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Genuine solutions and formal solutions with Gevrey type estimates of nonlinear partial differential equations

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Dedicated to Professor Hikosaburo Komatsu *on his 60th birthday*

Abstract. Let $L(u) = L(z, \partial^\alpha u; |\alpha| \le m)$ be a nonlinear partial differential operator defined in a neighbourhood Ω of $z = 0$ in $\boldsymbol{C}^{n+1}$, where $z = (z_0, z') \in \boldsymbol{C} \times \boldsymbol{C}^n$. We consider a nonlinear partial differential equation $L(u) = g(z)$, which has *a formal solution* $\tilde{u}(z)$ of the form

$$\tilde{u}(z) = z_0^q(\sum_{n=0}^{+\infty} u_n(z') z_0^{q_n}) \qquad u_0(z') \not\equiv 0,$$

where $q \in \boldsymbol{R}$ and $0 = q_0 < q_1 < \ldots < q_n < \ldots \to +\infty$, with

$$|u_n(z')| \le AB^{q_n}\Gamma(\frac{q_n}{\gamma_*} + 1) \qquad \gamma_* > 0,$$

which we often call the Gevrey type estimate. It is the main purpose to show under some conditions that there exists *a genuine solution* $u_{S_1}(z)$ with the asymptotic expansion $u_{S_1}(z) \sim \tilde{u}(z)$ as $z_0 \to 0$ in some sector S_1. We apply the results to formal solutions constructed in Ōuchi [7].

0. Contents

We summarize what we need and resluts in §1. We give notations and definitions and introduce several notions for nonlinear partial differential operators $L(u)$. In particular we define for $L(u)$ and $r \in \boldsymbol{R}$ the characteristic polygon $\Sigma_L^*(r)$. Other notions related to them, the characteristic indices

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etc., are also introduced and investigated. Secondly we treat a nonlinear equation $L(u) = g(z)$ with $g(z) \sim 0$ as $z_0 \to 0$ and consider the existence of $u(z) \sim 0$ as $z_0 \to 0$ (Theorems 1.8 and 1.9). Thirdly we consider a nonlinear equation $L(u) = g(z)$ stated in ABSTRACT and give a result of the existence of genuine solutions (Theorem 1.12) and apply it to formal solutions constructed in [7] (Theorems 1.16 and 1.17). The proofs of Theorems and Propositions stated in §1 are mainly given in §3 - §5.

In §2 we prepare majorant functions and function spaces and give their properties for our purposes.

In §3 and §4 we construct $u_{S'}(z) \sim 0$ in a sector S' satisfying $(L(u_{S'}) - g(z)) \sim 0$ with some exponential order, and show Theorems 1.8 and 1.9.

In §5 we give the proofs of Propositions and Theorems which are not yet shown in the preceding sections.

