J. Math. Sci. Univ. Tokyo
2 (1995), 501–561.

On a Remarkable Polyhedron Geometrizing the Figure Eight Knot Cone Manifolds

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Abstract. We define a one parameter family of polyhedra $P(t)$ that live in three dimensional spaces of constant curvature $C(t)$. Identifying faces in pairs in $P(t)$ via isometries gives rise to a cone manifold $M(t)$ (A cone manifold is much like an orbifold.). Topologically $M(t)$ is S^3 and it has a singular set that is the figure eight knot. As t varies, curvature takes on every real value. At $t = -1$ a phenomenon which we call spontaneous surgery occurs and the topological type of $M(t)$ changes. We discuss the implications of this.

0. Introduction

We consider in this paper a one parameter family of three dimensional cone manifolds whose underlying space is S^3 and whose singular curve is the rational knot $\frac{5}{3}$, more popularly known as the figure eight. We denote a member of the family by $\left(\frac{5}{3}, \alpha\right)$ where α is the cone angle at the singular set (i.e. the knot). The problem that we study and solve here, is that of giving a "concrete" geometric structure to these cone manifolds. The end result is a family of cone manifolds $\left(\frac{5}{3}, \alpha\right)$ defined for $\alpha \in [0, \pi]$. The geometric structure is hyperbolic for α belonging to the interval $\left[0, \frac{2\pi}{3}\right)$, Euclidean when α equals $\frac{2\pi}{3}$ and spherical for α in $\left(\frac{2\pi}{3}, \pi\right]$. We don't know what happens when α is greater than π but we conjecture that there is some kind of natural continuation of the family in which the members are spherical cone-manifolds.

1991 *Mathematics Subject Classification.* 57M50, 53C20.

*This research was supported by grant PB92-0236.

There appears in the course of the construction, spontaneously as we say, the manifold obtained by zero surgery in the figure eight. The "spontaneous surgery" produces a manifold that is a fibre bundle over S^1 with fibre a torus and Anosov monodromy given by the 2 by 2 matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This manifold has a "Sol" type geometric structure, as was shown by Thurston ([T]). We consider the core of the surgery as a singular curve, we think of this manifold as a cone manifold and we show that for cone angle in the interval $[0, 2\pi)$ the geometry is hyperbolic.

Thurston has shown, ([T]), in his celebrated "notes", that the manifold resulting from Dehn surgery on the figure eight knot has hyperbolic structure in many cases. In the course of his proof he considers a pair of ideal hyperbolic tetrahedra glued along their faces by isometries. The underlying topological space is the complement of the figure eight knot in S^3. Geometrically the structure is hyperbolic, but not complete. When he completes the hyperbolic structure he sometimes obtains, (He spells out exactly when), a compact hyperbolic manifold that is topologically the result of Dehn surgery on the figure eight knot. This gives a context in which to view our paper. Our point of view, though, is different from Thurston's.

We shall now describe how we obtain geometric structures. The idea is to construct polyhedra which we think of as lying in the Klein model for hyperbolic three space of curvature $-1/R^2$. In contradistinction to other authors, for example ([HKM]) in which an infinite family of manifolds in hyperbolic three space called Fibonacci manifolds, is constructed, we construct a *continuous* family of polyhedra in Klein models. At each moment the faces are identified in pairs by isometries and so define the cone manifolds. It happens that as the polyhedra vary continuously, the cone angle also varies continuously.

As a starting point we consider a certain crystallographic group, which we call G_8, acting on E^3. The orbit space is topologically S^3 and the image of the axes of rotation (they are all 3–fold) is the figure eight knot. This, which is well–known, is how the Euclidean orbifold structure, or cone manifold structure, is induced on the pair (S^3, figure eight knot). The cone angle is $\frac{2\pi}{3}$.

Our "remarkable polyhedron" is a certain Dirichlet domain for the group G_8. It has twelve faces, four pentagonal and eight triangular.

We consider the polyhedron as lying in the Klein model for hyperbolic space of constant curvature $-1/R^2$ when R equals infinity. We find that by changing the shape of the polyhedron slightly we are able to continuously decrease R, while maintaining most of the geometric structure, i.e. the combinatorial structure of the polyhedron remains the same, the symmetry group of the polyhedron remains the same, identifications of pairs of faces are still via isometries, (But hyperbolic isometries in the Klein model.), the underlying topological space is still S^3, and the singular set is still the figure eight knot. However the cone angle decreases as R decreases. This works fine all the way down to $R = 1$ at which point the cone angle becomes zero, the pentagons become quadrilaterals, and the vertices of the quadrilaterals resulting from the collapsed edges of the pentagon now lie in the sphere at infinity. The case $R = 1$ corresponds to the by now "standard" complete hyperbolic structure on the complement of the figure eight knot; except that we obtain this structure by identifying the faces of a certain dodecahedron rather than two tetrahedra of Thurston.

To our surprise, as R decreases below one, the polyhedron, which by now has a life of its own, but has metamorphosed to a dodecahedron with eight quadrilateral and four triangular faces is still inside the sphere at infinity. The construction makes sense all the way down to $R = 0$ but spontaneous zero surgery occurs when $R = 1$ and the underlying topological space is the aforementioned S^1 bundle. The singular set is still a knot and as R decreases from one to zero the cone angle in the new singular set increases from zero to 2π. At $R = 0$ everything disappears.

Returning to the original Dirichlet domain, the case $\alpha = \frac{2\pi}{3}$, we now consider it to lie in a kind of Klein model for a space of constant positive curvature $1/R^2$ when R equals infinity. By now we have defined the polyhedron parametrically and it is just a matter of seeing what happens as the parameters change. The construction makes sense as R decreases from infinity to zero, the combinatorial structure of the polyhedron remains the same, the underlying topological space remains S^3 and the singular set remains the figure eight knot. As R decreases monotonically from infinity to zero the cone angle increases monotonically from $\frac{2\pi}{3}$ to π.

Most people who work in hyperbolic or spherical three space of constant curvature almost instinctively normalize the curvature to plus or minus one. There are good reasons for doing this. One can speak of "the" volume of a hyperbolic manifold or identify hyperbolic isometries with $PSL(2, \mathbb{C})$. But if one does this one cannot envision continuous families of cone manifolds in which curvature changes from positive to negative and the Euclidean case is a point in a continuum.

On the other hand, in our example when cone angle α lies in the interval $\left[0, \frac{2\pi}{3}\right)$ and curvature is negative, or in the interval $\left(\frac{2\pi}{3}, \pi\right]$ and curvature is positive we can simply change scale in order to obtain a continuous family of cone manifolds in a fixed hyperbolic or spherical space of curvature minus or plus one. The map $\overrightarrow{v}$ into $(1/R)\,\overrightarrow{v}$ on $\mathbb{R}^3$ sends the Klein model for the space of curvature $\pm R$ to the Klein model for the space of curvature ± 1. This map preserves angles, in particular cone angles, but not lengths. When we speak of length or volume for a particular cone manifold we mean its length or volume in the corresponding normalized space with curvature ± 1.

We are able to compute cone angle and length of the singular set. This information, together with Schläffli's formula for the variation of the volume of a polyhedron in a space of constant curvature enables us to give an integral formula for the volumes of the cone manifolds in our one parameter family. (The normalizing map $\overrightarrow{v} \longrightarrow (1/R)\,\overrightarrow{v}$ crushes the polyhedron to a point with volume zero in the Euclidean case. Thus with volume a known differentiable function of cone angle α for $\alpha \in \left[0, \frac{2\pi}{3}\right]$ or $\alpha \in \left[\frac{2\pi}{3}, \pi\right]$, its derivative being given by Schläffli's formula and volume at $\alpha = \frac{2\pi}{3}$ being zero an integral formula naturally arises.)

We give a table of volumes for certain cone angles in the interval $\left(0, \frac{2\pi}{3}\right)$ and display a graph.

We also do this in the case of spontaneous surgery and in the spherical case.

Finally, we'd like to make the following observation. Consider the normalized hyperbolic polyhedron that gives rise to the figure eight knot orbifold or cone manifold with cone angle $\frac{2\pi}{n}$, $n \geqq 4$. Paste n copies of this polyhedron together cyclicaly around one of the two edges that gives rise to the singularity. The result is a fundamental domain for the (hyperbolic) n–fold cyclic covering of the figure eight. These manifolds are the Fibonacci manifolds studied by Helling, Kim, and Mennicke. (See [HLM$_3$]). But the fundamental domains we obtain are very different from theirs. Our methods give a very effective procedure for computing volumes when hyperbolic manifolds can be represented as members of a one parameter family of cone manifolds. In another paper (See [HLM$_3$]) we studied the arithmeticity of the figure eight knot orbifolds. We can now calculate the volumes of these orbifolds in two different ways; using our formulas or using arithmetic methods. (See [V] p. 108; compare [M$_3$]).

Analogous methods work for the manifold obtained by zero surgery on the figure eight (spontaneous surgery). This manifold is topologically, as we stated previously, a fibre bundle over S^1 with torus as fibre. Geometrically it is a cone manifold, sometimes orbifold, with singular set of two components. When the singularity has form $\frac{2\pi}{n}$ this orbifold admits coverings which are hyperbolic manifold that fiber over S^1.(Compare [J].)

1. The figure eight knot as a Euclidean orbifold

The purpose of this section is to obtain a representation of the figure eight knot as a Euclidean orbifold by identifying pairs of faces in a certain polyhedron. (The knot itself will be a subset of the set of edges.)

The polyhedron will be very symmetric and will be a Dirichlet domain for a certain crystallographic group. Our first problem, then will be to define this group. Actually, we need to define several crystallographic groups.

Consider then, the tesselation of E^3 by unit cubes whose vertices have integer coordinates. The cube with the origin for a vertex that lies in the positive octant will be called *the main cube.* Let T_x, T_y, T_z, be the unit translations in the positive x, y, and z directions and let T_{2x}, T_{2y}, T_{2z} be the corresponding squares of T_x, T_y, T_z, that is the translations by two. All the groups we wish to define will be subgroups of the orientation preserving symmetry group G_0 of the tesselation, and all will contain the subgroup T_E (E is for even) generated by T_{2x}, T_{2y}, and T_{2z}.

The group G_0 is the semi-direct product of the normal subgroup generated by the translations and the orientation preserving symmetry group of the main cube. This last group has twenty-four elements so that the volume of a fundamental domain for G_0 equals $1/24$, and the volume of any finite index subgroup of G_0 will equal the index divided by twenty-four.

Next, we bisect the faces of the main cube by defining six edges as in figure 1.

Translating the main cube around by all possible elements of the translation subgroup we obtain three sets of parallel lines, the lines in each set

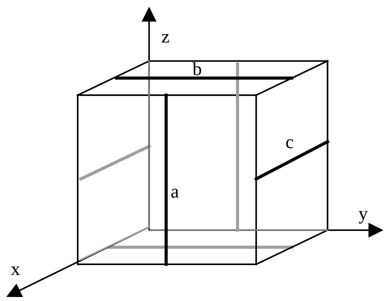


Figure 1

coming from the bisecting edges and being parallel to the x, y, and z axes respectively.

Let G_B (the B stands for Borromean rings) be the subgroup generated by all 180° rotations with these lines as axes. Any such 180° rotation preserves the tesselation so $G_B \leqq G_0$. Two successive 180° rotations about parallel lines a distance of one apart is a translation by two, so that $T_E \leqq G_B$. In fact G_B is generated by the normal subgroup T_E and the 180° rotations in the axes a, b, c of figure 1 so that $G_B/T_E \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. (Verify, for example, that 180° rotation in a followed by T_{2x}^{-1} is 180° rotation about the axis parallel to a on the back side of the main cube, etc.) Thus we see that T_E has index eight in G_B, and since a $2 \times 2 \times 2$ set of cubes is a fundamental domain for T_E, the main cube itself is a fundamental domain for G_B.

The orbit space E^3/G_B is topologically S^3 and the image of the axes of rotation is the Borromean rings. The purpose of figure 2, in which we identify pairs of faces on the main cube, two pairs at a time, is to help you to see this. Thus we have demonstrated the known result (See [T], [D]) that the Borromean rings are a Euclidean orbifold with each component having singularity of type $n = 2$. (In a Euclidean (hyperbolic, spherical) orbifold points off the singular set have neighborhoods isometric to neighborhoods in Euclidean (hyperbolic, spherical) space. Points on the singular set have neighborhoods that are like the orbit spaces of finite groups of Euclidean (hyperbolic, spherical) isometries acting on Euclidean (hyperbolic, spherical) space. For us, in this paper, the space dimension will always be three, the isometries will always be rotations and the finite groups will always be

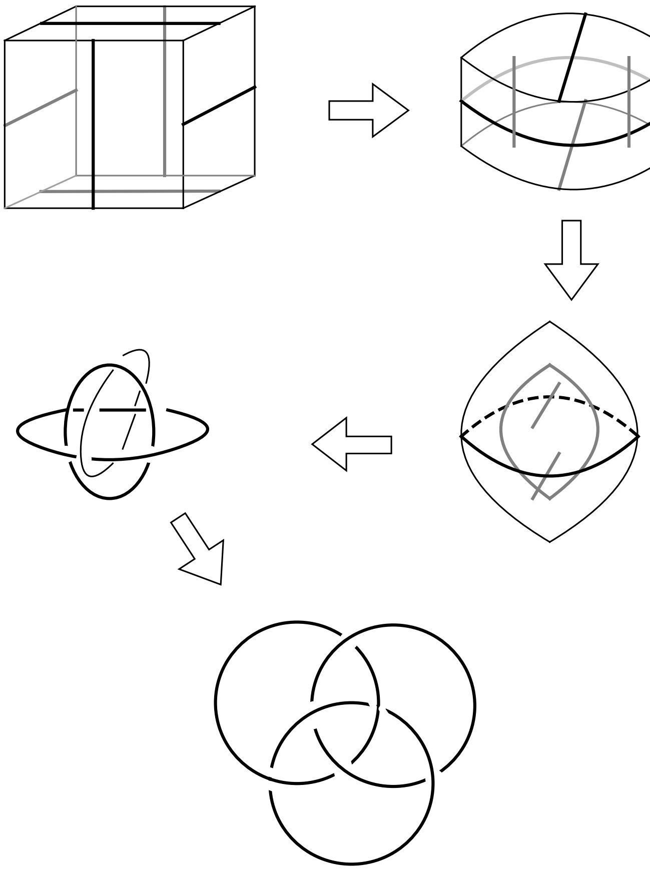


Figure 2

cyclic. Points on the singular set will be of type n if the cyclic group has order n.)

Next let A be the 3-fold rotation about an axis through the points $(0,0,1)$ and $(1,1,0)$ so that A is a symmetry of the main cube and so that A appears to be a clockwise rotation of the main cube to an observer at $(1,1,0)$ looking toward $(0,0,1)$. (This strange choice of axis for A is made so that certain drawings which appear later are more clear in perspective.) Then A normalizes the group G_B because it leaves invariant the set of axes of $180°$ rotation for G_B. Let G_L, (L is for link), be the group generated by A and G_B. Then the index of G_B in G_L is three and the volume of

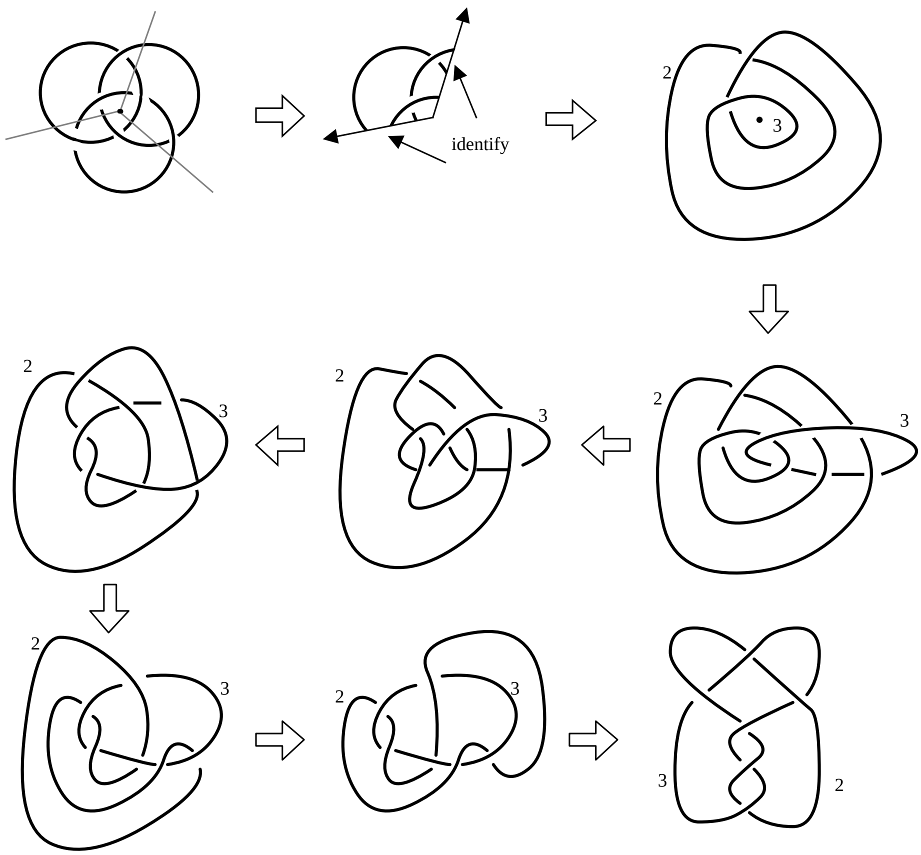


Figure 3

a fundamental domain for G_L is 1/3. The orbit space E^3/G_L is the orbit space $S^3 = E^3/G_B$ under the action of the 3-fold rotation, topologically just S^3 again. The image of the axes of rotation is the image of the Borromean rings plus the image of the 3-fold axis of rotation in the 3 to 1 branched covering $S^3 \longrightarrow S^3$, where the map is the orbit space map under the 3-fold rotation that permutes the components of the Borromean rings. This link, the link 6^2_2 in ([Ro]), is depicted in figure 3.

Thus we have shown the also well known fact that the link 6^2_2 is also a Euclidean orbifold, in which one component has singularity type 2, the other type 3.

The two fold branched covering of the link 6_2^2, as Euclidean orbifold, with branch set the component with singularity of type 2 is also a Euclidean orbifold. As a topological space it is S^3 because it is the double branched covering of the trivial knot. The singularity in the pre-image of the branch set disappears. The pre-image of the component with singularity type 3 is the figure eight knot. To see this refer to figures 3 and 4.

