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On invertibility of the windowed Radon transform

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Abstract. The idea of the Windowed Radon Transform (WRT) was first introduced by G.Kaiser. This transform is interpreted as a generalization of several transforms. In this article, taking WRT for a generalized wavelet transform, we show that its invertion formula holds in various senses.

§0. Introduction

In this paper we study invertibility of the transform called the windowed Radon transform (WRT). This is a tranform defined as

$$R_h f(x, A) \equiv \int_{\boldsymbol{R}^d} \overline{h(t)} f(x + At) dt,$$

where h is defined on $\boldsymbol{R}^d$, f is on $\boldsymbol{R}^n$ and $A = vRJ$, $v > 0$, $R \in SO(n)/SO(n-d)$, J is an inclusion. This concept was originally introduced by G.Kaiser as the windowed x-ray transform (WXT) as a generalization of Analytic-Signal transform to extend physical fields to complex space-time (cf. [Ka1] and [Ka2]). After that he and R.Streater [KS] generalized it to WRT so as to be considered as a generalization of the wavelet transforms. In case of $n = d$ by a suitable change of variables we can show that WRT is almost the same as the wavelet transform (Proposition 1.6 below), which implies that WRT is a generalization of a wavelet transform. Further, this can be taken as generalization of other transforms. Therefore the study of this transform seems to be of interest. Here in this paper we take WRT for a generalization of the wavelet transform. First we give the definition of

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WRT and argue its relationship with other transforms. Then, restricting window functions, we unify the inversion formulas which were established by G.Kaiser-R.Streater (Theorem 2.1 below). Their theorem made use of all the data $\{R_h f(x,A)|\ x \in \boldsymbol{R}^n, v > 0, R \in SO(n)/SO(n-d)\}$ for invertion, but here we claim that $\{R_h f(x,A)|\ x \in \boldsymbol{R}^n, v > 0, R \in SO(d)\}$ are enough to reconstruct f. Additionary, we generalize and extend their meanings. Though the formula established by Kaiser-Streater holds in a very weak sense (Theorem 2.0), in fact, a little improvement impies the validity as the limit in the mean of L^2 norm (Proposition 2.4) and pointwise (Theorems 2.5, 2.7 and 2.8). Our invertion formula holds as the limit in the mean of L^2 norm, as mentioned above, however, it does not as L^1 mean, which we show by constructing counterexamples (Section 3).

Before finishing Introduction, the author would like to express his truthful appreciation to Professor Akira Kaneko for having valuable discussions and continuous encouragement.

§1. Notations

First of all, let us define windowed Radon transforms (WRT).

DEFINITION 1.1. Let $1 \leq d \leq n$. The d dimensional windowed Radon transform $R_h f(x,A)$ of a function f defined on $\boldsymbol{R}^n$ is

$$R_h f(x,A) \equiv \int_{\boldsymbol{R}^d} \overline{h(t)} f(x+At)dt, \tag{1.1}$$

where h is a function defined on $\boldsymbol{R}^d$ and

$$A \in \{A = vRJ \,\big|\, J : \boldsymbol{R}^d \mapsto \boldsymbol{R}^n \text{ (inclusion)},\ R \in SO(n)/SO(n-d),\ v > 0,\\ J(x') = (x',0)\}.$$

By $\bar{h}$ we mean the complex conjugate of h. We write $R_h f(x,A) = f_h(x,A)$. Later we will prove that $R_h f \in L^2$ in a sense for $f \in L^2(\boldsymbol{R}^n)$.

Throughout this paper, we use the expression $A = vRJ$.

DEFINITION 1.2. For a function f defined on $\boldsymbol{R}^n$ we call

$$X_h f(x,\xi) \equiv \int_{\boldsymbol{R}} \overline{h(t)} f(x+t\xi)dt, \tag{1.2}$$

the windowed x-ray transform (WXT) with window h, where h is a function defined on $\boldsymbol{R}$ and $\xi \in \boldsymbol{R}^n$. From now on we utilize the notation $\xi = v\omega,\ v > 0,\ \omega \in S^{n-1}$.

WXT can be interpreted as 1 dimensional WRT, since we are capable of identifying $SO(n)/SO(n-1)$ with S^{n-1} .

Let us express WRT by Fourier transforms, for which we need some preparation.