1. Notations, definitions and theorems

Firstly we give usual notations and definitions: $\boldsymbol{C}$ means the set of the complex numbers. $z = (z_0, z_1, \dots, z_n) = (z_0, z_1, z'') = (z_0, z')$ is the coordinate of $\boldsymbol{C}^{n+1}$. $|z| = \max\{|z_i|;\ 0 \le i \le n\}$ and $\partial = (\partial_0, \partial_1, \dots, \partial_n) = (\partial_0, \partial')$, $\partial_i = \partial/\partial z_i$. $\boldsymbol{R}$ means the set of the real numbers and $\boldsymbol{R}_+ = \{x \in \boldsymbol{R}; x > 0\}$. The set of all nonnegative integers (resp. integers) is denoted by $\boldsymbol{N}$ (resp. $\boldsymbol{Z}$). For multi-index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) = (\alpha_0, \alpha')$, $|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_n = \alpha_0 + |\alpha'|$, $\partial^\alpha = \partial_0^{\alpha_0}\partial^{\alpha'} = \partial_0^{\alpha_0}\partial'^{\alpha'} = \prod_{i=1}^n \partial_i^{\alpha_i}$ and $z^\alpha = z_0^{\alpha_0} z_1^{\alpha_1} \dots z_n^{\alpha_n}$. We introduce notations for products of multi-indices. Let $A \in (\boldsymbol{N}^{n+1})^s$, where $A = (A_1, A_2, \dots, A_s)$ and $A_i = (A_{i,0}, A'_i) \in \boldsymbol{N} \times \boldsymbol{N}^n$. Then we define $s_A = s$, $k_A = \max\{|A_i|; 1 \le i \le s_A\}$, $k'_A = \max\{|A'_i|; 1 \le i \le s_A\}$, $|A| = \sum_{i=1}^{s_A} |A_i|$, and $l_A = \sum_{i=1}^{s_A} |A'_i|$. Let $A, B \in (\boldsymbol{N}^{n+1})^s$. If some rearrangement of the components A_i's coincides with B, we identify A with B. We denote by $\mathcal{N}^d$ the set of all different elements of $\cup_{s=1}^d (\boldsymbol{N}^{n+1})^s$. For a real number a, $[a]$ means the integral part of a. Let $\omega_0 = \{z_0; |z_0| \le R\}$ and $\omega = \{z' \in \boldsymbol{C}^n;\ |z'| \le R\}$. Let $S = \{z_0 \ne 0;\ \phi_- < \arg z_0 < \phi_+\}$ be a sector in $\boldsymbol{C}$. Put $\Omega = \omega_0 \times \omega$ and $\Omega_S = (S \cap \omega_0) \times \omega$. Let $S' = \{z_0 \ne 0;\ \phi'_- < \arg z_0 < \phi'_+\}$ and $\Omega' = \{z \in \boldsymbol{C}^{n+1}; |z| \le R'\}$. Then $S' \subset\subset S$ ($\Omega'_{S'} \subset\subset \Omega_S$) means $\phi_- < \phi'_- < \phi'_+ < \phi_+$ and $R' < R$. We often use the notation $S(\phi_-, \phi_+) = \{z_0;\ \phi_- < \arg z_0 < \phi_+, 0 < |z_0| \le r\}$ and $S(\theta) = S(-\theta, \theta)$ $(\theta > 0)$, where $r > 0$ is small if necessary. $\mathcal{O}(\Omega)$ $(\mathcal{O}(\Omega_S))$ is the set of all holomorhic functions on Ω (resp. Ω_S).

For the simplicity we often denote different constants by the same notations A, B, B', *etc.*, if confusions will not occur.

DEFINITION 1.1. (1). $\mathcal{F}$ is the set of all formal series $f(z) = \sum_{n=0}^{+\infty} f_n(z')z_0^{r_n}$, $f_n(z') \in \mathcal{O}(\omega)$, where ω depends on $f(z)$ and $r_0 < r_1 < \ldots < r_n <\to +\infty$.
(2). For $f(z) \in \mathcal{F}$, $\min\{r_n; f_n(z') \not\equiv 0\}$ is said to be the formal valuation of $f(z)$. If $f_n(z') \equiv 0$ for all $n \in \boldsymbol{N}$, then its formal valuation is $+\infty$.

DEFINITION 1.2. (1). Let $f(z) \in \mathcal{O}(\Omega_S)$. $f(z)$ is said to have the asymptotic expansion $f(z) \sim \sum_{n=0}^{+\infty} f_n(z'){z_0}^{r_n}$, if the following holds: for any sector S_0 ($S_0 \subset\subset S$) and any N,

$$|f(z) - \sum_{n=0}^{N-1} f_n(z')z_0^{r_n}| \leq C_N|z_0|^{r_N} \quad as\ z_0 \to 0\ \ in\ S_0. \tag{1.1}$$

$Asy(\Omega_S)$ is the totality of $f(z) \in \mathcal{O}(\Omega_S)$ which has the asymptotic expansion such as (1.1)
(2). Let $\gamma > 0$. $Asy^0_{\{\gamma\}}(\Omega_S)$ is the totality of $f(z) \in \mathcal{O}(\Omega_S)$ such that for any $S_0 \subset\subset S$

$$|f(z)| \leq C_0 \exp(-c_0|z_0|^{-\gamma}) \quad (c_0 > 0), \tag{1.2}$$

where c_0 depends on S_0.