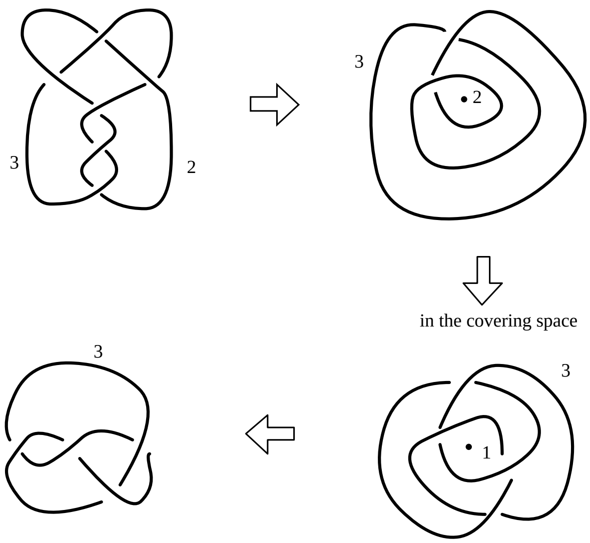


Figure 4

This shows that the figure eight knot is the singular set in a Euclidean orbifold with singularity of type 3; something that is well known. (See [T], [D]). This construction of the figure eight knot as pre-image of one component of the link 6_2^2 in a double branched covering over the other was used by Debby Goldsmith who developed a general technique for constructing fibred knots and links (See [Ro]). Its importance for us lies in the following construction.

Consider again the groups G_L, G_B, and T_E. The group G_B has a fixed point free subgroup of index two which we shall call H. To construct H observe that $G_B/T_E \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let φ be the composition of the homomorphism $G_B \longrightarrow G_B/T_E$ with the homomorphism $(a,b,c) \longrightarrow a+b+c$ of $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ to $Z/2$. Then H is defined to be the kernel of φ. The group H, which is just the group of *even* products of rotations, contains no rotations because all rotations are sent to 1 by the map φ.

Since the 3-fold rotation A sends a 2-fold axis for G_B to another 2-fold axis, it normalizes the group H. Let G_8 (for figure eight) be the group generated by A and H.

Counting indices we have $[G_L : G_B] = 3$, $[G_B : H] = 2$, $[G_8 : H] = 3$ and $[G_L : G_8] = 2$. Since the main cube is a fundamental domain for G_B and the volume of the main cube is 1, it follows that the volumes of fundamental domains for G_L, H, and G_8 are 1/3, 2, and 2/3 respectively.

Consider the 2 to 1 map $E^3/G_8 \longrightarrow E^3/G_L$. This is the double branched covering of $E^3/G_L = S^3$ branched over one of the two components of the link 6^2_2. It follows that as a topological space $E^3/G_8 = S^3$ and that the preimage of the other component of the link 6^2_2 is the figure eight knot. Lifting the Euclidean orbifold structure on E^3/G_L to E^3/G_8 we have shown that the figure eight knot as Euclidean orbifold is obtained as the orbit space of the crystallographic group G_8. The knot itself is the image of the 3-fold axes of rotation.

At this point we need to study the groups G_8, H, and G_L some more. Consider the element $XAX^{-1}A^{-1}$ of G_L where X is 180° rotation about an axis of G_B intersecting the main cube of figure 1. This element belongs to H because $AX^{-1}A^{-1}$ is a 180° rotation in G_B and the product of any two such 180° rotations is in H. The element XAX^{-1} belongs to G_8 because A belongs to G_8 and $XAX^{-1}A^{-1}$ belongs to $H \leqq G_8$. But the element XAX^{-1} is a 120° rotation in an axis that is the image of the axis of rotation of A under 180° rotation about an axis bisecting one of the faces of the main cube of figure 1.

In this way we see that the cubes adjacent to the main cube, and thus, iterating, all cubes in the tesselation, contain 3-fold axes of rotation for elements of G_8.

We would like to know the length of the figure eight knot singular set in the Euclidean orbifold E^3/G_8. To find this we must first define a certain fundamental domain for H. Since the main cube is a fundamental domain for G_B, and $[G_B : H] = 2$, and 180° rotations do not belong to H, we see that the main cube together with an adjacent cube constitute a fundamental domain for H. We take the adjacent cube and divide it into six congruent pyramids, each with square base. We then glue each of these six pyramids onto the six faces of the main cube using a different product of two 180° rotations in each of the six cases. In this way we obtain a fundamental domain for H that is a dodecahedron with rhombic faces. This fundamental domain is invariant under the rotation A, and is depicted in figure 5. Now we can obtain a fundamental domain for G_8 by intersecting the domain of figure 5 with any 120° wedge with axis equal to the axis of the rotation A.

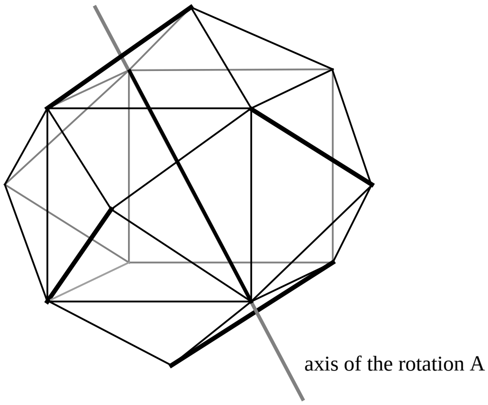


Figure 5

Unfortunately, (We learned this the hard way.), this domain does not have enough symmetry to allow us to push through certain calculations we need to make, and the face identifications are too complicated. However, we can obtain one useful fact from this domain.

The intersection of the domain of figure 5 with the axes of rotation of the elements of G_8 consists of one line segment, the diagonal of the main cube, of length $\sqrt{3}$, and six line segments each of length $\frac{1}{2}\sqrt{3}$ marked in figure 5. The intersection of the axes of rotation of elements of G_8, an appropriately chosen 120° wedge, and the domain of figure 5 consists of one line segment of length $\sqrt{3}$ and two line segments of length $\frac{1}{2}\sqrt{3}$. Hence the following proposition.

Proposition 1. *The orbit space of the action of the group G_8 on E^3 is a Euclidean orbifold of volume* 2/3. *The orbifold is topologically S^3 with* *singular set the figure eight knot. The singularity type is* 3 *and the length of the singular set is* $2\sqrt{3}$.

The fundamental domain for G_8 that we wish to construct will be a Dirichlet domain. Dirichlet domains for a discrete group G are constructed as follows. Pick a point P_0 not fixed by any element of G except the identity. For each non-identity element g of G let H_g be that closed half space that is bounded by the perpendicular bisector of the line segment $[P_0, g(P_0)]$ and that contains P_0. The Dirichlet domain $D(P_0)$ determined by P_0 is the intersection of all these closed half spaces. There are various sufficient conditions, satisfied in our case, that imply that such a domain is a finite polyhedron, but we will not explore them as we will see this explicitly anyway. That a Dirichlet domain really is a fundamental domain is not difficult to prove and the reader may wish to prove it himself.

We shall now go about choosing the point P_0 which defines the Dirichlet domain. At this point it will help the reader to refer to figure 6. Recall that the axis for the rotation A is the line $[(0,0,1),(1,1,0)]$, and that the line $(0,1,1/2)+t(1,0,0)$ is a 2-fold axis for an element, call it X, of G_B. The element $B = XAX^{-1}$ belongs to G_8 and is a 3-fold rotation with axis the line $[(1,1,1),(0,2,0)]$. The point P_0 is defined to be the midpoint of the unique line segment with endpoints on the axes of A and B and perpendicular to both these axes.

Since the half turn X of G_B interchanges the axes of A and B it leaves their unique mutual perpendicular invariant and the point P_0 must lie on the axis of X; that is, on the line $(0,1,1/2)+t(1,0,0)$.

Then P_0 turns out to be

$$P_0 = (3/4, 1, 1/2) \tag{1}$$

We shall also need to know the intersections of the common perpendicular with the axes of A and B which turn out to be $(3/4,3/4,1/4)$ and $(3/4,5/4,3/4)$ respectively.

It is convenient, at this point, to change coordinates so that P_0 becomes the origin, and so that various other coordinates become integers. Hence we translate P_0 to the origin and change scale by a factor of four. The new coordinates of points on the axes of A and B and points on their common perpendicular are indicated in brackets in figure 6.

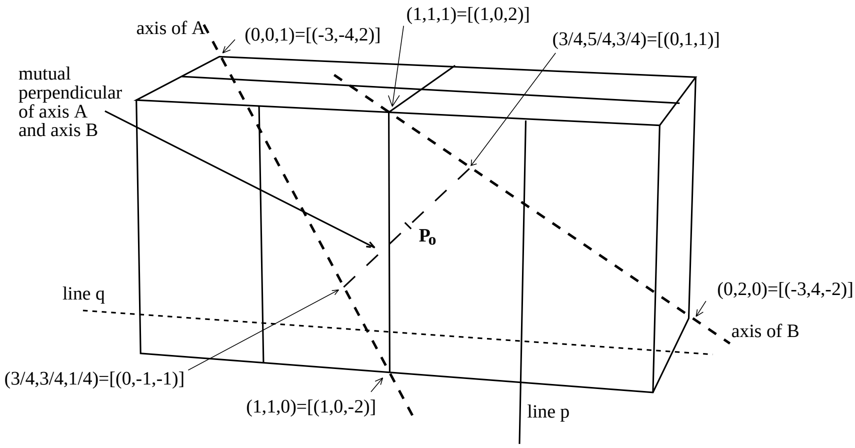


Figure 6

The new volume for a fundamental domain of G_8 is $2/3 \cdot 4^3 = 128/3$ and the new length of the figure eight knot in a fundamental domain is $8\sqrt{3}$.

The first four half spaces of the Dirichlet domain to compute will be those associated with the elements A, A^{-1}, B, and B^{-1}. The two bounding planes associated with A, A^{-1} contain the line $[(-3,-4,2),(1,0,-2)]$ because in the case of a rotation the perpendicular bisector of the line segment consisting of a point and its image always contains the axis of rotation. The two half spaces associated with A and A^{-1} intersect in an infinite wedge whose dihedral angle is $120°$. The projection of this wedge on a plane perpendicular to the axis of rotation is depicted in figure 7.

We shall need the equations of the two planes bounding the wedge determined by A and A^{-1}. The perpendicular from P_0, now the origin, to the axis of rotation of A intersects that axis in the vector $(0,-1,-1)$ (Again see figure 6). A vector parallel to the axis of rotation of A is the vector $(1,1,-1)$ and a vector perpendicular to both of these is $(2,-1,1)$. It happens that the vector $(2,-1,1)$ has length $\sqrt{3}$ times the length of $(0,-1,-1)$ and so the geometry of equilateral triangles, or $60°$ angles, forces it to lie in one of the bounding planes. (See figure 7, $(-2,1,-1)$ lies in the other.) Now that we know three points in a plane, $((-2,1,-1),(0,-1,-1),\text{and }(1,0,-2))$ we use calculus to determine its equation which is $x+y+2z+3=0$. Similarly the other plane bounding the wedge has equation $-x+2y+z+3=0$ and

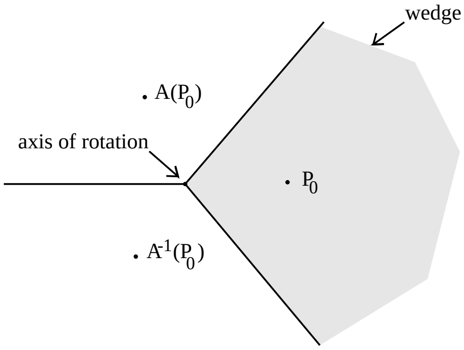


Figure 7

also similarly we compute the equations of the wedge determined by B and B^{-1}.

$$\begin{aligned} \text{Wedge } A \ \& \ A^{-1} \equiv \quad & x+y+2z+3 \geqq 0 \\ & -x+2y+z+3 \geqq 0 \\ \text{Wedge } B \ \& \ B^{-1} \equiv & -x-2y-z+3 \geqq 0 \\ & x-y-2z+3 \geqq 0 \end{aligned} \tag{2}$$

The two wedges intersect in a tetrahedron, which is depicted in figure 8. The vertices, which we label j,l,m and n, are found by solving wedge plane equations three at a time. We call this tetrahedron the main tetrahedron.

The vertices of the main tetrahedron are:

$$\{m=(3,2,-4), n=(-3,4,-2), l=(-3,-4,2), j=(3,-2,4)\} \tag{3}$$

The axis of A is ml, the axis of B is jn.

The main tetrahedron is not a fundamental domain because length $jn+$ length $lm = 12\sqrt{3}$ which is greater than $8\sqrt{3}$. (Which is what we would obtain if it were a fundamental domain.) So we must seek other elements of G_8 which define new half spaces that intersect the main tetrahedron.

Remenber that X is a half turn of G_B about the x-axis (See figure 6). If Y is any half turn in G_B then YX belongs to $H \leqq G_8$ so that $YX(P_0) = Y(P_0)$ can be used to define a half plane which may intersect the main tetrahedron and thus bring us closer to finding the Dirichlet domain determined by P_0.

Also X normalizes G_8 because $XgX^{-1} = g(g^{-1}Xg)X^{-1}$ and $(g^{-1}Xg)X^{-1}$ belongs to H. Moreover $X(g(P_0)) = XgX^{-1}(P_0)$ so that X leaves the set $\{g(P_0) \mid g \in G_8\}$ invariant and so X is a symmetry of the Dirichlet domain determined by P_0.

Let Z be the half turn in the line $t(0,1,1)$ (new coordinates) which is the mutual perpendicular of the axes of A and B. The isometry Z is not even a symmetry of the tesselation but $ZAZ^{-1} = A^{-1}$ and $ZBZ^{-1} = B^{-1}$. Since Z sends lines parallel to the x, y, and z axes to lines parallel to the x, z, and y axes respectively it follows that Z normalizes T_E. But G_8 is generated by T_E, A, and B, (Start with a $2 \times 2 \times 2$ fundamental domain for T_E and cut it down to size using A, B, and some translations by two) so that Z normalizes G_8.

Now Z also restricts to a symmetry of the Dirichlet domain because $ZgZ^{-1}(P_0) = Z(g(P_0))$.

The images of points under X and Z in the new coordinates are easy to compute; $X(x,y,z) = (x,-y,-z)$ and $Z(x,y,z) = (-x,z,y)$. At this point we refer the reader to figure 8.

Now consider the half turn about the line $(1,2,0) + t(0,0,1)$, which is a 2-fold axis in G_B (line p of figure 6). The origin, P_0, is sent to $(2,4,0)$ and the equation of the perpendicular bisecting plane is $x + 2y - 5 = 0$. Thus the vertex $n = (-3,4,-2)$ of the main tetrahedron lies on this plane.

Using calculus we compute the intersections of this plane with the edges of the main tetrahedron and display the results in figure 8.

The line jm is intersected in the point $(3,1,-2)$ and the line lm is intersected in the point $(7/3, 4/3, -10/3)$. Intersecting the corresponding half space with the main tetrahedron eliminates the line segment $[(7/3, 4/3, -10/3), m]$ which has length $2/3\sqrt{3}$.

Eliminating also the three images of this line segment under X, Z, and XZ has the effect of reducing the intersection of the proposed Dirichlet domain with the axes of rotation of A and B by $8/3\sqrt{3}$, from $12\sqrt{3}$ to $9\frac{1}{3}\sqrt{3}$, still greater than the desired $8\sqrt{3}$, so we do not yet have the Dirichlet domain.

Next, consider the half turn about the line $q = (-1,0,-2) + t(0,1,0)$, (See figure 6), followed by half turn about the line $p = (1,2,0) + t(0,0,1)$. This is an element of H. The origin is sent to $(4,4,-4)$ under this com-

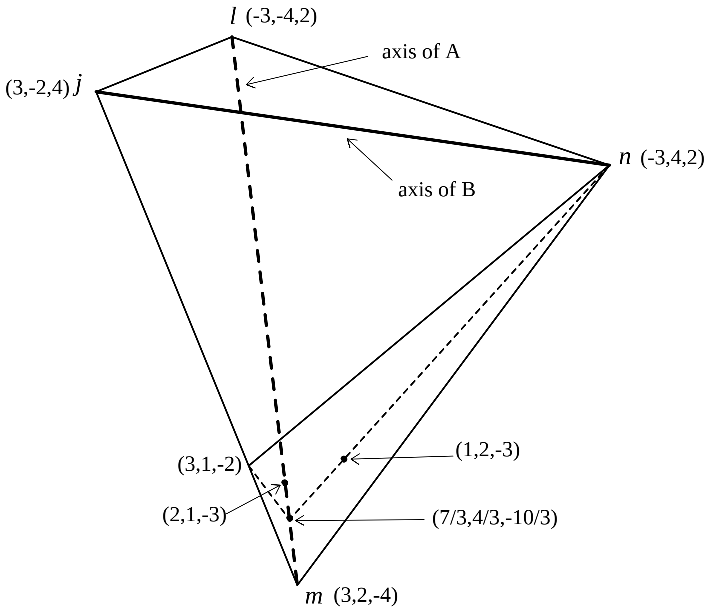


Figure 8

position. It follows that the equation of the planar perpendicular bisector of $[(0,0,0),(4,4,-4)]$ is $x+y-z-6=0$. The expression $x+y-z-6$ is negative at j, l and n, zero at $(3,1,-2)$ which lies on the line jm and in the plane $x+2y-5=0$, and positive at m. Using calculus we compute the intersection with lm, which is $(2,1,-3)$ and the intersection with $[(7/3,4/3,-10/3),n]$ which is $(1,2,-3)$.

The line segment from $(2,1,-3)$ to $(3,2,-4)$, which has length $\sqrt{3}$ has been eliminated. Eliminating also the images of this line segment under the action of X, Z and XZ reduces the total length of the intersection with the axes of A and B from $12\sqrt{3}$ to $8\sqrt{3}$ which is the correct length.

Thus the polyhedron obtained by intersecting the main tetrahedron with the images under, I, X, Z, and XZ of the half spaces $x+2y-5 \leqq 0$ and $x+y-z-6 \leqq 0$ has a good chance of being the Dirichlet domain we seek.