DEFINITION 1.3. Let h_1 and h_2 belongs to $\mathcal{S}'(\boldsymbol{R}^d)$, and $\frac{\widehat{h_1}(\xi)\overline{\widehat{h_2}(\xi)}}{|\xi|^d} \in L^1$. We say (h_1, h_2) is admissible if

$$N_{h_1,h_2} \equiv \frac{1}{\omega_{d-1}} \int_{\boldsymbol{R}^d} \frac{\widehat{h_1}(\xi)\overline{\widehat{h_2}(\xi)}}{|\xi|^d} d\xi \tag{1.3}$$

converges, where $\hat{h}(\xi) \equiv \mathcal{F}_{x\to\xi} h(\xi) = \int e^{-2\pi i x\cdot\xi} h(x)dx$ is the Fourier transform of h and $\omega_{d-1} = \int_{S^{d-1}} d\omega$. Especially, h is called admissible if

$$N_h \equiv \frac{1}{\omega_{d-1}} \int_{\boldsymbol{R}^d} \frac{|\hat{h}(\xi)|^2}{|\xi|^d} d\xi \tag{1.4}$$

is convergent.

Now let us express f_h by Fourier transforms. Here we do not mind the conditions to be assigned on f, h. Define

$$\hat{h}_{x,A}(\xi) \equiv e^{-i\xi\cdot x} \hat{h}({}^t A\xi), \tag{1.5}$$

where ${}^tA$ is the transposed transform of A (if A is an $n \times d$ matrix then ${}^tA$ is a $d \times n$ matrix), and

$$h_{x,A}(y) \equiv \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^n} e^{i\xi\cdot(y-x)} \hat{h}({}^tA\xi) d\xi. \tag{1.6}$$

This defines a generalized wavelet. We have

$$
\begin{aligned}
(1.7) \qquad R_h f(x, A) &= \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^d} dt \int_{\boldsymbol{R}^n} e^{i\xi\cdot(x+At)} \overline{h(t)} \hat{f}(\xi) d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^n} e^{i\xi\cdot x} \overline{\hat{h}({}^t A\xi)} \hat{f}(\xi) d\xi \\
&= \frac{1}{(2\pi)^n} \langle \hat{f}, \hat{h}_{x,A} \rangle \\
&= \langle f, h_{x,A} \rangle
\end{aligned}
$$

where $\langle f, g \rangle = \int f\bar{g}$.

WRT is a generalization of several transforms. In order to state the relation between WRT and other transforms, let us define some transforms.

DEFINITION 1.4. Let f be a function on $\boldsymbol{R}^n$ and ξ be a d dimensional plane ($1 \leq d \leq n-1$). The d dimensional Radon transform $Rf(\xi)$ of f is

$$
(1.8) \qquad Rf(\xi) \equiv \int_\xi f(x) dx,
$$

where dx is Euclidean measure on ξ. When $d = 1$ we write Xf instead of Rf, which we call x-ray transform of f.

If $h \equiv 1$ and then $R_h f(x, RJ) = Rf(\{x + RJ\boldsymbol{R}^d\})$, $X_h f(x, \omega) = Xf(\{x + \boldsymbol{R}\omega\})$. As for the Radon transform, confer [He] Chapter I.

Next we consider wavelet transforms.

DEFINITION 1.5. Consider f, h are functions on $\boldsymbol{R}^n$. The wavelet transform of f with window h is

$$
(1.9) \qquad W_h f(x, v, R) \equiv v^{-n/2} \int_{\boldsymbol{R}^n} f(y) \overline{h\left(\frac{R^{-1}(y-x)}{v}\right)} dx,
$$

for $v > 0,\ x \in \boldsymbol{R}^n,\ R \in SO(n)$.

PROPOSITION 1.6. *As for n dimensional WRT we have*

$$
(1.10) \qquad R_h f(x, vR) = v^{-n/2} W_h(x, v, R).
$$

PROOF. This is almost trivial.

$$
\begin{aligned}
R_h f(x, vR) &= \int_{\boldsymbol{R}^n} \overline{h(t)} f(x + vRt) dt \\
&= v^{-n} \int_{\boldsymbol{R}^n} h\left(\frac{R^{-1}(y-x)}{v}\right) f(y) dy \\
&= v^{-n/2} W_h f(x, v, R). \ \square
\end{aligned} \tag{1.11}
$$

In case $d = n$ (1.6) becomes

$$
\begin{aligned}
h_{x,vR}(y) &= \frac{1}{(2\pi)^n} \int e^{i\xi\cdot(y-x)} \hat{h}({}^t A\xi) d\xi \\
&= \frac{v^{-n}}{(2\pi)^n} \int e^{i\eta\cdot A^{-1}(y-x)} \hat{h}(\eta) d\eta \\
&= v^{-n} h\left(\frac{R^{-1}(y-x)}{v}\right).
\end{aligned} \tag{1.6$'$}
$$

This equation and (1.7) gives another proof of Proposition 1.6 for $f \in L^2$ and admissible h.

