \times i\mathring{T}^*N \simeq \partial\tilde{L}_+ \simeq \{(s, x'; i\sigma, i\xi') \in (\mathbb{R}\cup\{\infty\})\times N\times(i\mathbb{R}^n\backslash\{0\});\, \sigma = 0\}$, $F = \{(s, x'; i\sigma, i\xi');\, \sigma \leq 0\}$.

It is compatible with the trace homomorphism

$$\lambda_*(\mathcal{BO}|_{\mathrm{int}(L_+)})|_{\partial L_+} \to \mathcal{B}_{\mathbb{R}\times N}$$

$$f(\zeta, x') \mapsto f(is+0, x').$$

By using this boundary value morphism and the real quantized Legendre transform, we see that $E_j f$ defines a j-null solution, which we denote by $\tilde{E}_j f(x)$.

Moreover, there exists a nonzero constant C_k such that

$$D_1^k(\tilde{E}_j f)(+0, x') = C_k D_n^{\frac{k}{\lambda+1}} E_{jk}(x', D')f(x').$$

So we have a better understanding of j-null solutions from the viewpoint of boundary value problems.

Although a rigorous proof must resort to [Kat 2], it is possible to give a formal explanation to this formula. Recall the following formulas in [Kat 1] p.358.

$$\beta_n^+(f(x')\delta(t)) = \frac{1}{2\pi} D_n f(x'),$$
$$\beta_n^+ D_t (\beta_n^+)^{-1} = -i\zeta D_n.$$

They imply that microlocally

$$\begin{aligned} \beta_n^+(f(x') \cdot \frac{t_+^{\frac{k}{\lambda+1}}}{\Gamma(\frac{k}{\lambda+1}+1)}) &= \beta_n^+(f(x') D_t^{-(\frac{k}{\lambda+1}+1)} \delta(t)) \\ &= (-i\zeta D_n)^{-(\frac{k}{\lambda+1}+1)} \cdot \frac{1}{2\pi} D_n f(x'),\ k \geq 0. \end{aligned}$$

Since $t = x_1^{\lambda+1}/(\lambda+1)$, we obtain the following correspondence:

$$(x_1^k)_+ \cdot \frac{(\lambda+1)^{-\frac{k}{\lambda+1}}}{\Gamma(\frac{k}{\lambda+1}+1)} \cdot f(x') \mapsto \frac{1}{2\pi}(-i\zeta D_n)^{-(\frac{k}{\lambda+1}+1)} D_n f(x').$$

The k-th trace of the left hand side is

$$\frac{k!(\lambda+1)^{-\frac{k}{\lambda+1}}}{\Gamma(\frac{k}{\lambda+1}+1)} f(x').$$

Now the relationship between the k-th trace and the coefficient of $\zeta^{-(\frac{k}{\lambda+1}+1)}$ is obtained.

In certain cases, this approach is really powerfull. In PARTS 1 and 2, we will present a very detailed analysis of some second and third order operators.

Next, we give a result about a particlular class of m-th order hyperbolic operator.

Let us consider

$$P(x, D) = D_1^m + \sum_{l=1}^m \sum_q b_{ql} x_1^q D_1^{m-l} D_n^{\frac{q+l}{\lambda+1}}.$$

where b_{ql} is a complex constant which vanishes unless $q = 0, 1, 2, \ldots, \lambda l$, $q \equiv \lambda l \bmod \lambda + 1$. Then

$$\sigma(P) = \xi_1^m + \sum_{l=1}^{m} b_{\lambda l, l} x_1^{\lambda l} \xi_1^{m-l} \xi_n^l.$$

Moreover, in this case Q is an ordinary differential operator of Fuchs type without parameter (x', ξ'). That is, Q has the same form as L in Lemma 2. We assume that

$$\sigma(P)(x, \xi) = (\xi_1 - x_1^{\lambda} a_1 \xi_n) \ldots (\xi_1 - x_1^{\lambda} a_m \xi_n),\ a_j \in \mathbb{R}, a_j \neq a_k\ (j \neq k).$$

Set $\alpha_j(x, \xi') = a_j \xi_n$. According to [K-K] and [Kat 1bis], we have the isomorphism

$$\begin{aligned} \text{b.v. } : (\Gamma_{\{x_1>0\}} \mathcal{C}_M^P)_p &\to \overset{m}{\oplus} \mathcal{C}_{N,p'} \\ u &\mapsto \Big(D_1^k u(+0, x') \Big)_{k=0}^{m-1}. \end{aligned}$$

Here $p' = \rho(p)$ and ρ is the projection $N \underset{M}{\times} iT^*M \to iT^*N$, $N = \{x_1 = 0\} \subset M$. Assume that for each j, there is a characteristic exponent $\notin \mathbb{Z}$ at $\zeta = i\alpha_j \xi_n^{-1} = ia_j$. Our result is

Theorem 6. *(i) There is an isomorphism*

$$N_j^+ : \overset{m-1}{\oplus} \mathcal{C}_{N,p'} \to Null(j, +).$$

(ii) The image of $Null(j, +) \subset (\Gamma_{\{x_1>0\}} C_M^P)_p$ under b.v. is characterized by a relationship written in terms of microdifferential operators of fractional order.

Proof. There exists a solution $v_c^j(\zeta)$ to $Qv = 0$ in the right half plane which is not holomorphic at ia_j for $c = m$ and holomorphic there for $c = 1, 2, \ldots, m-1$. We may assume that v_c^j's are linearly independent. According to [Kat 1], $v_c^j(\zeta) f(x'), f \in \mathcal{C}_{N,p'}$, defines an element of $(\Gamma_{\{x_1>0\}} \mathcal{C}_M^P)_p$. Thus we have constructed

$$N_j^{+,c} : \mathcal{C}_{N,p'} \to (\Gamma_{\{x_1>0\}} \mathcal{C}_M^P)_p \quad .$$

Since $N_j^{+,c} f$ is j-null if $1 \leq c \leq m-1$, we can define

$$\begin{aligned} N_j^+ \ : \qquad \overset{m-1}{\oplus} \mathcal{C}_{N,p'} &\to Null(j,+) \\ (f_1,\dots,f_{m-1}) &\mapsto \sum_{c=1}^{m-1} (N_j^{+,c} f)(x). \end{aligned}$$

On the other hand, we can define

$$\begin{aligned} \tilde{N}_j^+ \ : \qquad \overset{m}{\oplus} \mathcal{C}_{N,p'} &\to (\Gamma_{\{x_1>0\}} \mathcal{C}_M^P)_p \\ (f_1,\dots,f_{m-1},f_m) &\mapsto \sum_{c=1}^{m} (N_j^{+,c} f)(x). \end{aligned}$$

Obviously $\tilde{N}_j^+ = (N_j^+, N_j^{+,m})$. We define B_j^+ by using the following commutative diagram

$$\begin{tikzcd} \overset{m}{\oplus}\mathcal{C}_{N,p'} \arrow[r, "\tilde{N}_j^+"] \arrow[d, equal] & (\Gamma_{\{x_1>0\}} \mathcal{C}_M^P)_p \arrow[d, "\wr \text{b.v.}"] \\ \overset{m}{\oplus}\mathcal{C}_{N,p'} \arrow[r, "\tilde{B}_j^+"'] & \overset{m}{\oplus}\mathcal{C}_{N,p'}. \end{tikzcd}$$

We show that $\tilde{B}_j^+$ is an isomorphism. Expand $v_c^j(\zeta)$ at $\zeta = \infty$ into the following form:

$$v_c^j(\zeta) = \sum_{k=0}^{\infty} v_{c,k}^j \zeta^{-1-\frac{k}{\lambda+1}}.$$

Then $\tilde{B}_j^+$ is represented by

$$\begin{aligned} \tilde{B}_j^+ = \operatorname{diag}(C_0, C_1 D_n^{\frac{1}{\lambda+1}}, C_2 D_n^{\frac{2}{\lambda+1}}, \dots, C_{m-1} D_n^{\frac{m-1}{\lambda+1}}) \\ \cdot \begin{pmatrix} v_{1,0}^j & \cdots & v_{m,0}^j \\ v_{1,1}^j & \cdots & v_{m,1}^j \\ \vdots & & \vdots \\ v_{1,m-1}^j & \cdots & v_{m,m-1}^j \end{pmatrix} \end{aligned}$$

The second matrix is invertible because it comes from m linearly independent solutions.(See Lemma 3.) The first one is obviously invertible. So $\tilde{B}_j^+$ is an isomorphism.

By using the commutativity of the diagram, we see that $\tilde{N}_j^+$ is an isomorphism. Now let us prove (i). N_j^+ is obviously injective. Surjectivity follows because $(N_j^{+,m} f_m)(x) \neq 0$ on b_j^+ if $f_m \neq 0$ and $\tilde{N}_j^+$ is surjective. Next, let us prove (ii). We denote by B_j^+ the restriction of $\tilde{B}_j^+$ on $\overset{m-1}{\oplus}\, \mathcal{C}_{N,p'} = \{(f_1, \ldots, f_{m-1}, 0) \in \overset{m}{\oplus} \mathcal{C}_{N,p'}\}$. We have

$$B_j^+ = \mathrm{diag}(C_0, C_1 D_n^{\frac{1}{\lambda+1}}, C_2 D_n^{\frac{2}{\lambda+1}}, \ldots, C_{m-1} D_n^{\frac{m-1}{\lambda+1}})$$
$$\cdot \begin{pmatrix} v_{1,0}^j & \cdots & v_{m-1,0}^j \\ v_{1,1}^j & \cdots & v_{m,1}^j \\ \vdots & & \vdots \\ v_{1,m-1}^j & \cdots & v_{m-1,m-1}^j \end{pmatrix}$$

The rank of the second matrix is $m-1$. Since the components of the two matrices are all commutative, we can use the same argument as in the usual linear algebra. □

Finally, we introduce a notion which will be important in PARTS 1 and 2.

Set

$$Sol(j, \pm) := \{u \in (\Gamma_{\{\pm x_1 > 0\}} \mathcal{C}_M^P)_p\,;\, u = 0 \,\mathrm{on}\, b_k\, (k \neq j)\}$$

An element of it is called a *j-pure* solution (in $\pm x_1 > 0$). In other words, a solution is j-pure if and only if it is k-null for all $k \neq j$. Obviously, a null solution is a sum of pure solutions. The study of pure solutions is more difficult than that of null solutions.

Part 1 Second order case

§1 statement of the theorems

Let

$$P(x, D) = D_1^2 - \frac{1}{i}(\beta_1 + \beta_2) x_1 D_1 D_n - \beta_1 \beta_2 x_1^2 D_n^2 - \frac{2}{i} \gamma D_n$$
$$+ \sum_{l=0}^{\text{finite}} \alpha_{-l}(x_1^2, x', D') x_1^l D_1^l$$

be a microdifferential operator defined in a neighborhood of $p \in \{(x, i\xi) \in iT^*M; x_1 = \xi_1 = 0, \xi_n > 0\}$ such that $\operatorname{ord}\alpha_{-l} \leq -l-1$ and α_{-l} is a polynomial in $t = \frac{1}{2}x_1^2$ and x_n. Here we write $x = (x_1, \ldots, x_n) = (x_1, x') \in \mathbb{R}^n = M$. We also assume that β_1 and β_2 are purely imaginary constants with $\frac{\beta_1}{i} > \frac{\beta_2}{i}$. The principal symbol of P is $\sigma(P) = (\xi_1 - \frac{\beta_1}{i}x_1\xi_n)(\xi_1 - \frac{\beta_2}{i}x_1\xi_n)$. P is microhyperbolic and has double characteristics over the initial surface $N = \{x; x_1 = 0\}$. $\operatorname{Char}(P)$, the (purely imaginary) characteristic variety, is the union of two hypersurfaces $\xi_1 = \pm\frac{\beta_j}{i}x_1\xi_n\,(j = 1, 2)$, which have an non-involutory intersection $\{x_1 = \xi_1 = 0\} \ni p$. Let b_j be the bicharacteristic strip of $\{\xi_1 - \frac{\beta_j}{i}x_1\xi_n = 0\}$ issuing from p, and $b_j^{\pm}$ be its intersection with $\{(x; i\xi dx); \pm x_1 > 0\}$. Since P has simple characteristics in $x_1 \neq 0$, we can apply the propagation theorem in [S-K-K]. That is, if a microfunction u satisfies $Pu = 0$ in $\pm x_1 > 0$, then $b_j^{\pm} \subset \operatorname{supp} u$ or $b_j^{\pm} \cap \operatorname{supp} u = \phi$. Moreover, the general theory on microhyperbolic operators due to [K-K] implies that we have the commutative diagram:

$$
\begin{array}{ccc}
\mathcal{C}^P_{M,p} & \xrightarrow{\sim} & (\Gamma_{\{\pm x_1 > 0\}}\mathcal{C}^P_M)_p \\
\downarrow\wr & & \downarrow\wr \\
\overset{2}{\oplus}\mathcal{C}_{N,p'} & =\!=\!= & \overset{2}{\oplus}\mathcal{C}_{N,p'}
\end{array}
$$

$$
p' = \rho(p), \rho : N \underset{M}{\times} iT^*M \to iT^*N,
$$

where $\mathcal{C}^P_M$ is the kernel sheaf of P, the horizontal arrow is the restriction, and the vertical arrows are the initial and the boundary value morphisms. Set

$$
Sol(j, \pm) = \{u \in (\Gamma_{\{\pm x_1 > 0\}}\mathcal{C}^P_M)_p; u = 0 \,\text{on}\, b_k^{\pm}\,(k \neq j)\}.
$$

An element of $Sol(j, \pm)$ is called a *j-pure solution.* Assume

$$
(*): \quad c \underset{\text{def}}{=} \frac{\frac{3}{2}\beta_1 - 2\beta_2 + \gamma}{\beta_1 - \beta_2} \notin \frac{1}{2}\mathbb{Z} = \{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots\}.
$$

Then we have the following three theorems.

THEOREM A. *(boundary value problem with purity)*
The map

$$
Sol(j, \pm) \to \mathcal{C}_{N,p}
$$

$$u \mapsto u(+0, x')$$

is an isomorphism. Moreover, if $\alpha_{-l} = 0$ for all l, (*) can be replaced by a weaker condition

$$(*)': \quad c \notin (\frac{3}{2} + \mathbb{N}) \cup (2 - \mathbb{N}), \quad \mathbb{N} = \{1, 2, 3, \dots\}.$$

THEOREM B. *(characterization of j-pure solutions by a relationship between their boundary values)*

There exists a microdifferential operator $P_j^{\pm}(x', D')$ of order $\frac{1}{2} - \mathbb{N}_0$ with the following property.(Here $\mathbb{N}_0$ is the set of non-negative integers.) : An element u of $(\Gamma_{\{\pm x_1 > 0\}} \mathcal{C}_M^P)_p$ is j-pure if and only if

$$D_1 u(\pm 0, x') = P_j^{\pm}(x', D')[u(\pm 0, x')].$$

Moreover if $\alpha_{-l} = 0$ for all l, then (*) can be replaced by (*)'.

THEOREM C. *(branching of singularities: See [O] Corollary 3.7)*

Let $u(x)$ be an element of $\mathcal{C}_{M,p}^P$. If u is pure and $u \neq 0$ in $\pm x_1 > 0$, then $b_1^{\mp} \cup b_2^{\mp}$ is contained in $\operatorname{supp} u$. Moreover, if $\alpha_{-l} = 0$ for all l, we have the following two cases not included in (*).

(i) $c \in \frac{5}{2} - \mathbb{N}$

If u is 1-pure and $u \neq 0$ in $\pm x_1 > 0$, then $b_1^{\mp} \cup b_2^{\mp}$ is contained in $\operatorname{supp} u$. If u is 2-pure in $\pm x_1 > 0$, then it is 2-pure also in $\mp x_1 > 0$.

(ii) $c \in 1 + \mathbb{N}$

If u is 1-pure in $\pm x_1 > 0$, then it is 1-pure also in $\mp x_1 > 0$. If u is 2-pure and $u \neq 0$ in $\pm x_1 > 0$, then $b_1^{\mp} \cup b_2^{\mp}$ is contained in $\operatorname{supp} u$.

We can treat another kind of perturbation. The constant γ can be replaced by a microdifferential operator. Let the coordinate of p' be $(\dot{x}_2, \dots, \dot{x}_n; i\dot{\xi}' dx')$ and $\tilde{\gamma} = \tilde{\gamma}(x', D')$ be a microdifferential operator of order ≤ 0 defined near p'. $\tilde{\gamma}$ has an expansion of the form

$$\tilde{\gamma}(x', D') = \sum_{j=0}^{\infty} \gamma_j(x'', D')(x_n - \dot{x}_n)^j$$

$$x'' = (x_2, \dots, x_{n-1}).$$

Let $\hat{\gamma} = \hat{\gamma}(x, D)$ be defined by

$$\hat{\gamma}(x, D) = \sum_{j=0}^{\infty} \gamma_j(x'', D')(\frac{1}{2}x_1 D_1 D_n^{-1} + x_n - \dot{x}_n)^j.$$

It is an operator of order ≤ 0 defined in a neighborhood of p. Set $\mathbb{C} \ni \gamma = \sigma_0(\tilde{\gamma})(p') = \sigma_0(\gamma_0)(p') = \sigma_0(\hat{\gamma})(p)$ and $c = \frac{\frac{3}{2}\beta_1 - 2\beta_2 + \gamma}{\beta_1 - \beta_2}$. Let us consider the operator

$$P(x, D) = D_1^2 - \frac{1}{i}(\beta_1 + \beta_2)x_1 D_1 D_n - \beta_1\beta_2 x_1^2 D_n^2 - \frac{2}{i} D_n \hat{\gamma}(x, D).$$

Purity and the related mappings are defined in the usual way. In this situation,we have the following results.

THEOREM A'.
If c satisfies (*)', then the map

$$Sol(j, \pm) \to \mathcal{C}_{N,p}$$

is an isomorphism.

THEOREM B'.
If c satisfies (*)', then there exists a microdifferential operator $P_j^{\pm}(x', D')$ of order $\in \frac{1}{2} - \mathbb{N}_0$, which has the following property: An element of $(\Gamma_{\{x_1>0\}}\mathcal{C}_M^P)_p$ is j-pure if and only if

$$D_1 u(\pm 0, x') = P_j^{\pm}[u(\pm 0, x')].$$

THEOREM C'. *(See [O] Corollary 3.7)*
Assume $c \notin \frac{1}{2}\mathbb{Z}$. Then we have: If u is pure and $u \neq 0$ in $\pm x_1 > 0$, then $b_1^{\mp} \cup b_2^{\mp}$ is contained in supp$u$.

§2 proof of the unperturbed case

We are going to construct a $\mathbb{C}$-linear mapping

$$\begin{aligned} E_j^{\pm} : \mathcal{C}_{N,p'} &\to Sol(j,\pm) \\ f(x') &\mapsto (E_j^{\pm} f)(x). \end{aligned}$$

Here $p' = \rho(p)$ and $\rho : iT^*M \underset{M}{\times} N \to iT^*N$ is the pull-back of the inclusion map $N \hookrightarrow M$.