It is important to understand this polyhedron, (In fact it does turn out to be the Dirichlet domain.) and we refer the reader at this point to figure 9 which is drawn from the perspective of an observer on the positive x-

axis far from the origin. That is, points (x, y, z) are plotted as (y, z) with horizontal y-axis and vertical z-axis.

The main polyhedron intersects the bisecting planes $x + 2y - 5 = 0$ and $x + y - z - 6 = 0$ and their images under it, X, Z, and XZ, only in points with positive y and negative z coordinate or negative y and positive z coordinate, that is, only in points in the lower right or upper left of figure 9. In the lower right we have drawn all relevant points lines and intersections. In the upper left we have drawn only that part of the polyhedron remaining after intersection with the half spaces. To go from a point in the lower right to the corresponding point in the upper left just change the signs of the y and z coordinates.

The next step is to compute the volume of the polyhedron pictured in figure 9. This can be done using only a formula for the volume of a tetrahedron, given its vertices. Let $V(x, y, z, w)$ be the volume of a tetrahedron with vertices x, y, z, w, and refer to figure 9 for the labels of vertices in this computation.

Then the volume of the remaining polyhedron equals $V(l, j, m, n) - 2([V(a, b, m, n) + V(c, d, m, n) - V(e, f, m, n)] + V(a, e, g, b) + V(c, h, f, d)) = V(l, j, m, n) - 4V(a, b, m, n) + 2V(e, f, m, n) - 4V(a, e, g, b) = 128/3$. (This can actually be checked by hand in a reasonable amount of time. The volume of a tetrahedron is $\mid (u \times v) \cdot w \mid /6$ where u, v, and w are three vectors with a common tail at one of the vertices of the tetrahedron and heads at the other three vertices. And $(u \times v) \cdot w$ is obtained by writing the coordinates of the vectors in the rows of a 3×3 matrix and computing the determinant.)

Since the volume of the polyhedron of figure 9 equals the volume of a fundamental domain it must be a fundamental domain. Any other bisecting plane containing an interior point of this polyhedron would give rise to a polyhedron of volume less than 128/3 after intersection with the half space.

2. The Dirichlet domain in the Klein model

Our aim in this section is to take the Dirichlet domain polyhedron and to continuously change its dihedral angles, so as to obtain a one parameter family of "cone manifolds" in Klein models for hyperbolic spaces of constant curvature.

According to Thurston's, now well known, geometrization conjecture, all

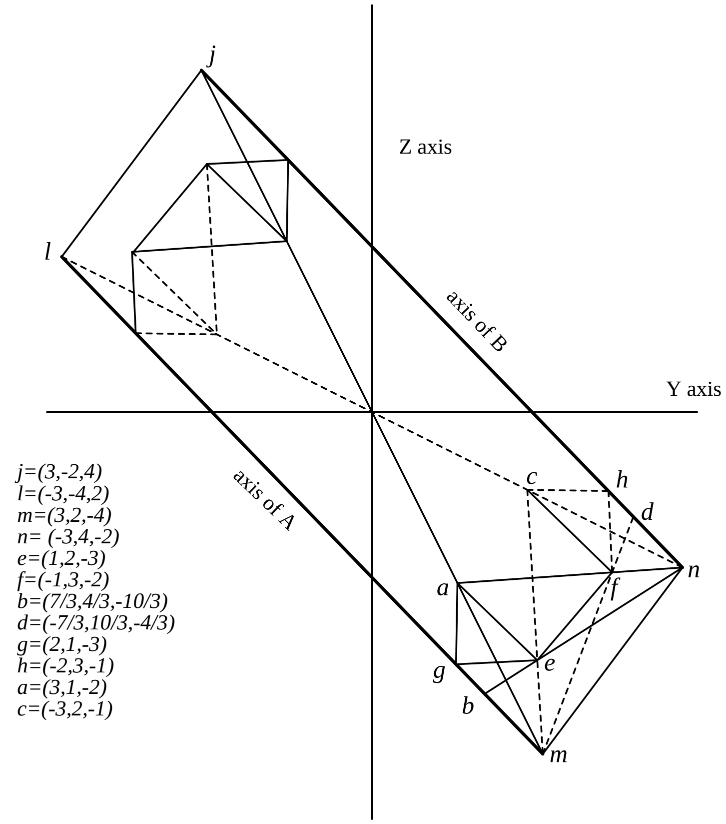


Figure 9

compact oriented 3-manifolds are expected to break down into "geometric" pieces with each piece belonging to one of exactly eight geometries. Three of these eight geometries are the spherical, Euclidean, and hyperbolic geometries each with constant curvature. Most of the researchers that we know working in this field immediately normalize the spherical and hyperbolic cases so that constant curvature means, in effect, curvature equal to plus or minus one. An advantage of this is, for example, that one can speak of "the" Poincare model.

But we believe this point of view can lead one astray since if the only curvatures are 0, $+1$, and -1, there is no possibility for smooth transitions between "manifolds" with these curvatures. In this section we shall construct a one parameter family of "cone manifolds", where the parameter is curvature and it changes smoothly from negative to positive through zero.

"Cone manifold" is a slight generalization of "orbifold". A cone manifold is a PL manifold together with a, possibly empty, codimension two locally flat, submanifold called the singular set. (In this paper we only care about dimensions two and three so the singular set will consist of isolated points in dimension two and curves, but not graphs, in dimension three.) There is a geometric model which is some spherical, Euclidean, or hyperbolic space of constant curvature, where the constant is any real number. Points off the singular set have neighborhoods homeomorphic to neighborhoods in the model. Points on the singular set have neighborhoods homeomorphic to neighborhoods constructed as follows: take an angle α wedge in the model. (A wedge is the intersection of two half spaces, the angle α is the dihedral angle.) Then identify the two boundaries of the wedge, using the natural rotation by α, to form a topological space. Points on the singular set have neighborhoods homeomorphic to neighborhoods in this topological space. The homeomorphism carries the singular set to the axis of rotation in the topological space. Transition functions are isometries.

The difference between our definition of a cone manifold and the common definitions of orbifold in dimension two and three are:

1. The curvature off the singular set in a cone manifold is constant, as in an orbifold, but the constant isn't necessarily 0 or ± 1.
2. In dimension three the singular set in a cone manifold is a curve, not, as is sometimes the case in an orbifold, a graph.
3. The "cone angle" is any angle α, $0 < \alpha \leqq 2\pi$. In an orbifold this angle is always $2\pi/n$.

One way to construct a 3-dimensional cone manifold (the method we shall use) is to start with a polyhedron in a space of constant curvature such that the faces are partitioned into pairs and there is an identifying isometry for each pair. Form the topological space using the isometries for identifications. This will be a cone manifold provided that:

1. No edge is identified with its inverse in the equivalence relation induced by the identifications, and the identifications of wedges along faces are cyclic for each equivalence class of edges.
2. The cone angle at each edge, (the sum of the dihedral angles about that edge), adds to $\leqq 2\pi$.

3. Each vertex has a neighborhood that is a "cone on a sphere". (A vertex might have a neighborhood that is a cone on a torus for example, in which case the topological space is not a manifold at all.)
4. There are either two edges, or no edges, emanating from each vertex with cone angles not equal to 2π. If two edges, the cone angles must be equal, say to α, and the edges must be "lined up". (The vertex has a neighborhood that looks like an α wedge with the half planes identified.)

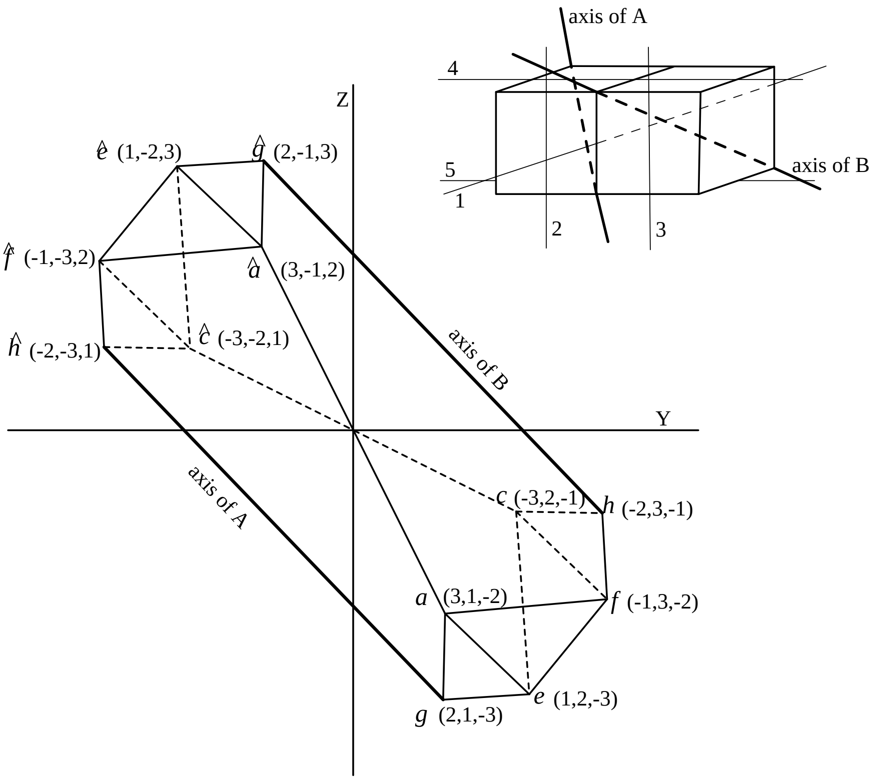


Figure 10

Before we can make further progress we must find the face identifications in the Dirichlet domain that determine the Euclidean orbifold structure for the figure eight knot. At this point we refer the reader to figure 10 for a picture of the polyhedron from the point of view of an observer on the positive x-axis with its vertices labelled with the labelling we shall use in the rest of the paper. Certain half turn axes are labelled $1, 2, 3, 4, 5$ in figure 10 also to help identify half turns.

To find the face identifications we look for elements of G_8 that do not move points in the polyhedron too far. Successive half turns about "perpendicular" axes separated by a distance of two seems a good place to start. Let H_j be a half turn about the axis j of figure 10. Then H_1 is just a half turn in the x-axis and $H_1(x, y, z) = (x, -y, -z)$. H_3 is a half turn in the axis $(1, 2, 0) + t(0, 0, 1)$ and $H_3(x, y, z) = (2 - x, 4 - y, z)$ so that for example $H_3H_1(\widehat{e}) = H_3H_1(1, -2, 3) = (1, 2, -3) = e$, so that $H_3H_1(x, y, z) = (2 - x, 4 + y, -z)$, and H_3H_1 sends triangle $\widehat{e}\,\widehat{a}\,\widehat{f}$ to triangle efa. In a similar fashion we find the rest of the triangular identifications in the list which follows shortly.

The pentagon $\widehat{g}hfa\widehat{a}$ with base $\widehat{g}h$ has right angles at its base $\widehat{g}h$ and is symmetric with respect to a reflection in the perpendicular bisector of $\widehat{g}h$. The rotations A and B are the elements of G_8 that identify the four pentagonal faces in pairs.

List of Identifications.

$$
\begin{aligned}
H_3H_1 &: \quad \Delta\widehat{e}\,\widehat{a}\widehat{f} \longrightarrow \Delta efa \\
H_5H_1 &: \quad \Delta\widehat{c}\,\widehat{e}\widehat{f} \longrightarrow \Delta ecf \\
H_3H_5 &: \quad \Delta hcf \longrightarrow \Delta gea \\
H_2H_4 &: \quad \Delta\widehat{f}\widehat{c}\widehat{h} \longrightarrow \Delta\widehat{a}\,\widehat{e}\,\widehat{g} \\
A &: \quad \text{Pentagon } \widehat{h}ga\widehat{a}\widehat{f} \longrightarrow \text{Pentagon } \widehat{h}gec\widehat{c} \\
B &: \quad \text{Pentagon } \widehat{g}hc\widehat{c}\,\widehat{e} \longrightarrow \text{Pentagon } \widehat{g}hfa\widehat{a}
\end{aligned} \tag{4}
$$

The face identifications induce an equivalence relation on the set of edges, and the set of vertices. These are as follows:

Edges.

$$
\begin{aligned}
&\widehat{e}\,\widehat{a} = \; ef = \widehat{cf}\\
&\widehat{ef} = \; ea = cf\\
&\widehat{ch} = \widehat{f}\;\widehat{h} = \; \widehat{a}\,\widehat{g} = \widehat{e}\,\widehat{g}\\
&ag = \; eg = ch = fh\\
&\widehat{a}a = \; ce = \widehat{e}\,\widehat{c}\\
&af = \; \widehat{f}\widehat{a} = \widehat{c}c\\
&\widehat{g}h\\
&g\widehat{h}
\end{aligned}
\tag{5}
$$

Vertices.

$$
\begin{aligned}
&\widehat{g} = \widehat{h}\\
&g = h\\
&\widehat{a} = f = \widehat{f} = a = e = c = \widehat{e} = \widehat{c}
\end{aligned}
\tag{6}
$$

Thus the resulting complex has 6 faces, 8 edges, 3 vertices, and 1 3-cell giving an Euler characteristic of zero, which is comforting as it is supposed to be homeomorphic to S^3.

The next step is to define a polyhedron very much like the polyhedron of figure 10. It will end up having exactly the same configuration of vertices edges and faces, as well as the same symmetry group as the Euclidean polyhedron of figure 10. However this polyhedron will live inside a sphere, the sphere at infinity for the Klein model of a hyperbolic space of constant curvature.

Also, it will be convenient to have the axes of symmetry for this polyhedron be the x, y, and z axes.

We begin by choosing a, b, c and R positive numbers. (We also use a, b, and c to refer to vertices, for example in the list of identifications, but this should cause no problems because a vertex is not a number. It's best to think of R, which will be the radius of the sphere at infinity in the Klein model, and will not concern us for a while, as being much larger than a, b, and c. Refer to figure 11.)

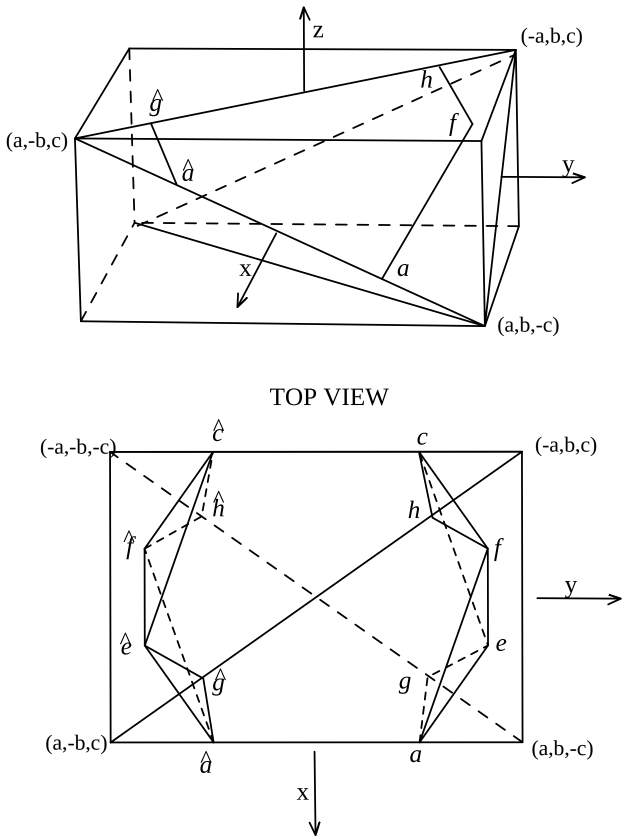


Figure 11

Define the *main tetrahedron* to be the tetrahedron with vertices $\{(-a,b,c),(a,-b,c),(a,b,-c),(-a,-b,-c)\}$ which lies inside the "box" $\{(\pm a,\pm b,\pm c)\}$. The bases of the pentagons will lie along the lines $[(-a,b,c),(a,-b,c)]$ and $[(a,b,-c),(-a,-b,-c)]$. The pentagons we shall define, in analogy with the pentagons of figure 11, will lie in the faces of the main tetrahedron, will be symmetric with respect to reflection in the perpendicular bisector of the base edge, and will have one of the two apex edges lying along an edge of the main tetrahedron.

These conditions completely define three of the five vertices of each pentagon, but not the two base vertices. The perpendicular bisector of the base of the front pentagon and of the line segment $[(-a,b,c),(a,-b,c)]$ has equation $ax - by = 0$.

The equation of the plane containing the front face of the main tetrahedron is $bcx + acy + abz = abc$. The apex of the front pentagon lies in both these planes and has x coordinate equal to a. Hence the apex of the front pentagon, denoted by a in analogy with figure 10 is the point $(a, a^2/b, -ca^2/b^2)$. One of the other vertices is the point $\widehat{a}$, obtained by rotating point a 180 degrees about the x-axis. Thus $\widehat{a} = (a, -a^2/b, ca^2/b^2)$. The front pentagon vertex f is obtained by reflecting $\widehat{a}$ in the perpendicular bisecting plane $ax - by = 0$. Thus $f = ((ab^2 - 3a^3)/(a^2 + b^2), (3a^2b^2 - a^4)/b(a^2 + b^2), ca^2/b^2)$.

With R, and thus the sphere at infinity chosen, we can measure angles in the Klein model. The base vertices of the pentagon $\widehat{g}$ and h are defined by the condition that $\widehat{a}$ and f belong to the planes perpendicular to axis $\widehat{g}h$ through the points $\widehat{g}$ and h respectively. The other three pentagons are the images of $\widehat{a}afh\widehat{g}$ under half turns in the x, y and z axes. Since every vertex of the polyhedron belongs to a pentagon, the polyhedron itself has been defined, and its vertices are labelled exactly as in figure 10 but we hold off for a moment before making identifications. So far, the construction only depends on the choice of a,b,c and R.

If we are successful in satisfying the conditions for one set of a,b,c and R then a constant multiple of everything will give another example. Therefore we can normalize one coordinate so we set $b = 1$. (Surprisingly, normalizing a or c instead leads to substantially more complicated computations.)