We treat a nonlinear partial differential operator with order m:

$$\begin{aligned} L(u) &:= L(z, \partial^\alpha u; |\alpha| \leq m) \\ &= \sum_{s=1}^{M} \sum_{\{A;\ s_A=s\}} z_0^{e_A} b_A(z) \prod_{i=1}^{s} \partial'^{A'_i}(z_0\partial_0)^{A_{i,0}}u. \end{aligned} \tag{1.3}$$

The coefficients $b_A(z)$'s are in $\mathcal{F}$ or $Asy(\Omega_S)$. If $b_A(z) \in \mathcal{F}$, $L(u)$ is said to be *formal.* In any case the formal valuation of $b_A(z)$ is 0 if $b_A(z) \not\equiv 0$. For A with $b_A \equiv 0$ we put $e_A = +\infty$. We suppose that $L(u)$ is a polynomial of $\{\partial^\alpha u; |\alpha| \leq m\}$ with degree M. But some definitions and results will be hold for operators of non polynomial type.

Now put

$$\text{(1.4)} \qquad \begin{cases} \mathcal{L}(z,\partial) = \displaystyle\sum_{\{A;\ s_A=1\}} z_0^{e_A} b_A(z) \partial'^{A'_i} (z_0\partial_0)^{A_{i,0}} u, \\ M(u) = L(z, \partial^\alpha u; |\alpha| \le m) - \mathcal{L}(z,\partial)u. \end{cases}$$

$\mathcal{L}(z,\partial)$ is the linear part of $L(u)$ and $M(u)$ is the nonlinear part of $L(u)$. We write often $\mathcal{L}(z,\partial)$ in the following form:

$$\text{(1.5)} \qquad \mathcal{L}(z,\partial) = \sum_{k=0}^{m} \sum_{l=0}^{k} z_0^{e(k,l)} b_{k,l}(z,\partial')(z_0\partial_0)^{k-l},$$

where $b_{k,l}(z,\xi')$ is homogeneous with order l with respect to ξ' and $\mathcal{L}(z,\partial)$ is an operator with order $\le m$.

We proceed to define the characteristic polygon for $\mathcal{L}(z,\partial)$ and $L(u)$. Put

$$\text{(1.6)} \qquad \begin{cases} e_{k,\mathcal{L}} = \min\{e(k,l);\ 0 \le l \le k\} \\ l_{k,\mathcal{L}} = \max\{l; e(k,l) = e_{k,\mathcal{L}}\} \end{cases}$$

and for $r \in \boldsymbol{R}$

$$\text{(1.7)} \qquad e_{k,L}(r) = \min\{s_A r + e_A; A \in \mathcal{N}^M \textit{ with } k_A = k\}.$$