2-1 construction of $E_j^{\pm}$

Let us consider

$$P(x,D) = D_1^2 - \frac{1}{i}(\beta_1+\beta_2)x_1D_1D_n - \beta_1\beta_2x_1^2D_n^2 - \frac{2}{i}\gamma D_n,$$

where γ is a complex constant. In $x_1 > 0$, we have $\mathrm{Ker}P = \mathrm{Ker}\frac{1}{4}x_1^2P$. We perform the change of variables $t = \frac{1}{2}x_1^2$ in the latter operator. By using $x_1D_1 = 2tD_t$ and $x_1^2D_1^2 = x_1D_1(x_1D_1 - 1)$, we obtain

$$\begin{cases} \frac{1}{4}x_1^2D_1^2 = tD_t(tD_t - \frac{1}{2}) \\ \frac{1}{4}x_1^3D_1D_n = tD_n \cdot tD_t \\ \frac{1}{4}x_1^4D_n^2 = (tD_n)^2. \end{cases}$$

Hence

$$\frac{1}{4}x_1^2P = tD_t(tD_t - \frac{1}{2}) + i(\beta_1+\beta_2)tD_ntD_t - \beta_1\beta_2(tD_n)^2 - \frac{\gamma}{i}tD_n.$$

Next, we apply the quantized Legendre transform $\mathcal{L}$ with respect to (t,x'). ($\mathcal{L}$ is denoted by β_n^+ in [Kat1]). $\mathcal{L}$ is a quantized contact transformation defined by

$$\begin{cases} \mathcal{L}D_t\mathcal{L}^{-1} = -i\zeta D_n, \quad \mathcal{L}D_k\mathcal{L}^{-1} = D_k \, (k \neq 1) \\ \mathcal{L}t\mathcal{L}^{-1} = -iD_\zeta D_n^{-1}, \mathcal{L}x_k\mathcal{L}^{-1} = x_k \, (k \neq 1, n) \\ \mathcal{L}x_n\mathcal{L}^{-1} = x_n + D_\zeta\zeta D_n^{-1}. \end{cases}$$

In particular, we have

$$\begin{cases} \mathcal{L}tD_t\mathcal{L}^{-1} = -D_\zeta\zeta = -(\zeta D_\zeta + 1) \\ \mathcal{L}tD_n\mathcal{L}^{-1} = -iD_\zeta. \end{cases}$$

Here ζ is the dual variable of (the complexification of) t.

Then the Legendre image, denoted by $Q(\zeta, D_\zeta)$, is

$$\begin{aligned} Q &= (-\zeta D_\zeta - 1)(-\zeta D_\zeta - \frac{3}{2}) + i(\beta_1 + \beta_2) \cdot iD_\zeta(\zeta D_\zeta + 1) \\ &\quad - \beta_1\beta_2(-D_\zeta^2) - \frac{\gamma}{i}(-iD_\zeta) \\ &= (\zeta^2 D_\zeta^2 + \frac{7}{2}\zeta D_\zeta + \frac{3}{2}) - (\beta_1 + \beta_2)(\zeta D_\zeta^2 + 2D_\zeta) + \beta_1\beta_2 D_\zeta^2 + \gamma D_\zeta \\ &= (\zeta - \beta_1)(\zeta - \beta_2)D_\zeta^2 + [\frac{7}{2}\zeta - 2(\beta_1 + \beta_2) + \gamma]D_\zeta + \frac{3}{2}. \end{aligned}$$

$-Q$ is transformed into Gauss hypergeometric operator $G = G(\frac{3}{2}, 1, c; z, D)$ if we introduce a new independent variable z by $\zeta = (-\beta_1+\beta_2)z+\beta_1$, where

$$G = G(\frac{3}{2}, 1, c; z, D) = z(1-z)D^2 + [c - (\frac{3}{2} + 1 + 1)z]D - \frac{3}{2} \cdot 1, \quad D = D_z$$

$$c = \frac{\frac{3}{2}\beta_1 - 2\beta_2 + \gamma}{\beta_1 - \beta_2}.$$

Its Riemann scheme is

$$\left\{ \begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & 1 & ; z \\ 1-c & c-\frac{5}{2} & \frac{3}{2} & \end{matrix} \right\}$$

LEMMA 1.

Let $u(z)$ be a solution to $Gu = 0$. If it is holomorphic both at $z = 0, 1$, then it vanishes identically.

PROOF. u is analytically continued to the entire complex plane, and its exponent at $z = \infty$ is 1. Apply Liouville's theorem. $\square$

We want to find a solution $v_j(z)\,(j = 1, 2\,\text{respectively})$ in the upper half plane, not vanishing identically, such that v_j is holomorphic at $z = 1, 0$ respectively. (Hence singular at $z = 0, 1$ respectively). Moreover its expansion coefficients at $z = \infty$ will be necessary in the next section. This is a kind of connection problem. It is solved by using well-known formulas. We quote from [I-K-S-Y]. Assume that

$$c \notin (\frac{3}{2} + \mathbb{N}) \cup (2 - \mathbb{N}), \mathbb{N} = \{1, 2, 3, \dots\},$$

and, in the upper half plane, set $0 < \arg z < \pi$.

By choosing six different paths in the Euler integral representaion, we obtain six solutions $F_1(z), \dots, F_6(z)$ that have the following properties:

$$F_1 + F_2 + F_3 = 0, \tag{1}$$

$$F_1 - F_4 + F_5 = 0, \tag{2}$$

$$\varepsilon(-c)F_2 - F_5 + F_6 = 0, \tag{3}$$

$$-\varepsilon(-c)F_3 + F_4 + F_6 = 0, \tag{4}$$

$$F_1(z) = \frac{\Gamma(\frac{5}{2} - c)\Gamma(c-1)}{\Gamma(\frac{3}{2})} z^{-\frac{3}{2}} F(\frac{3}{2}, \frac{5}{2} - c; \frac{3}{2}; \frac{1}{z}), \tag{5}$$

$$F_2(z) \quad \text{is holomorphic at} \quad z = 0, \tag{6}$$

$$F_3(z) \quad \text{is holomorphic at} \quad z = 1, \tag{7}$$

$$F_6(z) = 2i\varepsilon(-\frac{1}{2}c)F(1, 2 - c; \frac{1}{2}; \frac{1}{z}), \tag{8}$$

where $\varepsilon(\cdot) = \exp(2\pi i \cdot)$ and F is the Gauss hypergeometric series. With the notation above, we define

$$v_1(z) = 2\varepsilon(-c)F_3(z), \quad v_2(z) = -2\varepsilon(-c)F_2(z).$$

Let us calculate their expansion coefficients at $z = \infty$. From $(1), \dots, (4)$, we obtain

$$(1 - \varepsilon(-c))F_1 - 2\varepsilon(-c)F_3 + 2F_6 = 0$$

and

$$(1 + \varepsilon(-c))F_1 + 2\varepsilon(-c)F_2 + 2F_6 = 0.$$

Hence

$$v_1(z) = (1 - \varepsilon(-c))F_1(z) + 2F_6(z),$$

$$v_2(z) = (1 + \varepsilon(-c))F_1(z) + 2F_6(z).$$

When we expand $v_j(z)$ into the form

$$\text{(9)} \qquad v_j(z) = \sum_{n=0}^{\infty} v_{j,-1-\frac{n}{2}} z^{-1-\frac{n}{2}} \quad \text{at} \quad z = \infty,$$

$$(0 < \arg z < \pi)$$

we see easily that

$$\text{(10)} \qquad \begin{pmatrix} v_{1,-1} & v_{2,-1} \\ v_{1,-\frac{3}{2}} & v_{2,-\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} 4i\varepsilon(-\frac{1}{2}c) & 0 \\ 0 & \frac{\Gamma(\frac{5}{2}-c)\Gamma(c-1)}{\Gamma(\frac{3}{2})} \end{pmatrix} \cdot \begin{pmatrix} (1 & 1) \\ 1 - \varepsilon(-c) & 1 + \varepsilon(-c) \end{pmatrix}.$$

Now we come back to the ζ-plane. Since there is a correspondence

$$\zeta = \beta_1, \beta_2, \infty \longleftrightarrow z = 0, 1, \infty,$$

$$\operatorname{Re}\zeta > 0 \longleftrightarrow \operatorname{Im} z > 0 \quad (0 < \arg z < \pi),$$

$v_j(z)\,(j = 1, 2)$ defines a solution to Q in the right half plane $\subset \mathbb{C}_\zeta$ which is singular at β_j and holomorphic at β_{3-j}. We denote it by $V_j(\zeta)$. Let us calculate its expansion coefficients at $\zeta = \infty$, $\operatorname{Re}\zeta > 0$. $z = \frac{\zeta - \beta_1}{-\beta_1 + \beta_2}$ leads to

$$z^{-1} = \frac{-\beta_1 + \beta_2}{\zeta} \sum_{k=0}^{\infty} (\frac{\beta_1}{\zeta})^k$$

and

$$z^{-\frac{3}{2}} = (-\beta_1 + \beta_2)^{\frac{3}{2}} \zeta^{-\frac{3}{2}} \left(1 + \frac{3}{2}\frac{\beta_1}{\zeta} + O(\zeta^{-2})\right).$$

Here

$$-\frac{\pi}{2} < \arg \zeta < \frac{\pi}{2}, \quad \arg(-\beta_1 + \beta_2) = -\pi/2.$$

When we expand $V_j(\zeta)$ into the form

$$V_j(\zeta) = \sum_{n=0}^{\infty} V_{j,-1-\frac{n}{2}} \zeta^{-1-\frac{n}{2}} \,\text{at}\, \zeta = \infty, \,\mathrm{Re}\, \zeta > 0 \quad (-\frac{\pi}{2} < \arg \zeta < \frac{\pi}{2}),$$

we have

$$\begin{pmatrix} V_{j,-1} \\ V_{j,-\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} -\beta_1 + \beta_2 & 0 \\ 0 & (-\beta_1 + \beta_2)^{\frac{3}{2}} \end{pmatrix} \begin{pmatrix} v_{j,-1} \\ v_{j,-\frac{3}{2}} \end{pmatrix}$$

Combining this and (10), we obtain

$$\begin{aligned} &\begin{pmatrix} V_{1,-1} & V_{2,-1} \\ V_{1,-\frac{3}{2}} & V_{2,-\frac{3}{2}} \end{pmatrix} \\ &= \begin{pmatrix} 4i\varepsilon(-\frac{1}{2})(-\beta_1 + \beta_2) & 0 \\ 0 & \frac{\Gamma(\frac{5}{2}-c)\Gamma(c-\frac{1}{2})}{\Gamma(\frac{3}{2})}(-\beta_1 + \beta_2)^{\frac{3}{2}} \end{pmatrix} \\ &\cdot \begin{pmatrix} 1 & 1 \\ 1 - \varepsilon(-c) & 1 + \varepsilon(-c) \end{pmatrix} \end{aligned} \tag{11}$$

Let us define E_j^+. According to [Kat1], $V_j(\zeta)f(x') \in \mathcal{CO}_+^\infty$ gives an element of *Sol* $(j,+)$, which we denote by $(E_j^+ f)(x)$. To define E_j^-, we change the sign of the time variable by introducing $\bar{x}_1 = -x_1$. Since $D_1 = -D_{\bar{x}_1}$, we have

$$P(x, D) = P(-\bar{x}_1, x', -D_{\bar{x}_1}, D') = P(\bar{x}_1, x', D_{\bar{x}_1}, D')$$

That is, P does not change its form. We can apply the same argument as above and the definition of E_j^- is obvious. Note that $\xi_1 - \beta_j x_1 \xi_n = -(\bar{\xi}_1 - \beta_j \bar{x}_1 \xi_n)$, where $\bar{\xi}_1$ is the dual of $\bar{x}_1$.

2-2 boundary values of $E_j^\pm$

Let $f(x'), g(x') \in \mathcal{C}_{N,p'}$. According to [Kat2], we have

$$(12)\qquad \begin{pmatrix} (E_j^+ f)(+0,x') \\ (D_1 E_j^+ f)(+0,x') \end{pmatrix} = \begin{pmatrix} 2\pi V_{j,-1} f(x') \\ 2\sqrt{2\pi}(\frac{D_n}{i})^{\frac{1}{2}} V_{j,-\frac{3}{2}} f(x') \end{pmatrix} (j=1,2)$$

Denote by L^+ the morphism

$$L^+ : \overset{2}{\oplus}\mathcal{C}_{N,p'} \to \overset{2}{\oplus}\mathcal{C}_{N,p'}$$

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} (E_1^+ f + E_2^+ g)(+0,x') \\ D_1(E_1^+ f + E_2^+ g)(+0,x') \end{pmatrix}.$$

Combination of (11) and (12) yields

$$L^+ \begin{pmatrix} f \\ g \end{pmatrix} = A \begin{pmatrix} 1 & 1 \\ 1-\varepsilon(-c) & 1+\varepsilon(-c) \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

$$A = \begin{pmatrix} 2\pi & 0 \\ 0 & 2\sqrt{2\pi}(\frac{D_n}{i})^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} 4i\varepsilon(-\frac{1}{2})(-\beta_1+\beta_2) & 0 \\ 0 & \frac{\Gamma(\frac{5}{2}-c)\Gamma(c-\frac{1}{2})}{\Gamma(\frac{3}{2})}(-\beta_1+\beta_2)^{\frac{3}{2}} \end{pmatrix}.$$

We may forget the explicit form of A. All we'll need is the fact that $A = \operatorname{diag}(A_1, A_2 D_n^{\frac{1}{2}})$ where A_1 and A_2 are nonzero constants. In particular, L^+ is an isomorphism. Next, denote by L^- the morphism

$$L^- : \overset{2}{\oplus}\mathcal{C}_{N,p'} \to \overset{2}{\oplus}\mathcal{C}_{N,p'}.$$

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} (E_1^- f + E_2^- g)(-0,x') \\ D_1(E_1^- f + E_2^- g)(-0,x') \end{pmatrix}.$$

Obviously,

$$\begin{pmatrix} E_1^- f + E_2^- g \\ D_{\bar{x}_1}(E_1^- f + E_2^- g)(\bar{x}_1, x')|_{\bar{x}_1 \to +0} \end{pmatrix}$$

is represented by the same matrix as L^+. Since $D_{\bar{x}_1} = -D_1$, L^- has a slightly different representation:

$$L^- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} L^+ = A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1-\varepsilon(-c) & 1+\varepsilon(-c) \end{pmatrix}.$$

L^- is an isomorphism, of course.

2-3 end of the proofs (of the unperturbed case)

We have the following commutative diagram:

$$
\text{(CD1)} \qquad
\begin{array}{ccc}
(\Gamma_{\{\pm x_1>0\}}\mathcal{C}_M^P)_p = \oplus_{j=1}^2 Sol(j,\pm) & \xleftarrow{E^\pm} & \overset{2}{\oplus}\mathcal{C}_{N,p'} \\
\text{b.v.}\wr\downarrow & & \downarrow\wr L^\pm \\
\overset{2}{\oplus}\mathcal{C}_{N,p'} & = & \overset{2}{\oplus}\mathcal{C}_{N,p'}
\end{array}
$$

where the horizontal arrow is the map

$$E^\pm = \oplus_{j=1}^2 E_j^\pm : {}^t(f(x'), g(x')) \mapsto (E_1^\pm f)(x) + (E_2^\pm g)(x).$$

The first vertical arrow is

$$u(x) \mapsto {}^t(u(\pm 0, x'), D_1 u(\pm 0, x')),$$

and it is well-known to be an isomorphism. Therefore $E^\pm$ is an isomorphism. Since $E^\pm = \oplus_{j=1}^2 E_j^\pm$, each $E_j^\pm$ is an isomorphism. In this way, we have arrived at the important identification:

$$E_1^\pm : \mathcal{C}_{N,p'} \oplus 0 \xrightarrow{\sim} Sol(1,\pm),$$

$$E_2^\pm : 0 \oplus \mathcal{C}_{N,p'} \xrightarrow{\sim} Sol(2,\pm).$$

From (CD1), we obtain the following commutative diagram:

$$
\begin{array}{ccc}
Sol(1,\pm) & \xleftarrow{\sim} & \mathcal{C}_{N,p'} \\
\downarrow & & \downarrow \\
\mathcal{C}_{N,p'} & = & \mathcal{C}_{N,p'}
\end{array}
$$

where the first horizontal arrow is the identification above, the left vertical arrow is

$$u \mapsto u(\pm 0, x'),$$

and the right vertical arrow is $f(x') \mapsto A_1 f(x')$. This implies the latter part of Theorem A.

Next, we prove Theorem B. We want to characterize the image of $Sol(j,\pm)$ under b.v. Because of (CD1), it is $L^\pm(\mathcal{C}_{N,p'} \oplus 0)$ if $j=1$ and $L^\pm(0 \oplus \mathcal{C}_{N,p'})$ if $j=2$. Here

$$L^\pm \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} A \begin{pmatrix} 1 \\ 1-\varepsilon(-c) \end{pmatrix} f$$

and

$$L^\pm \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} A \begin{pmatrix} 1 \\ 1+\varepsilon(-c) \end{pmatrix} g.$$

Theorem B follows immediately.

Finally, let us prove Theorem C. We have the isomorphisms below:

$$\begin{tikzcd}
\oplus_{j=1}^2 Sol(j,-) \arrow[r, "b.v.\sim"] & \overset{2}{\oplus}\mathcal{C}_{N,p'} & \oplus_{j=1}^2 Sol(j,+) \arrow[l, "b.v.\sim"'] \\
\overset{2}{\oplus}\mathcal{C}_{N,p'} \arrow[u, "E^-\wr"] \arrow[r, "L^-\sim"] & \overset{2}{\oplus}\mathcal{C}_{N,p'} \arrow[u, equal] & \overset{2}{\oplus}\mathcal{C}_{N,p'} \arrow[u, "\wr E^+"'] \arrow[l, "L^+\sim"']
\end{tikzcd}$$

where $E^\pm$ is the direct sum of the identification maps $E_j^\pm, \quad j=1,2$. We set $B=(L^-)^{-1}L^+$ and call it *the branching matrix.* It is easy to see that

$$B = \begin{pmatrix} 1 & 1 \\ 1-\varepsilon(-c) & 1+\varepsilon(-c) \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1-\varepsilon(-c) & 1+\varepsilon(-c) \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon(c) & 1+\varepsilon(c) \\ 1-\varepsilon(c) & \varepsilon(c) \end{pmatrix}.$$

The identification above enables us to reduce the problem of branching to the study of the branching matrix B. We have only to know when a certain component of B is (not) zero. The proof of Theorem C is now complete.

A nonzero constant is an elliptic microdifferential operator of order 0. Even if it is perturbed in the lower order terms, it remains elliptic. This observation will be important in the following section.

§3 proof of the perturbed case

In this section we assume that

$$c \notin \frac{1}{2}\mathbb{Z} = \{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2} \dots\}.$$

We only explain the construction of E_2^+. The remaining three maps are constructed in the same way.

3-1 right inverse

We make some preparation for the symbol calculus in the next subsection. $G(z, D_z)$ is an ordinary differential operator of Fuchs type with three regular singular points $z = 0, 1, \infty$. Its Riemann scheme is

$$\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & 1 & z \\ 1-c & c-\frac{5}{2} & \frac{3}{2} & \end{matrix}\right\}$$

and no logarithmic term appears. The exponent of the Wronskian is $-c$ at $z = 0$ and $c - \frac{7}{2}$ at $z = 1$. Let $\Omega \subset \mathbb{C}_z$ be a domain as in Figure 1.