Next, we observe that in the Euclidean case all vertices have the same distance, $\sqrt{14}$, from the origin. Imposing this condition in the hyperbolic case is too strong, (We tried.), but we do impose the conditions that all vertices to be identified, a, e, f, c,$\widehat{a}$,$\widehat{e}$,$\widehat{f}$,$\widehat{c}$ have the same distance from the origin. This implies that if x, y, z, w are selected from this set of vertices then $[x, y]$ is Euclidean congruent to $[z, w]$ if and only if it is hyperbolic congruent to $[z, w]$. In the list of identifications, we find that ae is identified with $\widehat{f}\widehat{e}$ and that $\widehat{f}\widehat{e}$ is congruent to ef by a symmetry of the polyhedron (half turn in the z-axis). Hence we must have ef hyperbolic congruent, and therefore Euclidean congruent to ea. We have already computed the coordinates of a and f, $a = (a, a^2, -ca^2)$ and $f = ((a - 3a^3)/(a^2 + 1), (3a^2 - a^4)/(a^2 + 1), ca^2)$ and e is obtained from f by half turn in the y-axis so that $e = (-(a - 3a^3)/(a^2 + 1), (3a^2 - a^4)/(a^2 + 1), -ca^2)$. The vector

$ef = (2/(a^2+1))(a-3a^3, 0, ca^2(a^2+1))$ and the vector $ea = (2/(a^2+1))(a-a^3, a^4-a^2, 0)$. Setting length $ea =$ length ef we find:

$$c = \sqrt{5-10a^2+a^4}/(a^2+1) \tag{7}$$

We are only interested in the case where c is real and positive. This occurs for $0 < a^2 < 5-2\sqrt{5}$ and $a^2 > 5+2\sqrt{5}$. But if $a^2 > 5+2\sqrt{5}$ the pentagon vertex a is outside the box because its y coordinate is a^2 which is greater than one. Hence from now on we shall assume that $0 < a^2 < 5-2\sqrt{5}$ and that c is determined by a from equation 7.

The identifications given in the list of identifications (4) can be done using hyperbolic isometries. The pentagons can be identified via reflections in planes through the origin. Identified triangles not containing vertices g,h,$\widehat{g}$,$\widehat{h}$ are Euclidean congruent, and therefore hyperbolic congruent. The four triangles with vertices g,h,$\widehat{g}$,$\widehat{h}$ are Euclidean and hyperbolic isosceles (Use a reflection in a plane through the origin.) and are cyclically permuted by the half turns in the axes, which are hyperbolic isometries.

So far $0 < a^2 < 5-2\sqrt{5}$, and any R with the sphere of radius R containing the polyhedron we have constructed will turn out to be a cone manifold with singular set the figure eight knot. The only additional conditions that need to be checked are that the sum of the dihedral angles around an equivalence class of edges is what it should be.

The cone angle at the edges $\widehat{g}h$ and $g\widehat{h}$ is sure to be some angle less than π and the figure eight knot will thus be contained in the singular set. By construction, the four dihedral angles at $ag = eg = ch = fh$ and $\widehat{c}\widehat{h} = \widehat{f}\,\widehat{h} = \widehat{a}\widehat{g} = \widehat{e}\,\widehat{g}$ are right angles so the *cone angle* at these edges equals 2π and they don't belong to the singular set. If the following conditions on dihedral angles are satisfied, (they come from (5)) the orbifold will be a hyperbolic orbifold with singular set the figure eight knot. (We use the same notation for an edge as its dihedral angle.)

$$\begin{aligned} \widehat{e}\,\widehat{a} + ef + \widehat{c}\widehat{f} &= 2\pi \\ \widehat{e}\widehat{f} + ea + cf &= 2\pi \\ \widehat{a}a + ce + \widehat{e}\widehat{c} &= 2\pi \\ af + \widehat{f}\widehat{a} + \widehat{c}c &= 2\pi \end{aligned} \tag{8}$$

At this point we make use of the symmetries of the polyhedron (Refer to figure 10). They imply that

$$
\begin{aligned}
ef &= \widehat{\widehat{e}f}\\
\widehat{\widehat{a}f} &= \widehat{e}\,\widehat{c} = af = ec\\
\widehat{e}\,\widehat{a} &= \widehat{f\widehat{c}} = ae = fc\\
a\widehat{a} &= c\widehat{c}
\end{aligned}
\tag{9}
$$

Using equations (9) in equations (8) we see that the first two equations of (8) are equivalent and the last two equations of (8) are equivalent and that the systems (8) and (9) are equivalent to:

$$
\begin{aligned}
ea + ef/2 &= \pi\\
af + a\widehat{a}/2 &= \pi
\end{aligned}
\tag{10}
$$

To satisfy these equations, given $0 < a^2 < 5 - 2\sqrt{5}$ we shall have to choose R correctly.

At this point, in order to facilitate computations in the Klein model (See Appendix A), we switch to homogeneous coordinates $((x, y, z, t) = (\lambda x, \lambda y, \lambda z, \lambda t)$ for any $\lambda \neq 0$ and $\mathbb{R}^3$ embeds in $\mathbb{R}P^3$ by $(x, y, z) \longrightarrow (x, y, z, R))$. Points in the Klein model determined by the sphere of radius R are those points (x, y, z) or (x, y, z, R) with $x^2 + y^2 + z^2 < R^2$. The plane given by the equation $ax + by + cz = d$ and $t = R$, has the pole (Ra, Rb, Rc, d) (in homogeneous coordinates). For two vectors in $\mathbb{R}^4$ we define $\langle \overrightarrow{v}, \overrightarrow{w} \rangle$, $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$, $\overrightarrow{w} = (w_1, w_2, w_3, w_4)$ by

$$
\langle \overrightarrow{v}, \overrightarrow{w} \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3 - v_4 w_4 \tag{11}
$$

Of course $\langle \overrightarrow{v}, \overrightarrow{w} \rangle$ is not well defined for homogeneous vectors but it is useful for determining the dihedral angle between planes. The following formula, proved in Appendix A, gives the cosine of the dihedral angle $\ominus$ between two intersecting planes in the Klein model with poles $\overrightarrow{v}$ and $\overrightarrow{w}$.

$$
cos\ominus = \frac{-\langle \overrightarrow{v}, \overrightarrow{w} \rangle}{\sqrt{\langle \overrightarrow{v}, \overrightarrow{v} \rangle}\sqrt{\langle \overrightarrow{w}, \overrightarrow{w} \rangle}} \tag{12}
$$

At this point let the homogeneous vectors $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ be the poles of the planes $a\widehat{a}$-origin, $a\widehat{a}f$, and efa. We had previously computed $a = (a, a^2, -ca^2)$, $\widehat{a} = (a, -a^2, ca^2)$, $f = ((a - 3a^3)/(a^2+1), (3a^2 - a^4)/(a^2+1), ca^2)$ and $e = (-(a-3a^3)/(a^2+1), (3a^2-a^4)/(a^2+1), -ca^2)$.

It is easy, if tedious, to find the equation of a plane given three points in it. We find the equations for the planes in question

$$\begin{aligned} \text{Plane } a\widehat{a} \text{ -origin}: \quad & cy + z = 0 \\ \text{Plane } a\widehat{a}f: \quad & cx + acy + az = ac \\ \text{Plane } efa: \quad & ca(1+a^2)x + c(1+a^2)y + (3a^2-1)z \\ & = 3ca^2 - ca^4 \end{aligned} \tag{13}$$

So that taking the outward normals

$$\begin{aligned} u &= (0, -c, -1, 0) \\ v &= (Rc, Rac, Ra, ac) \\ w &= (Rca(1+a^2), Rc(1+a^2), R(3a^2-1), ca^2(3-a^2) \end{aligned} \tag{14}$$

Now consider again the second condition $af + \frac{1}{2}a\widehat{a} = \pi$. Since $\frac{1}{2}a\widehat{a}$ is precisely the dihedral angle formed by the planes $a\widehat{a}$ origin and $a\widehat{a}f$, necessarily an acute angle, and since af is the dihedral angle between $a\widehat{a}f$ and efa, the second condition of (10) is precisely equivalent to the conditions, cosine af is $\leqq 0$ and $\cos^2 af = \cos^2(\frac{1}{2}a\widehat{a})$. But this last equality is equivalent to

$$\frac{\langle u, v\rangle^2}{\langle u, u\rangle\langle v, v\rangle} = \frac{\langle v, w\rangle^2}{\langle v, v\rangle\langle w, w\rangle} \tag{15}$$

We compute

$$\begin{aligned} \langle u, u\rangle &= (1+c^2) \\ \langle u, v\rangle &= -Ra(1+c^2) \\ \langle w, w\rangle &= R^2c^2(1+a^2)^3 + R^2(3a^2-1)^2 - c^2a^4(a^2-3)^2 \\ \langle v, w\rangle &= 2R^2ac^2(1+a^2) + R^2a(3a^2-1) + c^2a^3(a^2-3) \end{aligned} \tag{16}$$

Equation (15) reduces to $\langle w, w\rangle\langle u, v\rangle^2 - \langle u, u\rangle\langle v, w\rangle^2 = 0$. After some algebraic manipulation (Recall a,c,$R > 0$, and $a < 1$.) we obtain:

$\langle w,w\rangle\langle u,v\rangle^2 - \langle u,u\rangle\langle v,w\rangle^2 = a^2c^2(a^2-3)(1+c^2)(-a^2+R^2)\cdot(a^2c^2(a^2-3)+R^2[-2+3a^2+a^4+c^2(1+a^2)^2])$

Only the expression in the last parentheses is important. Thus we set:

$$R^2 = \frac{c^2(3a^2-a^4)}{c^2(1+a^2)^2-2+3a^2+a^4}. \tag{17}$$

And using (7) we can express R^2 as a function of a alone

$$R^2 = \frac{a^2(a^4-10a^2+5)}{(a^2+1)^2(1-2a^2)}. \tag{18}$$

We recall for the reader that we assume that a satisfies $0 < a^2 < 5-2\sqrt{5}$ and we note that a^4-10a^2+5 has zeroes at $5\pm 2\sqrt{5}$, and that $R^2 \leqslant 0$ for $1/2 \leqq a^2 \leqq 5-2\sqrt{5}$. The values $1/2 < a^2 < 5-2\sqrt{5}$ will turn out to be interesting, but for the moment we assume $0 < a^2 < 1/2$.

Next, we turn to the signs of the cosines of the two dihedral angles in question. These are determined by $\langle u,v\rangle$ and $\langle v,w\rangle$. The first equals $-Ra(1+c^2)$ which is obviously negative and the second is obtained by using (17), (18), and (7) in (16). After factoring we obtain: $\langle v,w\rangle = \frac{2a^3(a^4-10a^2+5)(3-a^2)(a^2-1)^2}{(a^2+1)^3(1-2a^2)}$. This expression is positive for $0 < a^2 < 1/2$, so that the second of the two equations (10) is satisfied.

At this point we turn to the first of the equations; $ea + ef/2 = \pi$.

The dihedral angle $ef/2$ is formed by the planes through e, f, and the origin and the plane efa. From the coordinates for e and f we compute that this first plane has equation $ca(a^2+1)x + (3a^2-1)z = 0$ and we have already computed the equation of plane efa which is $ca(1+a^2)x + c(1+a^2)y + (3a^2-1)z = ca^2(3-a^2)$.

The dihedral angle ea is formed by the plane efa and the plane containing a and perpendicular in the Klein model to the line $\widehat{gh}$, which is one of the edges of the main tetrahedron, contains the points $(-a,-1,-c)$ and $(a,1,-c)$, and has parametric equation $(0,0,-c)+t(a,1,0) = (ta,t,-c)$.

Suppose the plane perpendicular to $\widehat{gh}$ has equation $\alpha x+\beta y+\gamma z = \delta$ and therefore pole $(\alpha,\beta,\gamma,\delta/R)$ in homogeneous coordinates and $(R^2/\delta)(\alpha,\beta,\gamma)$ in Euclidean coordinates. The condition that the plane be perpendicular to the line in the Klein model is that the line contains the pole of the plane. Hence we have $\alpha = (\delta/R^2)ta$, $\beta = (\delta/R^2)t$, and $\gamma = (-\delta/R^2)c$.

The condition that $a = (a, a^2, -ca^2)$ belongs to the plane implies that $t = \frac{R^2 - c^2a^2}{2a^2} = \frac{c^2a^2}{1-2a^2}$ (by (7) and (18), so that the pole of the plane, in homogeneous coordinates equals $(a(R^2 - c^2a^2), (R^2 - c^2a^2), -2ca^2, 2a^2R)$. The equation of the plane is

$$ax + y - \frac{1-2a^2}{a^2c}z = 1 \tag{19}$$

We also recall here that $g = (ta, t, -c)$ with

$$t = \frac{(R^2 - a^2)2a^2}{(a^2+1)(R^2 - c^2a^2)} = \frac{3a^2 - 1}{a^2(a^2+1)} \tag{20}$$

We define, once again, homogeneous vectors $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{w}$ for the poles of these three planes.

Pole ef -origin
$\equiv \overrightarrow{u} = (-ca(a^2+1), 0, -(3a^2-1), 0)$.
Pole efa
$\equiv \overrightarrow{v} = (Rca(1+a^2), Rc(1+a^2), R(3a^2-1), ca^2(3-a^2))$.
Pole of the perpendicular plane
$\equiv \overrightarrow{w} = ((R^2 - c^2a^2)a, R^2 - c^2a^2, -2ca^2, 2a^2R)$.

The conditions we must check are that $\langle w, w\rangle\langle u, v\rangle^2 - \langle u, u\rangle\langle v, w\rangle^2 = 0$ for the choices of c and R that we have made in (7) and (17), and that $\langle v, w\rangle \geqq 0$.

Again we compute

$$\begin{aligned}
\langle u, u\rangle &= c^2a^2(a^2+1)^2 + (3a^2-1)^2; \\
\langle u, v\rangle &= -Rc^2a^2(a^2+1)^2 - R(3a^2-1)^2; \\
\langle w, w\rangle &= (R^2 - c^2a^2)^2a^2 + (R^2 - c^2a^2)^2 + 4c^2a^4 - 4R^2a^4; \\
\langle v, w\rangle &= R(R^2 - c^2a^2)ca^2(1+a^2) + R(R^2 - c^2a^2)c(1+a^2) \\
&\quad + R(3a^2-1)(-2ca^2) - 2a^2Rca^2(3-a^2).
\end{aligned}$$

And we see that $\langle w, w\rangle\langle u, v\rangle^2 - \langle u, u\rangle\langle v, w\rangle^2 = 0$ is in fact satisfied by a tedious calculation.

Also we compute that $\langle v, w\rangle = 2Rca^2(1-a^2)^3/(1-2a^2)$ which is positive for $a^2 < 1/2$.

3. The hyperbolic and Euclidean cases; the interval $1/3 \leqq a^2 \leqq 1/2$

In the previous section we gave a construction of a certain polytope, and, hopefully, cone manifold, which depended on parameters a, c, and R. The construction only made sense for $0 < a^2 < 5 - 2\sqrt{5} = .5278\cdots$. For values of a satisfying this condition we found that

$$c = \sqrt{a^4 - 10a^2 + 5}/(1 + a^2) \tag{7}$$

$$R^2 = a^2(a^4 - 10a^2 + 5)/((1 + a^2)^2(1 - 2a^2)) \tag{18}$$

Since $R^2 < 0$ when $a^2 > 1/2$, we wish to restrict our interest to the case $0 < a^2 \leqq 1/2$ for the moment.

The value of R was determined by the first condition on dihedral angles; $ea + ef/2 = \pi$ (10). Then it turned out that the second condition on dihedral angles $af + a\widehat{a}/2 = \pi$ was also satisfied. However (See figure 10) we are implicitly assuming that, for this value of R, the half-space that contains the origin and whose planar boundary contains the point $\widehat{a}$ and is perpendicular to the line $\widehat{g}h$ intersects the half-space that contains the origin and whose planar boundary contains the point f and is perpendicular to the line $\widehat{g}h$ in a wedge that has nontrivial intersection with the line $\widehat{g}h$.

We also are implicitly assuming that the sphere of radius R contains the polytope we have defined. We now investigate these assumptions.

The line $\widehat{g}h$, which is the line $[(a, -1, c), (-a, 1, c)]$ has the parameterization $(0, 0, c) + t(-a, 1, 0) = (-at, t, c)$. Points on this line with $t < 0$ lie on the same side of its intersection with the z-axis as does $(a, -1, c)$.

Comparing with (20) we see that $\widehat{g}$ is the point on the line $\widehat{g}h = (-at, t, c)$ corresponding to a value of t equal to:

$$t = \frac{-2a^2(R^2 - c^2)}{(1 + a^2)(R^2 - c^2a^2)} = \frac{(1 - 3a^2)}{a^2(1 + a^2)}. \tag{21}$$

We see that t is negative if and only if $a^2 > 1/3$.

Since the implicit assumption about the half-spaces and the line $\widehat{g}h$ is precisely equivalent to the condition $t < 0$ we shall make the assumption that $1/3 \leqq a^2$ for the rest of this section.

Next we verify that for $1/3 < a^2 < 1/2$, the sphere of radius R contains the polytope. For this, it suffices to show that it contains the vertices of the polytope. The vertices a, e, c, f, $\widehat{a}$, $\widehat{c}$, $\widehat{f}$, $\widehat{e}$ all have the same square distance from the origin which equals $a^2(1 + a^2 + a^2c^2)$ (from the coordinates of $\widehat{a}$).

The points g, $\widehat{g}$, h, $\widehat{h}$ also all have the same square distance from the origin. The coordinates of $\widehat{g}$ equal $(-at, t, c)$ for the value of t equal $(1 - 3a^2)/a^2(1 + a^2)$. Hence the square of the distance of $\widehat{g}$ from the origin is $(a^8 - a^6 + 8a^4 - 5a^2 + 1)/a^4(1 + a^2)^2$. Doing some algebraic computation we compute that distance square of $\widehat{g}$ from the origin less distance square of $\widehat{a}$ from the origin equals:

$$(1 - 2a^2)\left(1 - a^2\right)^4 / a^4 \left(1 + a^2\right). \tag{22}$$

This quantity is positive for $a^2 < 1/2$. Hence we need only compare R^2 with the square distance of $\widehat{g}$ from the origin.

Again, computing R^2 minus the square of the distance of $\widehat{g}$ from the origin we find that this equals

$$\frac{\left(3a^2 - 1\right)\left(a^2 - 1\right)^4}{\left(1 + a^2\right)^2 a^4 \left(1 - 2a^2\right)}. \tag{23}$$

Since this last quantity is positive for a^2 in the interval $(1/3, 1/2)$ the last obstacle has been surmounted and we have succeeded in defining a one parameter family of cone manifolds; the parameter is a, with $1/3 < a^2 < 1/2$.