Define $\Pi(a,b) = \{(x,y) \in \boldsymbol{R}^2; x \le a,\ y \ge b\}$ and $\Pi(a,+\infty) = \emptyset$. Put $\Sigma_L^*(r)$ = *the convex hull of* $\cup_{k=0}^m \Pi(k, e_{k,L}(r))$ and $\Sigma_{\mathcal{L}}^*$ = *the convex hull of* $\cup_{k=0}^m \Pi(k, e_{k,\mathcal{L}})$. The boundary of $\Sigma_L^*(r)$ ($\Sigma_{\mathcal{L}}^*$) consists of a vertical half line $\Sigma_{0,L}^*(r)$ (*resp.* $\Sigma_{0,\mathcal{L}}^*$), a horizontal half line $\Sigma_{p_r,L}^*(r)$ (*resp.* $\Sigma_{p,\mathcal{L}}^*$) and segments $\Sigma_{i,L}^*(r)$, $1 \le i \le p_r - 1$ (*resp.* $\Sigma_{i,\mathcal{L}}^*$, $1 \le i \le p-1$). The set of vertices of $\Sigma_L^*(r)$ ($\Sigma_{\mathcal{L}}^*$) consists of p_r (*resp.* p) points $(k_{i,L}(r), e_{k_{i,L}}(r))$, $0 \le k_{p_r-1,L}(r) < k_{p_r-2,L}(r) < \ldots < k_{1,L}(r) < k_{0,L}(r) = m$ (*resp.* $(k_{i,\mathcal{L}}, e_{k_{i,\mathcal{L}}})$, $0 \le k_{p-1,\mathcal{L}} < k_{p-2,\mathcal{L}} < \ldots < k_{1,\mathcal{L}} < k_{0,\mathcal{L}} \le m$) (see Figure 1). Let $\gamma_{i,L}(r)$ ($\gamma_{i,\mathcal{L}}$) be the slope of $\Sigma_{i,L}(r)$ (*resp.* $\Sigma_{i,\mathcal{L}}$). Then $0 = \gamma_{p_r,L}(r) < \gamma_{p_r-1,L}(r) < \ldots < \gamma_{1,L}(r) < \gamma_{0,L}(r) = +\infty$) (*resp.* $0 = \gamma_{p,\mathcal{L}} < \gamma_{p-1,\mathcal{L}} < \ldots < \gamma_{1,\mathcal{L}} < \gamma_{0,\mathcal{L}} = +\infty$).

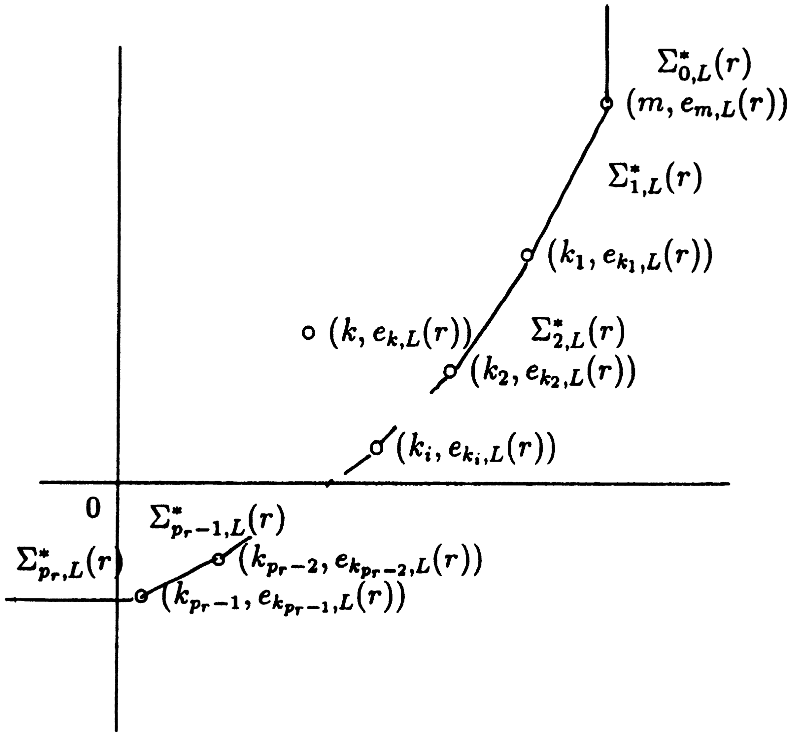

Fig. 1. Characteristic polygon

Define for $1 \le i \le p$

$$\mathcal{L}_i(z,\partial) = \sum_{\left\{\substack{(k,l);\ e(k,l)=e_{k,\mathcal{L}} \\ e_{k_{i-1},\mathcal{L}}-e_{k,\mathcal{L}}=\gamma_{i,\mathcal{L}}(k_{i-1}-k)}\right\}} z^{e(k,l)} b_{k,l}(z,\partial')(z_0\partial_0)^{k-l}, \tag{1.8}$$

which is a linear operator corresponding to the segment $\Sigma^*_{i,\mathcal{L}}$ and we often denote $\mathcal{L}_i(z,\partial)$ by $\mathcal{L}_i(z, z_0\partial_0, \partial')$.