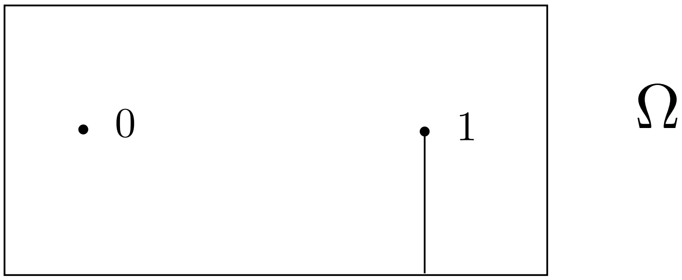

Figure 1

$G(z, D)$ induces a linear mapping:

$$G : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega)$$

We are going to construct a right inverse G^{-1} by using the method of variation of parameters. Let F_1, F_2 be the series solutions of exponent $0, 1-c$ respectively defined near $z = 0$. Let W be their Wronskian.

$$W = \det\begin{pmatrix} F_1 & F_2 \\ F_1' & F_2' \end{pmatrix}.$$

It is easy to see that

$$f(z) \mapsto (G^{-1}f)(z) = -F_1(z)\int_0^z \frac{F_2(y)}{y(1-y)W(y)} f(y)dy + F_2(z)\int_0^z \frac{F_1(y)}{y(1-y)W(y)} f(y)dy$$

gives a right inverse of G. Here the integrals are taken in the sense of Riemann-Liouville. We want to obtain some estimate on the integral operator G^{-1}. We say that a function f has *exponent (p, q)* at $z = a$ if f has the form

$$f(z) = z^p f_1(z) + z^q f_2(z), \quad (p - q \notin \mathbb{Z})$$

where f_1 and f_2 are holomorphic at $z = a$ and $f_1(a), f_2(a) \neq 0$. Set, for δ, $0 < \delta \ll 1$,

$$K_\delta = \{z \in \Omega; \mathrm{dist}(z, \partial\Omega) \leq \delta\}$$

PROPOSITION 2.
There exist positive constants $\tilde{c}$ and C, not depending on δ, such that for all $f \in \mathcal{O}(\Omega)$, we have

$$\sup_{K_\delta}|G^{-1}f| \leq C\delta^{-\tilde{c}} \sup_{K_\delta}|f|$$

In the proof, we see that $\tilde{c} = [|\mathrm{Re}\, c - \frac{5}{2}|] + 1$, where $[a]$ is the smallest integer not exceeding a.

PROOF. We consider the second term in the definition of G^{-1}. (The first term is easier to deal with.) Let us introduce the following notation:

$$(Jf)(z) = F_2(z)\int_0^z \frac{F_1(y)}{y(1-y)W(y)} f(y)dy.$$

We will deduce an estimate on J in several steps. □

LEMMA 3. *Fix a sufficiently small constant $R > 0$. Then, there exists a constant $C_1 > 0$, independent of δ, such that*

$$\sup_{|z| \leq \frac{R}{2}} \left| z^{1-c} \int_0^z y^{1-c} g(y)dy \right| \leq C_1 \sup_{K_\delta} |g(y)|$$

for all $g(z) \in \mathcal{O}(\Omega)$.

PROOF. We may assume that $\{|z| \leq 2R\} \subset \mathrm{int} K_\delta$ for any δ, $0 < \delta \ll 1$. So $g(z)$ has a Taylor expansion

$$g(z) = \sum_{n=0}^{\infty} g_n z^n \quad \text{in} \quad \{|z| < 2R\}$$

Since Riemann-Liouville integral can be carried out term by term ([I-K-S-Y]), we have

$$\int_0^z y^{c-1} g(y) dy = z^c \sum_{n=0}^{\infty} \frac{1}{c+n} g_n z^n.$$

$$z^{1-c} \int_0^z y^{c-1} g(y) dy = \sum_{n=0}^{\infty} \frac{1}{c+n} g_n z^{n+1}, \tag{13}$$

By the way, the assumption $c \notin \frac{1}{2}\mathbb{Z}$ implies that there exists a constant C_c such that

$$\left| \frac{1}{c+n} \right| \leq C_c \quad \text{for all} \quad n = 0, 1, 2, \ldots.$$

Moreover, Cauchy's estimate shows that

$$|g_n| \leq \frac{1}{R^n} \sup_{|y|=R} |g(y)|.$$

Therefore, in view of (13), we obtain, in $|z| < R$,

$$\begin{aligned} \left| z^{1-c} \int_0^z z^{c-1} g(y) dy \right| &\leq \sum_{n=o}^{\infty} C_c \cdot \frac{1}{R^n} \sup_{|y|=R} |g(y)| \cdot |z|^{n+1} \\ &= C_c \frac{|z|}{1 - \frac{|z|}{R}} \sup_{|y|=R} |g(y)|. \end{aligned}$$

This leads to the lemma because we have

$$\sup_{|z| \leq \frac{R}{2}} \frac{|z|}{1 - \frac{|z|}{R}} = R. \ \Box$$

LEMMA 4.
There exists a constant $C_2 > 0$, independent of δ, such that

$$\sup_{K_\delta}\left|z^{1-c}\int_0^z y^{c-1}g(y)dy\right| \leq C_2 \sup_{K_\delta}|g(y)|$$

for all $g(z) \in \mathcal{O}(\Omega)$.

PROOF. What remains is the estimate for $z \in \{|z| > \frac{R}{2}\} \cap K_\delta$. We write the function in question as the sum of two terms.

$$z^{1-c}\int_0^z = z^{1-c}\int_0^{\frac{Rz}{2|z|}} + z^{1-c}\int_{\frac{Rz}{2|z|}}^z .$$

We can apply Lemma 3 to the first term. In fact,

$$z^{1-c}\int_0^{\frac{Rz}{2|z|}} = (\frac{2|z|}{R})^{1-c} \cdot (\frac{Rz}{2|z|})^{1-c}\int_0^{\frac{Rz}{2|z|}},$$

where the first factor is bounded in $\Omega \cap \{|z| > R/2\}$ and the second factor is estimated by using Lemma 1. Let us consider the second term. We may assume that the length of the path of integration $\subset K_\delta$ from $\frac{Rz}{2|z|}$ to z is estimated by a constant $C^1_{R,\Omega} > 0$ independent of δ. Additionally, in $\{|z| > \frac{R}{2}\} \cap K_\delta$, z^{1-c} is estimated by a constant $C^2_{R,\Omega} > 0$, independent of δ. Therefore we have

$$\left|z^{1-c}\int_{\frac{Rz}{2|z|}}^z\right| \leq C^1_{R,\Omega}C^2_{R,\Omega}\sup_{K_\delta}|g(y)|$$

in $\{|z| > \frac{R}{2}\} \cap K_\delta$. □

LEMMA 5. *Put $c' = |Re\, c - \frac{5}{2}| \geq 0$. There exists a constant $C_J > 0$, independent of δ, such that for all $f \in \mathcal{O}(\Omega)$, we have*

$$\sup_{K_\delta}|Jf| \leq C_J\delta^{-c'}\sup_{K_\delta}|f|.$$

PROOF.
We have

$$\frac{F_1(y)}{y(1-y)W(y)} = y^{c-1}G(y),$$

where $G(y)$ is holomorphic in Ω, or more precisely, in the universal covering space of $\mathbb{C}\backslash\{1\}$, and has exponent $(-c+\frac{5}{2}, 0)$ at $y=1$. Obviously,

$$\frac{F_1(y)}{y(1-y)W(y)} f(y) = y^{c-1} \times G(y)f(y)$$

and we consider Gf as g in Lemma 2. Since

$$\sup_{K_\delta}|Gf| \leq \sup_{K_\delta}|G|\sup_{K_\delta}|f| \leq C_G\delta^{\min(-\mathrm{Re}\, c+\frac{5}{2},0)}\sup_{K_\delta}|f|,$$

Lemma 4 implies that

$$\sup_{K_\delta}\left|z^{1-c}\int_0^z y^{c-1}G(y)f(y)dy\right| \leq C_2C_G\delta^{\min(-\mathrm{Re}\, c+\frac{5}{2},0)}\sup_{K_\delta}|f| \tag{14}$$

On the other hand, $F_2(z)/z^{1-c}$ is holomorphic in Ω and has exponent $(0, c-\frac{5}{2})$ at $z=1$. So, there exists a constant $C_3 > 0$ independent of δ such that

$$\sup_{K_\delta}\left|\frac{F_2(z)}{z^{1-c}}\right| \leq C_3\delta^{\min(\mathrm{Re}\, c-\frac{5}{2},0)}. \tag{15}$$

Combination of (14) and (15) yields the lemma, because

$$\min(-\mathrm{Re}\, c+\frac{5}{2},0)+\min(\mathrm{Re}\, c-\frac{5}{2},0) = \min(\pm(\mathrm{Re}\, c-\frac{5}{2})) = -c'. \ \square$$

PROOF OF PROPOSITION 2 CONTINUED. The first term in the definition of G^{-1} satisfies the same estimate as Lemma 5, with a larger C, if necessary. Then the proposition follows immediately, because $0 \geq -c' > -\tilde{c}$. $\square$

3-2 successive approximation

Let us consider

$$P(x,D) = D_1^2 - \frac{1}{i}(\beta_1+\beta_2)x_1D_1D_n - \beta_1\beta_2x_1^2D_n^2 - \frac{2}{i}\gamma D_n + \sum_{l=0}^{\text{finite}} \alpha_{-l}(x_1^2,x',D')x_1^lD_1^l.$$

As in §2, we put $t=\frac{1}{2}x_1^2$ in $\frac{1}{4}x_1^2P(x,D)$ and use the quantized Legendre transform $\mathcal{L}$. Let us calculate the contribution of the perturbation term

$$P'(x,D) = \sum_{l=0}^{\text{finite}} \alpha_{-l}(x_1^2,x',D')x_1^lD_1^l.$$

First, we consider $x_1^2\cdot x_1^lD_1^l$. It is easy to see that

$$x_1^2\cdot x_1^lD_1^l = 2t\cdot 2tD_t(2tD_t-1)\dots(2tD_t-l+1).$$

LEMMA 6. *Let $W(\mathbb{C})$ be the Weyl algebra of a variable t and V be the subalgebra generated by t and $\vartheta = tD_t$. Then we have $t^jV \subset Vt^j \subset W(\mathbb{C})t^j$. $(j=0,1,2,\dots)$*

PROOF. Obviously $[t,\vartheta]=-t$, so $t\vartheta\in Vt$. Hence the case $j=1$ is proved. The remaining cases are proved by induction. □

This lemma ($j=1$) implies that

$$\frac{1}{4}x_1^2\cdot x_1^lD_1^l \in \mathcal{D}_{t,w'}t.$$

Therefore $\frac{1}{4}x_1^2P'(x,D)$ belongs to $\mathcal{E}_{t,x'}t\cap\mathcal{E}_{t,x'}(-1)$ and is a polynomial in t and x_n by the assumption on α_{-l}. Its image under $\mathcal{L}$, denoted by $Q'(\zeta,x',D_\zeta,D')$, belongs to $\mathcal{E}_{\zeta,x'}D_\zeta\cap\mathcal{E}_{\zeta,x'}(-1)$ and is a polynomial in D_ζ and ζ. More precisely, it has the form

$$Q'(\zeta,x',D_\zeta,D') = \sum_{m=1}^{\text{finite}}\sum_{j=0}^{m-1}\tilde{\alpha}_{m,j}(x',D')\zeta^jD_\zeta^m \in \mathcal{E}(-1).$$

where $\operatorname{ord}\tilde{\alpha}_{m,j} \leq -m-1$. If we write it in terms of the other complex variable $z = (\zeta - \beta_1)/(-\beta_1 + \beta_2)$, Q' is transformed into

$$G'(z, x', D_z, D') = \sum_{m=1}^{\bar{m}} \sum_{j=0}^{m-1} \alpha_{m,j}(x', D') z^j D_z^m \in \mathcal{E}(-1).$$

Here $\bar{m}$ is a positive integer and $\alpha_{m,j}$ is a microdifferential operator defined in a neighborhood of $p' = (x'; i\xi' dx') = \rho(p) \in iT^*N$, $N = \mathbb{R}^{n-1}$. Thus $-\frac{1}{4}x_1^2 P$ is transformed into $G - G'$, where G is the Gauss hypergeometric operator in §2. We will construct a microdifferential operator $E(z, x', D')$ of order 0 that satisfies

$$(G - G')(z, x', \partial_z, D')E = 0, \quad \partial_z = [D_z, \cdot].$$

In addition, we require that E should be defined in (a neighborhood $\subset \mathbb{C}_z$ of $\{z; \operatorname{Im} z \geq 0, z \neq 1\}$) $\times$ (a conic neighborhood $\subset iT^*N$ of $p' = \rho(p)$) and that

$$E \in z^{-1}\mathcal{E}(0) + z^{-3/2}\mathcal{E}(0) \quad \text{at} \quad z = \infty$$

where $\mathcal{E}(0)$ is regarded as a sheaf on $T^*(\mathbb{P}^1 \times \mathbb{C}^{n-1})$. There is another requirement to be explained in 3-3. Put

$$\begin{cases} E_0(z) = v_2(z), \quad \text{where } v_2 \text{ is defined in } 2-1, \\ E_{k+1}(z, x', D') = G^{-1}[G'(z, x', \partial_z, D')E_k(z, x', D')], \quad k = 0, 1, 2, \dots. \end{cases}$$

Here G^{-1} and G' are mappings on $\mathcal{O}_z \otimes_{\mathbb{C}} \mathcal{E}_{x'}$, to which E_k belongs. We want to show that $E = \sum_{k \geq 0} E_k$ converges in $\mathcal{E}_{x'}\mathcal{O}_z$. We have only to prove it when z belongs to a fixed Ω, where Ω is as in 3-1. Obviously

$$\begin{aligned} E_k(z, x', D') &= (G^{-1}G')^k E_0(z) \\ &= \sum_{(m_k, \dots, m_1)} \sum_{(j_k, \dots, j_1)} (G^{-1}\alpha_{m_k, j_k} z^{j_k} \partial_z^{m_k}) \\ &\qquad \dots (G^{-1}\alpha_{m_1, j_1} z^{j_1} \partial_z^{m_1}) E_0(z), \end{aligned}$$

where $(m_k, \dots, m_1)$ runs through the set $\{1, \dots, \bar{m}\} \times \dots \times \{1, \dots, \bar{m}\}$ (k times) and $(j_k, \dots, j_1)$ through $\{0, \dots, m_k - 1\} \times \dots \times \{0, \dots, m_1 - 1\}$.

Therefore

$$\begin{aligned}E_k(z,x',D') = &\sum_{(m_k,\dots,m_1)}\sum_{(j_k,\dots,j_1)} \alpha_{m_k,j_k}\cdots\alpha_{m_1,j_1}\\ &\otimes (G^{-1}z^{j_k}\partial_z^{m_k})\cdots(G^{-1}z^{j_1}\partial_z^{m_1})E_0\\ &\in \mathcal{E}_{x'}(-(m_k+\cdots+m_1)-k)\otimes_{\mathbb{C}}\mathcal{O}_z\subset\mathcal{E}_{z,x'}(0).\end{aligned}$$

We will show the convergence of $\sum E_k$ for $z\in\Omega$ in three steps. They are:
STEP 1 estimate of $(G^{-1}z^{j_k}\partial_z^{m_k})\cdots(G^{-1}z^{j_1}\partial_z^{m_1})E_0(z)$,
STEP 2 estimate of $\alpha_{m_k,j_k}\cdots\alpha_{m_1,j_1}$,
STEP 3 convergence of $\sum E_k$.

STEP 1

PROPOSITION 7. *With the notation of 3-1, there exists a constant C' independent of δ, such that*

$$\sup_{K_\delta}\left|(G^{-1}z^{j_k}\partial_z^{m_k})\cdots(G^{-1}z^{j_1}\partial_z^{m_1})E_0(z)\right|$$

$$\le {C'}^{k+1}\{\tilde{c}+(m_k+\cdots+m_1)\}!\delta^{-(k+1)\tilde{c}-(m_k+\cdots+m_1)}$$

for all $k\ge 0$ and all $\delta, 0<\delta\ll 1$, where $\tilde{c}$ is the one in Proposition 2. We refer to this inequality as $(*)_{k,\delta}$.

PROOF. It is true for $k=0$. We proceed by induction on k. Assume that $(*)_{k,\delta}$ is true for all sufficiently small δ. Take $\delta' = (1+\frac{1}{\tilde{c}+(m_k+\cdots+m_1)})^{-1}<\delta$. $(*)_{k,\delta'}$, which is true by assumption, states that

$$\begin{aligned}&\sup_{K_{\delta'}}\left|(G^{-1}z^{j_k}\partial_z^{m_k})\cdots(G^{-1}z^{j_1}\partial_z^{m_1})E_0(z)\right|\\ &\le {C'}^{k+1}\{\tilde{c}+(m_k+\cdots+m_1)\}!\delta^{-(k+1)\tilde{c}-(m_k+\cdots+m_1)}\\ &\qquad\times(1+\frac{1}{\tilde{c}+(m_k+\cdots m_1)})^{(k+1)\tilde{c}+(m_k+\cdots+m_1)}.\end{aligned}$$

Here

$$\begin{aligned}\text{the last factor} &\le \left(1+\frac{1}{\tilde{c}+(m_k+\cdots+m_1)}\right)^{\tilde{c}+(m_k+\cdots m_1)}\\ &\quad\cdot\left(1+\frac{1}{\tilde{c}+(m_k+\cdots+m_1)}\right)^{k\tilde{c}}\\ &\le e\{(1+\frac{1}{k})^k\}^{\tilde{c}}\\ &\le e^{\tilde{c}+1}\end{aligned}$$

because $m_k, \ldots, m_1 \ge 1$. Next, we employ Cauchy's estimate. A circle with center in K_δ and radius $\delta/\{\tilde{c}+(m_k+\cdots+m_1)+1\}$ is contained in $K_{\delta'}$. Therefore

$$\begin{aligned}&\sup_{K_\delta}\left|\partial_z^{m_{k+1}}(G^{-1}z^{j_k}\partial_z^{m_k})\cdots(G^{-1}z^{j_1}\partial_z^{m_1})E_0(z)\right|\\ &\le m_{k+1}!\{\tilde{c}+(m_k+\cdots+m_1)+1\}^{m_{k+1}}\delta^{-m_{k+1}}\\ &\quad\times C'^{k+1}\{\tilde{c}+(m_k+\cdots+m_1)\}!\delta^{-(k+1)\tilde{c}-(m_k+\cdots+m_1)}\times e^{\tilde{c}+1}\\ &\le m_{k+1}!e^{\tilde{c}+1}C'^{k+1}\{\tilde{c}+(m_{k+1}+m_k+\cdots+m_1)\}!\\ &\quad\times\delta^{-(k+1)\tilde{c}-(m_{k+1}+m_k+\cdots+m_1)}\end{aligned}$$

Here remark that $m_{k+1}! \le \bar{m}\,!$, (independent of k). By the way $|z^{j_{k+1}}|$ is bounded by a positive constant C'' independent of k. Then we finish the proof by choosing $C' > C \cdot C'' \cdot \bar{m}!e^{\tilde{c}+1}$, where C is the constant in Proposition 2. □

PROPOSITION 8.
There is a constant C_δ such that

$$\sup_{K_\delta}\left|(G^{-1}z^{j_k}\partial_z^{m_k})\cdots(G^{-1}z^{j_1}\partial_z^{m_1})E_0(z)\right| \le (m_k+\cdots+m_1)!C_\delta^{k+1}.$$

PROOF. First, we have

$$\delta^{-(m_k+\cdots+m_1)} \le (\delta^{-\bar{m}})^k.$$

Secondly, since there is a constant $C_{\tilde{c}} > 1$ such that

$$\begin{aligned}\frac{(\tilde{c}+l)!}{l!} &= \text{a polynomial in } l \text{ of degree } \tilde{c} \\ &\le C_{\tilde{c}}^{l+1}\end{aligned}$$

for any positive integer l, we have

$$\frac{\{\tilde{c}+(m_k+\cdots+m_1)\}!}{(m_k+\cdots+m_1)!} \le C_{\tilde{c}}^{(m_k+\cdots+m_1)+1} \le (C_{\tilde{c}}^{\bar{m}})^k C_{\tilde{c}}.$$

Thus the present proposition follows from the preceding one. □

A holomorphic function $f(z)$ in Ω can be regarded as a microdifferential operator in (z, x'), and its formal norm $N_0^{K_\delta}(f;T)$ is defined. Here T is an indeterminate.