It is of interest to compute the cone angle. This equals twice the angle, call it $\ominus$, between the planes $\widehat{g}h$ origin and $\widehat{g}ha$. The formulas for these planes are respectively $x + ay = 0$ and $cx + acy + az = ac$.

Let $u = (1, a, 0, 0)$ and $v = (Rc, Rac, Ra, ac)$ be the poles of these planes. The angle $\ominus$, necessarily acute, is determined by the formula $\cos^2\ominus = \langle u, v\rangle / \langle u, u\rangle\langle v, v\rangle$. Computing this last expression and using the values we have obtained for R^2 and c^2 and the formula for the cosine of a double angle, we obtain:

$$\text{Cone angle} = \text{arc cos}\,\frac{3 - 6a^2 - a^4}{2(a^2 - 1)^2}. \tag{24}$$

In the interval $1/3 \leqq a^2 \leqq 1/2$, we see that R is a monotone increasing function, (Consider R as a function of a and compute the sign of the first derivative.) that $R = 1$ at $a^2 = \frac{1}{3}$ and that $R = \infty$ at $a^2 = \frac{1}{2}$.

Also we see that the cone angle increases monotonely from zero at $a^2 = 1/3$ to $2\pi/3$ at $a^2 = 1/2$.

Since R is a monotone increasing function of a so that a is also a monotone increasing function of R, $1 \leqq R \leqq \infty$, we could as well use R as the parameter. In the Klein model the curvature turns out to be simply the negative reciprocal of R^2 so that we have defined a 1-parameter family of cone manifolds whose curvature increases from -1 (at $a^2 = 1/3$) to 0 at $a^2 = 1/2$.

The case $a^2 = 1/2$ is the Euclidean case and the polytope corresponding to $a^2 = 1/2$ is isometric to the polytope defined in section one, except for a change of scale.

The case $a^2 = 1/3$ corresponds to the hyperbolic case with knot "at infinity". We shall study this case in more detail. The parameters are $a = 1/\sqrt{3}$ and, solving (7) and (18) for c and R, $c = 1$ and $R = 1$. The pentagon $a\widehat{a}\widehat{f}\ \widehat{h}g$ has collapsed to a quadrilateral $a\widehat{a}fg$, $(\widehat{h} = g)$, with the vertex g at infinity. The coordinates are $a = \left(1/\sqrt{3}, 1/3, -1/3\right)$, $\widehat{a} = \left(1/\sqrt{3}, -1/3, 1/3\right)$, $f = (0, 2/3, 1/3)$, $h = \widehat{g} = (0, 0, 1)$. The rest of the vertices of the polyhedron are obtained by rotating these coordinates about the x, y, and z axes by 180 degrees. The polyhedron is pictured in figure 12. There are four quadrilateral faces and eight triangular faces.

In Thurston's original construction (See [T]) the complement of the figure eight knot was constructed using two solid tetrahedra with pairs of faces identified. These tetrahedra were Euclidean equilateral tetrahedra with vertices in the unit sphere.

Their dihedral angles were 60° when considered as hyperbolic tetrahedra in the Klein model with the unit Euclidean sphere being the sphere at infinity. We see no apparent connection between our polytope and Thurston's two tetrahedra, but we feel that there must be some way of chopping the polyhedron of figure 12 up and reglueing to get two tetrahedra with vertices at infinity. For the convenience of the reader who wishes to attempt this we have labelled the edges of the polyhedron in figure 12b with the cosines of the dihedral angles (in hyperbolic space), and with the Euclidean lengths in figure 12a.

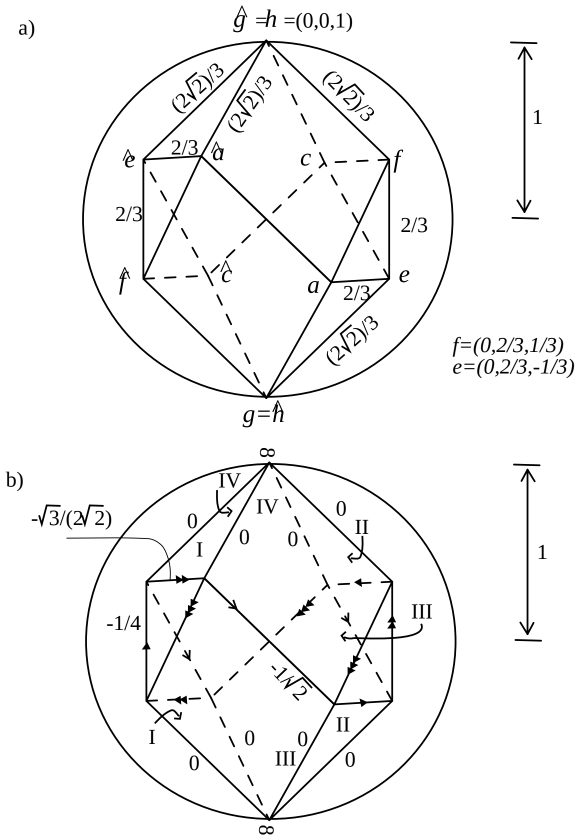


Figure 12

We sumarize the above in the final proposition of this section

Proposition 3.1. *For $1/3 \leqq a^2 \leqq 1/2$ the topological cone-manifold determined by a^2 is S^3. The singular set is the figure eight knot. The angle increases from 0 ($a^2 = 1/3$) to $2\pi/3$ ($a^2 = 1/2$), while the curvature (constant) increases from -1 to 0. Therefore, the geometry for $a^2 = 1/2$ is Euclidean. For $a^2 = 1/3$ we have the complete hyperbolic structure of finite volumen in the complement of the figure eight knot.*

4. Spontaneous surgery, the case $0 < a^2 < \frac{1}{3}$

We saw in section four that when a^2 decreased to $\frac{1}{3}$ the points $\widehat{g}$ and

h coalesced to a single point, that R decreased to one so that this point was on the sphere at infinity and that the pentagonal faces of the polytope became quadrilaterals (See figure 12). In this section we shall investigate what happens when a^2 decreases beyond $\frac{1}{3}$.

Recall that when $a^2 > \frac{1}{3}$, the line $\widehat{g}h$ has parametric equation $(-at, t, c)$; $-\infty < t < \infty$. This line is the intersection of two planes, the planes that contain the pentagons when $a^2 > \frac{1}{3}$. (See figures 10 and 11). These are the planes $\widehat{g}ha$ and $\widehat{g}h\widehat{c}$ and they have equations $cx + acy + az = ac$ and $-cx - acy + az = ac$ respectively. (They contain the points $(a, -1, c)$, $(-a, 1, c)$ and $(a, 1, -c)$ $[(-a, -1, -c)$ resp.$])$. We shall refer to these planes in this section as the "pentagonal planes" even though they don't contain any pentagons when $a^2 < \frac{1}{3}$.

Another plane that was important in defining the polytope was the plane that contained the point $\widehat{a} = (a, -a^2, ca^2)$ and was perpendicular to the line $\widehat{g}h$ in the Klein model. (Again see figures 10 and 11). Comparing with (19), it turns out that the equation of this plane is:

$$-ax + y + \frac{1 - 2a^2}{a^2 c} z = 1 \tag{25}$$

In this section we shall refer to this plane and its image under 180° rotation in the z–axis as the "perpendicular planes" although they aren't perpendicular to each other. The other perpendicular plane has equation $ax - y + [(1 - 2a^2)/a^2 c]z = 1$.

The two perpendicular planes intersect in a line which we call the "singular line". The singular line has parametric equation

$$\left(t, at, \frac{a^2 c}{1 - 2a^2}\right); -\infty < t < \infty \tag{26}$$

When $\frac{1}{3} \leqq a^2 \leqq \frac{1}{2}$ each face of the polytope determines a plane and each plane determines a half space, the one containing the origin. The polytope itself is simply the intersection of these half spaces.

We shall use the same planes, and the same half spaces to define the polytope when $0 < a^2 < \frac{1}{3}$ although it is not obvious that the polytope lies inside the sphere at infinity. We now investigate this.

When $\frac{1}{3} < a^2 < \frac{1}{2}$ the line $\widehat{g}h = (-at, t, c)$ lies below the singular line $\left(t, at, \frac{a^2 c}{1-2a^2}\right)$ so that the singular line lies outside the polytope and has no interest for us. However when $0 < a^2 < \frac{1}{3}$ the singular line lies below the line $\widehat{g}h$ and contains an edge of the polytope. The line $\widehat{g}h$ now lies outside the polytope and the points $\widehat{g}$, h, g and $\widehat{h}$ are no longer vertices of the polytope.

Four new vertices are created by the intersection of the singular line with the two pentagonal planes and image of these two vertices under 180° rotation about the x axis. We call these new vertices p, q, $\widehat{p}$ and $\widehat{q}$. We compute the coordinates of p from the equations of the singular line and one of the pentagonal planes.

$$p = \left(t_0, at_0, \frac{a^2 c}{1 - 2a^2} \right)$$

$$t_0 = \frac{a\left(1 - 3a^2\right)}{\left(1 + a^2\right)\left(1 - 2a^2\right)} \tag{27}$$

The "new" polytope, $0 < a^2 < \frac{1}{3}$, is invariant under 180° rotations about the x, y, and z axes as was the "old" polytope, $\frac{1}{3} \leqq a^2 \leqq \frac{1}{2}$. In figure 13 we have drawn the intersection of the half spaces defined by the two pentagonal planes and the two perpendicular planes for a^2 equal to $\frac{1}{3} - \epsilon$, $\frac{1}{3}$, and $\frac{1}{3} + \epsilon$, from the point of view of an observer on the positive z–axis. The "new" polytope (figure 16) has triangular faces afe, $\widehat{a}\widehat{f}\widehat{e}$, cfe, and $\widehat{c}\widehat{f}\widehat{e}$ that correspond exactly to the same triangular of the "old" polytope. The "old" triangular faces aeg, cfh, $\widehat{a}\,\widehat{e}\,\widehat{g}$, and $\widehat{c}\widehat{f}\,\widehat{h}$ correspond to new quadrilateral faces $ae\widehat{p}\,\widehat{q}$, $cfpq$, $\widehat{a}\,\widehat{e}pq$, $\widehat{c}\widehat{f}\widehat{p}\,\widehat{q}$ respectively and the old pentagonal faces $a\widehat{a}f\widehat{g}h$, $\widehat{e}\widehat{c}c\widehat{g}h$, $a\widehat{a}\widehat{f}g\widehat{h}$, $e\widehat{c}cg\widehat{h}$ correspond to the new quadrilateral faces $a\widehat{a}fp$, $\widehat{e}\widehat{c}cq$, $a\widehat{a}\widehat{f}\widehat{p}$, $c\widehat{c}e\widehat{q}$ respectively.

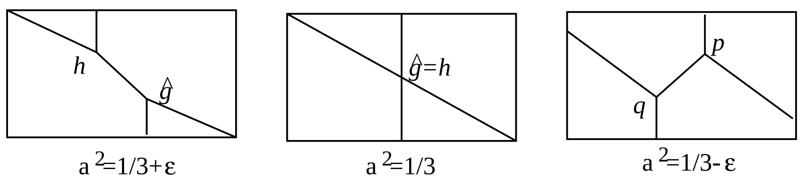


Figure 13

Soon we will verify that the "new" polytope lies inside the sphere at infinity when $0 < a^2 < \frac{1}{3}$. Assume for the moment we have already done this.

We identify faces in the “new” polytope exactly as we did in the old one.(See figure 16.) The identifications are still isometries because the new polytope retains the same symmetry as the old one.

We assert that we thus obtain a one parameter family of cone manifolds for $0 < a^2 < \frac{1}{3}$. The vertices a, e, c, f, $\widehat{a}$, $\widehat{e}$, $\widehat{c}$, $\widehat{f}$ are identified, exactly as in the old polytope so the “neighborhood is a cone on a sphere” condition is satisfied for them. Also, except for $\widehat{g}h$ and $g\widehat{h}$, which have disappeared, all the old edges correspond exactly to new edges and the “sum of the dihedral angles 360°” condition is satisfied for these new edges exactly as it was satisfied for the corresponding old edges.

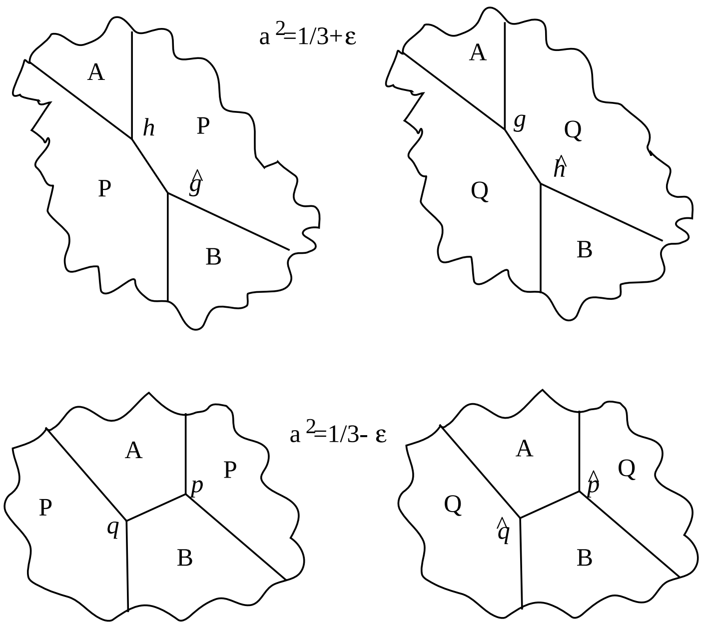


Figure 14

To understand what happens for the new vertices p,q,$\widehat{p}$,$\widehat{q}$ and new edges pq and $\widehat{p}\,\widehat{q}$ refer to figure 14 which shows neighborhoods of $\widehat{g}h$ and $g\widehat{h}$ for $a^2 = \frac{1}{3}+\epsilon$ and the corresponding neighborhoods of pq and $\widehat{p}\,\widehat{q}$ for $a^2 = \frac{1}{3}-\epsilon$. Faces of the new and old polytopes that are identified are labelled by the same letter A, B, P, or Q.

Gluing the P's together first we see that pq has a neighborhood that looks like $S^1\times$ two dimensional wedge and similarly $\widehat{p}\,\widehat{q}$ has such a neighborhood. Now gluing the two $S^1\times$ two dimensional wedges together along A and B shows that pq equals $\widehat{q}\,\widehat{p}$ and that pq has a topological solid torus neighborhood. Also we see that the sum of the dihedral angles about pq equals twice the dihedral angle between A and B which is less than $360°$. Hence pq gives rise to the new singular set and the new cone angle is twice the dihedral angle between the "perpendicular" planes defined earlier.

To see that the polytope lies inside the sphere at infinity we simply compute the square of the Euclidean distance of each vertex from the origin and compare it with R^2. The vertices a, e, c, f, $\widehat{a}$, $\widehat{e}$, $\widehat{c}$, $\widehat{f}$ all have the same distance from the origin as $a = (a, a^2, -ca^2)$ which has square distance $a^2(1 + a^2 + c^2a^2)$. The new vertices p, q, $\widehat{p}$, $\widehat{q}$ all have the same distance from the origin as p whose coordinates are given by (27). Using formulas (7), (18), and (27) we can compute all relevant square distances from the origin which we list below

$$\begin{aligned} R^2 &= \frac{a^2\left(a^4 - 10a^2 + 5\right)}{\left(a^2+1\right)^2\left(1-2a^2\right)} \\ \text{square distance of } a &= \frac{a^2\left(1 + 8a^2 - 7a^4 + 2a^6\right)}{\left(1+a^2\right)^2} \\ \text{square distance of } p &= \frac{a^2\left(1 - 7a^4 + 10a^6\right)}{\left(1+a^2\right)^2\left(1-2a^2\right)^2} \end{aligned} \tag{28}$$

It is now a routine, if tedious computation to verify that a and p and therefore all the vertices of the polyhedron lie inside the sphere of radius R.

Next we turn to the problem of determining the topological type of the cone manifold when $0 < a^2 < \frac{1}{3}$.

Stare awhile at figure 10. Consider the union of the two top pentagons $\widehat{g}hfa\widehat{a}$ and $g\widehat{h}\widehat{f}\widehat{a}a$. It is a "topological" octogon with boundary $hfag\widehat{h}\widehat{f}\widehat{a}\widehat{g}$. When identifications are made $\left(\text{for } \frac{1}{3} < a^2 < \frac{1}{2}\right)$ to obtain the cone manifold, (See(5)), that is topologically S^3, the following identifications are made in the octogon; $af = \widehat{f}\widehat{a}$, $ag = fh$, $\widehat{a}\,\widehat{g} = \widehat{f}\,\widehat{h}$.

The octogon becomes an orientable surface (actually a torus with a hole) with boundary the figure eight knot, $g\widehat{h}\cup\widehat{g}h$. This surface is a Seifert surface for the figure eight knot and (This is the point of the construction.) we can

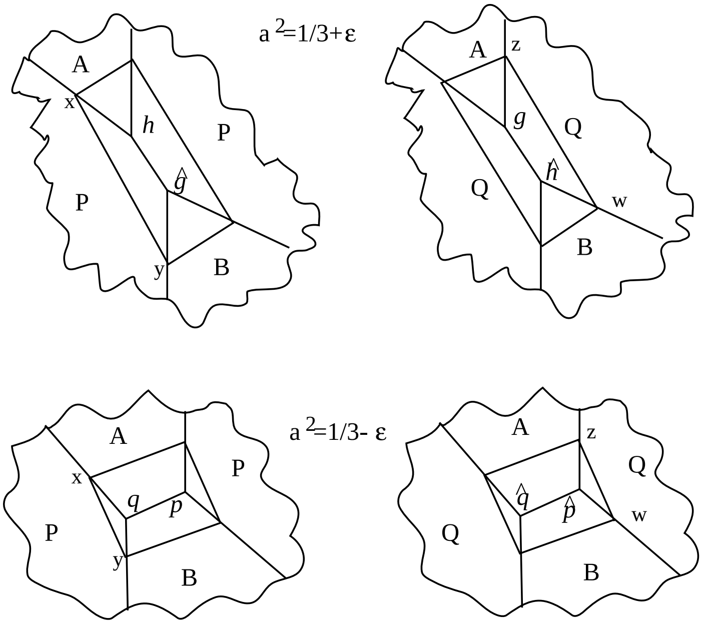


Figure 15

obtain a longitude for the figure eight knot by displacing the line segments $\widehat{g}h$ and $g\widehat{h}$ *within* the top pentagons, $\widehat{g}hfa\widehat{a}$ and $g\widehat{h}\widehat{f}\widehat{a}a$.