Before finishing this section, we mention analytic-signal transforms (AST). Denote the Schwartz space by $\mathcal{S}$. AST of $f \in \mathcal{S}(\boldsymbol{R}^n)$ is a function $\tilde{f}$ on $\boldsymbol{C}^n$ defined by

$$
\tilde{f}(x + iy) \equiv \frac{1}{2\pi i} \int_{\boldsymbol{R}} \frac{1}{t - i} f(x + ty) dt.
$$

This is written by $X_h f(x, y)$ with $h(t) = (2\pi(1 - it))^{-1}$. For more about AST confer [Ka2], [Ka3] and [KS].

§2. Inversion formulas

In this section, we prove inversion formulas for WRT which hold in several senses. To begin with, let us introduce inversion formulas proved by G.Kaiser-R.Streater, which depend on the dimension of d.

THEOREM 2.0. ([KS] subsection 2.2 and 2.3.)

(i) the case of $d = 1$

Assume $f \in L^2(\boldsymbol{R}^n)$. For admissible $h \in L^1(\boldsymbol{R})$

$$f(x) = \frac{2}{N_h} \int_{\boldsymbol{R}^n \times \boldsymbol{R}^n} |v|^{-n} h_{y,v}(x) X_h f(y,v) dy dv$$

holds in L^2 weak sense, where $h_{y,v}(x) = (2\pi)^{-n} \int e^{i\xi\cdot(x-y)} \hat{h}(\xi \cdot v) d\xi$

(ii) the case of $d > 1$

Assume that $f \in \mathcal{S}(\boldsymbol{R}^n)$ and that $h \in \mathcal{S}(\boldsymbol{R}^d)$ satisfies

$$N^{-1} \equiv \int_{\boldsymbol{R}_+ \times SO(n)} v^{-1} |\hat{h}(v^t J^t R_1)|^2 dv dR < \infty,$$

where ${}^t J : \boldsymbol{R}^n \to \boldsymbol{R}^d$ is the projection and R_1 is the first row of R. Then there holds

$$f(x) = N \int_{\boldsymbol{R}^n \times \boldsymbol{R}_+ \times SO(n)} v^{-1} h_{y,A}(x) f_h(y,A) dy dv dR$$

in L^2 weak sense.

Here, it is stated that the data $\{R_h f(x,A) |\ x \in \boldsymbol{R}^n, v > 0, R \in SO(n)/SO(n-d)\}$ are utilized, but we prove that $\{R_h f(x,A) |\ x \in \boldsymbol{R}^n, v > 0, R \in SO(d)\}$ are enough to reconstruct f. Futhermore, we unify their formulas into one and have succeeded in corresponding the admissible condition in Theorem 2.0 (ii) to the counterpart of wavelet transforms.

THEOREM 2.1. *Let $1 \le d \le n$ and Ξ be a d plane in $\boldsymbol{R}^n$, $f \in L^2(\boldsymbol{R}^n)$ and $(h_1, h_2) \in \mathcal{S}'(\Xi)$, be admissible. Then we have the following reconstruction formula;*

$$\begin{aligned} f(x') &= N_{h_1,h_2}{}^{-1} \int_{\boldsymbol{R}^n \times \boldsymbol{R}_+ \times SO(d)} v^{-1} f_{h_1}(x,A) h_{2,x,A}(x') dx dv dR \\ &=: R_{h_2}^{-1} R_{h_1} f(x'), \end{aligned} \tag{2.1}$$

which holds weakly in L^2 where dR is the Haar measure on $SO(d)$ being the rotation group on Ξ.

Proof. It is sufficient to prove the case of $\Xi = \boldsymbol{R}^d$, x' being expressed as $(x', 0, \cdots, 0) \in \boldsymbol{R}^n$. Let $g \in L^2(\boldsymbol{R}^n)$. For simplicity we write

$$d\mu(x, A) \equiv \frac{1}{N_{h_1,h_2}} v^{-1} dx dv dR \tag{2.2}$$

and

$$\mathcal{X} \equiv \boldsymbol{R}^n \times \boldsymbol{R}_+ \times SO(d). \tag{2.3}$$