PROPOSITION 9.

$$\begin{aligned}&N_0^{K_\delta}((G^{-1}z^{j_k}\partial_z^{m_k})\cdots(G^{-1}z^{j_1}\partial_z^{m_1})E_0(z);T) \\ &\le 2(m_k+\cdots+m_1)!C_{\delta/2}^{k+1}\frac{1}{1-\frac{2T}{\delta}}.\end{aligned}$$

PROOF. Use Cauchy's estimate. The path of integration should by centered in K_δ and with radius $\delta/2$. □

STEP 2

First, we prepare some generalities.

LEMMA 10. *Let $P(x,D)$ be a microdifferential operator of order $\le -m < 0$ defined in a neighborhood of a compact set $\omega \subset T^*\mathbb{C}_x^n$, where m is a positive integer. Then we have*

$$N_0^\omega(P;T) \ll \frac{(2n)^{-m}}{m!}T^{2m}N_{-m}^\omega(P;T)$$

Proof. By definition,

$$N_0^\omega(P;T) = \sum_{k,\alpha,\beta} \frac{2(2n)^{-k}k!}{(|\alpha|+k)!(|\beta|+k)!} \sup_\omega \left| D_x^\alpha D_\xi^\beta p_{-k}(x,\xi) \right| T^{2k+|\alpha+\beta|},$$

where $P = \sum_{k\geq 0} p_{-k}$ and p_{-k} is the homogeneous part of degree $-k$. There is no contribution by the terms corresponding to $k = 0, 1, 2, \ldots, m-1$. Hence, if we put $l = k - m$,

$$N_0^\omega(P;T) = \sum_{l\geq 0,\alpha\beta} \frac{2(2n)^{-(l+m)}(l+m)!}{(|\alpha|+l+m)!(|\beta|+l+m)!} \times \sup_\omega \left| D_x^\alpha D_\xi^\beta p_{-(l+m)}(x,\xi) \right| T^{2(l+m)+|\alpha+\beta|} \quad .$$

We have only to prove that

$$\frac{2(2n)^{-(l+m)}(l+m)!}{(|\alpha|+l+m)!(|\beta|+l+m)!} \leq \frac{(2n)^{-m}}{m!} \frac{2(2n)^{-l}l!}{(|\alpha|+l)!(|\beta|+l)!}.$$

This inequality is obtained by the calculation below.

$$\frac{2(2n)^{-(l+m)}(l+m)!}{(|\alpha|+l+m)!(|\beta|+l+m)!} \times \frac{(|\alpha|+l)!(|\beta|+l)!}{2(2n)^{-l}l!}$$

$$\leq (2n)^{-m} \times \frac{1}{(|\alpha|+l+m)\cdots(|\alpha|+l+1)} \times \frac{(l+m)\cdots(l+1)}{(|\beta|+l+m)\cdots(|\beta|+l+1)}$$
$$\leq (2n)^{-m} \times \frac{1}{m!} \times 1. \ \square$$

Lemma 11. *Let $P_1(x,D), \ldots, P_k(x,D)$ be microdifferential operators of order $\leq -m_1, \ldots, -m_k$ respectively, where $m_1, \ldots, m_k$ are positive integers. Then we have*

$$N_0^\omega(P_k \cdots P_1;T) \ll \frac{(2n)^{-(m_k+\cdots+m_1)}}{(m_k+\cdots+m_1)!} T^{2(m_k+\cdots+m_1)} N_{-m_k}(P_k;T) \cdots N_{-m_1}(P_1;T).$$

PROOF. Since $\mathrm{ord}(P_k \cdots P_1) \leq -(m_k + \cdots + m_1)$, the preceding lemma implies that

$$\begin{aligned}
&N_0(P_k \cdots P_1; T) \\
&\ll \frac{(2n)^{-(m_k+\cdots+m_1)}}{(m_k + \cdots + m_1)!} T^{2(m_k+\cdots+m_1)} N_{-(m_k+\cdots+m_1)}(P_k \cdots P_1; T).
\end{aligned}$$

Moreover, according to [Bou-Kr], we have

$$N_{-(m_k+\cdots+m_1)}(P_k \cdots P_1; T) \ll N_{-m_k}(P_k; T) \cdots N_{-m_1}(P_1; T) \ \square$$

In the lemma above, let $P_1, \ldots P_k$ be our $\alpha_{m_1,j_1}, \ldots, \alpha_{m_k,j_k}$ respectively. Regard them as operators of n variables (z, x'). Then we have

$$\begin{aligned}
&N_0(\alpha_{m_k,j_k} \cdots \alpha_{m_1,j_1}; T) \\
&\ll \frac{1}{(m_k + \cdots + m_1 + k)!} \left(\frac{T^2}{2n}\right)^{m_k+\cdots+m_1+k} \\
&\quad \times N_{-m_k-1}(\alpha_{m_k,j_k}; T) \cdots N_{-m_1-1}(\alpha_{m_1,j_1}; T).
\end{aligned}$$

STEP3

Combining Proposition 9 and the estimate immediately above, we obtain

$$\begin{aligned}
&N_0^{K_\delta \times \omega}(\alpha_{m_k,j_k}(x', D') \cdots \alpha_{m_1,j_1}(x', D')(G^{-1} z^{j_k} \partial_z^{m_k}) \\
&\qquad \cdots (G^{-1} z^{j_1} \partial_z^{m_1}) E_0(z); T) \\
&\ll \frac{2}{1 - \frac{2T}{\delta}} \frac{1}{k!} C_{\delta/2}^{k+1} \\
&\quad \times \{(\frac{T^2}{2n})^{m_k+1} N_{-m_k-1}(\alpha_{m_k,j_k}; T)\} \cdots \{(\frac{T^2}{2n})^{m_1+1} N_{-m_1-1}(\alpha_{m_1,j_1}; T)\}.
\end{aligned}$$

Here $\omega \ni p'$ is a compact set of $T^*\mathbb{C}^{n-1}$ in a neighborhood of which $\alpha_{m_k,j_k}, \dots, \alpha_{m_1,j_1}$ are defined. Since

$$\begin{aligned}
E_k(z, x', D') &= \left(\sum_{m=1}^{\bar{m}} \sum_{j=0}^{m-1} G^{-1} \alpha_{m,j}(x', D') z^j \partial_z^m \right)^k E_0(z) \\
&= \sum_{(m_k,\dots,m_1)} \sum_{(j_k,\dots,j_1)} \alpha_{m_k,j_k} \cdots \alpha_{m_1,j_1} (G^{-1} z^{j_k} \partial_z^{m_k}) \\
&\qquad\qquad \cdots (G^{-1} z^{j_1} \partial_z^{m_1}) E_0(z) \\
&\in \mathcal{E}(0),
\end{aligned}$$

we have

$$\begin{aligned}
&N_0(E_k; T) \\
&\ll \sum_{(m_k,\dots,m_1)} \sum_{(j_k,\dots,j_1)} \frac{2}{k!} \frac{1}{1 - \frac{2T}{\delta}} C_{\delta/2}^{k+1} \\
&\qquad \times \{ (\frac{T^2}{2n})^{m_k+1} N_{-m_k-1}(\alpha_{m_k,j_k}; T) \} \\
&\qquad \cdots \{ (\frac{T^2}{2n})^{m_1+1} N_{-m_1-1}(\alpha_{m_1,j_1}; T) \} \\
&\ll \frac{2}{k!} \frac{1}{1 - \frac{2T}{\delta}} C_{\delta/2}^{k+1} [\sum_{m=1}^{\bar{m}} \sum_{j=0}^{m-1} (\frac{T^2}{2n})^{m+1} N_{-m-1}(\alpha_{m,j}; T)]^k.
\end{aligned}$$

Now the convergence of $E = \sum_{k \geq 0} E_k$ is clear. Here remark that its principal part is $E_0(z) = v_2(z)$.

Next, we have to study the behaviour of E near $z = \infty$.

LEMMA 12. *Let $f(z), g(z)$ be holomorphic functions in the upper half plane such that $G(z, \partial_z) g(z) = f(z)$. Assume that in a neighborhood of $z = \infty$, f is a finite sum of functions of exponent $2, \frac{5}{2}, 3, \frac{7}{2}, \dots$. Then g is a finite sum of functions of exponent $1, \frac{3}{2}, 2, \frac{5}{2}, 3, \dots$.*

PROOF. This is a consequence of PART 0 Lemma 2. An alternative proof is the use of the variation of parameters method. Let F_1, F_2 be two linearly independent homogeneous solutions and W be their Wronskian. $\frac{F_j(y)}{y(1-y)W(y)}f(y)$ is a sum of terms of exponent $\frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$. So $\int^z \frac{F_j(y)}{y(1-y)W(y)}f(y)dy$ is a sum of terms of exponent $0, 1/2, 1, 3/2, 2, \ldots$ at $z=\infty$. Note that no logarithmic term appears. Since F_{3-j} is of exponent $(1, 3/2)$, the lemma follows immediately. □

LEMMA 13. *Let $f(z)$ be of exponent α at $z=\infty$. Then $z^j \partial_z^m f(z)$ is of exponent $\alpha+m-j$, or larger by a positive integer.*

PROOF. Easy. □

We will use this lemma in the case $m \geq 1, 0 \leq j \leq m-1$. Then the exponent increases because $m-j \geq 1$. Combining Lemmas 12 and 13, we conclude that $(G^{-1}z^{j_k}\partial_z^{m_k}) \cdots (G^{-1}z^{j_1}\partial_z^{m_1})E_0(z)$ is a finite sum of functions of exponent $1, 3/2, 2, 5/2, 3, \ldots$ at $z=\infty$. Therefore $E(z, x', D')$ can be written

$$E(z, x', D') = z^{-1}E'(z, x', D') + z^{-3/2}E''(z, x', D'),$$

where E' and E'' are formal microdifferential operators in a neighborhood of $z=\infty$. In fact, we have

LEMMA 14. *$E'(z, x', D')$ and $E''(z, x', D')$ are microdifferential operators. (That is, they satisty a suitable growth condition.)*

PROOF. E satisfies the growth condition of microdifferential operators in the universal covering space of $\{1 \ll |z| < \infty\}$. We can derive the lemma by using the sublemma below (with $\lambda = 1/2$) and the lemma of Schwarz. □

SUBLEMMA. *Let $D = \{z \in \mathbb{C}; 0 < |z| < r\}$ be a punctured disk, $\tilde{D}$ its universal covering and λ a non-integer. Then the sum $\mathcal{O}(D) + z^\lambda \mathcal{O}(D) \subset \mathcal{O}(\tilde{D})$ is a direct sum. Moreover, if K is a compact set in D and $\tilde{K}$ is the closure of $\bigcup_{0 \leq \theta \leq 2\pi} e^{i\theta}K \subset \tilde{D}$, then, there exists a constant $C = C_{\lambda,K}$ such that : For $f(z) = g(z) + z^\lambda h(z)$ with $g, h \in \mathcal{O}(D)$, we have*

$$\sup_K |g| \leq C \sup_{\tilde{K}} |f|, \quad \sup_K |h| \leq C \sup_{\tilde{K}} |f|$$

Proof. Consider the variation of f,

$$\begin{aligned}\mathrm{Var} f(z) &= f(e^{2\pi i}z) - f(z)\\ &= (e^{2\pi i\lambda} - 1)z^{\lambda}h(z)\end{aligned}$$

Note that $e^{2\pi i\lambda} - 1 \neq 0$ by the assumption on λ. Obviously we have

$$\sup_{K}|\mathrm{Var} f(z)| \leq 2\sup_{\tilde{K}}|f| \ \square$$

3-3 construction of $E_j^{\pm}$

First, by using Lemma 14, we see that at $z = \zeta = \infty$, $E(z, x', D')f(x')$ belongs to (the inverse image under $\zeta = (-\beta_1 + \beta_2)z + \beta_1$ of) $\mathcal{CO}_+^{\infty}$ for any $f(x') \in \mathcal{C}_{N,p'}$, since $\mathcal{CO}_+^{\infty}$ is an $\mathcal{E}$-Module.

Next,we are interested in the behaviour of $E(z, x', D')$ at $z = 1$.

Proposition 15. *$E(z, x', D')f(x')$ belongs to (the inverse image under $\zeta = (-\beta_1 + \beta_2)z + \beta_1$ of) $\mathcal{CO}_+^{\infty}$ at $z = 1$.*

Proof. We construct a defining function which is holomorphic in $\{\mathrm{Im}\, z > 0\}\times$ (an infinitesimal wedge ω in $\mathbb{C}_{z'}^{n-1}$). We employ the action of Bony-Schapira. We may assume that $p' = (0'; idx_n)$ and choose $z_n = i\sigma$ as the initial surface of the action. Let $F(z')$ be a defining function of $f(x')$, which is holomorphic in a flat domain $\omega \subset \mathbb{C}^{n-1}$ as in [B-S] p.107. By virtue of the flabbiness of $\mathcal{C}$ and the remark in [B-S]p.99,l.7-9, we may work in a domain where F is bounded, thus satisfying the assumption of [B-S] Proposition 2.4.3. By Lemma 11, we have

$$\begin{aligned}&N_0(\alpha_{m_k,j_k}\cdots\alpha_{m_1,j_1};T)\\ &\ll \frac{1}{(m_k+\cdots+m_1+k)!}\\ &\quad\times\left(\frac{T^2}{2(n-1)}\right)^{m_k+\cdots+m_1+k} N_{-m_k-1}(\alpha_{m_k,j_k};T)\cdots N_{-m_1-1}(\alpha_{m_1,j_1};T).\end{aligned}$$

Because there are only a finite number of $\alpha_{m,j}$'s, there is a constant $B > 0$ such that

$$N_{-m-1}(\alpha_{m,j};T) < B \quad \text{for all } m \text{ and } j.$$

Therefore, there is a constant $A > 0$ such that

$$N_0(\alpha_{m_k,j_k}\cdots\alpha_{m_1,j_1};T) \ll \frac{A^{k+1}}{(k+m_k+\cdots+m_1)!}$$

for any choice of $(m_k, j_k), \dots, (m_1, j_1)$. Let us derive an estimate like the one in [B-S] p.94. We see easily that there is a constant M_0 such that the homogeneous part of degree $(-l)$ of $\alpha_{m_k,j_k}\cdots\alpha_{m_1,j_1}$ is estimated by $\frac{A^{k+1}}{(k+m_k+\cdots+m_1)!}M_0^{l+1}l!$. Then [B-S] Proposition 2.4.3 implies that there is a constant C such that

$$\left|(\alpha_{m_k,j_k}\cdots\alpha_{m_1,j_1})_\Sigma F(z')\right| \le C\frac{A^{k+1}}{(k+m_k+\cdots+m_1)!}d_I(z')^{-\alpha}d_J(z')^{-\beta}.$$

in ω. Combining this with Proposition 8, we obtain

$$\sup_{z\in K_\delta}\left|(G^{-1}z^{j_k}\partial_z^{m_k})\cdots(G^{-1}z^{j_1}\partial_z^{m_1})E_0(z)\times(\alpha_{m_k,j_k}\cdots\alpha_{m_1,j_1})_\Sigma F(z')\right|$$

$$\le C\frac{(AC_\delta)^{k+1}}{k!}d_I(z')^{-\alpha}d_J(z')^{-\beta}.$$

Set

$$S_k(z,z') \underset{\mathrm{def}}{=} (\sum_{m=1}^{\bar m}\sum_{j=0}^{m-1}G^{-1}\alpha_{m,j}(z',D')_\Sigma z^j\partial_z^m)^k E_0(z)F(z').$$

Since the summation above consists of $\bar m(\bar m-1)/2 \le \bar m^2$ terms,

$$\begin{aligned}\sup_{z\in K_\delta}\left|S_k(z,z')\right| &\le \sum_{(m_k,\dots,m_1)}\sum_{(j_k,\dots,j_1)}C\frac{(AC_\delta)^{k+1}}{k!}d_I(z')^{-\alpha}d_J(z')^{-\beta}\\ &\le CAC_\delta d_I^{-\alpha}d_J^{-\beta}\frac{1}{k!}(\bar m^2AC_\delta)^k.\end{aligned}$$

This proves that $\sum_k S_k$ converges in $\{\operatorname{Im} z > 0\}\times\omega$ locally uniformly. This completes the proof. $\square$

Again according to [Kat 1], Ef, or rather its counterpart in ζ-variable, defines a 2-pure solution. We denote it by $(E_2^+f)(x)$, $x\in\mathbb{R}^n$. All the other $E_j^\pm$'s are defined similarly. A special emphasis is laid on the fact that the principal part of E is E_0. There's no contribution of the perturbation terms in this respect.

3-4 end of the proofs

In this subsection, we prove the remaining parts of Theorems A, B and C. The mappings $L^{\pm}, E^{\pm}$ and B are defined and calculated in the same way as before. Because of the remark at the end of the preceding subsection, the principal part (= the 0-th order part) remains the same as the unperturbed case. This preserves the ellipticity of the components of the above mappings.

§4 proof of the case γ is an operator

4-1 substitution of operators into a convergent power series

Proposition 16. *Let $S(w_1, w_2) = \sum_{j,k\geq 0} a_{jk}(w_1 - \dot{w}_1)^j (w_2 - \dot{w}_2)^k$ be a convergent power series, and $P = P(z, D) \in \mathcal{E}_{\mathbb{C}^n}(0)$ be a microdifferential operator of order ≤ 0 defined in a neighborhood of $p \in T^*\mathbb{C}^n$. If $\sigma_0(P)(p) = \dot{w}_1$, then*

$$S(P, w_2) = \sum_{j,k\geq 0} a_{jk}(P - \dot{w}_1)^j (w_2 - \dot{w}_2)^k \in \mathcal{E}_{\mathbb{C}^{n+1}}(0)$$

is a well-defined microdifferential operator. Moreover we have

$$S(P, w_2) = \sum_{j\geq 0} \left(\sum_{k\geq 0} a_{jk}(w_2 - \dot{w}_2)^k \right) (P - \dot{w}_1)^j.$$

Proof. We use the formal norm $N_0(\cdot, t)$, which we denote by $\|\cdot\|$ for brevity. We have

$$\|S\| \ll \sum_{j,k} |a_{jk}| \|P - \dot{w}_1\|^j \|w_2 - \dot{w}_2\|^k < \infty. \square$$

Remark 17. The last expression in the above proposition justifies analytic continuation in the w_2−direction.

Example 18.

The hypergeometric function $F(a,b;c;w)$ is a holomorphic function in $\{(c,w); c\neq 0,-1,-2,\dots,|w|<1\}$. We can define $F(a,b;P(z,D);w)$ for $P\in\mathcal{E}(0)$ if $\sigma_0(P)$ avoids $0,-1,-2,\dots$.