Now consider figure 15 which shows neighborhoods of the singular set in the topological cone manifolds $M(a^2 = \frac{1}{3} + \epsilon)$ and $M(a^2 = \frac{1}{3} - \epsilon)$. The neighborhood N in both manifolds is the union of two prisms, where a prism is topologically a triangle cross [0,1], but the prisms are glued together differently in the two cases. Removing N from both manifolds we obtain S^3 minus a solid torus neighborhood of the figure eight knot in both cases. And in both cases N is a solid torus.

It follows that $M(a^2 = \frac{1}{3} - \epsilon)$ was obtained from $M(a^2 = \frac{1}{3} + \epsilon)$ by a Dehn surgery. (Removing a solid torus and sewing it in differently.)

We have shown, a couple of paragraphs ago, that the circle $xy \cup zw$ bounds an orientable surface in S^3- solid torus neighborhood of the figure eight knot. In the manifold $M(a^2 = \frac{1}{3} - \epsilon)$ this circle bounds the topological disc that is the union of the two triangles xqy and $z\widehat{p}w$. (xq is identified with $z\widehat{p}$, yq with $w\widehat{p}$). The next proposition now follows immediately.

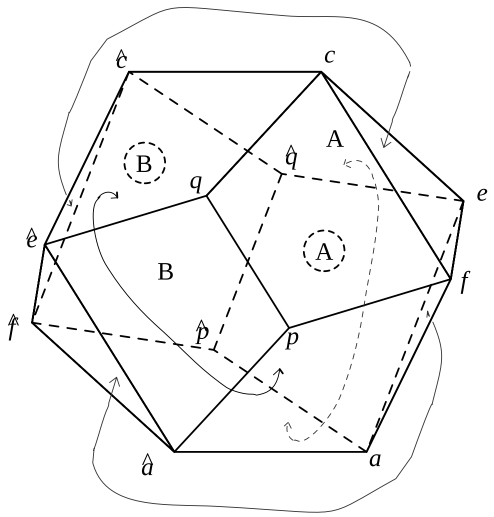


Figure 16

PROPOSITION 4.1. *For* $0 < a^2 < \frac{1}{3}$ *the topological cone manifold determined by* a^2 *is obtained by doing* $(0,1)$ *surgery on the figure eight knot.*

In the above proposition $(0,1)$ surgery means gluing a disc in the solid torus to a longitude of the knot.

The figure eight knot is a fibered knot with fibre a torus with a hole (See [BZ]). When $(0,1)$ surgery is done on the figure eight, discs are glued to the holes in the tori and the fibration extends to the surgered manifold. The surgered manifold thus becomes a fibre bundle over S^1 with fibre a torus.

The cone angle at the singular curve $pq = \widehat{p}\,\widehat{q}$ is the sum of the dihedral angles at pq and $\widehat{pq}$ or, what is the same thing, twice the dihedral angle θ at pq. The dihedral angle θ, the angle between the two "perpendicular" planes, can be computed from equation (25) and Proposition A.1. The poles of the two "perpendicular" planes are $\left(-a, 1, \left(1-2a^2\right)/(a^2c), 1/R\right)$ and $\left(a, -1, \left(1-2a^2\right)/(a^2c), 1/R\right)$. Hence

$$\cos\theta = \frac{(1+a^2) - (1-2a^2)^2/(a^4c^2) - 1/R^2}{(1+a^2) + (1-2a^2)^2/(a^4c^2) + 1/R^2} \tag{29}$$

Using (7) and (18) this simplifies to

$$\cos\theta = \frac{-1 + 4a^2 + 4a^4 - 16a^6 + a^8}{(a^2-1)^4} \tag{30}$$

As a^2 decreases from $\frac{1}{3}$ to 0 the angle θ increases from 0 to 180° and the cone angle increases from 0 to 360°. We summarize the above in the final proposition of this section.

PROPOSITION 4.2. *For* $0 < a^2 < \frac{1}{3}$ *the topological cone manifold determined by* a^2 *is a torus bundle over* S^1. *As* a^2 *decreases from* $\frac{1}{3}$ *to zero the cone angle at the singular curve increases from* 0 *to* 360°.

We shall study the case $a^2 = 0$ in more detail. When $a \to 0$ the radius $R \to 0$ and therefore we have to change coordinates to see the limiting cone-manifold. If we put $u = \frac{x}{R^2}$, $v = \frac{y}{R^2}$ and $w = \frac{z}{R^2}$, the sphere $x^2+y^2+z^2 = R^2$ goes to $u^2 + v^2 + w^2 = \frac{1}{R^2}$ so that when $a \to 0$, the sphere radius tends to infinity, and in the limit, the sphere is just the plane at infinity. However the limiting polyhedron degenerates into a segment, which can be interpreted as the developping map of the circle S^1, i.e. the base of the torus bundle. Therefore we have to proceed with more care. Putting

$$u = \frac{x}{R}, \quad v = \frac{y}{R^2}, \quad w = \frac{z}{R^2}$$

the sphere $x^2 + y^2 + z^2 = R^2$ goes to $u^2 + R^2(v^2 + w^2) = 1$ which is an ellipsoid cutting $u = 0$ in the circle $v^2 + w^2 = \frac{1}{R^2}$. In this way, when $R \to 0$, the ellipsoid degenerates into the pair of planes $u^2 = 1$ and the limiting polyhedron is shown in Figure 17 together with the identifications and coordinates. The intersection of this polyhedron with $u = constant$ is an hexagon with opposite parallel sides identified, i.e. a torus. It is easy to obtain the monodromy of the torus bundle which is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This manifold has a Sol geometry (see [Sc]), but this geometry is *not* the limiting geometry of our polyhedron. In fact it can be checked that the two homographies fixing $u^2 = 1$ and identifying $\widehat{ea}\widehat{f}$ with efa, and $\widehat{ce}\widehat{f}$ with ecf, resp. *do not commute.*

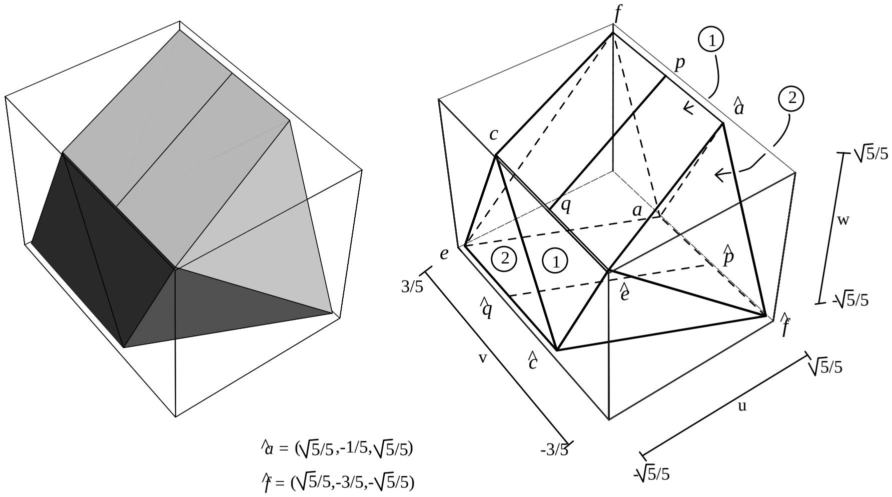


Figure 17

5. The spherical case; $\frac{1}{2} \leqq a^2 \leqq 5 - 2\sqrt{5} = .527...$

First of all, we observe that the polytope, as defined in Section 2, is defined for a^2 in this range. The main tetrahedron is defined by its corner (a, b, c), where $a \approx \frac{1}{\sqrt{2}}$ varies very little, b is a constant equal to one, and c, given by (7), decreases from $\frac{1}{3}$ to 0 as a^2 increases from $\frac{1}{2}$ to $5 - 2\sqrt{5}$. To get an idea of what the polytope looks like for a^2 in this range the reader should look at figures 9, 10, and 11 and think of the scale on the z–axis as increasing without bound.

The problem is that R^2, as given by (18) is negative for a^2 in this range. In fact R^2 increases from $-\infty$ to 0 as a^2 increases from $\frac{1}{2}$ to $5 - 2\sqrt{5}$.

The key idea is to consider the polytope as lying in the Klein model for a sphere of radius R. We redefine the bilinear form $b\left(\left(\overrightarrow{v_1}, t_1\right), \left(\overrightarrow{v_2}, t_2\right)\right) = \overrightarrow{v_1} \cdot \overrightarrow{v_2} - t_1 t_2$ by defining

$$
\begin{aligned}
&\widehat{b}\left(\left(\overrightarrow{v_1}, t_1\right), \left(\overrightarrow{v_2}, t_2\right)\right) = \overrightarrow{v_1} \cdot \overrightarrow{v_2} + t_1 t_2 \\
&\widehat{q}\left(\left(\overrightarrow{v}, t\right)\right) = \widehat{b}\left(\left(\overrightarrow{v}, t\right), \left(\overrightarrow{v}, t\right)\right).
\end{aligned} \tag{31}
$$

Of course, $\widehat{b}$ and $\widehat{q}$ are just the usual scalar product and square norm on R^4.

Since $H_R^3 = \left\{(\overrightarrow{v}, t) \mid q(\overrightarrow{v}, t) = -R^2\right\}$, define $S_R^3 = \left\{(\overrightarrow{v}, t) \mid \widehat{q}(\overrightarrow{v}, t) = R^2\right\}$ and S_R^3 is just the sphere of radius R.

The "Klein model" K for the sphere of radius R is the hyperplane $K = \{(\overrightarrow{v}, t) \mid t = R\}$. Unlike in the hyperbolic case, we don't need to be concerned with whether the polytope lies inside the sphere of radius R in K as projection from the origin (which is not conformal) defines a $1-1$ correspondence between K and the upper hemisphere of S_R^3. The metric on K is obtained by pulling back the metric on S_R^3. As in the hyperbolic case reflections in planes through the origin and rotation about axes through the origin are both Euclidean and spherical isometries.

Now we shall review the steps in the definition of the polytope and see how the spherical case differs from the hyperbolic. The reader should refer to figures 9, 10, and 11 and to Section 2.

The definition of the "box" with vertices $\{(\pm a, \pm 1, \pm c)\}$ is exactly the same. The definition of the points a, c, e, f, $\widehat{a}$, $\widehat{c}$, $\widehat{e}$, $\widehat{f}$ is exactly the same and the condition that they all lie the same Euclidean distance from the origin is exactly the same. This means that c is determined once again by (7) and so the box is well defined for a^2 in the range $\frac{1}{2} \leqq a^2 \leqq 5 - 2\sqrt{5}$.

Now suppose $R > 0$. The points g, h, $\widehat{g}$, $\widehat{h}$ are defined, as before, by dropping four perpendicular planes to the lines $[(a, -1, c), (-a, 1, c)]$ and $[(-a, -1, -c), (a, 1, -c)]$ through the points a, c, $\widehat{a}$, $\widehat{c}$ respectively. (Hopefully, the fact that we use a and c for both points and coordinates causes no confusion). However, the notion of perpendicular is different in the hyperbolic and spherical models.

The equations of planes $a\widehat{a}$-origin, $a\widehat{a}f$ and efa are exactly the same (equations 13 and 14). If a plane in K has equation $\alpha x + \beta y + \gamma z = \delta$, and $t = R$ then in homogeneous coordinates the equation is $\alpha x + \beta y + \gamma z - (\delta/R)t = 0$.

A line is perpendicular to this plane, in the spherical case, if and only if the line goes through the pole, where the pole has homogeneous coordinates $(\alpha, \beta, \gamma, -\delta/R)$. In the hyperbolic case the pole would have had coordinates $(\alpha, \beta, \gamma, \delta/R)$.

The poles of the planes $a\widehat{a}$ origin, $a\widehat{a}f$, and efa are the same as in (14) except that the sign of the last coordinate is changed.

$$\begin{aligned}
\widehat{u} &= (0, -c, -1, 0) \\
\widehat{v} &= (Rc, Rac, Ra, -ac) \\
\widehat{w} &= \left(Rca\left(1+a^2\right), Rc\left(1+a^2\right), R\left(3a^2-1\right), -ca^2\left(3-a^2\right)\right)
\end{aligned} \tag{32}$$

If a pair of planes with poles $(\overrightarrow{p_1}, t_1)$ and $(\overrightarrow{p_2}, t_2)$ in the spherical case intersect with dihedral angle θ then θ is given by

$$\pm\cos\theta = \frac{\widehat{b}\left((\overrightarrow{p_1}, t_1), (\overrightarrow{p_2}, t_2)\right)}{\sqrt{q\left(\overrightarrow{p_1}, t_1\right) q\left(\overrightarrow{p_2}, t_2\right)}}. \tag{33}$$

The formulas that are the analogue of (16) are as follows:

$$\begin{aligned}
\widehat{b}(\widehat{u}, \widehat{u}) &= \left(1+c^2\right) \\
\widehat{b}(\widehat{u}, \widehat{v}) &= -Ra\left(1+c^2\right) \\
\widehat{b}(\widehat{w}, \widehat{w}) &= R^2c^2\left(1+a^2\right)^3 + R^2\left(3a^2-1\right)^2 + c^2a^4\left(a^2-3\right)^2 \\
\widehat{b}(\widehat{v}, \widehat{w}) &= 2R^2ac^2\left(1+a^2\right) + R^2a\left(3a^2-1\right) - c^2a^3\left(a^2-3\right).
\end{aligned} \tag{34}$$

The analoge of (15) reduces to:

$$\widehat{b}(\widehat{w}, \widehat{w})\left[\widehat{b}(\widehat{u}, \widehat{v})\right]^2 - \widehat{b}(\widehat{u}, \widehat{u})\left[\widehat{b}(\widehat{v}, \widehat{w})\right]^2 = 0. \tag{35}$$

After plugging (34) into (35), using (7), and doing a lot of algebraic manipulation the left hand side of (35) factors nicely and (35) becomes

$$\begin{aligned}
&a^2c^2\left(a^2-3\right)\left(1+c^2\right)\left(a^2+R^2\right) \\
&\times\left[a^2c^2\left(3-a^2\right) + R^2\left(2a^2-1\right)\left(a^2-3\right)\right] = 0.
\end{aligned} \tag{36}$$

Again, only the last factor matters and we obtain

$$R^2 = \frac{a^2c^2}{2a^2-1} = \frac{a^2\left(a^4-10a^2+5\right)}{\left(a^2+1\right)^2\left(2a^2-1\right)}. \tag{37}$$

We see that this is exactly the same as the R^2 obtained in the hyperbolic case except for a crucial change of sign when $a^2 > \frac{1}{2}$.

The face and edge identifications are made exactly as in (4) and (5) and the identifications are made via isometries as in the hyperbolic case. We obtain a cone manifold with singular set the figure eight knot.

We compute the cone angle in analogy with the hyperbolic case. The cone angle equals twice the angle, call it θ, between the planes $\widehat{g}h$ origin and $\widehat{g}hc$. These planes have equations $x + ay = 0$ and $cx + acy + az = ac$ and poles $\widehat{u} = (1, a, 0, 0)$ and $\widehat{v} = (Rc, Rac, Ra, -ac)$.

Then $\cos\theta$ is given by $\widehat{b}\,(\widehat{u}, \widehat{v})\;/\;[\widehat{q}\,(\widehat{u})\,\widehat{q}\,(\widehat{v})]^{\frac{1}{2}}$. We use this expression for $\cos\,\theta$, (7), (37) and the formula for the cosine of a double angle to obtain

$$\text{Cone angle} = \text{arc cos}\,\frac{3 - 6a^2 - a^4}{2(a^2 - 1)^2}. \tag{38}$$

This is exactly the same as (24), the formula for the cone angle in the hyperbolic case.

The derivative of the function $f(a) = \left(3 - 6a^2 - a^4\right)/2(a^2 - 1)^2$ is negative for $0 < a < 1$ so that the function is decreasing.

Summarizing:

PROPOSITION 5.1. *For $\frac{1}{2} \leqq a^2 \leqq 5 - 2\sqrt{5}$ the topological cone-manifold determined by a^2 is S^3. The singular set is the figure eight knot. The angle increases from $2\pi/3$ to π, while the curvature (constant) increases from* 0 *to* 1*.*

The case $a^2 = 5 - 2\sqrt{5}$ corresponds to the angle π, and we shall study this case in more detail.

When $a^2 \to 5 - 2\sqrt{5}$ the polyhedron (which lives in $t = R$) collapses into the plane $z = 0$. Therefore to visualize this limiting polyhedron we project the polyhedron for the different values of a^2 into the sphere S_1^3 of radius one:

$$x^2 + y^2 + z^2 + t^2 = 1$$

Then, when $a^2 \to 5 - 2\sqrt{5}$, the vertices of the polyhedron all tend to points located in the great circle $C = \{z = 0\,,\, t = 0\}$ of S_1^3. The polyhedron degenerates into a lens bounded by two half-spheres at distance $\frac{\pi}{5}$ with common boundary C. The lines $\widehat{gh}$, $\widehat{g}h$ are great half-circles of these half-spheres and the points g, h (resp. $\widehat{g}, \widehat{h}$) are at distance $\frac{2\pi}{5}$ in C (see Figure 18). The π-rotations around $\widehat{gh}$ and $\widehat{g}h$ generate a dihedral action of D_{10} in

S^3_1. The quotient of S^3_1 under the action of the cyclic group $C_5 \leq D_{10}$ is the familiar lens space $L(5,3)$. The quotient of S^3 under the action of D_{10} is the orbifold 5/3 with angle π. This orbifold is the result of self-identifying the boundary of the lens by reflexion in the edges $g\widehat{h}$ and $\widehat{g}h$ (compare [BS] p.39).

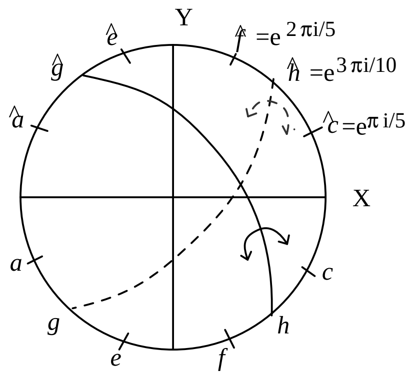


Figure 18

6. The volume of cone-manifolds

The key to computing the volume of the cone manifolds we have constructed is the Formula of Schl*ä*ffli, (See A.2). In a one parameter family of polytopes in a space of constant curvature K, $KdV = (1/2)\sum \ell_i d\alpha_i$, where V is volume, the sum is taken over all the edges, ℓ_i is the length of the ith edge and α_i is its dihedral angle. The volume of a cone manifold is the volume of the polyhedron from which it is constructed before identifications are made.