Definition 1.3. (1) $\Sigma^*_{\mathcal{L}}$ is called "the characteristic polygon" of linear partial differential operator $\mathcal{L}(z,\partial)$ and $\Sigma^*_L(r)$ is called the characteristic polygon with valuation r, shortly "r-characteristic polygon", of $L(u)$.
(2) $\gamma_{i,\mathcal{L}}$ is called "the i-th characteristic index" of $\mathcal{L}(z,\partial)$ and $\gamma_{i,\mathcal{L}}(r)$ is called the i-th characteristic index of $L(u)$ with respect to valuation r,

shortly "the i-th r-characteristic index". In particular $\gamma_{p-1,\mathcal{L}}$ ($\gamma_{p_r-1,L}(r)$) is called "the minimal irregularity" (resp. "the minimal irregularity with respect to valuation r") and denoted by $\gamma_{\min,\mathcal{L}}$ (resp. $\gamma_{\min,L}(r)$).

It follows from the definition that the *r-characteristic polygon* $\Sigma^*_{\mathcal{L}}(r)$ of the linear operator $\mathcal{L}(z,\partial)$ is equal to $\Sigma^*_{\mathcal{L}} + (0,r)$.

Remark 1.4. Characteristic polygons and characteristic indices for a linear partial differential operator $\mathcal{L}(z,\partial)$ were defined in Ōuchi [6] with other useful notions. The definitions in [6] are slightly different from those in this paper. But the difference is not essential. The characteristic indices were denoted by σ_i in [6] and it holds that $\gamma_{i,\mathcal{L}} = \sigma_i - 1$.

Definition 1.5. (1) A nonlinear partial differential operator $L(u)$ with order m is said to be linearly nondegenerate, if its linear part $\mathcal{L}(z,\partial)$ is also an operator with order m.
(2) If the r-characteristic polygon $\Sigma^*_M(r)$ of $M(u)$, which is the nonlinear part of $L(u)$ (see (1.4)), is included in the interior of the r-characteristic polygon of $\mathcal{L}(z,\partial)$, then the nonlinear operator $L(u)$ is said to have the strongly linear part with respect to valuation r.
(3) If $L(u)$ has the strongly linear part for any valuation $r > \rho$, then it is said to have the strongly linear part with respect to valuation ρ_+.

Proposition 1.6. *(1) Put $R(u) = L(z_0^r u)$. Then $\Sigma^*_L(r'+r) = \Sigma^*_R(r')$ for any r', $\mathcal{R}_i(z, z_0\partial_0, \partial') = z_0^r \mathcal{L}_i(z, z_0\partial_0, \partial')$ for $1 \le i \le p-1$ and $\mathcal{R}_p(z, z_0\partial_0, \partial') = z_0^r \mathcal{L}_p(z, z_0\partial_0 + r, \partial')$. Moreover if $L(u)$ has the strongly linear part with respect to valuation r, then $R(u)$ has the strongly linear part with respect to valuation 0.*
(2) $L(u)$ is linearly nondegenerate if and only if there is a ρ such that it has the strongly linear part with respect to valuation ρ_+.

Let $v(z) \in \mathcal{F}$, $v(z) = \sum_{n=0}^{+\infty} v_n(z') z_0^{r_n}$. Define a formal operator

$$L^v(u) := L(u+v) - L(v) \tag{1.9}$$

and its linear part is denoted by $\mathcal{L}^v(z,\partial)$, which we call the linearization of $L(u)$ at $u = v(z)$. Put

$$v^*_{-1}(z) = 0, \quad v^*_l(z) = \sum_{n=0}^{l} v_n(z') z_0^{r_n} \quad for \ \ l \in \boldsymbol{N}. \tag{1.10}$$

We can also define $L^{v_N^*}$ such as (1.9) and $\mathcal{L}^{v_N^*}$.