Example 19. (microdifferential connection formula)

The classical connection formula for hypergeometric functions asserts that

$$\begin{aligned}
&F(\frac{3}{2},1;c;w)\\
&=\frac{c-1}{c-\frac{5}{2}}F(\frac{3}{2},1,\frac{7}{2}-c;1-w)\\
&\quad+\frac{\Gamma(c)\Gamma(\frac{5}{2}-c)}{\Gamma(\frac{3}{2})}(1-w)^{c-\frac{5}{2}}F(c-\frac{3}{2},c-1;c-\frac{3}{2};1-w)
\end{aligned}$$

If $\sigma_0(P)\notin\frac{1}{2}\mathbb{Z}$, we can replace c by $P(z,D)$. We obtain

$$\begin{aligned}
&F(\frac{3}{2},1;P;w)\\
&=\frac{P-1}{P-\frac{5}{2}}F(\frac{3}{2},1,\frac{7}{2}-P;1-w)\\
&\quad+\frac{\Gamma(P)\Gamma(\frac{5}{2}-P)}{\Gamma(\frac{3}{2})}(1-w)^{P-\frac{5}{2}}F(P-\frac{3}{2},P-1;P-\frac{3}{2};1-w)
\end{aligned}$$

In the example above, we encountered an operator of the form $w^{P(z,D)}$, which is defined by using Proposition 16. On the other hand, in [Tah] and [O], this kind of operator is defined by

$$\begin{aligned}
w^{P(z,D)}&=\exp\left(P(z,D)\log w\right)\\
&=\sum_{l\geq 0}\frac{1}{l!}\left\{P(z,D)\log w\right\}^l.
\end{aligned}$$

Proposition 20. *Our definition coincides with that of [Tah] and [O].*

PROOF. Let w^P be defined by

$$w^{P(z,D)} = \sum_{j,k} a_{jk} \left(P(z,D) - \dot{w}_1\right)^j \left(w - \dot{w}_2\right)^k$$

where $w_2^{w_1} = \sum_{j,k} a_{jk} \left(w_1 - \dot{w}_1\right)^j \left(w_2 - \dot{w}_2\right)^k$ is a convergent power series (in the classical sense). Set

$$w_1 = (w_1 - \dot{w}_1) + \dot{w}_1, \quad \log w_2 = \sum_{m \geq 0} b_m (w - \dot{w}_2)^m,$$

then

$$\begin{aligned} &\sum_{l \geq 0} \frac{1}{l!} \left[\{(w_1 - \dot{w}_1) + \dot{w}_1\} \sum_m b_m (w - \dot{w}_2)^m \right]^l \\ &= w_2^{w_1} \\ &= \sum a_{jk} (w_1 - \dot{w}_1)^j (w_2 - \dot{w}_2)^k. \end{aligned}$$

Moreover we have

$$\begin{aligned} &N_0 \left(\sum_{l \geq 0} \frac{1}{l!} (P \log w)^l \right) \\ &\ll \sum_{l \geq 0} \frac{1}{l!} \left[\{N_0(P - \dot{w}_1) + N_0(\dot{w}_1)\} \sum_{m \geq 0} |b_m| N_0 (w - \dot{w}_2)^m \right]^l \\ &< \infty. \end{aligned}$$

Therefore we may rearrange the order of the sum in the same way as in the classical case and obtain

$$\sum_{l \geq 0} \frac{1}{l!} (P \log w)^l = \sum_{j,k} a_{jk} (P - \dot{w}_1)^j (w - \dot{w}_2)^k. \ \square$$

LEMMA 21. *Let U be a conic open set of $iT^*\mathbb{R}^n_x$ and $P(x,D)$ be a 0-th order microdifferential operator defined there. $w^{P(x,D)}$ is defined in $\{Re\, w >$*

$0\} \times U$. *Then, for any microfunction* $f(x)$ *in* U, $w^{P(x,D)}f(x)$ *is an element of* $\mathcal{CO}_+^{\infty}(\{Re\,w \geq 0\} \times U)$.

PROOF. Although this fact is well-known to specialists, there seems to be no published proof. Here we give a sketch of a proof based on the action of Bony-Schapira. We borrow some notation from them. We construct a defining function which is holomorphic in $\{\mathrm{Re}\,w > 0\}\times$(an infinitesimal wedge in $\mathbb{C}_z^n$). We may assume that U is a neighborhood of $(0, idx_n)$ and we choose $z_n = i\sigma$ as the initial surface of the action. Let $F(z)$ be a defining function of $f(x)$. We have only to prove the convergence of

$$(w^P)_\Sigma F = \sum_{l \geq 0} \frac{1}{l!} (\log w)^l \left(P(z,D)_\Sigma\right)^l F(z).$$

We may assume that F satisfies the assumption of [B-S] Proposition 2.4.3 without loss of generality by virtue of the flabbiness of the sheaf of microfunctions and the remark in [B-S] p.99 l.7-9. We have

$$\left|(P(z,D)_\Sigma)^l\right| \leq C^l d_I(z)^{-\alpha} d_J(z)^{-\beta}.$$

Then the convergence follows. Here the factor $1/l!$ is essential. □

Example 22.
Put $\zeta = (-\beta_1 + \beta_2)z + \beta_1$, for complex variables ζ and x. Here β_1 and β_2 are purely imaginary and $\beta_1/i > \beta_2/i$. Let $x' \in \mathbb{R}^{n-1}$ be a real coordinate and $f(x')$ be a microfunction. We see easily that $F(3/2, 1; c(x', D'); z)f(x')$ belongs to $\mathcal{CO}_+^{\infty}$. In fact, $\mathcal{CO}_+^{\infty}$ is an $\mathcal{E}-$Module and we know that $(1 - z)^{c-\frac{5}{2}} f(x')$ belongs to $\mathcal{CO}_+^{\infty}$.

4-2 end of the proofs

We calculate in the same way as in the beginning of 2-1. Then G should be replaced by

$$G(3/2, 1; \tilde{c}(x', D'); z, D) = z(1-z)D^2 + \{\tilde{c}(x', D') - (\frac{3}{2} + 1 + 1)z\}D - \frac{3}{2} \cdot 1,$$

$$\text{with} \qquad \tilde{c} = \frac{\frac{3}{2}\beta_1 - 2\beta_2 + \tilde{\gamma}}{\beta_1 - \beta_2}.$$

Here we have used the fact that

$$\begin{aligned}\tilde{\gamma}(x, D) &= \sum_j \gamma_j(x'', D')(\frac{1}{2}x_1 D_1 D_n^{-1} + x_n - \dot{x}_n)^j \\ &= \sum_j \gamma_j(x'', D')(tD_t D_n^{-1} + x_n - \dot{x}_n)^j\end{aligned}$$

is transformed under $\mathcal{L}$ into

$$\begin{aligned}&\sum_j \gamma_j\{(-\zeta D_\zeta - 1)D_n^{-1} + x_n + D_\zeta \zeta D_n^{-1} - \dot{x}_n\}^j \\ &= \sum_j \gamma_j(x_n - \dot{x}_n)^j = \tilde{\gamma}(x', D').\end{aligned}$$

Obviously we have

$$G(3/2, 1; \tilde{\gamma}; z, \partial_z)F(3/2, 1; \tilde{\gamma}; z) = 0, \quad \text{etc.}$$

Therefore we may replace γ, c in §2 by $\tilde{\gamma}, \tilde{c}$. $L^{\pm}$ and B (in the present context) are calculated easily. For example, we have

$$B = \begin{pmatrix} \varepsilon(\tilde{c}(x', D')) & 1 + \varepsilon(\tilde{c}(x', D')) \\ 1 - \varepsilon(\tilde{c}(x', D')) & \varepsilon(\tilde{c}(x', D')) \end{pmatrix}.$$

To prove Theorem C', we have to prove the ellipticity of all the components. We have

$$\sigma_0(\varepsilon(\tilde{c}(x', D'))) = \varepsilon(\sigma_0(\tilde{c}(x', D'))),$$

$$\sigma_0(\tilde{c})(p') = c.$$

Hence $\sigma_0(\varepsilon(\tilde{c}(x', D')))(p') = \varepsilon(c) \neq 0$. The other components are dealt with in the same way. Theorems A' and B' are proved similarly.

Part 2 third order case

§1 the statement of the theorems

Let

$$P(x,D) = D_1^3 - x_1^2 D_n^2 D_1 + 2(a-b) D_n D_1 + \{2(a+b)-3\} x_1 D_n^2 + \sum_{l=0}^{\text{finite}} \alpha_{-l}(x_1^2, x', D') x_1^{l+1} D_1^l$$

be a microdifferential operator defined in a neighborhood of $p \in \{(x, i\xi dx) \in iT^*M; x_1 = \xi_1 = 0, \xi_n > 0\}$, such that $\operatorname{ord}\alpha_{-l} \le -l-1$ and that α_{-l} is a polynomial in $t = \frac{1}{2}x_1^2$ and x_n. Here we write $x = (x_1, \ldots, x_n) = (x_1, x') \in \mathbb{R}^n = M$. The principal symbol of P, denoted by $\sigma(P)(x,\xi)$, is factorized in the form $\sigma(P) = (\xi_1 - x_1\xi_n)\xi_1(\xi_1 + x_1\xi_n)$. P is microhyperbolic and has triple characteristics over the initial surface $N = \{x_1 = 0\}$. $\operatorname{Char}(P)$, the (purely imaginary) characteristic variety, is the union of three hypersurfaces $\xi_1 = 0, \pm x_1\xi_n$, which have a non-involutory intersection $\{x_1 = \xi_1 = 0\} \ni p$. Let b_j be the bicharacteristic strip of $\{\xi_1 = x_1\xi_n\}, \{\xi_1 = 0\}, \{\xi_1 = -x_1\xi_n\}$ for $j = 1, 2, 3$ respectively, issuing from p, and $b_j^{\pm}$ be its intersection with $\{(x; i\xi dx); \pm x_1 > 0\}$. We set, as in the second order case,

$$Sol(j, \pm) = \{u \in (\Gamma_{\{\pm x_1 > 0\}} \mathcal{C}_M^P)_p; u = 0 \text{ on } b_k^{\pm} (k \ne j)\}.$$

An element of $Sol\,(j, \pm)$ is called a *j-pure solution* in $\pm x_1 > 0$. First we give the following three theorems, assuming

$$(\sharp): \quad \alpha_{-l} = 0 \quad \text{for all } l$$

Set

$$Z = \{(a,b) \in \mathbb{C}^2; a = 0, -1, -2, \ldots, \quad \text{or} \quad b = 0, -1, -2, \ldots, \quad \text{or} \quad a + b = 3/2, 5/2, 7/2, \ldots\}.$$

Let $p' = \rho(p)$, where ρ is the projection $N \underset{M}{\times} iT^*M \to iT^*N, N = \{x_1 = 0\} \subset M$.

THEOREM D. *(boundary value problem with purity)*

If $(a,b) \notin Z$, then the map

$$Sol(j,\pm) \to \mathcal{C}_{N,p'}$$

$$u \mapsto D_1 u(+0,x')$$

is an isomorphism.

Remark.

There is an open dense subset of $\mathbb{C}^2 \backslash Z$ such that if (a,b) belongs to it, then the mappings

$$Sol(j,\pm) \to \mathcal{C}_{N,p}$$

$$u \mapsto u(+0,x')$$

$$u \mapsto D_1^2 u(+0,x')$$

are isomorphisms.

Theorem E. *(characterization of j-pure solutions by a relationship between their boundary values)*

If $(a,b) \notin Z$, then there exist microdifferential operators $P_j^{\pm(0)}(x',D')$ and $P_j^{\pm(2)}(x',D')$ of half integer order that have the following property: An element of $(\Gamma_{\{\pm x_1>0\}}\mathcal{C}_M^P)_p$ is j-pure if and only if

$$\begin{cases} u(\pm 0,x') = P_j^{\pm(0)}(x',D')\{D_1 u(\pm 0,x')\} \\ D_1^2 u(\pm 0,x') = P_j^{\pm(2)}(x',D')\{D_1 u(\pm 0,x')\} \end{cases}$$

Theorem F. *(branching of singularities)*

(1) There is an open dense subset of $\mathbb{C}^2 \backslash Z$ such that if (a,b) belongs to it, we have: Let $u(x)$ be an element of $\mathcal{C}_{M,p}^P$. If u is pure and $u \neq 0$ in $\pm x_1 > 0$, then $b_1^{\mp} \cup b_2^{\mp} \cup b_3^{\mp}$ is contained in $\operatorname{supp} u$.

(2) Assume that $b \in \mathbb{N} = \{1,2,3,\dots\}$ and $a+b = 1/2, -1/2, -3/2, -5/2, \dots$, then we have;

(2-1) If u is 1-pure and $u \neq 0$ in $\pm x_1 > 0$, then u is 1-pure in $\mp x_1 > 0$.

(2-2) If u is 2-pure and $u \neq 0$ in $\pm x_1 > 0$, then $b_1^{\mp} \cup b_2^{\mp} \subset \operatorname{supp} u$ and $u = 0$ on $b_3^{\mp}$.

(2-3) If u is 3-pure and $u \neq 0$ in $\pm x_1 > 0$, then $b_1^{\mp} \cup b_3^{\mp} \subset \operatorname{supp} u$ and $u = 0$ on $b_2^{\mp}$.

(3) Assume that $(a, b) \in \mathbb{N} \times \mathbb{N}$, then we have;

(3-1) If u is 1-pure and $u \neq 0$ in $\pm x_1 > 0$, then $b_1^{\mp} \cup b_2^{\mp} \subset \operatorname{supp} u$ and $u = 0$ on $b_3^{\mp}$.

(3-2) If u is 2-pure and $u \neq 0$ in $\pm x_1 > 0$, then u is 2-pure in $\mp x_1 > 0$.

(3-3) If u is 3-pure and $u \neq 0$ in $\pm x_1 > 0$, then $b_2^{\mp} \cup b_3^{\mp} \subset \operatorname{supp} u$ and $u = 0$ on $b_1^{\mp}$.

(4) Assume that $a \in \mathbb{N}$ and $a + b = 1/2, -1/2, -3/2, -5/2, \ldots$, then we have;

(4-1) If u is 1-pure and $u \neq 0$ in $\pm x_1 > 0$, then $b_1^{\mp} \cup b_3^{\mp} \subset \operatorname{supp} u$ and $u = 0$ on $b_2^{\mp}$.

(4-2) If u is 2-pure and $u \neq 0$ in $\pm x_1 > 0$, then $b_2^{\mp} \cup b_3^{\mp} \subset \operatorname{supp} u$ and $u = 0$ on $b_1^{\mp}$.

(4-3) If u is 3-pure and $u \neq 0$ in $\pm x_1 > 0$, then u is 3-pure in $\mp x_1 > 0$.

Next, we remove the condition ($\sharp$) and consider the case α_{-l} is not necessarily 0. We have the following three results. Set

$$\tilde{Z} = \{(a, b) \in \mathbb{C}^2; a \in \mathbb{Z} \quad \text{or} \quad b \in \mathbb{Z} \quad \text{or} \quad a + b + \frac{1}{2} \in \mathbb{Z}\}.$$

THEOREM D'.
There is an open dense subset of $\mathbb{C}^2 \backslash \tilde{Z}$ such that if (a, b) belongs to it, then, the mappings

$$\begin{aligned} Sol(j, \pm) &\to \mathcal{C}_{N,p'} \\ u &\mapsto u(+0, x') \\ u &\mapsto D_1 u(+0, x') \\ u &\mapsto D_1^2 u(+0, x') \end{aligned}$$

are isomorphisms.

THEOREM E'. *There is an open dense subset of $\mathbb{C}^2 \backslash \tilde{Z}$ such that if (a, b) belongs to it, then the same conclusion as Theorem E holds.*

THEOREM F'. *There is an open dense subset of $\mathbb{C}^2 \setminus Z$ such that if (a, b) belongs to it, then the same conclusion as Theorem F(1) holds.*

REMARK. It is a generic condition that (a, b) belongs to an open dense subset. So in the following proofs, we sometimes say "for a generic (a, b)", or "generically" instead of mentioning an open dense subset. Those generic conditions will be the avoidance by (a, b) of the zeroes of holomorphic functions $\not\equiv 0$.

Finally we state some results about the case a and b are replaced by 0-th order microdifferential operators. Let the coordinate of p' be $(\dot{x}_2, \dots, \dot{x}_n; i\dot{\xi}' dx')$ and $\tilde{a} = \tilde{a}(x', D')$, $\tilde{b} = \tilde{b}(x', D')$ be microdifferential operators of order ≤ 0 defined near p' which are commutative: $[\tilde{a}, \tilde{b}] = 0$. They have an expansion of the form

$$\begin{cases} \tilde{a}(x', D') = \sum_{j=0}^{\infty} a_j(x'', D')(x_n - \dot{x}_n)^j, \\ \tilde{b}(x', D') = \sum_{j=0}^{\infty} b_j(x'', D')(x_n - \dot{x}_n)^j, \\ x'' = (x_2, \dots, x_{n-1}). \end{cases}$$

Let $\hat{a} = \hat{a}(x, D)$ and $\hat{b} = \hat{b}(x, D)$ be defined by

$$\hat{a}(x, D) = \sum_{j=0}^{\infty} a_j(x'', D')(\frac{1}{2} x_1 D_1 D_n^{-1} + x_n - \dot{x}_n)^j,$$

$$\hat{b}(x, D) = \sum_{j=0}^{\infty} b_j(x'', D')(\frac{1}{2} x_1 D_1 D_n^{-1} + x_n - \dot{x}_n)^j.$$

They are operators of order ≤ 0 defined in a neighborhood of p. Set

$$a = \sigma_0(\tilde{a})(p') = \sigma_0(a_0)(p') = \sigma_0(\hat{a})(p)$$

$$b = \sigma_0(\tilde{b})(p') = \sigma_0(b_0)(p') = \sigma_0(\hat{b})(p).$$

Let us consider the operator

$$\begin{aligned} P(x, D) = {} & D_1^3 - x_1^2 D_n^2 D_1 \\ & + 2 D_n D_1 \{\hat{a}(x, D) - \hat{b}(x, D)\} \end{aligned}$$

$$+ x_1 D_n^2 \{2\hat{a}(x, D) + 2\hat{b}(x, D) - 3\}.$$

Purity and the related mappings are defined in the usual way. In this situation, we have the following theorems D", E" and F".

THEOREM D". *The map The same statement as Theorem D is true.*

REMARK. The same statement as the Remark following Theorem D is true. We can take the same open dense subset.

THEOREM E". *The same statement as Theorem E is true.*

THEOREM F". *The same statement as Theorem F (1) is true. (We may take the same open dense subset.)*

§2 Jordan-Pochhammer operator and Euler integral representation

Let us consider the following ordinary differential equation of Fuchs type.

$$\begin{aligned} J[y] =& (x - p_1)(x - p_2)(x - p_3)y''' \\ & - \{(\lambda_1 - 3)(x - p_2)(x - p_3) + (\lambda_2 - 3)(x - p_3)(x - p_1) \\ & \qquad + (\lambda_3 - 3)(x - p_1)(x - p_2)\}y'' \\ & - 2\{(\lambda_2 + \lambda_3 - 3)(x - p_1) + (\lambda_3 + \lambda_1 - 3)(x - p_2) \\ & \qquad + (\lambda_1 + \lambda_2 - 3)(x - p_3)\}y' \\ & - 2(\lambda_1 + \lambda_2 + \lambda_3 - 3)y = 0 \end{aligned}$$

We refer to this equation as (JP).

Later, we will set

$$\lambda_1 = a,\ \lambda_2 = \frac{3}{2} - (a + b),\ \lambda_3 = b$$

$$p_1 = i,\ p_2 = 0,\ p_3 = -i$$

LEMMA 1. *If* $\lambda_j \neq 0, -1, -2, -3, \ldots,$ (*for* $j = 1, 2, 3$), $\lambda_1 + \lambda_2 + \lambda_3 - 4 \neq 1, 2, 3, \ldots,$ *then*

$$y_j(x) = \int_{p_j}^{\infty} (u - p_1)^{\lambda_1 - 1}(u - p_2)^{\lambda_2 - 1}(u - p_3)^{\lambda_3 - 1}(u - x)^{-1} du$$

is a solution to (JP). Here the integral is taken in the sense of finite part, if necessary.