If several edges of a polytope are identified and the resulting identified edge is <u>not</u> part of the singular set then the sum of the corresponding dihedral angles is 2π and, since the differential of the constant 2π equals zero, these edges make no contribution to dV. Hence, in our case, $KdV = (1/2)\sum \ell_i d\alpha_i$ where the sum is taken over the edges that become the singular set, only one or two edges. (This is a Theorem of Hogdson, see [H]).

The formula of Schl*ä*ffli applies to a one parameter family in a space of constant curvature. We can arrange for constant curvature by simply changing scale. Consider the map $\overrightarrow{v} \longrightarrow \lambda\overrightarrow{v}$; $\lambda > 0$, from the Klein model of either hyperbolic or spherical space with sphere at infinity of radius R to hyperbolic or spherical space with sphere of radius λR. This map induces the identity map on real projective 3–space and so preserves projective equations of planes, poles of planes and dihedral angles between planes. It is thus a conformal map.

It is not, however, an isometry since it sends H^3_R onto $H^3_{\lambda R}$ (S^3_R onto $S^3_{\lambda R}$, resp.). Thus distances are multiplied by the factor λ, curvature by the factor λ^2 and volume by the factor λ^3.

We have already derived expressions for the cone angles α_i in all cases, (24),(30),(38), and now we must derive expressions for the edge length of the singular set to carry out the program.

For $\frac{1}{3} < a^2 < \frac{1}{2}$, the length of the singular set is twice the length of line segment $\widehat{g}h$ of figure 10 or 11. Observe that line $\widehat{g}h$ is perpendicular to the planes $\widehat{a}\,\widehat{e}\,\widehat{g}$ and cfh. In fact, for example, $\widehat{g}$ was defined by the condition that it lie on the line $[(a,1,-c),(-a,-1,-c)]$ and be contained in a plane containing $\widehat{a}$ and perpendicular to this line. It follows that length $\widehat{g}h$ is equal to the distance between planes $\widehat{a}\,\widehat{e}\,\widehat{g}$ and cfh. The poles for these planes, p_1 and p_2, were found in Section 2 (by formula (19)). They turn out to be $\left(a\left(R^2-c^2a^2\right), R^2-c^2a^2, -2ca^2, 2a^2R\right)$, $\left(-a\left(R^2-c^2a^2\right), -\left(R^2-c^2a^2\right), -2ca^2,\ 2a^2R\right)$.

We can compute length $\widehat{g}h$ from Proposition A.1. After applying (7) and (18) to simplify the expression for $\delta\left(\overrightarrow{p_1}, \overrightarrow{p_2}\right)$ and applying Proposition A.2 we find that

$$(39) \qquad \cosh\left(\frac{\text{length}\,\widehat{g}h}{R}\right) = \frac{-1+4a^2+4a^4-16a^6+a^8}{\left(a^2-1\right)^4}.$$

When $0 < a^2 < \frac{1}{3}$ the singular set is determined by the intersection of the two pairs of "perpendicular" planes. One pair intersects in the singular line $\left(t, at, a^2c/\left(1-2a^2\right)\right)$, $-\infty < t < \infty$ in the Klein model. The singular line goes through the poles of the planes $\widehat{g}ha$ and $\widehat{g}hc$ which are respectively $\overrightarrow{p_1} = (c, ac, a, ac/R)$ and $\overrightarrow{p_2} = (-c, -ac, a, ac/R)$ when $t = \pm R^2/(2a)$. (See second paragraph of Section 4.)

It follows that the singular line is perpendicular to the planes $\widehat{g}ha$ and $\widehat{g}hc$ and that the singular line segment pq defined by the intersections with the planes has length equal to the distance between the planes. There is another singular line $\widehat{p}\,\widehat{q}$ but it gets identified with pq (See figures 14, 15 and the surrounding text.) so we don't need to compute its length. After the usual simplification using (7) and (18) we see from Proposition A.1 that

$$\cosh\left(\frac{\text{length } pq}{R}\right) = \frac{3 - 6a^2 - a^4}{2\left(a^2 - 1\right)^2}. \tag{40}$$

When $\frac{1}{2} < a^2 < 5 - 2\sqrt{5}$, we once again have that line segment $\widehat{g}h$ is perpendicular to planes $\widehat{a}\,\widehat{e}\,\widehat{g}$ and cfh. The poles of these planes (computed in Section 2), are

$$\begin{aligned}\overrightarrow{p_1} &= \left(a\left(R^2 - c^2a^2\right), R^2 - c^2a^2, -2ca^2, -2a^2R\right), \quad \text{and}\\ \overrightarrow{p_2} &= \left(-a\left(R^2 - c^2a^2\right), -\left(R^2 - c^2a^2\right), -2ca^2, -2a^2R\right).\end{aligned}$$

(But note the change of sign of the fourth coordinate in the spherical case.) It follows, once again that the length of $\widehat{g}h$ equals the distance between the planes, equals the spherical distance between their poles.

The poles of the planes, considered as lines in $\mathbb{R}^4$ given by homogeneous coordinates intersect the sphere of radius R in points and the spherical distance between the points can be found with ordinary trigonometry. Hence if θ is the angle between the vectors $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$ we have

$$\cos\theta = \frac{-\left(a^2+1\right)\left(R^2 + c^2a^2\right)^2 + 4c^2a^4 + 4a^4R^2}{\left(a^2+1\right)\left(R^2 + c^2a^2\right)^2 + 4c^2a^4 + 4a^4R^2}. \tag{41}$$

Using (7), the spherical case R^2 given by (37), and trigonometry we obtain

$$\text{length}\,\widehat{g}h = R\,\text{arc}\cos\frac{-1 + 4a^2 + 4a^4 - 16a^6 + a^8}{\left(a^2 - 1\right)^4}. \tag{42}$$

We are about to summarize these results but first it is convenient to define a pair of functions and state a proposition concerning them. Thus define;

$$\begin{aligned} f(x) &= \frac{3 - 6x - x^2}{2(x-1)^2}\\ g(x) &= \frac{-1 + 4x + 4x^2 - 16x^3 + x^4}{(x-1)^4}. \end{aligned} \tag{43}$$

PROPOSITION 6.1. *The function $f(x)$ is strictly decreasing on the interval $[0, 5-2\sqrt{5} \approx .527]$. It takes the following values: $f(0) = \frac{3}{2}$, $f\left(\frac{1}{3}\right) = 1$, $f\left(\frac{1}{2}\right) = -\frac{1}{2}$, $f\left(5-2\sqrt{5}\right) = -1$.*

Also:

$$\frac{d}{dx}(\text{arc }\cos(f(x))) = \frac{8x}{(1-x)\sqrt{(x^2-10x+5)(3x-1)(x+1)}}.$$

The function $g(x)$ is strictly increasing on the interval $\left[0, \left(2\sqrt{10}-5\right)/3 \approx .44\right]$ and strictly decreasing on the interval $\left[\left(2\sqrt{10}-5\right)/3, 5-2\sqrt{5}\right]$. It takes the following values: $g(0) = -1$, $g\left(\frac{1}{3}\right) = 1$, $g\left(\frac{1}{2}\right) = 1$, $g\left(5-2\sqrt{5}\right) = -1$.

Also:

$$\frac{d}{dx}(\text{arc }\cos(g(x))) = \frac{2\left(3x^2+10x-5\right)}{(1-x)\sqrt{(3x-1)(2x-1)(x+1)\left(x^2-10x+5\right)}}. \square$$

For $0 < a^2 < 5-2\sqrt{5}$ we have defined a cone manifold. This cone manifold was Euclidean for $a^2 = \frac{1}{2}$, hyperbolic with curvature $-\frac{1}{R^2}$ for $0 < a^2 < \frac{1}{2}$ and spherical with curvature $\frac{1}{R^2}$ for $\frac{1}{2} < a^2 < 5-2\sqrt{5}$ where R is given by (18) or (37); in the Euclidean case $R = \infty$.

For any a^2 in the interval let $M\left(a^2\right)$ be the image of the cone manifold we have defined under the map $\overrightarrow{v} \longrightarrow \frac{1}{R}\overrightarrow{v}$ on $\mathbb{R}^4$. That is, we change scale so that $M\left(a^2\right)$ has curvature -1 for $0 < a^2 < \frac{1}{2}$ and $+1$ for $\frac{1}{2} < a^2 < 5-2\sqrt{5}$.

Now let $x = a^2$ and define $V(x)$ to be the volume of the cone manifold $M(x)$, let $\ell(x)$ be the length of the singular set in $M(x)$ and let $\alpha(x)$ be the cone angle in $M(x)$.

We summarize the computations we have made earlier in this section in the following proposition. By arc cosh we mean the inverse hyperbolic cosine function. Recall that $\cosh(t) = \frac{1}{2}\left(e^t + e^{-t}\right)$ so that arc cosh has domain $[1, \infty]$ and range $[0, \infty]$.

PROPOSITION 6.2.

$$\textit{For} \quad 0 < x < \frac{1}{3}; \quad \begin{aligned} \ell(x) &= \text{arc }\cosh(f(x)) \\ \alpha(x) &= 2\,\text{arc }\cos(g(x)) \\ dV(x) &= -\,\text{arc }\cosh(f(x))\, d(\text{arc }\cos(g(x))) \end{aligned}$$

$$For \qquad \frac{1}{3} < x < \frac{1}{2} \qquad \begin{aligned} \ell(x) &= 2\,\text{arc cosh}\,(g(x)) \\ \alpha(x) &= \text{arc cos}\,(f(x)) \\ dV(x) &= -\,\text{arc cosh}\,(g(x))\,d(\text{arc cos}\,(f(x))) \end{aligned}$$

$$For \qquad \frac{1}{2} < x < 5 - 2\sqrt{5} \qquad \begin{aligned} \ell(x) &= 2\,\text{arc cos}\,(g(x)) \\ \alpha(x) &= \text{arc cos}\,(f(x)) \\ dV(x) &= \text{arc cos}\,(g(x))\,d(\text{arc cos}\,(f(x))) \end{aligned}$$

The proof of Proposition 6.2 follows directly from the formula of Schl*ä*ffli and the computations we have made. □

Since the volume of $M\left(a^2 = \frac{1}{2}\right)$, the Euclidean case, is zero after normalization (the map $\overrightarrow{v} \longrightarrow \frac{1}{R}\overrightarrow{v}$ is the zero map and the polytope is sent to a point.), the last two proposition give a formula for the volume.

THEOREM 6.3. *The volume $V(x)$ of the normalized cone-manifold $M(x)$ is as follows:*

1. $\frac{1}{3} \leqq x \leqq \frac{1}{2}$

$$V(x) = \int_x^{\frac{1}{2}} \text{arc cosh}\left(1 + \frac{2(1-3x)(2x-1)(x+1)}{(1-x)^4}\right) \frac{8x\;dx}{(1-x)\sqrt{(x^2-10x+5)\,(3x-1)(x+1)}}$$

2. $\frac{1}{2} \leqq x \leqq 5 - 2\sqrt{5}$

$$V(x) = +\int_{\frac{1}{2}}^{x} \text{arc cos}\left(1 + \frac{2(1-3x)(2x-1)(x+1)}{(1-x)^4}\right) \frac{8x\;dx}{(1-x)\sqrt{(x^2-10x+5)\,(3x-1)(x+1)}}$$

3. $0 < x \leqq \dfrac{1}{3}$

$$V(x) = V\left(\frac{1}{3}\right) + \int_x^{\frac{1}{3}} \operatorname{arc\,cosh}\left(1 + \frac{(1-3x)(1+x)}{2(1-x)^2}\right) \frac{2\left(3x^2+10x-5\right)dx}{(1-x)\sqrt{(3x-1)(2x-1)(x+1)\left(x^2-10x+5\right)}} \quad \square$$

In the normalized hyperbolic case, $\frac{1}{3} \leqq a^2 < \frac{1}{2}$, the cone angle increases continuously from 0 to $\frac{2\pi}{3}$. For cone angles $\theta = \frac{2\pi}{n}$, the normalized to curvature -1 cone manifolds that we constructed are orbifolds in the usual sense. These orbifolds are arithmetic exactly for $n = 4, 5, 6, 8, 12, \infty$. (See [HKM] and [$\mathrm{HLM}_3$]; see also [V], [$\mathrm{HLM}_1$], [$\mathrm{HLM}_2$], [R] and [MR] for definitions and results on arithmeticity.)

In this range the cone angle $\alpha(x)$ is determined by the formula $\cos(\alpha(x)) = f(x) = \left(3 - 6x - x^2\right)/(2(x-1)^2)$. (Recall $x = a^2$.) We can easily solve for x in terms of $\cos\alpha(x)$.

$$x = \frac{\left(2\sqrt{3 - 2\cos\alpha(x)} + 2\cos\alpha(x) - 3\right)}{1 + 2\cos\alpha(x)} \tag{44}$$

We have computed the volumes for the values $n = 4, 6, 8, 12, \infty$ (See table II in Appendix B.)

We observe that $V(\infty) = 2.02988...$ is precisely double 1.01494... which is the volume of the ideal tetrahedron with dihedral angles all sixty degrees given in [M]. This squares with the observation made long ago by Thurston in his "notes" ([T]) that the complement of the figure eight knot, with a complete hyperbolic structure, could be obtained by gluing together two of these ideal tetrahedra in a certain way.

COROLLARY 6.4.

(i) *The volume of the ideal tetrahedron with dihedral angles of* $60°$ *is* $\frac{1}{2}V(\frac{1}{3})$.

(ii) $V(\tan^2 \frac{\pi}{5}) = \frac{\pi^2}{5}$

Proof. We only need to see (ii). For $a^2 = 5 - 2\sqrt{2} = \tan^2 \frac{\pi}{5}$ we obtain the cone manifold with singular set the figure eight knot and with angle π. The 2-fold branched covering of this orbifold is the lens space $L(5,2)$ whose universal cover (of 5 sheets) is S^3. The volume of S^3 is $2\pi^2$. □

Remarks. 1. If we take our polyhedron for the angle $2\pi/n$, $n \geqslant 3$ we can fit n of them cyclically around one of the long edges ($h\widehat{g}$ or $g\widehat{h}$ in figure 10). This non convex polyhedron is a fundamental domain for the n-fold cyclic covering of S^3 branched over the figure eight knot, i.e. the Fibonacci manifolds of [HKM]. (See [HLM_3] and figure 17.) We remark that our construction is totally different of the one made in [HKM]. The volumes of these manifolds have been computated by [MV] for $n \leqslant 50$. They coincide with our computations in Appendix B.

2. The manifold M obtained by expontaneous surgery in the figure eight knot is a torus bundle over S^1. Call Σ the singular curve (core of the surgery). Then $(M, 2\pi/n)$, $n \geqslant 2$ is a hyperbolic orbifold. The fundamental domain appears in figure 16. The angle around pq (and around $\widehat{pq}$) is π/n. The orbifold $(M, 2\pi/n)$ admits manifold coverings by [Se]. This coverings are examples of surface bundles over S^1 which are hyperbolic manifolds (Compare [J]). Since $(M, 2\pi/n)$ is arithmetic precisely for $n = 2, 3$ (see [HLM_4]) this gives examples of arithmetic and non arithmetic hyperbolic surface bundles. (Compare [BMR].)

Appendix A

Computations in the Klein model

The aim of this appendix is to recall formulae for distance and angles in the Klein model of hyperbolic space. All this is well–known (See [T] and [Vi]) but there are a couple of subtle points that we want to fix up here.

Typically one sees hyperbolic 3–space as the set of interior points of a quadric of ellipsoid type in real projective 3–space $\mathbb{R}P^3$. One defines the group of motions as the group of collineations of $\mathbb{R}P^3$ fixing the ellipsoid as a set. Then a distance is defined in terms of cross–ratios. While all this is very beautiful it is not practical for computational purposes.

We intend here to retain this model as a tool to *visualize* another more practical model; the vector model of [Vi] or hyperboloid model of [T] which is very suitable for calculations and definitions.

The scalar product $b(\overrightarrow{v}, \overrightarrow{v}') = b((x,y,z,t),(x',y',z',t')) = xx' + yy' + zz' - tt'$ turns $\mathbb{R}^4$ into a pseudo-Riemannian manifold denoted by $\mathbb{R}^{3,1}$. The smooth submanifold $H_R^3 = \left\{\overrightarrow{v} \in \mathbb{R}^4 \,|\, b(\overrightarrow{v}, \overrightarrow{v}) = -R^2, t > 0\right\}$ is then a Riemannian submanifold of $\mathbb{R}^{3,1}$. One now finds the group of isometries of H_R^3. It is induced by the group of automorphisms of $\mathbb{R}^4$ fixing b. Then one computes the curvature of H_R^3 which turns out to be constant of value $-1/R^2$. The Riemannian submanifold H_R^3 is sometimes called the hyperboloid model of *hyperbolic* 3*–space of curvature* $-1/R^2$ and it is sometimes called the sphere of imaginary radius iR to emphasize the analogy with S_R^3, the sphere with radius R and curvature $1/R^2$ which is obtained by redefining b in the obvious way.

Before introducing the Klein model we establish some notation. For $\overrightarrow{v} \in \mathbb{R}^4$, we denote by $\langle\overrightarrow{v}\rangle$ the set $\{\lambda\overrightarrow{v} | \lambda \in \mathbb{R}\}$. Then we denote the point $\langle\overrightarrow{v}\rangle \cap H_R^3$ by v if $b(\overrightarrow{v}, \overrightarrow{v}) < 0$; we denote $\left\{w \in H_R^3 \,|\, b(\overrightarrow{v}, \overrightarrow{w}) = 0\right\}$ by $H_{\overrightarrow{v}}$ and we denote $\left\{w \in H_R^3 | b(\overrightarrow{v}, \overrightarrow{w}) \leqq 0\right\}$ by $H_{\overrightarrow{v}}^+$ if $b(\overrightarrow{v}, \overrightarrow{v}) > 0$. Thus v is the point of H_R^3 defined by a vector inside the light-cone $L = \left\{\overrightarrow{v} \in R^{3,1} \,|\, b(\overrightarrow{v}, \overrightarrow{v}) = 0\right\}$; $H_{\overrightarrow{v}}$ is a plane of H_R^3; and $H_{\overrightarrow{v}}^+$ is a half–space of H_R^3.