Proposition 1.7. *Suppose that $L^v(u)$ is linearly nondegenerate. Then there is an $N_0 \in \boldsymbol{N}$ such that if $N \geq N_0$, $\Sigma^*_{\mathcal{L}^v} = \Sigma^*_{\mathcal{L}^{v_N^*}}$ and $\Sigma^*_{L^v}(r) = \Sigma^*_{L^{v_N^*}}(r)$ for any r. Moreover there is a ρ such that if $N \geq N_0$,*

$$\Sigma^*_{L^v}(r) = \Sigma^*_{L^{v_N^*}}(r) = \Sigma^*_{\mathcal{L}^v}(r) = \Sigma^*_{\mathcal{L}^{v_N^*}}(r) \quad for \ \ r > \rho. \tag{1.11}$$

The proofs of Propositions 1.6 and 1.7 are given in §5.

Now consider an equation

$$L(u) = g(z) \in Asy^0_{\{\gamma_{i,\mathcal{L}}\}}(\Omega_S), \tag{$\{\mathrm{Eq}^0\}$}$$

where we assume $p \geq 2$ and $1 \leq i \leq p-1$. We try to find a solution $u(z)$ of $\{\mathrm{Eq}^0\}$ with exponential decay. We suppose that the coefficients b_A's of $L(u)$ are in $Asy(\Omega_S)$ and give conditions to state results.

Condition 0. *$L(u)$ is linearly nondegenerate.*

We introduce a condition for $\mathcal{L}_i(z, \partial)$

Condition 1-{i}. *The following holds for $\mathcal{L}_i(z, \partial)$:*

$$\begin{cases} (1) & l_{k_{i-1},\mathcal{L}} > l_{k,\mathcal{L}} \\ & for \ \ k \in \{k;\ k < k_{i-1},\ e_{k_{i-1},\mathcal{L}} - e_{k,\mathcal{L}} = \gamma_{i,\mathcal{L}}(k_{i-1} - k)\}, \\ (2) & b_{k_{i-1}, l_{k_{i-1}},\mathcal{L}}(0, \xi') \not\equiv 0. \end{cases}$$

The following Theorem 1.8 is fundamental in this paper.

Theorem 1.8. *Suppose that for $L(u)$ Condition-0 and Condition-1-$\{i\}$ hold. Let $S' = S(\phi_-, \phi_+)$ be a sector such that $S' \subset\subset S$.*

(1) If $2 \leq i \leq p-1$, for any S' with $\phi_+ - \phi_- < \pi/\gamma_{i-1,\mathcal{L}}$ there is a $u_{S'}(z) \in Asy^0_{\{\gamma_{i,\mathcal{L}}\}}(\Omega'_{S'})$ such that

$$L(u_{S'}) - g(z) = g_{S'}(z) \in Asy^0_{\{\gamma_{i-1,\mathcal{L}}\}}(\Omega'_{S'}). \tag{1.12}$$

(2) If $i = 1$, for any S' with $\phi_+ - \phi_- < \pi/\gamma_{1,\mathcal{L}}$ there is a solution $u_{S'}(z) \in Asy^0_{\{\gamma_{1,\mathcal{L}}\}}(\Omega'_{S'})$ of $\{\mathrm{Eq}^0\}$.
Here Ω' is a neighbourhood of $z = 0$.

THEOREM 1.9. *Suppose that Condition-0 and Condition-1-$\{i\}$ hold for all $1 \le i \le i_*$. Consider*

$$L(u) = g(z) \in Asy^0_{\{\gamma_{i_*,\mathcal{L}}\}}(\Omega_S). \tag{1.13}$$

Let $S_1 = S(\phi_{1,-}, \phi_{1,+})$ be a sector with $S_1 \subset\subset S$ and $\phi_{1,+} - \phi_{1,-} < \pi/\gamma_{1,\mathcal{L}}$. Then there is a solution $u_{S_1}(z) \in Asy^0_{\{\gamma_{i_*,\mathcal{L}}\}}(\Omega'_{S_1})$ of (1.13), where Ω' is a neighbourhood of $z = 0$.