Proof. Although several textbooks (e.g. [Huk], [I-K-S-Y]) treat Jordan-Pochhammer equations, ours does not belong to the class solved in them. Therefore, we give an independent proof. See [M] and [I]. Since the finite part is holomorphic in $\lambda_1, \lambda_2, \lambda_3$, we may assume that $\operatorname{Re}\lambda_j > 0\,(j = 1, 2, 3)$ and that $\operatorname{Re}(\lambda_1 + \lambda_2 + \lambda_3) - 4 < 0$ without loss of generality. We set $y = y_j(x)$ in the left hand side of (JP) and write it in terms of powers of $u - x$ by using $x = u - (u - x)$. We have

$$\begin{aligned}
&(x-p_1)(x-p_2)(x-p_3)\\
&=\{(u-p_1)-(u-x)\}\{(u-p_2)-(u-x)\}\{(u-p_3)-(u-x)\}\\
&=(u-p_1)(u-p_2)(u-p_3)\\
&\quad-\{(u-p_1)(u-p_2)+(u-p_2)(u-p_3)+(u-p_3)(u-p_2)\}(u-x)\\
&\quad+(3u-p_1-p_2-p_3)(u-x)^2-(u-x)^3,
\end{aligned}$$

$$\begin{aligned}
&(\lambda_1-3)(x-p_2)(x-p_3)+\cdots\\
&=(\lambda_1-3)\{(u-p_2)-(u-x)\}\{(u-p_3)-(u-x)\}+\cdots\\
&=(\lambda_1-3)(u-p_2)(u-p_3)+(\lambda_2-3)(u-p_3)(u-p_1)\\
&\quad+(\lambda_3-3)(u-p_1)(u-p_2)\\
&\quad-\{(\lambda_1-3)(2u-p_2-p_3)+(\lambda_2-3)(2u-p_3-p_1)\\
&\qquad+(\lambda_3-3)(2u-p_1-p_2)\}(u-x)\\
&\quad+(\lambda_1+\lambda_2+\lambda_3-9)(u-x)^2,
\end{aligned}$$

$$\begin{aligned}
&(\lambda_2+\lambda_3-3)(x-p_1)+\cdots\\
&=(\lambda_2+\lambda_3-3)\{(u-p_1)-(u-x)\}+\cdots\\
&=\{(\lambda_2+\lambda_3-3)(u-p_1)+(\lambda_3+\lambda_1-3)(u-p_2)\\
&\qquad+(\lambda_1+\lambda_2-3)(u-p_3)\}\\
&\quad-(2\lambda_1+2\lambda_2+2\lambda_3-9)(u-x)
\end{aligned}$$

Moreover

$$D_x^n y_j(x) = n! \int_{p_j}^{\infty} (u-p_1)^{\lambda_1-1}(u-p_2)^{\lambda_2-1}(u-p_3)^{\lambda_3-1}(u-x)^{-1-n} du.$$

From the equalities above, we have

$$\begin{aligned} J[y_j] = \int_{p_j}^{\infty} &(u-p_1)^{\lambda_1-1}(u-p_2)^{\lambda_2-1}(u-p_3)^{\lambda_3-1}(u-x)^{-4} \\ &\times \{c_0 + c_1(u-x) + c_2(u-x)^2 + c_3(u-x)^3\} du, \end{aligned}$$

where

$$\begin{aligned} c_0 &= 6(u-p_1)(u-p_2)(u-p_3), \\ c_1 &= -6\{(u-p_1)(u-p_2)+\cdots\} - 2\{(\lambda_1-3)(u-p_2)(u-p_3)+\cdots\} \\ &= -2\{\lambda_1(u-p_2)(u-p_3) + \lambda_2(u-p_3)(u-p_1) + \lambda_3(u-p_1)(u-p_2)\}, \\ c_2 &= 6(3u-p_1-p_2-p_3) \\ &\quad + 2\{(\lambda_1-3)(2u-p_2-p_3) + (\lambda_2-3)(2u-p_3-p_1) \\ &\qquad + (\lambda_3-3)(2u-p_1-p_2)\} \\ &\quad - 2\{(\lambda_2+\lambda_3-3)(u-p_1) + (\lambda_3+\lambda_1-3)(u-p_2) \\ &\qquad + (\lambda_1+\lambda_2-3)(u-p_3)\} \\ &= 0, \\ c_3 &= -6 - 2(\lambda_1+\lambda_2+\lambda_3-9) \\ &\quad + 2(2\lambda_1+2\lambda_2+2\lambda_3-9) \\ &\quad - 2(\lambda_1+\lambda_2+\lambda_3-3) \\ &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &J[y_j] \\ &= \int_{p_j}^{\infty} (u-p_1)^{\lambda_1-1}(u-p_2)^{\lambda_2-1}(u-p_3)^{\lambda_3-1}(u-x)^{-4} \\ &\quad \times [6(u-p_1)(u-p_2)(u-p_3) \\ &\qquad - 2\{\lambda_1(u-p_2)(u-p_3) + \lambda_2(u-p_3)(u-p_1) \end{aligned}$$

$$
\begin{aligned}
&\qquad + \lambda_3(u-p_1)(u-p_2)\}(u-x)]du \\
&= \int_{p_j}^{\infty} [-2(u-p_1)^{\lambda_1}(u-p_2)^{\lambda_2}(u-p_3)^{\lambda_3}\frac{\partial}{\partial u}(u-x)^{-3} \\
&\qquad - 2\{\lambda_1(u-p_2)(u-p_3)+\cdots\} \\
&\qquad \cdot (u-p_1)^{\lambda_1-1}(u-p_2)^{\lambda_2-1}(u-p_3)^{\lambda_3-1}(u-x)^{-3}]du \\
&= [-2(u-p_1)^{\lambda_1}(u-p_2)^{\lambda_2}(u-p_3)^{\lambda_3}(u-x)^{-3}]_{p_j}^{\infty} \\
&\quad + 2\int_{p_j}^{\infty} \frac{\partial}{\partial u}\{(u-p_1)^{\lambda_1}(u-p_2)^{\lambda_2}(u-p_3)^{\lambda_3}\}(u-x)^{-3}du \\
&\quad - 2\int_{p_j}^{\infty} \{\lambda_1(u-p_2)(u-p_3)+\cdots\} \\
&\qquad \cdot (u-p_1)^{\lambda_1-1}(u-p_2)^{\lambda_2-1}(u-p_3)^{\lambda_3-1}(u-x)^{-3}du \\
&= 0.
\end{aligned}
$$

Here we have used integration by parts. □

If $\lambda_1+\lambda_2+\lambda_3=\frac{3}{2}$, it is easy to see that the Riemann scheme of (JP) is

$$
\left\{
\begin{array}{ccccc}
p_1 & p_2 & p_3 & \infty & \\
0 & 0 & 0 & 1 & x \\
1 & 1 & 1 & 3/2 & \\
\lambda_1-1 & \lambda_2-1 & \lambda_3-1 & 2 &
\end{array}
\right\}
$$

and that ∞ is a non-logarithmic singularity. Moreover, if $\lambda_j \notin \mathbb{Z}$, then p_j is non-logarithmic.

Lemma 2. *An entire solution to (JP) vanishes identically.*

Proof. The characteristic exponents at ∞ are larger than 1. Use Liouville's theorem. □

Hereafter, we consider the case

$$
\lambda_1 = a,\ \lambda_2 = \frac{3}{2} - (a+b),\ \lambda_3 = b,
$$

$$p_1 = i,\ p_2 = 0,\ p_3 = -i,$$

$$(a,b) \notin Z \underset{\text{def}}{=} \{(a,b) \in \mathbb{C}^2; a = 0,-1,-2,\dots \text{or}\, b = 0,-1,-2,\dots \text{or}$$
$$a+b = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots\}.$$

Then, our operator is

$$\begin{aligned} &Q(x,D) \\ &= (x^3+x)D^3 + \{\frac{15}{2}x^2 - i(a-b)x + a + b + \frac{3}{2}\}D^2 \\ &\quad + \{12x - 2i(a-b)\}D + 3. \end{aligned}$$

PROPOSITION 3. *Take the path of integration from p_j to ∞ in $\mathrm{Re}\, u \leq 0$. Then y_j is holomorphic in $\mathrm{Re}\, x > 0$. Moreover, it is holomorphically extended to $x = a_k\, (k \neq j)$, but not to a_j.*

PROOF. y_j is obviously holomorphic in the right half plane and at $x = a_k$. If it is holomorphic at a_j, then it is entire. The preceding lemma implies that it vanishes identically. But this is not the case as will be seen when we calculate the expansion coefficients of y_j at $x = \infty$. $\square$

Let us calculate the expansion coefficients of y_j $(j = 1,2,3)$ at $x = \infty$, $\mathrm{Re}\, x > 0$. We will need the coefficients of $x^{-1}, x^{-\frac{3}{2}}, x^{-2}$ in the next section. For convenience, set $\tau = i/x$. Obviously,

$$\mathrm{Re}\, x > 0,\ x = \infty \quad (-\frac{\pi}{2} < \arg x < \frac{\pi}{2}) \Leftrightarrow \mathrm{Im}\, \tau > 0,\ \tau = 0 \quad (0 < \arg \tau < \pi).$$

$$\tau^{3/2} = e^{3\pi i/4} x^{-3/2},\ \tau^2 = -1/x^2.$$

We consider the expansion coefficients at

$$\tau = 0,\ \mathrm{Im}\, \tau > 0.$$

In the following three propositions, we give the coefficients of $\tau, \tau^{\frac{3}{2}}, \tau^2$.

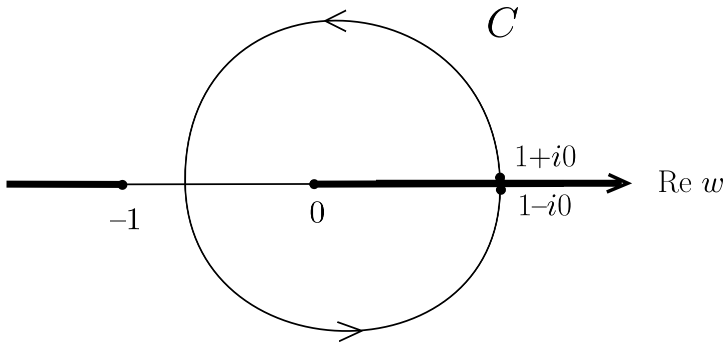

Figure 2

Let C be the path in Figure 2. Here the w-plane has a cut in $\{w; w \geq 0 \,\text{or}\, w \leq -1\}$.

In the following three propositions, the integrands are continuous on C and

$$0 \leq \arg w \leq 2\pi,\ -\pi \leq \arg(1+w) \leq \pi,\ -\pi \leq \arg(1-w) \leq \pi.$$

Proposition 4.

There exists a nonzero constant C_1 such that $V_1(x) \underset{\text{def}}{=} C_1 y_1(x)$ is expanded into the form

$$V_1(x) = p\tau + 1 \cdot \tau^{3/2} + q\tau + \cdots,$$

$$\text{where} \qquad \begin{aligned} p &\underset{\text{def}}{=} \frac{1}{2\pi i}\int_C (1-w)^{a-1} w^{-1/2} (1+w)^{b-1} dw, \\ q &\underset{\text{def}}{=} \frac{1}{2\pi i}\int_C (1-w)^{a-1} w^{-3/2} (1+w)^{b-1} dw. \end{aligned}$$

Remark that p and q are holomorphic in $\{(a,b); a \neq 0, -1, -2, \dots\}$.

Proposition 5.

There exists a nonzero constant C_2 such that $V_2(x) \underset{\text{def}}{=} C_2 y_2(x)$ is expanded into the form

$$V_2(x) = r\tau + 1 \cdot \tau^{3/2} + s\tau^2 + \cdots ,$$

$$\text{where} \qquad r \underset{\text{def}}{=} \frac{1}{2\pi} \int_C w^{-\frac{1}{2}} (1-w)^{-(a+b)+\frac{1}{2}} (1-2w)^{b-1} dw,$$
$$s \underset{\text{def}}{=} \frac{-1}{2\pi} \int_C w^{-\frac{3}{2}} (1-w)^{-(a+b)+\frac{3}{2}} (1-2w)^{b-1} dw.$$

Remark that r and s are holomorphic in $\{(a,b); a+b \neq 3/2, 5/2, 7/2, \dots\}$.

PROPOSITION 6.
There exists a nonzero constant C_3 such that $V_3(x) \underset{\text{def}}{=} C_3 y_3(x)$ is expanded into the form

$$V_3(x) = t\tau + 1 \cdot \tau^{3/2} + u\tau^2 + \cdots$$

$$\text{where} \qquad t \underset{\text{def}}{=} \frac{1}{2\pi} \int_C (1+w)^{a-1} w^{-1/2} (1-w)^{b-1} dw,$$
$$u \underset{\text{def}}{=} -\frac{1}{2\pi} \int_C (1+w)^{a-1} w^{-3/2} (1-w)^{b-1} dw.$$

Remark that t and u are holomorphic in $\{(a,b); b \neq 0, -1, -2, \dots\}$.

PROOF OF PROPOSITION 4.

$$y_1(x) = \int_i^\infty (u-i)^{a-1} u^{\frac{1}{2}-(a+b)} (u+i)^{b-1} (u-x)^{-1} du.$$

Put $u = i/w$, $x = i/\tau$. Then

$$\int_{u=i}^{\infty} = \int_{w=1}^{0}, \quad du = -\frac{i}{w^2} dw.$$

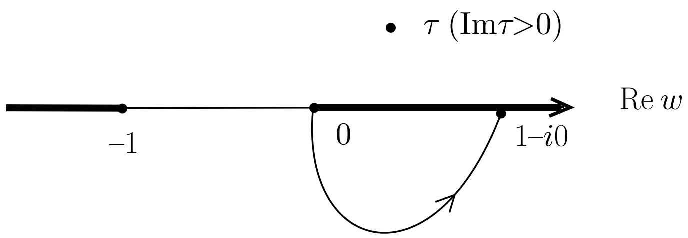

Figure 3

The path of integration was taken in $\mathrm{Re}\, u \leq 0$, which corresponds to $\mathrm{Im}\, w \leq 0$ (Figure 3).

We have

$$\begin{aligned} &y_1(x) \\ &= \text{const} \int_0^1 (\frac{1}{w} - 1)^{a-1} w^{a+b-\frac{1}{2}} (\frac{1}{w} + 1)^{b-1} (\frac{1}{w} - \frac{1}{\tau})^{-1} \frac{dw}{w^2} \\ &= \text{const} \times \tau \int_0^1 w^{\frac{1}{2}} (1-w)^{a-1} (1+w)^{b-1} (\tau - w)^{-1} dw \\ &= \text{const} \times \tau \int w^{\frac{1}{2}} (1-w)^{a-1} (1+w)^{b-1} (\tau - w)^{-1} dw. \end{aligned}$$

In the last expression, the path is the left one in Figure 4. Here remark that τ is outside the path.

The left path is homologous to the right one. The integration around τ is calculated by means of Cauchy's formula. (Take care of the orientation.)

$$\begin{aligned} \frac{y_1(x)}{\text{const}} = \tau [&- 2\pi i \tau^{1/2} (1-\tau)^{a-1} (1+\tau)^{b-1} \\ &+ \int_C w^{1/2} (1-w)^{a-1} (1+w)^{b-1} (\tau - w)^{-1} dw]. \end{aligned}$$

Let us calculate

$$I \underset{\text{def}}{=} \int_C w^{1/2} (1-w)^{a-1} (1+w)^{b-1} (\tau - w)^{-1} dw$$

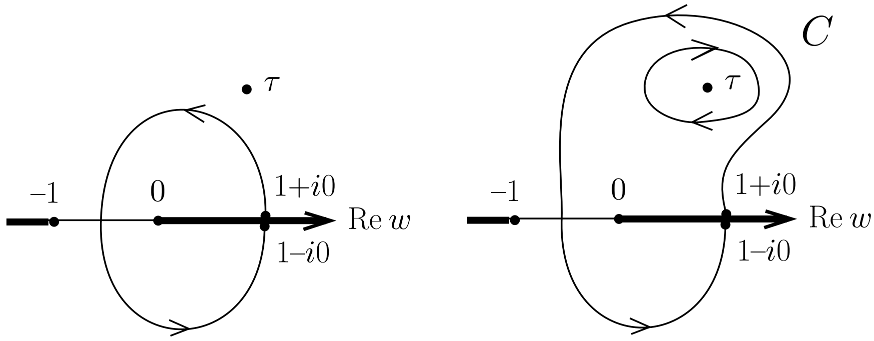

Figure 4

by deforming C. If $|\tau| \ll 1$, then on the path of integration,

$$(\tau - w)^{-1} = -w^{-1}(1 - \frac{\tau}{w})^{-1} = -w^{-1} \sum_{n=0}^{\infty} \left(\frac{\tau}{w}\right)^n.$$

Hence

$$I = I(\tau, a) = -\int_C w^{-1/2}(1-w)^{a-1}(1+w)^{b-1} \sum_{n=0}^{\infty} \left(\frac{\tau}{w}\right)^n dw.$$

We can change the order of the integration and the infinite sum at least if $\operatorname{Re} a > 0$. The proposition follows in this case.

On the other hand, $I = I(\tau, a)$ is holomorphic in τ and $a \neq 0, -1, -2, \ldots$. Here we take finite part at $w = 1$. Taylor coefficients with respect to τ is calculated by

$$\frac{1}{2\pi i} \oint \frac{1}{\tau^m} I(\tau, a) d\tau.$$

This is holomorphic in $a \neq 0, -1, -2, \ldots$. Therefore the proposition is proved for $a \neq 0, -1, -2, \ldots$ by analytic continuation with respect to a. $\square$

PROOF OF PROPOSITION 5.

$$y_2(x) = \int_0^{\infty} (u-i)^{a-1} u^{\frac{1}{2}-(a+b)} (u+i)^{b-1} (u-x)^{-1} du.$$

Set $x = i/\tau$ as before. Moreover, set $u = i - \tilde{w}$ first, and then $\tilde{w} = i/w$. The paths of integration is in $\operatorname{Re}\tilde{w} \geq 0$ and $\operatorname{Im} w \geq 0$ respectively. We have

$$\begin{aligned} y_2(x) &= \int_i^\infty (-\tilde{w})^{a-1}(i-\tilde{w})^{-(a+b)+\frac{1}{2}}(2i-\tilde{w})^{b-1}(i-x-\tilde{w})^{-1}(-d\tilde{w}) \\ &= \text{const}\int_0^1 w^{-a+1}(1-\frac{1}{w})^{-(a+b)+\frac{1}{2}}(2-\frac{1}{w})^{b-1}(\frac{1-\tau}{\tau}+\frac{1}{w})^{-1}\frac{dw}{w^2} \\ &= \text{const}\frac{\tau}{1-\tau}\int_0^1 w^{\frac{1}{2}}(1-w)^{-(a+b)+\frac{1}{2}}(1-2w)^{b-1}(w+\frac{\tau}{1-\tau})^{-1}. \end{aligned}$$

Set $\theta = \frac{\tau}{1-\tau}$, $|\tau| \ll 1, \operatorname{Im}\tau > 0$ and deform the path of integration as in Figure 5. Here -C is the path obtained by reversing the orientation of C.