Noting that $\mathcal{F}^{-1}_{\xi\to x} f$ means Fourier inversion transform, we have

$$\begin{aligned}
&\int_{\mathcal{X}} f_{h_1}(x, A)\overline{g_{h_2}(x, A)} d\mu(x, A) \\
&\quad = \int_{\mathcal{X}} \mathcal{F}^{-1}_{p\to x}\{\overline{\widehat{h_1}({}^tAp)}\hat{f}(p)\}\overline{\mathcal{F}^{-1}_{p\to x}\{\widehat{h_2}({}^tAp)\overline{\hat{g}(p)}\}} d\mu(x, A) \\
&\quad = \frac{1}{(2\pi)^n}\int_{\mathcal{X}} \overline{\widehat{h_1}({}^tAx)}\widehat{h_2}({}^tAx)\hat{f}(x)\overline{\hat{g}(x)} d\mu(x, A),
\end{aligned} \tag{2.4}$$

where $y = (y', 0, \ldots, 0)$, y' is a unit vector in $\boldsymbol{R}^d$. Here we let $d\mu(x, A) = dx d\rho(A)$. Since $d\rho$ is invariant under dilations and rotations we have

$$\begin{aligned}
&\int_{\boldsymbol{R}_+\times SO(d)} \overline{\widehat{h_1}({}^tAx)}\widehat{h_2}({}^tAx) d\rho(A) \\
&\quad = \int_{\boldsymbol{R}_+\times SO(d)} \overline{\widehat{h_1}({}^tAy)}\widehat{h_2}({}^tAy) d\rho(A) \\
&\quad = \frac{1}{N_{h_1,h_2}}\int_{\boldsymbol{R}_+} v^{-1} dv \int_{SO(d)} \overline{\widehat{h_1}(vR^{-1t}Jy)}\widehat{h_2}(vR^{-1t}Jy) dR \\
&\quad = \frac{1}{N_{h_1,h_2}\omega_{d-1}}\int_{\boldsymbol{R}_+} v^{-1} dv \int_{S^{d-1}} \overline{\widehat{h_1}(v\theta)}\widehat{h_2}(v\theta) d\theta \\
&\quad = \frac{1}{N_{h_1,h_2}\omega_{d-1}}\int_{\boldsymbol{R}^d} \frac{\overline{\widehat{h_1}(t)}\widehat{h_2}(t)}{|t|^d} dt \\
&\quad = 1,
\end{aligned} \tag{2.5}$$

where ${}^tJ : \boldsymbol{R}^n \to \boldsymbol{R}^d$ is the projection. Up here, this proof is almost the same as Kaiser-Streater's, however, (of course, there were a few differences in detail), we can proceed futher. Considering this and (2.4) gives us

$$\int_{\boldsymbol{R}^n} \left(R_{h_2}^{-1} R_{h_1} f\right)(x)\overline{g(x)}dx = \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi \\ = \int_{\boldsymbol{R}^n} f(x)\overline{g(x)}dx. \tag{2.6}$$

Futhermore it has just been asserted that

$$\int_{\mathcal{X}} R_{h_1} f(x,A)\overline{R_{h_2} g(x,A)}d\mu(x,A) = \int_{\boldsymbol{R}^n} f(x)\overline{g(x)}dx. \ \square \tag{2.7}$$

REMARK.

(1) Note that in case of $n = d = 1$, we also have a modified invertion formulas above by replacing integral interval $\boldsymbol{R}_+$ in v with $\boldsymbol{R}$ and the measure $dv/|v|$ with $dv/(2|v|)$, which, from now on, we apply without mentioning.

(2) Theorem 2.1 requires less information for $R_h f$ than Theorem 2.0, which in his forthcoming paper, the author has intension to utilize to establish invertion formulas of the Radon transform with weight and to apply the problems of tomography with some improvements.

(2.7) implies the Plancherel type formula for WRT;

COROLLARY 2.2. *Under the same condition on $f, g, (h_1, h_2)$ as Theorem 2.1 we have the formula of Plancherel type as follows*

$$\int_{\mathcal{X}} f_{h_1}(x,A)\overline{g_{h_2}(x,A)}d\mu(x,A) = \int_{\boldsymbol{R}^n} f(x)\overline{g(x)}dx.$$

Remark that under the assumption of Theorem 2.1 f_h is meaningless in pointwise sense, for f_h is not necessarily of absolute convergence. As an element of $L^2(\mathcal{X})$, however, f_h makes sense. Thus we have from Corollary 2.2;

THEOREM 2.3. *For admissible h, R_h is an isometry of $L^2(\boldsymbol{R}^n)$ to $L^2(\mathcal{X})$.*

In addition, the reconstruction formula (2.1) holds in other senses, in [KS], however, only a weak sense was proved (Theorem 2.0). Now we show the invertibility as limit in the mean of L^2 norm. The counterpart of the following proposition for the wavelet transform was established by I.Daubechies [D].