Now we proceed to the main purpose of this paper, that is, the investigation of the relation between solutions of formal series and genuine solutions of nonlinear partial differential equations. For this purpose Theorems 1.8 and 1.9 are available. Let us introduce function spaces.

DEFINITION 1.10. Let $\mathcal{S}$ be a finitely generated additive semi-group, $\mathcal{S} = \{q_i;\ i \in \boldsymbol{N}\}$, $0 = q_0 < q_1 < \ldots < q_i <\to +\infty$.
(1) $\mathcal{F}_S$ is the set of all $f(z) \in \mathcal{F}$ such that $f(z) \sim \sum_{n=0}^{+\infty} f_n(z') z_0^{q_n}$.
(2) $Asy^{\mathcal{S}}_{\{\kappa\}}(\Omega_S)$ $(0 < \kappa \le +\infty)$ is the set of all $f(z) \in \mathcal{O}(\Omega_S)$ with asymtotic expansion $f(z) \sim \sum_{n=0}^{+\infty} f_n(z') z_0^{q_n}$ in the following sense: for any sector S' $(S' \subset\subset S)$

$$|f(z) - \sum_{n=0}^{N-1} f_n(z') z_0^{q_n}| \le AB^{q_N} |z_0|^{q_N} \Gamma(\frac{q_N}{\kappa} + 1) \quad for\ z_0 \in S'. \tag{1.14}$$

If $\kappa = +\infty$ and $\mathcal{S} = \boldsymbol{N}$, then $Asy^{\boldsymbol{N}}_{\{+\infty\}}(\Omega_S)$ is holomorphic in a neighbourhood of $z = 0$. In the following of this section $\mathcal{S}$ means a finitely generated additive semi-group, $\mathcal{S} = \{q_i;\ i \in \boldsymbol{N}\}$. We have

PROPOSITION 1.11. *Let $\{f_n(z')\}$ $(n \in \boldsymbol{N})$ be a sequence in $\mathcal{O}(\omega')$ with*

$$|f_n(z')| \le AB^{q_n} \Gamma(\frac{q_n}{\kappa} + 1) \quad (0 < \kappa < +\infty). \tag{1.15}$$

Let $S := S(\phi_-, \phi_+) = \{z_0; \phi_- < \arg z_0 < \phi_+,\ 0 < |z_0| \le r_0\}$ *be a sector with* $\phi_+ - \phi_- \le \pi/\kappa$ *and a small* $r_0 > 0$. *Then there is a* $f(z) \in Asy^{\mathcal{S}}_{\{\kappa\}}(\Omega_S)$ *such that for any* $S' \subset\subset S$

$$(1.16) \qquad |f(z) - \sum_{n=0}^{N-1} f_n(z') z_0^{q_n}| \le A_1 B_1^{q_N} |z_0|^{q_N} \Gamma(\frac{q_N}{\kappa} + 1) \quad for\ z_0 \in S',$$

where A_1 *and* B_1 *depend on* S'.

The proof of Proposition 1.11 will be given in §5. Let

$$(1.17) \qquad \begin{aligned} L(u) &:= L(z, \partial^\alpha u; |\alpha| \le m) \\ &= \sum_{s=1}^{M} \{ \sum_{\{A;\ s_A = s\}} z_0^{e_A} b_A(z) \prod_{i=1}^{s_A} z_0^{A_{i,0}} \partial_0^{A_{i,0}} \partial^{A'_i} u \}, \end{aligned}$$

where $b_A(z) \in Asy^{\mathcal{S}}_{\{\kappa\}}(\Omega_S)$ $(0 < \kappa \le +\infty)$, $b_A(z) \sim \sum_{n=0}^{+\infty} b_{A,n}(z') z_0^{q_n}$, and its formal valuation is 0 if $b_A(z) \not\equiv 0$. The representation of $L(u)$ in (1.17) is different from (1.3) in order to cite the results in Ōuchi [7], which is not essential.