The set of lines $\langle\overrightarrow{v}\rangle$ form $\mathbb{R}P^3$, real projective space, and we can identify H_R^3 with the set of points of $\mathbb{R}P^3$ lying in the light cone L. This is sometimes called the projective model of H_R^3.

At this point we introduce the Klein model of H_R^3. The points of the Klein model, K_R^3 lie inside the sphere of radius R in $\mathbb{R}^3$. The points outside the sphere of radius R have meaning also. They are called ultrainfinite points. $\mathbb{R}^3$ is embedded in RP^3 by the map $(x,y,z) \longrightarrow (x,y,z,R) \longrightarrow (x : y : z : R)$ (the last are projective coordinates.). Thus points of K_R^3 are sent to points inside the light-cone, and there is a natural $1-1$ correspondence between the points of K_R^3 and the points of H_R^3 via the points $\langle\overrightarrow{v}\rangle$ of $\mathbb{R}P^3$. The Riemannian structure of H_R^3 is then pulled back to K_R^3 giving it the structure of a Riemannian manifold of curvature $-1/R^2$ (See figure A1).

To give the formulas for distances and angles in K_R^3 we first want to understand the sets $H_{\overrightarrow{v}}$, $H_{\overrightarrow{v}}^+$ in K_R^3. The set $H_{\overrightarrow{v}}$ in K_R^3 is the polar plane of a point $\overrightarrow{v}$ of $\mathbb{R}^3$ at ultrainfinity, i.e. a point such that $b(\overrightarrow{v}, \overrightarrow{v}) > 0$ (See figure A2). Thus $H_{\overrightarrow{v}}^+$ is one of the half–spaces determined by $H_{\overrightarrow{v}}$ in K_R^3, but there is no projective way to dictate which one. To overcome this problem we can normalize the homogeneous coordinates of the image

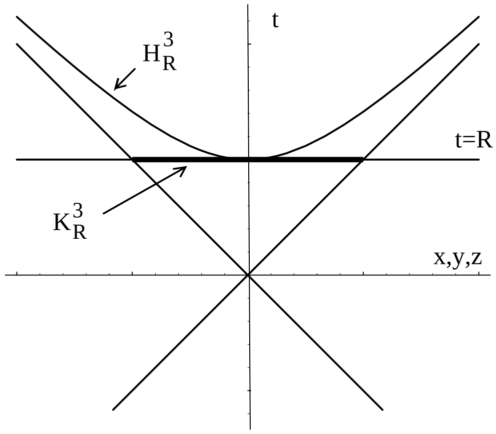


Figure A.1

of $\overrightarrow{v} = (x : y : z : t)$ by requiring $t \geqq 0$ and we denote the normalized homogeneous coordinates of $\overrightarrow{v}$ by $\overrightarrow{v} = [x, y, z, t]$, then, if $\overrightarrow{v}$ is normalized in this way, one can check that $H^+_{\overrightarrow{v}}$ is the half–space of K^3_R which is *away from* $\langle \overrightarrow{v} \rangle$. This rule does not work for $\overrightarrow{v} = [a, b, c, 0]$ because $\langle \overrightarrow{v} \rangle$ is at infinity in $\mathbb{R}^3$. In this case $H_{\overrightarrow{v}}$ is the plane $ax + by + cz = 0$ in $\mathbb{R}^3$ which is Euclidean orthogonal to (a, b, c) and passes through $(0, 0, 0)$. Then one can check just using calculus that $H^+_{\overrightarrow{v}}$ is the half–space of $\mathbb{R}^3$ which is away from (a, b, c) (See figure A3).

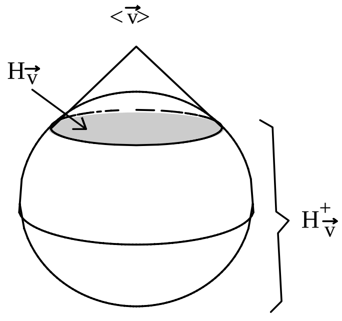


Figure A.2

Given $\langle \overrightarrow{v} \rangle$, $\langle \overrightarrow{w} \rangle$ at ultrainfinity; i.e. $b(\overrightarrow{v}, \overrightarrow{v}) > 0$, $b(\overrightarrow{w}, \overrightarrow{w}) > 0$, we can consider $H^+_{\overrightarrow{v}} \cap H^+_{\overrightarrow{w}}$. If the line $(\langle \overrightarrow{v} \rangle, \langle \overrightarrow{w} \rangle)$ does not cut K^3_R then its

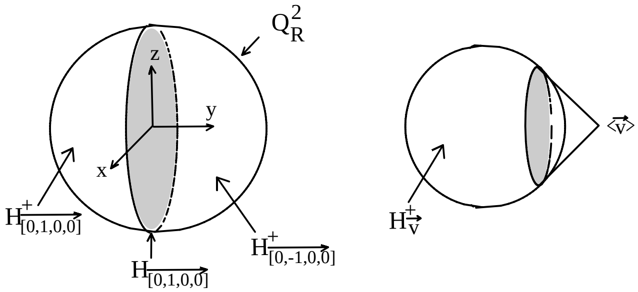


Figure A.3. $(a,b,c) = (0,1,0); v = [0,1,0,0], -v = [0,-1,0,0]$.

polar line is $H_{\overrightarrow{v}} \cap H_{\overrightarrow{w}}$, and we say that $H^+_{\overrightarrow{v}} \cap H^+_{\overrightarrow{w}}$ is a *dihedron with base* $H_{\overrightarrow{v}} \cap H_{\overrightarrow{w}}$. To measure the *angle* $\left(H^+_{\overrightarrow{v}} \cap H^+_{\overrightarrow{w}}\right)$ of this dihedron, take a point $\langle \overrightarrow{u} \rangle \in H_{\overrightarrow{v}} \cap H_{\overrightarrow{w}}$ at ultrainfinity. Then $H_{\overrightarrow{u}}$ cut $H_{\overrightarrow{v}} \cap H_{\overrightarrow{w}}$ orthogonally and $H^+_{\overrightarrow{v}} \cap H^+_{\overrightarrow{w}} \cap H_{\overrightarrow{u}}$ is an ordinary angle in the hyperbolic plane $H_{\overrightarrow{u}}$. This angle does not depend on the choice of $\overrightarrow{u}$ and is called the *dihedral angle.*

Similarly given $\langle \overrightarrow{v} \rangle, \langle \overrightarrow{w} \rangle$ at ultrainfinity, if the line $(\langle \overrightarrow{v} \rangle, \langle \overrightarrow{w} \rangle)$ cut K^3_R then its polar line $H_{\overrightarrow{v}} \cap H_{\overrightarrow{w}}$ is outside K^3_R, and the distance $d\left(H_{\overrightarrow{v}}, H_{\overrightarrow{w}}\right)$ is measured along $(\langle \overrightarrow{v} \rangle, \langle \overrightarrow{w} \rangle)$ which is orthogonal to both $H_{\overrightarrow{v}}$ and $H_{\overrightarrow{w}}$.

We can also define $d\left(v, H_{\overrightarrow{w}}\right)$ in this way.

To compute these distances and angles define the function

$$\delta\left(\overrightarrow{v}, \overrightarrow{w}\right) = \frac{b\left(\overrightarrow{v}, \overrightarrow{w}\right)}{\sqrt{b\left(\overrightarrow{v}, \overrightarrow{v}\right)}\sqrt{b\left(\overrightarrow{w}, \overrightarrow{w}\right)}}$$

where $\sqrt{r} = i\sqrt{-r}$ if r is negative, and $\sqrt{r}$ is positive if r is positive. Then we have ([T]):

PROPOSITION A1. *Let $\langle \overrightarrow{v} \rangle, \langle \overrightarrow{w} \rangle$ be points of $\mathbb{R}P^3$ with normalized homogeneous coordinates. Let d_R denote distance in K^3_R. Then*

i. *If $b\left(\overrightarrow{v}, \overrightarrow{v}\right) < 0$, $b\left(\overrightarrow{w}, \overrightarrow{w}\right) < 0$, then*

$$\delta\left(\overrightarrow{v}, \overrightarrow{w}\right) = +\cosh\frac{d_R(v,w)}{R} \tag{A1}$$

ii. *If $b\left(\overrightarrow{v}, \overrightarrow{v}\right) < 0$, $b\left(\overrightarrow{w}, \overrightarrow{w}\right) > 0$, then*

$$\delta\left(\overrightarrow{v}, \overrightarrow{w}\right) = \varepsilon i \sinh\frac{d_R\left(v, H_{\overrightarrow{w}}\right)}{R} \tag{A2}$$

and the sign ε *is positive if and only if* $v \in H^+_{\overrightarrow{w}}$

iii. *If* $b(\overrightarrow{v}, \overrightarrow{v}) > 0$, $b(\overrightarrow{w}, \overrightarrow{w}) > 0$ *and* $|\delta(\overrightarrow{v}, \overrightarrow{w})| \geqq 1$, *then*

$$\delta(\overrightarrow{v}, \overrightarrow{w}) = \varepsilon \cosh \frac{d_R(H_{\overrightarrow{v}}, H_{\overrightarrow{w}})}{R} \tag{A3}$$

and ε *is positive if and only if either* $H^+_{\overrightarrow{v}} \subset H^+_{\overrightarrow{w}}$ *or* $H^+_{\overrightarrow{w}} \subset H^+_{\overrightarrow{v}}$.

iv. *If* $b(\overrightarrow{v}, \overrightarrow{v}) > 0$, $b(\overrightarrow{w}, \overrightarrow{w}) > 0$ *and* $|\delta(\overrightarrow{v}, \overrightarrow{w})| \leqq 1$, *then*

$$\delta(\overrightarrow{v}, \overrightarrow{w}) = -\cos\left(H^+_{\overrightarrow{v}} \cap H^+_{\overrightarrow{w}}\right). \tag{A4}$$

Remarks.

(1) If $\delta(\overrightarrow{v}, \overrightarrow{w}) = \pm 1$, $H_{\overrightarrow{v}} \cap H_{\overrightarrow{w}}$ is a point in Q^2_R and we say that $H_{\overrightarrow{v}}$ and $H_{\overrightarrow{w}}$ are parallel (angle and distance equal zero).

(2) If precisely one of $\langle\overrightarrow{v}\rangle, \langle\overrightarrow{w}\rangle$ is not normalized, the signs in the above proposition change.

We now state the so–called formula of Schläffli (See [Vi], [C],[M$_2$]).

Proposition A2. *(the formula of Schläffli)*

Let P *be a one parameter family of convex polyhedra depending smoothly on one or more parameters, in a space of constant curvature* K. *Let* ℓ_i, $i = 1, \ldots, n$ *be the length of the* ith *edge and let* α_i *be the dihedral angle at the* ith *edge. Let* $V(P)$ *be the volume of* P.

Then the following equation between differential forms in the parameter space holds:

$$K dV(P) = \frac{1}{2} \sum_{i=1}^{n} \ell_i d\alpha_i \ \square$$

Appendix B

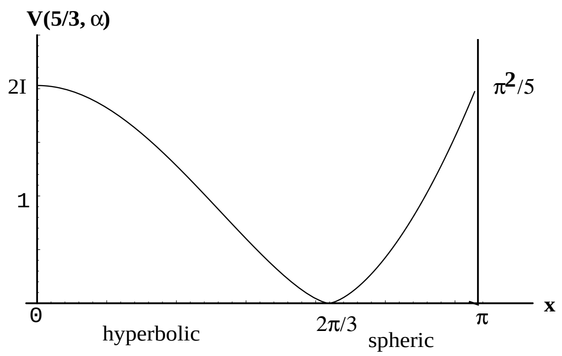
Volumes (tables and plots)

Let $V(5/3, \alpha)$ represent the volume of the cone-manifold $(5/3, \alpha)$ for angles $0 \leq \alpha \leq \pi$. Let $W(M, \beta)$ represent the volume of (M, β) , i.e. the cone-manifold obtained by $0-$surgery in the knot $5/3$. It is a torus bundle over S^1 with monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. The singular set is a section of M and β is the angle $0 \leq \beta \leq 2\pi$. For $\alpha = \beta = 0$, we have the complete hyperbolic structure of finite volume in $S^3 - 5/3$. For $\alpha = \pi$ we have the spherical structure of $5/3$ with angle π. For $\beta = 2\pi$ the hyperbolic structure degenerates. For $\alpha = 2\pi/3$ the hyperbolic geometry of $5/3$ degenerates to euclidean and then turns to spherical.

Plot I plots the volume $V(5/3, \alpha)$ against the parameter α ranging between 0 and π. For the value $\alpha = \frac{2\pi}{3}$ the geometry is euclidean.

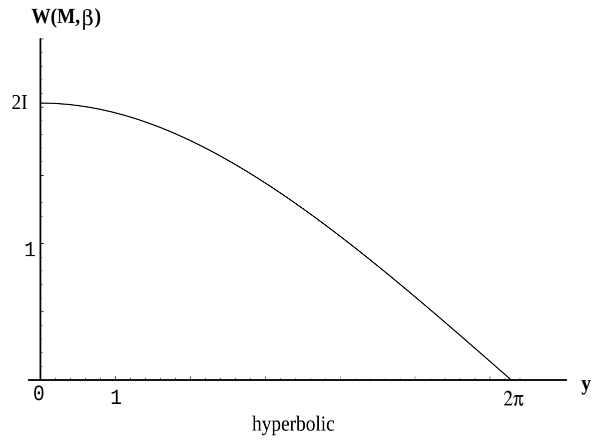
Plot II plots the volume $W(M, \beta)$ against the parameter β ranging between 0 and 2π

I = volume of ideal tetrahedron = 1.01494...



Plot I

The following tables of volumes and plots I and II were obtained using the program MATHEMATICA ([W]).



Plot II

VOLUMES OF SPHERICAL CONE-MANIFOLDS $(5/3, \alpha)$

n	$V(5/3, \frac{(n-1)\pi}{n})$	n	$V(5/3, \frac{(n-1)\pi}{n})$
4	0.21487656	24	1.58185558
5	0.448216429	25	1.59680134
6	0.638624655	26	1.610652
7	0.790807083	27	1.6235234
8	0.913575334	28	1.63551563
9	1.01410666	29	1.6467156
10	1.09768274	30	1.65719918
11	1.16813441	31	1.66703286
12	1.22826229	32	1.67627515
13	1.28014365	33	1.68497777
14	1.32534416	34	1.69318658
15	1.3650626	35	1.70094241
16	1.40023052	36	1.70828169
17	1.43158178	37	1.71523706
18	1.45970176	38	1.72183785
19	1.48506278	39	1.72811048
20	1.50804994	40	1.73407879
21	1.5289802	41	1.73976439
22	1.54811678	42	1.74518688
23	1.56567998	43	1.75036411

Volumes of hyperbolic cone-manifolds $(5/3, \alpha)$

n	$V(5/3, \frac{2\pi}{n})$	n	$V(5/3, \frac{2\pi}{n})$
4	0.507470803	42	2.0105734
5	0.937206855	43	2.01145795
6	1.22128746	44	2.01228315
7	1.41175465	45	2.01305419
8	1.54386328	46	2.0137757
9	1.63860068	47	2.01445182
10	1.70857095	48	2.01508629
11	1.76158141	49	2.01568246
12	1.80263322	50	2.01624334
13	1.83503266	51	2.01677166
14	1.86102868	52	2.01726989
15	1.88219043	53	2.01774027
16	1.89963778	54	2.01818485
17	1.91418624	55	2.01860546
18	1.92644053	56	2.01900381
19	1.93685631	57	2.01938143
20	1.94578205	58	2.01973974
21	1.95348776	59	2.02008003
22	1.96018518	60	2.02040349
23	1.96604221	61	2.02071121
24	1.97119331	62	2.0210042
25	1.97574718	63	2.02128337
26	1.97979243	64	2.02154959
27	1.98340188	65	2.02180364
28	1.98663579	66	2.02204625
29	1.98954437	67	2.02227811
30	1.99216976	68	2.02249983
31	1.99454748	69	2.022712
32	1.99670767	70	2.02291515
33	1.99867603	71	2.0231098
34	2.00047458	72	2.02329641
35	2.00212226	73	2.02347541
36	2.00363548	74	2.02364722
37	2.00502844	75	2.02381221
38	2.00631355	76	2.02397074
39	2.00750163	77	2.02412315
40	2.00860222	78	2.02426974

Volumes of hyperbolic cone-manifolds (M,β)

n	$W(M, \frac{2\pi}{n})$	n	$W(M, 2\frac{\pi}{n})$
2	1.39340228	42	2.02826857
3	1.7302584	43	2.02834278
4	1.85746199	44	2.02841198
5	1.91829403	45	2.02847663
6	1.95190401	46	2.0285371
7	1.97237158	47	2.02859376
8	1.98573948	48	2.02864691
9	1.99494326	49	2.02869685
10	2.00154627	50	2.02874382
11	2.00644238	51	2.02878805
12	2.01017237	52	2.02882976
13	2.01307884	53	2.02886913
14	2.01538733	54	2.02890634
15	2.01725118	55	2.02894153
16	2.01877759	56	2.02897485
17	2.02004332	57	2.02900644
18	2.02110449	58	2.02903641
19	2.02200289	59	2.02906487
20	2.02277017	60	2.02909191
21	2.02343066	61	2.02911764
22	2.02400327	62	2.02914214
23	2.02450294	63	2.02916547
24	2.02494154	64	2.02918772
25	2.02532863	65	2.02920895
26	2.02567197	66	2.02922923
27	2.0259779	67	2.0292486
28	2.02625168	68	2.02926713
29	2.02649765	69	2.02928486
30	2.02671946	70	2.02930183
31	2.02692017	71	2.02931809
32	2.02710237	72	2.02933367
33	2.02726827	73	2.02934863
34	2.02741975	74	2.02936297
35	2.02755845	75	2.02937675
36	2.02768576	76	2.02938999
37	2.02780288	77	2.02940272
38	2.02791089	78	2.02941496
39	2.02801071	79	2.02942674
40	2.02810313	80	2.02943808
41	2.02818888	81	2.029449

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(Received February 24, 1995)

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