PROPOSITION 2.4. *Assume $f \in L^2(\boldsymbol{R}^n)$ and (h, h') is admissible. Define*

$$\mathcal{X}_{\alpha,\beta,\gamma} \equiv \{|x| < \gamma\} \times \{\alpha \leq v \leq \beta\} \times SO(d).$$

Then there holds

$$\lim_{\alpha\to 0,\beta\to\infty,\gamma\to\infty} \left\| f - \int_{\mathcal{X}_{\alpha,\beta,\gamma}} f_h(x,A){h'}_{x,A}(x')d\mu(x,A) \right\|_{L^2(\boldsymbol{R}^n)} = 0. \tag{2.8}$$

PROOF. Denoting the complement of a set G by G^c, we obtain

$$\begin{aligned}
&\left\| f - \int_{\mathcal{X}_{\alpha,\beta,\gamma}} f_h(x,A){h'}_{x,A}(x')d\mu(x,A) \right\|_{L^2(\boldsymbol{R}^n)} \\
&= \sup_{\|g\|_{L^2}=1} \left| \langle f - \int_{\mathcal{X}_{\alpha,\beta,\gamma}} f_h(x,A){h'}_{x,A}(x')d\mu(x,A),\ g\rangle \right| \\
&= \sup \left| \int_{{\mathcal{X}_{\alpha,\beta,\gamma}}^c} \overline{f_h(x,A)} g_{h'}(x,A)d\mu(x,A) \right| \\
&\leq \sup \left(\int_{{\mathcal{X}_{\alpha,\beta,\gamma}}^c} |f_h(x,A)|^2 d\mu(x,A) \right)^{1/2} \left(\int_{\mathcal{X}} |g_{h'}(x,A)|^2 \right)^{1/2}.
\end{aligned}$$

The term in the first factor of the rightist hand side tends to 0 as $\alpha \to 0$, $\beta \to \infty$, $\gamma \to \infty$ and the term in the second one is 1 by Corollary 2.2. Hence the proof is finished. □

I.Daubechies [D] noted for wavelet tramsforms that although the counterpart for Proposition 2.4 holds in L^2 sense it does not in the sense of L^1.

We claim as the same for WRT. For more detail, we shall give similar discussion in section 3. Next we show that our invertion formula holds pointwise. The following theorem is our original formula in the sense that it seems that there is no counterpart for the wavelet transform written in the manuscript.

THOREM 2.5. *Let $h, h' \in L^1(\boldsymbol{R}^d), (h, h')$ be admissible and $f \in \mathcal{S}(\boldsymbol{R}^n)$. Then it is asserted that*

$$R_{h'}^{-1} R_h f(x) = f(x).$$

PROOF. We have

$$\begin{aligned}
(2.9) \qquad R_{h'}^{-1} R_h f(x) &= \int_{\mathcal{X}} d\mu(y, A) {h'}_{y,A}(x) \int_{\boldsymbol{R}^d} f(y + At)\overline{h(t)} dt \\
&= \int_{\mathcal{X}} d\mu(y, A) {h'}_{y,A}(x) \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^n} \hat{f}(\xi) e^{i\xi \cdot y} \overline{\hat{h}({}^t A\xi)} d\xi \\
&= \frac{1}{(2\pi)^{2n}} \int_{\boldsymbol{R}^n} \hat{f}(\xi) d\xi \int_{\mathcal{X}} e^{i\xi \cdot y} \overline{\hat{h}({}^t A\xi)} d\mu(y, A) \\
&\quad \cdot \int_{\boldsymbol{R}^n} e^{i\eta \cdot (x-y)} \widehat{h'}({}^t A\eta) d\eta \\
&= \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^n} \hat{f}(\xi) d\xi \int_{\boldsymbol{R}^+ \times SO(d)} \overline{\hat{h}({}^t A\xi)} d\rho(A) \\
&\quad \cdot \int_{\boldsymbol{R}^n} e^{i\eta \cdot x} \widehat{h'}({}^t A\eta) \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^n} e^{i(\xi - \eta) \cdot y} dy \\
&\equiv \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^n} \hat{f}(\xi) H(\xi) d\xi,
\end{aligned}$$

where $d\rho$ was defined in the proof of Theorem 2.1, and we have let

$$H(\xi) := \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^+ \times SO(d)} \overline{\hat{h}({}^t A\xi)} d\rho(A) \int_{\boldsymbol{R}^n} e^{i\eta \cdot x} \widehat{h'}({}^t A\eta) d\eta \int_{\boldsymbol{R}^n} e^{i(\xi - \eta) \cdot y} dy.$$