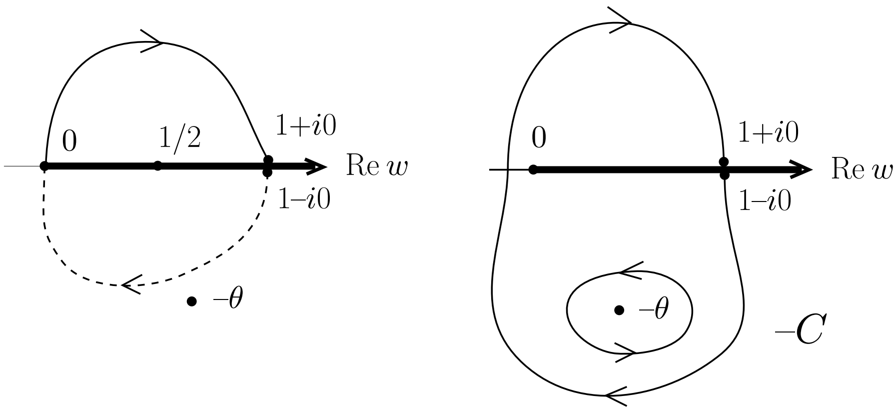

Figure 5

We have

$$\begin{aligned} \frac{y_2(x)}{\text{const}} = \theta[&2\pi i(-\theta)^{\frac{1}{2}}(1+\theta)^{-(a+b)+\frac{1}{2}}(1+2\theta)^{b-1} \\ &- \int_C w^{\frac{1}{2}}(1-w)^{-(a+b)+\frac{1}{2}}(1-2w)^{b-1}(w+\theta)^{-1}dw] \\ &(\pi < \arg(-\theta) < 2\pi). \end{aligned}$$

Here $(-\theta)^{\frac{1}{2}} = i\theta^{\frac{1}{2}}$, $0 < \arg\theta < \pi$. We expand the right hand side in powers of θ. Then the coefficients of θ, $\theta^{\frac{3}{2}}$, θ^2 are I_1, 2π and I_2, where

$$I_1 = -\int_C w^{-\frac{1}{2}}(1-w)^{-(a+b)+\frac{1}{2}}(1-2w)^{b-1}dw,$$
$$I_2 = \int_C w^{-\frac{3}{2}}(1-w)^{-(a+b)+\frac{1}{2}}(1-2w)^{b-1}dw$$

Since $\theta = \frac{\tau}{1-\tau} = \tau + \tau^2 + \tau^3 + \cdots$, $\theta^{\frac{1}{2}} = \tau^{\frac{1}{2}}(1 + \frac{1}{2}\tau + \cdots)$, the coefficients of τ, $\tau^{\frac{3}{2}}$ are I_1, -2π. On the other hand, because

$$I_1\theta + I_2\theta^2 = I_1(\tau + \tau^2 + \tau^3 + \cdots) + I_2(\tau^2 + 2\tau^3 + \cdots),$$

the coefficient of τ^2 is

$$\begin{aligned} I_1 + I_2 &= \int_C (-w+1)w^{-\frac{3}{2}}(1-w)^{-(a+b)+\frac{1}{2}}(1-2w)^{b-1}dw \\ &= \int_C w^{-\frac{3}{2}}(1-w)^{-(a+b)+\frac{3}{2}}(1-2w)^{b-1}dw. \ \square \end{aligned}$$

PROOF OF PROPOSITION 6.

$$y_3(x) = \int_{-i}^{\infty} (u-i)^{a-1}u^{\frac{1}{2}-(a+b)}(u+i)^{b-1}(u-x)^{-1}du.$$

Set $x = i/\tau$ as usual. In addition, we perform a change of variables $u = -i/w$. Then the path of integration would be in $\operatorname{Im} w \geq 0$. We have

$$\begin{aligned} y_3(x) &= \text{const} \int_1^0 (\frac{1}{w}+1)^{a-1}w^{(a+b)-\frac{1}{2}}(\frac{1}{w}-1)^{b-1}(\frac{1}{w}+\frac{1}{\tau})^{-1}\frac{dw}{w^2} \\ &= \text{const} \times \tau \int_0^1 w^{\frac{1}{2}}(1-w)^{b-1}(1+w)^{a-1}(w+\tau)^{-1}dw. \end{aligned}$$

We deform the path as in Figure 6.

Then we have

$$\begin{aligned} \frac{y_3(x)}{\text{const}} = \tau\,[&2\pi i(-\tau)^{\frac{1}{2}}(1+\tau)^{b-1}(1-\tau)^{a-1} \\ &- \int_C w^{\frac{1}{2}}(1-w)^{b-1}(1+w)^{a-1}(w+\tau)^{-1}dw], \end{aligned}$$

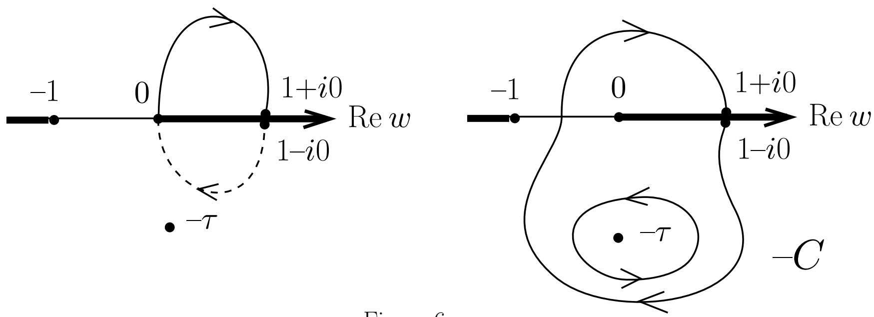

Figure 6

We expand it into a power series in τ. Since $(-\tau)^{\frac{1}{2}} = i\tau^{\frac{1}{2}}$, the coefficient of $\tau^{\frac{3}{2}}$ is -2π. Next, by using

$$(w+\tau)^{-1} = w^{-1}\sum_{n=0}^{\infty}\left(-\frac{\tau}{w}\right)^n,$$

we see that the coefficient of τ is

$$-\int_C w^{-\frac{1}{2}}(1-w)^{b-1}(1+w)^{a-1}dw,$$

and that the coefficient of τ^2 is

$$\int_C w^{-\frac{3}{2}}(1-w)^{b-1}(1+w)^{a-1}dw. \ \square$$

In the following section, we use the complex variable $\zeta = x$. ζ is to be the dual variable of t.

§3 proof of the unperturbed case

Let us consider

$$-\frac{1}{8}x_1^3P(x,D) = -\frac{1}{8}x_1^3[D_1^3 - x_1^2D_n^2D_1 + 2(a-b)D_nD_1 + \{2(a+b)-3\}x_1D_n].$$

Set $t = \frac{1}{2}x_1^2$ and apply the quantized Legendre transform $\mathcal{L}$. Since

$$\frac{1}{2}x_1D_1 = tD_t,\ tD_t \mapsto -(\zeta D_\zeta + 1),\ tD_n \mapsto -iD_\zeta,$$

we have

$$\begin{aligned}
-\frac{1}{8}x_1^3D_1^3 &= -\frac{1}{8}x_1D_1(x_1D_1-1)(x_1D_1-2)\\
&= -\frac{1}{8}\cdot 2tD_t(2tD_t-1)(2tD_t-2)\\
&= -tD_t(tD_t-\frac{1}{2})(tD_t-1)\\
&\mapsto (\zeta D_\zeta+1)(\zeta D_\zeta+\frac{3}{2})(\zeta D_\zeta+2)\\
&= \zeta^3D_\zeta^3+\frac{15}{2}\zeta^2D_\zeta^2+12\zeta D_\zeta+3,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{8}x_1^5D_n^2D_1 &= (\frac{1}{2}x_1^2D_n)^2\frac{1}{2}x_1D_1 = (tD_n)^2tD_t\\
&\mapsto -D_\zeta^2(-1)(\zeta D_\zeta+1) = D_\zeta^2(\zeta D_\zeta+1)\\
&= \zeta D_\zeta^3+3D_\zeta^2,
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{4}(a-b)x_1^3D_nD_1 &= -(a-b)\cdot\frac{1}{2}x_1^2D_n\cdot\frac{1}{2}x_1D_1\\
&= -(a-b)tD_n\cdot tD_t\\
&\mapsto -(a-b)(-iD_\zeta)(-1)(\zeta D_\zeta+1)\\
&= -i(a-b)D_\zeta(\zeta D_\zeta+1)\\
&= -i(a-b)(\zeta D_\zeta^2+2D_\zeta),
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{8}x_1^3\{2(a+b)-3\}x_1D_n^2 &= -\frac{1}{2}\{2(a+b)-3\}(\frac{1}{2}x_1^2D_n)^2\\
&= -\frac{1}{2}\{2(a+b)-3\}(tD_n)^2\\
&\mapsto -\frac{1}{2}\{2(a+b)-3\}(-iD_\zeta)^2\\
&= \frac{1}{2}\{2(a+b)-3\}D_\zeta^2.
\end{aligned}$$

Summing up, we obtain from $-\frac{1}{8}x_1^3P$

$$Q(a,b,\zeta,D_\zeta) \underset{\mathrm{def}}{=} (\zeta^3+\zeta)D_\zeta^3+\{\frac{15}{2}\zeta^2-i(a-b)\zeta+a+b+\frac{3}{2}\}D_\zeta^2$$

$$+\{12\zeta - 2i(a-b)\}D_\zeta + 3.$$

We encountered this operator in the previous section. $V_j(\zeta)$ is a solution to it.

In the same way as in the second order case, we can construct $E_j^{\pm}$ from $V_j(\zeta)$. $L^{\pm}$ and B are defined accordingly. Let the expansion of $V_j(\zeta)$ (j= 1,2,3) at $\zeta = \infty$, $\operatorname{Re}\zeta > 0$ be

$$V_j(\zeta) = \sum_{n=0}^{\infty} V_{j,-1-\frac{n}{2}} \zeta^{-1-\frac{n}{2}}.$$

Then, the matrix V, defined by

$$\begin{pmatrix} V_{1,-1} & V_{2,-1} & V_{3,-1} \\ V_{1,-\frac{3}{2}} & V_{2,-\frac{3}{2}} & V_{3,-\frac{3}{2}} \\ V_{1,-2} & V_{2,-2} & V_{3,-2} \end{pmatrix}$$

is

$$V = \begin{pmatrix} i & 0 & 0 \\ 0 & \exp(\frac{3}{4}\pi i) & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} p & r & t \\ 1 & 1 & 1 \\ q & s & u \end{pmatrix}.$$

$L^{\pm}$ is expressed by

$$L^{\pm} = \begin{pmatrix} 2\pi & 0 & 0 \\ 0 & \pm 2\sqrt{2\pi}\left(\frac{D_n}{i}\right)^{\frac{1}{2}} & 0 \\ 0 & 0 & 2\pi\frac{D_n}{i} \end{pmatrix} V.$$

Moreover, we have

$$\begin{aligned} B \underset{\text{def}}{=} (L^-)^{-1}L^+ &= \begin{pmatrix} p & r & t \\ 1 & 1 & 1 \\ q & s & u \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p & r & t \\ 1 & 1 & 1 \\ q & s & u \end{pmatrix} \\ &= \frac{-1}{d_1+d_2+d_3} \begin{pmatrix} d_1 - d_2 - d_3 & 2d_1 & 2d_1 \\ 2d_3 & -d_1 - d_2 + d_3 & 2d_3 \\ 2d_2 & 2d_2 & -d_1 + d_2 - d_3 \end{pmatrix}. \end{aligned}$$

where

$$d_1 = \begin{vmatrix} r & t \\ s & u \end{vmatrix},\ d_2 = \begin{vmatrix} p & r \\ q & s \end{vmatrix},\ d_3 = \begin{vmatrix} t & p \\ u & q \end{vmatrix}.$$

This is checked easily by using the observation that

$$pd_1 + rd_3 + td_2 = qd_1 + sd_3 + ud_2 = 0.$$

PROPOSITION 7.
d_1, d_2 and d_3 are holomorphic functions in

$$\mathbb{C}^2 \backslash Z = \{a \neq 0, -1, -2, \dots\} \cap \{b \neq 0, -1, -2, \dots\} \cap \{a + b \neq \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots\}.$$

Moreover, they do not vanish identically. (Hence generically $d_1, d_2, d_3 \neq 0$.)

PROOF. They are obviously holomorphic in $\mathbb{C}^2 \backslash Z$. The latter part of the proposition follows from Propositions 10, 14 and 15 below. □

PROPOSITION 8.
$\pm d_1 \pm d_2 \pm d_3$ does not vanish identically.

PROOF. This proposition follows immediately from Proposition 10 below. □

REMARK 9.

$$d_1 + d_2 + d_3 = -\begin{vmatrix} p & r & t \\ 1 & 1 & 1 \\ q & s & u \end{vmatrix}$$

never vanishes, because the components of the matrix are expansion coefficients of three linearly independent solutions.

PROPOSITION 10.
If $(a, b) \in \mathbb{N} \times \mathbb{N}$, $\mathbb{N} = \{1, 2, 3, \dots\}$, then $r = s = 0$. (Hence $d_1 = d_2 = 0$ and $d_3 = d_1 + d_2 + d_3 \neq 0$.)

PROOF. Let h be $1/2$ or $3/2$. Then

$$\begin{aligned} &-\frac{1}{2} \int_C w^{-h}(1-w)^{-(a+b)+h}(1-2w)^{b-1} dw \\ &= \int_0^1 w^{-h}(1-w)^{-(a+b)+h}(1-2w)^{b-1} dw. \end{aligned}$$

Since $(1-2w)^{b-1}$ is a polynomial of degree $b-1$, it suffices to prove

$$\int_0^1 w^{c-h}(1-w)^{-(a+b)+h}dw = 0 \quad \text{for} \quad c = 0, 1, 2, \ldots, b-1.$$

The left hand side is equal to

$$B(c-h+1, -a-b+h+1) = \frac{\Gamma(c-h+1)\Gamma(-a-b+h+1)}{\Gamma(-a-b+c+2)}.$$

Here the numerator is finite. The denominator is infinite because $-a-b+c+2$ is a nonpositive integer. □

Proposition 11.
Under the condition of the proposition above, we have

$$B = \frac{-1}{d_3}\begin{pmatrix} -d_3 & 0 & 0 \\ 2d_3 & d_3 & 2d_3 \\ 0 & 0 & -d_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition 12.
If $b \in \mathbb{N}$ and $a+b = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots,$ then $p = q = 0$.

Proof. Let h be $1/2$ or $3/2$. Since $(1+w)^{b-1}$ is a polynomial of degree $b-1$, it suffices to prove

$$\int_0^1 (1-w)^{a-1}w^{-h}w^c dw = 0 \quad \text{for} \quad c = 0, 1, 2, \ldots, b-1.$$

The left hand side is equal to

$$B(c-h+1, a) = \frac{\Gamma(c-h+1)\Gamma(a)}{\Gamma(a+c-h+1)}.$$

Here $a+c-h+1$ is an integer such that

$$a+c-h+1 \leq (\frac{1}{2}-b)+(b-1)-h+1 = \frac{1}{2}-h \leq 0. \ \square$$

PROPOSITION 13.
If $a \in \mathbb{N}$ *and* $a+b=\frac{1}{2},-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\ldots,$ *then* $t=u=0$.

PROOF. As functions of (a,b), p,q,t,u satisfies

$$p(b,a)=t(a,b), \quad q(b,a)=-u(a,b). \ \square$$

PROPOSITION 14.
Under the condition of Proposition 12, we have $d_2=d_3=0$. *Hence* $d_1=d_1+d_2+d_3\neq 0$ *and*

$$B=\frac{-1}{d_1}\begin{pmatrix} d_1 & 2d_1 & 2d_1 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_1 \end{pmatrix}=\begin{pmatrix} -1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

PROPOSITION 15.
Under the condition of Proposition 13, we have $d_1=d_3=0$. *Hence* $d_2=d_1+d_2+d_3\neq 0$ *and*

$$B=\frac{-1}{d_2}\begin{pmatrix} -d_2 & 0 & 0 \\ 0 & -d_2 & 0 \\ 2d_2 & 2d_2 & d_2 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & -1 \end{pmatrix}.$$

PROOF OF THEOREM F.
(1) follows from Proposition 7. (2) (3) and (4) follow from Propositions 14, 11 and 15 respectively. $\square$

PROOF OF THEOREM D.
Recall that

$$L^{\pm}=\begin{pmatrix} 2\pi & 0 & 0 \\ 0 & \pm 2\sqrt{2\pi}\left(\frac{D_n}{i}\right)^{\frac{1}{2}} & 0 \\ 0 & 0 & 2\pi\frac{D_n}{i} \end{pmatrix}\begin{pmatrix} i & 0 & 0 \\ 0 & \exp(\frac{3}{4}\pi i) & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$\cdot\begin{pmatrix} p & r & t \\ 1 & 1 & 1 \\ q & s & u \end{pmatrix}.$$

We have the following commutative diagram:

$$\begin{array}{ccc} (\Gamma_{\{\pm x_1>0\}}\mathcal{C}_M^P)_p = \oplus_{j=1}^3 Sol(j,\pm) & \xleftarrow{E^\pm \sim} & \overset{3}{\oplus}\mathcal{C}_{N,p'} \\ \text{b.v.}\wr\Big\downarrow & & \Big\downarrow\wr L^\pm \\ \overset{3}{\oplus}\mathcal{C}_{N,p'} & = & \overset{3}{\oplus}\mathcal{C}_{N,p'} \end{array}$$

where $E^\pm = \oplus_{j=1}^3 E_j^\pm$ and the first vertical arrow is

$$u(x) \mapsto (u(\pm 0, x'), D_1 u(\pm 0, x'), D_1^2 u(\pm 0, x')).$$

By $E^\pm$ we identify $Sol\,(j,\pm)$ with $\mathcal{C}_{N,p'} \oplus 0 \oplus 0$, $0 \oplus \mathcal{C}_{N,p'} \oplus 0$, $0 \oplus 0 \oplus \mathcal{C}_{N,p'}$ (j=1, 2, 3). So, in order to prove Theorem D, we have only to prove that the following maps $\in \mathrm{End}(\mathcal{C}_{N,p'})$ induced by $L^\pm$ are automorphisms.

$$\underline{j=1} \quad f \mapsto \text{the second component of } L^\pm \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{j=2} \quad f \mapsto \text{the second component of } L^\pm \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}$$

$$\underline{j=3} \quad f \mapsto \text{the second component of } L^\pm \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}.$$

For all j, these maps coincide with

$$f \mapsto \pm 2\sqrt{2\pi}\left(\frac{D_n}{i}\right)^{\frac{1}{2}} \cdot \exp(\frac{3}{4}\pi i) \cdot f.$$

They are obviously isomorphisms. The key is that 1 is an elliptic operator. □

Proof of the Remark after Theorem D.

We follow nearly the same argument as above. The main difference is that we need the non-vanishing of p, r and t in the Dirichlet case, and that of q, s and u in the case of $D_1^2 u(\pm 0, x')$ for a generic (a, b). They follow from the lemma below. □

LEMMA 16.
p, q, r, s, t and u are holomorphic functions in $(a, b) \in \mathbb{C}^2 \setminus Z$ which don't vanish identically.