Since $f \in \mathcal{S}$ it is allowed considering H as a tempered distribution, which provides that

$$\begin{aligned}
(2.9)' \qquad H(\xi) &= \int_{\boldsymbol{R}^+ \times SO(d)} \overline{\hat{h}({}^t A\xi)} d\rho(A) \int_{\boldsymbol{R}^n} e^{i\eta \cdot x} \widehat{h'}({}^t A\eta) \delta(\xi - \eta) d\eta \\
&= e^{i\xi \cdot x} \int_{\boldsymbol{R}^+ \times SO(d)} \overline{\hat{h}({}^t A\xi)} \widehat{h'}({}^t A\xi) d\rho(A).
\end{aligned}$$

By the argument below (2.5) we conclude that $H(\xi) = e^{i\xi\cdot x}$. Therefore putting this into (2.9) proves the theorem. $\square$

If we interpret all the Fourier invertion formulas in (2.9) as oscillatory integrals, Theorem 2.5 holds for $f \in L^1$. To prove this we need some preparation.

Lemma 2.6. *Let $\varphi, \hat{\varphi} \in L^1(\boldsymbol{R}^n)$, $\int \hat{\varphi} d\xi = 1$, and $f \in C(\boldsymbol{R}^n)$ be bounded. Then φ means of integral of $\frac{1}{(2\pi)^n}\int e^{i(\xi-\eta)\cdot y} f(\eta) dy d\eta$;*

$$\lim_{\epsilon\to 0}\frac{1}{(2\pi)^n}\int e^{i(\xi-\eta)\cdot y}\varphi(\epsilon y)dy f(\eta)d\eta$$

converge to $f(\xi)$ as a value for any ξ.

Proof. We write $\widehat{\varphi_\epsilon}(\xi) := \epsilon^n \widehat{\varphi}(\xi/\epsilon), \sup|f| = M$ then we have

$$\begin{aligned}(\widehat{\varphi_\epsilon} * f - f)(\xi) &= \int (f(\xi-\eta) - f(\xi))\epsilon^n \hat{\varphi}(\eta/\epsilon)d\eta \\ &= \int (f(\xi - \epsilon\eta) - f(\xi))\hat{\varphi}(\eta)d\eta\end{aligned}$$

The integrant is bounded by $2M|\hat{\varphi}(\eta)|$ therefore we can apply Lebesgue dominated convergence theorem and prove the Lemma. $\square$

Applying this lemma yields

Theorem 2.7. *Take h, h' as the same as Theorem 2.5. If we interpret all the Fourier invertion formula in* (2.9) *as oscillatory integrals, that is, we consider the integrals with respect to dy and $d\xi$ in the sense of Lemma 2.6, then we have for $f \in L^1(\boldsymbol{R}^n)$*

$$R_{h'}^{-1}R_h f(x) = f(x).$$

for any $x \in \boldsymbol{R}^n$

In case that $d = n$ we have another invertion formula, which is an extension of [D] Proposition 2.4.2. This extension is likely not to have been published.

THEOREM 2.8. *Assume* $f \in L^2 \cap L^\infty$, $h, h' \in L^1$, $|\xi|^{\frac{n+1}{2}} \widehat{h'}(\xi) \in L^2$, (h, h') *is admissible, and* f *is continuous at* x. *Put*

$$\mathcal{X}_{S,T} \equiv \boldsymbol{R}^n \times \{S \le v \le T\} \times SO(n).$$

If there hold

$$\hat{M} \in C^n(\boldsymbol{R}^n), \ \left(\frac{\partial}{\partial r}\right)^n \hat{M}(r\omega) \in L^2(\boldsymbol{R}^n) \ \textit{for}$$

$$\hat{M}(\xi) = {N_{h,h'}}^{-1} \int_{|\xi|<|\eta|} \frac{\overline{\hat{h}(\eta)} \widehat{h'}(\eta)}{|\eta|^n} d\eta.$$

Then there holds pointwise at this point x *that*

$$f(x) = \lim_{S \to 0, T \to \infty} \int_{\mathcal{X}_{S,T}} d\mu(x', A) \int_{\boldsymbol{R}^n} dy f(x' + Ay) \overline{h(y)} {h'}_{x',A}(x). \tag{2.10}$$

PROOF. As was mentioned in $(1.6)'$ we have

$${h'}_{x',A}(x) = v^{-n} h' \left(\frac{R^{-1}(x - x')}{v} \right),$$

we let

$$\begin{aligned} f_{S,T}(x) &= \int_{\mathcal{X}_{S,T}} v^{-2n} d\mu(x', A) \times \\ &\quad \times \int_{\boldsymbol{R}^n} dy f(y) \overline{h\left(\frac{R^{-1}(y - x')}{v} \right)} h' \left(\frac{R^{-1}(x - x')}{v} \right) \\ &= \int_{\boldsymbol{R}^n} dy f(y) M_{S,T}(x - y), \end{aligned} \tag{2.11}$$

where

$$M_{S,T}(x) = \int_{\mathcal{X}_{S,T}} v^{-2n} d\mu(x', A) \overline{h\left(\frac{R^{-1}(-x')}{v} \right)} h' \left(\frac{R^{-1}(x - x')}{v} \right).$$

The Fourier transform $\hat{M}_{S,T}$ of $M_{S,T}$ is

$$
\begin{aligned}
\hat{M}_{S,T}(\xi) &= {N_{h,h'}}^{-1} \int_{\mathcal{X}_{S,T}} d\rho(A) \overline{\hat{h}(vR\xi)} \widehat{h'}(vR\xi) \\
&= {N_{h,h'}}^{-1} \int_{S|\xi|<|\eta|<T|\xi|} d\eta \frac{\overline{\hat{h}(\eta)} \widehat{h'}(\eta)}{|\eta|^n} \\
&\equiv \hat{M}(S\xi) - \hat{M}(T\xi).
\end{aligned} \tag{2.12}
$$

Considering that $|\xi|^{\frac{n+1}{2}} \widehat{h'}(\xi) \in L^2$ gives us

$$
\begin{aligned}
|\hat{M}(\xi)| &\leq {N_{h,h'}}^{-1} \left(\int_{|\xi|<|\eta|} \frac{|\hat{h}(\eta)|^2}{|\eta|^{3n+1}} d\eta \right)^{1/2} \left(\int |\eta|^{n+1} |\widehat{h'}(\eta)|^2 d\eta \right)^{1/2} \\
&\leq C|\xi|^{-n-\frac{1}{2}}
\end{aligned}
$$

hence we obtain $\hat{M} \in L^1 \cap L^\infty$. Therefore $M = \check{\hat{M}}$ is well defined. From the assumption we obtain

$$
\begin{aligned}
\int |M(x)| dx &\leq \left(\int \frac{dx}{1+|x|^{2n}} \right)^{1/2} \left(\int (1+|x|^{2n}) |M(x)|^2 dx \right)^{1/2} \\
&\leq C \left(\int |\hat{M}(\xi)|^2 + \left| \left(\frac{\partial}{\partial r} \right)^n \hat{M}(\xi) \right|^2 d\xi \right)^{1/2} < \infty.
\end{aligned}
$$

So $M \in L^1$ and the fact $\hat{M}(0) = 1$ implies that $\int M(x)dx = 1$. We rewrite (2.10) utilizing (2.11) so that

$$
f_{S,T}(x) = \int_{\boldsymbol{R}^n} \frac{1}{S^n} M\left(\frac{x-y}{S} \right) f(y) dy - \int_{\boldsymbol{R}^n} \frac{1}{T^n} M\left(\frac{x-y}{T} \right) f(y) dy.
$$

Because M is of integral 1, the first term tends to $f(x)$ as $S \to 0$. The second term is bounded by

$$
\begin{aligned}
&\left| \int \frac{1}{T^n} M\left(\frac{x-y}{T} \right) f(y) dy \right| \\
&\leq \left(\int \frac{1}{T^{2n}} \left| M\left(\frac{(x-y)}{T} \right) \right|^2 dy \right)^{1/2} \left(\int |f(y)|^2 dy \right)^{1/2} \\
&\leq CT^{-n}.
\end{aligned}
$$

This term tends to 0 as $T \to \infty$. $\square$

Utilizing Theorems 2.5 and 2.6 we can define WRT or wavelet transform for tempered distributions. In our next paper we shall try to define WRT and wavelet transform for generalized functions and discuss some topics around them.