PROOF.
$p, q, t, u \neq 0$ at $(a, b) = (1, 1)$. $r, s \neq 0$ if $b = 1$ and a is a half-integer. (Use formulas about the Beta and the Gamma functions). □

PROOF OF THEOREM E.
Use the same identification as in the proof of Theorem D. □

§4 proof of the perturbed case

4-1 the method of the variation of parameters

Let us consider

$$Q(x, D) = (x^3+x)D^3+\{\frac{15}{2}x^2-i(a-b)x+a+b+\frac{3}{2}\}D^2+\{12x-2i(a-b)\}D+3,$$

$$a, b \notin \mathbb{Z} \quad \text{and} \quad a+b \notin \frac{1}{2}+\mathbb{Z}.$$

Its Riemann scheme is

$$\left\{ \begin{array}{ccccc} i & 0 & -i & \infty & \\ 0 & 0 & 0 & 1 & x \\ 1 & 1 & 1 & \frac{3}{2} & \\ a-1 & \frac{1}{2}-(a+b) & b-1 & 2 & \end{array} \right\},$$

and all the singularities are non-logarithmic. Let $p = i, 0, -i$ and φ_1, φ_2 and φ_3 be solutions in a neighborhood of p. We assume that φ_1 and φ_2 are of exponent 0, 1 and that φ_3 is of exponent $a-1$, $\frac{1}{2}-(a+b)$, $b-1$ if $p = i, 0, -i$ respectively. Set

$$W(x) = \begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_1' & \varphi_2' & \varphi_3' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' \end{vmatrix}, \quad W_{jk}(x) = \begin{vmatrix} \varphi_j & \varphi_k \\ \varphi_j' & \varphi_k' \end{vmatrix}.$$

By using the classical method of the variation of parameters, we see easily that for any $v(x)$, holomorphic near p,

$$\begin{aligned} I_p(v) \underset{\text{def}}{=} \quad & \varphi_1(x) \int_p^x \frac{W_{23}(y)}{(y^3+y)W(y)} v(y)dy \\ & + \varphi_2(x) \int_p^x \frac{W_{31}(y)}{(y^3+y)W(y)} v(y)dy \\ & + \varphi_3(x) \int_p^x \frac{W_{12}(y)}{(y^3+y)W(y)} v(y)dy \end{aligned}$$

is a holomorphic function near p such that

$$Q[I_p(v)(x)] = v(x).$$

Moreover, we see that

$$I_p(v)(p) = \{I_p(v)\}'(p) = 0,$$

that is, $I_p(v)$ is of exponent $\in 2 + \mathbb{N}_0$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

4-2 a right inverse of Q in a domain containing two regular singular points

Let Ω be a domain $\subset \mathbb{C}_x$ as in Figure 7.

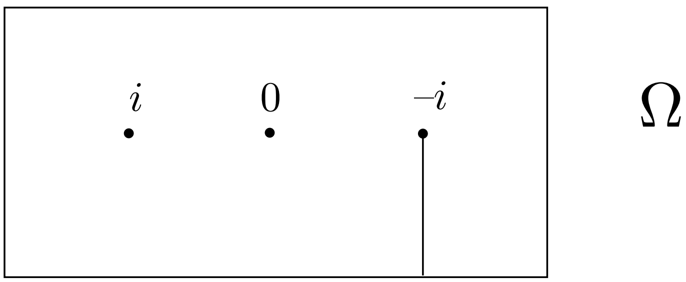

Figure 7

Obviously Q defines a linear mapping

$$Q:\,\mathcal{O}(\Omega)\to\mathcal{O}(\Omega).$$

We want to construct a right inverse of this. Let $v(x)$ be an element of $\mathcal{O}(\Omega)$. Although $I_i v$ is holomorphic near i, there's no guarantee that it should be holomorphic near 0. We have a similar trouble with $I_0 v$. In order to overcome this difficulty, we use the following trick.

Let

$$\Omega_i \underset{\text{def}}{=} \{x\in\Omega;\operatorname{Im}x>\frac{1}{3}\}\ni i,\quad \Omega_0 \underset{\text{def}}{=} \{x\in\Omega;\operatorname{Im}x<\frac{2}{3}\}\ni 0.$$

Obviously, these two domains constitute a covering of Ω and

$$I_i v - I_0 v \in \mathcal{O}^Q(\Omega_i\cap\Omega_0),$$

where $\mathcal{O}^Q$ is the kernel sheaf of $Q\in\mathcal{E}nd_{\mathbb{C}}(\mathcal{O})$.

If $\{F_1,F_2,F_3\}$ is a fundamental system of solutions to Q in $\Omega_i\cap\Omega_0$, then there exists a unique triple of constants $(\alpha,\beta,\gamma)\in\mathbb{C}^3$ such that

$$I_i v - I_0 v = \alpha F_1+\beta F_2+\gamma F_3.$$

Then obviously

$$(I_i v - I_0 v)' = \alpha F_1'+\beta F_2'+\gamma F_3'$$
$$(I_i v - I_0 v)'' = \alpha F_1''+\beta F_2''+\gamma F_3''.$$

Therefore

$$(*)\quad \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = W_F^{-1}\begin{pmatrix} I_i v - I_0 v\\ (I_i v - I_0 v)'\\ (I_i v - I_0 v)''\end{pmatrix},\quad\text{where}\quad W_F=\begin{pmatrix}F_1 & F_2 & F_3\\ F_1' & F_2' & F_3'\\ F_1'' & F_2'' & F_3''\end{pmatrix}.$$

Since the right hand side of (*) become a constant vector, it can be evaluated and estimated at any x. This observation will be useful later. We define functionals

$$\alpha,\beta,\gamma:\mathcal{O}(\Omega)\to\mathbb{C}$$
$$v\mapsto\alpha,\beta,\gamma$$

by means of (*). They mean obstructions. We have to kill them.

LEMMA 17.
For a generic (a,b),

$$M \underset{\text{def}}{=} \begin{pmatrix} \alpha(Q[1]) & \alpha(Q[x]) & \alpha(Q[x^2]) \\ \beta(Q[1]) & \beta(Q[x]) & \beta(Q[x^2]) \\ \gamma(Q[1]) & \gamma(Q[x]) & \gamma(Q[x^2]) \end{pmatrix}$$

is invertible.

PROOF. We prove that $M^t(\lambda,\mu,\nu) = \vec{0}$ implies ${}^t(\lambda,\mu,\nu) = \vec{0}$. The assumption is that

$$\alpha(Q[\lambda+\mu x+\nu x^2]) = \beta(Q[\lambda+\mu x+\nu x^2]) = \gamma(Q[\lambda+\mu x+\nu x^2]) = 0.$$

This means that $I_i(Q[\lambda+\mu x+\nu x^2])$ and $I_0(Q[\lambda+\mu x+\nu x^2])$ are patched together and define $\psi(x) \in \mathcal{O}(\Omega)$. We have $Q\psi = Q[\lambda+\mu x+\nu x^2]$. By the way, we proved before that $\mathcal{O}^Q(\Omega)$ is a one-dimensional space generated by $y_3(x)$. So there is a constant c such that

$$\psi - (\lambda+\mu x+\nu x^2) = cy_3(x).$$

Later we'll prove that c is generically 0. Once we have obtained this, it is clear that

$$\lambda+\mu x+\nu x^2 \in \mathrm{Image} I_i \cap \mathrm{Image} I_0.$$

Therefore it has a zero of order ≥ 2 at $x = i, 0$. Such a polynomial of degree ≤ 2 must vanish identically. Hence $\lambda = \mu = \nu = 0$.

Now what remains to be proved is that c is 0 for a generic (a,b). First we prove that if

$$(\#) \qquad 2iy_3(0) - y_3'(0) - 2iy_3(i) - y_3'(i) \neq 0$$

then $c = 0$. In fact, if $c \neq 0$,

$$\frac{\lambda+\mu x+\nu x^2}{c} + y_3(x) \left(= \frac{1}{c}\psi(x) \right)$$

has a zero of order ≥ 2 at $x = i, 0$. Set $\lambda' = -\lambda/c, \mu' = -\mu/c, \nu' = -\nu/c$. Then

$$\begin{aligned}
&\lambda' = y_3(0),\\
&\mu' = y_3'(0),\\
&\lambda' + i\mu' - \nu' = y_3(i),\\
&\mu' + 2i\nu' = y_3'(i).
\end{aligned}$$

From these we obtain

$$2iy_3(0) - y_3'(0) - 2iy_3(i) - y_3'(i) = 0,$$

which constradicts (#).

Finally, we prove that (#) holds for a generic (a, b). Since the left hand side of (#) is holomorphic in (a, b), we have only to prove that it is different from 0 for some (a, b).

Recall that

$$y_3(x) = \int_{-i}^{\infty} (u-i)^{a-1} u^{-(a+b)+\frac{1}{2}} (u+i)^{b-1} (u-x)^{-1} du.$$

Hence

$$y_3'(x) = \int_{-i}^{\infty} (u-i)^{a-1} u^{-(a+b)+\frac{1}{2}} (u+i)^{b-1} (u-x)^{-2} du.$$

Let us prove that if $(a, b) = (3, -\frac{7}{2})$, we have

$$y_3(0) = y_3(i) = y_3'(i) = 0,\ y_3'(0) \neq 0.$$

Set

$$B_3(p, q) = \int_{-i}^{\infty} (u-i)^p u^q (u+i)^{b-1} du.$$

Then at $(a, b) = (3, -\frac{7}{2})$,

$$y_3(0) = B_3(a-1, -(a+b) - \frac{1}{2}) = B_3(2, 0),$$

$$\begin{aligned} y_3'(0) &= B_3(a-1, -(a+b) - \frac{3}{2}) = B_3(2,-1), \\ y_3(i) &= B_3(a-2, -(a+b) + \frac{1}{2}) = B_3(1,1), \\ y_3'(i) &= B_3(a-3, -(a+b) + \frac{1}{2}) = B_3(0,1). \end{aligned}$$

By using a change of variables $u = -i/w$, we have

$$\begin{aligned} B_3(p,q) &= \text{const} \int_0^1 \left(\frac{w+1}{w}\right)^p w^{-q} \left(\frac{1-w}{w}\right)^{b-1} \frac{dw}{w^2} \\ &= \text{const} \int_0^1 (1+w)^p w^{-p-q-b-1} (1-w)^{b-1} dw. \end{aligned}$$

Hence at $(a,b) = (3, -\frac{7}{2})$,

$$\begin{aligned} y_3(0) &= \text{const} \int_0^1 (1+w)^2 w^{\frac{1}{2}} (1-w)^{-\frac{9}{2}} dw = 0, \\ y_3'(0) &= \text{const} \int_0^1 (1+w)^2 w^{\frac{3}{2}} (1-w)^{-\frac{9}{2}} dw \neq 0, \\ y_3(i) &= \text{const} \int_0^1 (1+w) w^{\frac{1}{2}} (1-w)^{-\frac{9}{2}} dw = 0, \\ y_3'(i) &= \text{const} \int_0^1 w^{\frac{3}{2}} (1-w)^{-\frac{9}{2}} dw = 0. \end{aligned}$$

This concludes the proof of the lemma. $\square$

We can define functionals $\lambda, \mu, \nu : \mathcal{O}(\Omega) \to \mathbb{C}$ by

$$\begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} = M^{-1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

It is easy to check that

$$(\natural) \qquad \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \circ Q \circ (1, x, x^2) \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} : \mathcal{O}(\Omega) \to \mathbb{C}^3.$$

Here the left hand side means

$$v \mapsto \begin{pmatrix} \alpha(Q[\lambda(v) + \mu(v)x + \nu(v)x^2]) \\ \beta(Q[\lambda(v) + \mu(v)x + \nu(v)x^2]) \\ \gamma(Q[\lambda(v) + \mu(v)x + \nu(v)x^2]) \end{pmatrix}.$$

Now set

$$\begin{aligned} \pi(v) &= v - Q[\lambda(v) + \mu(v)x + \nu(v)x^2] \\ &= v - \lambda(v)Q[1] - \mu(v)Q[x] - \nu(v)Q[x^2]. \end{aligned}$$

Then ($\natural$) implies that

$$I_i\pi(v) - I_0\pi(v) = 0.$$

We can define

$$\tilde{I}_{i,0} = \begin{cases} I_i\pi(v) & \text{on}\,\Omega_i, \\ I_0\pi(v) & \text{on}\,\Omega_0. \end{cases}$$

$\tilde{I}_{i,0} : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega)$ is a well-defined linear mapping. Next we define $I_{i,0} : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega)$ by

$$I_{i,0} = \tilde{I}_{i,0}(v) + \lambda(v) + \mu(v)x + \nu(v)x^2.$$

LEMMA 18.
$QI_{i,0}(v) = v$. That is, $I_{i,0}$ is a right inverse of Q.

PROOF. We have

$$\begin{aligned} QI_{i,0}(v) &= \pi(v) + Q[\lambda(v) + \mu(v)x + \nu(v)x^2] \\ &= v \ \square \end{aligned}$$

4-3 an estimate on the right inverse $I_{i,0}$

First, let us obtain an estimate on α, β and γ. Fix compact subsets $K_i \ni i$, $K_0 \ni 0$ of Ω such that $\mathrm{int}(K_i \cap K_0) \neq \phi$. Choose an arbitrary point $\dot{x}$ in int $(K_i \cap K_0)$. Then

$$\begin{pmatrix} \alpha(v) \\ \beta(v) \\ \gamma(v) \end{pmatrix} = W_F(\dot{x})^{-1} \begin{pmatrix} (I_iv - I_0v)(\dot{x}) \\ (I_iv - I_0v)'(\dot{x}) \\ (I_iv - I_0v)''(\dot{x}) \end{pmatrix}.$$

By the way, as in the second order case, we can prove that there exists a constant $C > 0$ such that

$$\sup_{K_i}|I_i v| \leq C\sup_{K_i}|v|,\ \sup_{K_0}|I_0 v| \leq C\sup_{K_0}|v|.$$

Hence, for a larger C, we have

$$|\alpha(v)|,\ |\beta(v)|,\ |\gamma(v)|\ \leq C \sup_{K_i\cup K_0}|v|.$$

Therefore, again for a larger C, we have

$$|\lambda(v)|,\ |\mu(v)|,\ |\nu(v)|\ \leq C \sup_{K_i\cup K_0}|v|.$$

Set $K_\delta = \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) \geq \delta\}\,(0 < \delta \ll 1)$. Then there exists a constant C' independent of δ such that

$$\sup_{K_\delta}|\pi(v)| \leq C\sup_{K_\delta}|v|.$$

Finally let us obtain an estimate on $I_{i,0}$. Set

$$K_\delta^{(i)} = K_\delta \cap \{\mathrm{Im}\, x \geq \frac{1}{2}\},\ K_\delta^{(0)} = K_\delta \cap \{\mathrm{Im}\, x \leq \frac{1}{2}\}.$$

We derive an estimate on $K_\delta^{(p)}$ $(p = 0, i)$ from the expression

$$I_{i,0} = I_p\pi(v) + \lambda(v) + \mu(v)x + \nu(v)x^2.$$

$$(p = 0, i)$$

There exist constants $\lambda, C_\Omega > 0$, independent of δ, such that

$$\sup_{K_\delta}|I_{i,0}(v)| \leq C_\Omega\delta^{-\lambda}\sup_{K_\delta}|v|.$$

(λ is determined by the characteristic exponents at $x = -i$, hence by b). This has the same form as the estimate on G^{-1} in the second order case.

8-4 the end of the proof

Let us construct E_1^+. Let us consider

$$P(x,D) = D_1^3 - x_1^2 D_n^2 D_1 + 2(a-b) D_n D_1 + \{2(a+b)-3\} x_1 D_n^2 + P'(x,D)$$

$$P'(x,D) = \sum_{l=0}^{\text{finite}} \alpha_{-l}(x_1^2, x', D') x_1^{l+1} D_1^l.$$

Set $t = \frac{1}{2}x_1^2$ and apply the quantized Legendre transform $\mathcal{L}$. From $x_1^3 P$, we obtain $Q + Q'$, where

$$\begin{aligned} Q &= Q(\zeta, D_\zeta) \\ &= (\zeta^3+\zeta)D_\zeta^3 + \{\frac{15}{2}\zeta^2 - i(a-b)\zeta + a + b + \frac{3}{2}\}D_\zeta^2 \\ &\quad + \{12\zeta - 2i(a-b)\}D_\zeta + 3, \end{aligned}$$

$$Q' = Q'(\zeta, x', D_\zeta, D') = \sum_{m=2}^{\text{finite}} \sum_{j=0}^{m-2} \alpha_{m,j}(x', D')\zeta^j D_\zeta^m,$$

$$\text{ord}\alpha_{m,j} \le -m-1.$$

Here we have used Part 1 Lemma 4.

With $I_{i,0}$ instead of G^{-1}, we can calculate in the same way as in the second order case. (We don't change ζ by another complex variable).

The other $E_j^\pm$'s are constructed similarly.

§5 proof of the case a and b are replaced by operators

In this case, when we perform the process as in the begining of §3, we obtain the operator

$$\begin{aligned} &(\zeta^3+\zeta)D_\zeta^3 \\ &\quad + \{\frac{15}{2}\zeta^2 - i\tilde{a}(x',D')\zeta + i\tilde{b}(x',D')\zeta + \tilde{a}(x',D') + \tilde{b}(x',D') + \frac{3}{2}\}D_\zeta^2 \\ &\quad + \{12\zeta - 2i\tilde{a}(x',D') + 2i\tilde{b}(x',D')\}D_\zeta + 3, \end{aligned}$$

which we denote by $Q\left(\tilde{a}(x',D'), \tilde{b}(x',D'), \zeta, D_\zeta\right)$. Recall that we have

$$Q(a,b,\zeta,\partial_\zeta)V_j(a,b,\zeta) = 0.$$

Here we write $V_j(a,b,\zeta)$ instead of $V_j(\zeta)$ to specify a,b. Remark that V_j is holomorphic not only in ζ but also in (a,b). So we can substitute the commutative pair of operators $(\tilde{a}(x',D'),\tilde{b}(x',D'))$ into (a,b) and obtain $V_j(\tilde{a}(x',D'),\tilde{b}(x',D'),\zeta)\in\mathcal{E}(0)$. Obviously, for all $f(x')\in\mathcal{C}_{N,p'}$, we have

$$Q(\tilde{a},\tilde{b},\zeta,\partial_\zeta)[V_j(\tilde{a}(x',D'),\tilde{b}(x',D'),\zeta)f(x')]=0.$$

We can easily construct $E_j^{\pm}$, $L^{\pm}$ and B in this context. For example, we have

$$L^{\pm}=\begin{pmatrix} 2\pi & 0 & 0 \\ 0 & \pm 2\sqrt{2\pi}\left(\frac{D_n}{i}\right)^{\frac{1}{2}} & 0 \\ 0 & 0 & 2\pi\frac{D_n}{i}\end{pmatrix}\begin{pmatrix} i & 0 & 0 \\ 0 & \exp(\frac{3}{4}\pi i) & 0 \\ 0 & 0 & -1\end{pmatrix}$$

$$\times\begin{pmatrix} \tilde{p}(x',D') & \tilde{r}(x',D') & \tilde{t}(x',D') \\ 1 & 1 & 1 \\ \tilde{q}(x',D') & \tilde{s}(x',D') & \tilde{u}(x',D')\end{pmatrix},$$

where

$$\tilde{p}(x',D')=p(\tilde{a}(x',D'),\tilde{b}(x',D'))$$

etc. It is obvious that

$$\sigma_0(\tilde{p}(x',D'))=p(\sigma_0(\tilde{a}(x',D')),\sigma_0(\tilde{b}(x',D'))),\quad \text{etc.}$$

This observation is used to prove the ellipticity of the components of $L^{\pm}$ and B. The remaining part of the proof of Theorems D", E" and F" is easy.

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