§3. Counterexamples

As was mentioned in Proposition 2.4, (2.8) holds in L^2 sense under suitable conditions. In the sense of L^1, however, this formula is not true, i.e,

$$\lim_{\alpha\to 0,\beta\to\infty,\gamma\to\infty} \left\| f - \int_{\mathcal{X}_{\alpha,\beta,\gamma}} f_h(x,A) {h'}_{x,A}(x') d\mu(x,A) \right\|_{L^1(\boldsymbol{R}^n)}$$

does not necessarily vanish. In this section we study this concretely.

Take $f \in \mathcal{S}$ to be positive, and h to be admissible. Letting

$$I_{\alpha,\beta,\gamma}(y) = \int_{\mathcal{X}_{\alpha,\beta,\gamma}} f_h(x,A) h_{x,A}(y) d\mu(x,A)$$

gives us

$$\begin{aligned} I_{\alpha,\beta,\gamma}(y) &\leq \int_{\mathcal{X}} |f|_{|h|}(x,A) |h_{x,A}|(y) d\mu(x,A) \\ &= \int_{\mathcal{X}} \mathcal{F}^{-1}_{p\to x}\{\overline{\widehat{|h|}({}^tAp)}\widehat{|f|}(p)\} \mathcal{F}^{-1}_{p\to x}\{e^{-ix\cdot y}\widehat{|h|}({}^tAp)\} d\mu(x,A) \\ &= \int_{\mathcal{X}} |\widehat{|h|}({}^tAx)|^2 e^{-ix\cdot y}\widehat{|f|}(x) d\mu(x,A) \\ &= |f|(y), \end{aligned}$$

where $|f|(x) = |f(x)|$. Therefore $I_{\alpha,\beta,\gamma} \in L^1(\boldsymbol{R}^n)$. Restricting $n = 2$ and $d = 1$, we construct a counterexample.

Take $h \in \mathcal{S}(\boldsymbol{R})$ such that $\hat{h}(p) = p$ for $|p| \leq 1$ and $\hat{h} \in C_0^\infty$, $f \in \mathcal{S}(\boldsymbol{R}^2)$.

$(2.9)'$ and Lemma 2.6 suggests for small α

$$I_{\alpha,\infty,\infty}(y) \approx \alpha f(y).$$

Hence we have

$$\lim_{\alpha\to 0} \|f - I_{\alpha,\infty,\infty}\|_{L^1(\boldsymbol{R}^n)} = \|f\|_{L^1(\boldsymbol{R}^n)}.$$

We can extend this counterexample to various cases.

In wavelet transform, another counterexample is stated in [D]; Let $n = 1$, $f, h \in L^1 \cap L^2$, h being admissible,

$$I_{\alpha,\beta,\gamma}(x) = \int_\alpha^\beta v^{-3} dv \int_{|x'|<\gamma} dx' h\left(\frac{x'-x}{v}\right) \int_{\boldsymbol{R}^n} dy f(y) \overline{h\left(\frac{y-x'}{v}\right)}.$$

Since this integral is bounded, we can change the order of integration. Admissible condition implies that the integration of h is 0. If $f - I_{\alpha,\beta,\gamma} \geq 0$ or ≤ 0 then

$$\|f - I_{\alpha,\beta,\gamma}\|_{L^1(\boldsymbol{R}^n)} \equiv \|f\|_{L^1(\boldsymbol{R}^n)}.$$

This counterexample does not apply to general WRTs.

References

[D] Daubechies, I., Ten Lectures on Wavelets, Rutgers University and AT&T Bell Laboratories, Philadelphia, Pensilvania, 1992.

[He] Helgason, S., Groupes and Geometric Analysis, Academic Press, Inc, Orlando, San Diego, San Francisco, New York, London, Toronto, Montreal, Sydney, Tokyo, São Paulo, 1984.

[Ka1] Kaiser, G., Generalized Wavelet Transforms I:The Windowed X-Ray Transform, Technical Reports Series no. 18, Univ. of Lowell, 1990, preprint.

[Ka2] Kaiser, G., Quantum Physics, Relativity, and Complex Spacetime, North-Holland, Amsterdam, New York, Oxford, Tokyo, 1990.

[Ka3] Kaiser, G., Wavelet Electrodynamics, part II: Electromagnetic Wavelets with Polarization, Applied and Computational Harmonic Analysis **1** (1994), 246–260.

[Ka4] Kaiser, G., A Friendly Guide to Wavelets, Birkhänser, Boston, 1994.

[KS] Kaiser, G. and R. Streater, Windowed Radon Transforms, Analytic Signals, and the Wave Equation, Wavelets-A Tutorial in Theory and Applications, 1992, pp. 399–441.

[SW] Stein, E. M. and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, New Jersey, 1971.

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