Now consider

$$\{\mathrm{Eq}\} \qquad L(u) = g(z) \qquad z_0^{-r} g(z) \in Asy^{\mathcal{S}}_{\{\gamma\}}(\Omega_S)\ (\gamma \le \kappa),$$

where $g(z) \sim z_0^r (\sum_{n=0}^{+\infty} g_n(z') z_0^{q_n})$. We treat a solution of {Eq} of formal series with formal valuation q

$$(1.18) \qquad \tilde{u}(z) = z_0^q (\sum_{n=0}^{+\infty} u_n(z') z_0^{q_n}) \in z_0^q \mathcal{F}_S.$$

We put a few assumptions on {Eq}.

Assumption 1. *There exists a formal solution* $\tilde{u}(z) = z_0^q (\sum_{n=0}^{+\infty} u_n(z') z_0^{q_n})$ *of* {Eq} *with formal valuation* q.

As before let $L^{\tilde{u}}(u)$ be an operator defined by $L^{\tilde{u}}(u) = L(u + \tilde{u}) - L(\tilde{u})$ and its linear part is denoted by $\mathcal{L}^{\tilde{u}}(z, \partial)$. $L^{\tilde{u}}(u)$ and $\mathcal{L}^{\tilde{u}}(z, \partial)$ are formal operators. We can also consider the characteristic polygons for $L^{\tilde{u}}(u)$ and $\mathcal{L}^{\tilde{u}}(z,\partial)$.

Assumption 2. *$L^{\tilde{u}}(u)$ is linearly nondegenerate.*

Assumption 3. *The coefficients $u_n(z')$ $(n \ge 0)$ of formal solution $\tilde{u}(z)$ satisfy $|u_n(z')| \le AB^{q_n}\Gamma(q_n/\gamma_{i_*,\mathcal{L}^{\tilde{u}}}+1)$ for some i_* $(1 \le i_* \le p-1)$ and $\gamma_{i_*,\mathcal{L}^{\tilde{u}}} \le \gamma$.*

Let $S_* = S(\phi_-, \phi_+)$ be a sector with $\phi_+ - \phi_- < \pi/\gamma_{i_*,\mathcal{L}^{\tilde{u}}}$ and $S_* \subset\subset S$. From Proposition 1.11 there is a $v(z) \in z_0^q\ Asy^{\mathcal{S}}_{\{\gamma_{i_*},\mathcal{L}^{\tilde{u}}\}}(\Omega_{S_*})$ such that $v(z) \sim z_0^q(\sum_{n=0}^{+\infty} u_n(z')z_0^{q_n})$ *in* S_*. Since $L^v(u) = L(u+v) - L(v)$ is a differential operator and the characteristic polygons of $L^v(u)$ and $L^{\tilde{u}}(u)$ are same, we use $L^v(u)$ instead of $L^{\tilde{u}}(u)$.

Assumption 4. $g_*(z) := g(z) - L(v) \in Asy^{0}_{\{\gamma_{i_*},\mathcal{L}^v\}}(\Omega_{S_*})$.

We can consider Condition-0 and Condition-1-$\{i\}$'s for the operator $L^v(u)$, that is, by replacing $L(u)$ by $L^v(u)$ and $\mathcal{L}(z,\partial)$ by $\mathcal{L}^v(z,\partial)$. We have

Theorem 1.12. *Suppose Assumptions 1-4 hold and that $L^v(u)$ satisfies Condition 1-(i)'s for all $1 \le i \le i_*$. Then for any sector $S_1 = S(\phi_{1,-}, \phi_{1,+})$ with $\phi_{1,+} - \phi_{1,-} < \pi/\gamma_{1,\mathcal{L}^v}$ and $S_1 \subset\subset S$ there exists a solution $u_{S_1}(z) \in Asy_{\gamma_{i_*},\mathcal{L}^v}(\Omega'_{S_1})$ of* {Eq} *with asymptotic expansion $u_{S_1}(z) \sim z_0^q(\sum_{n=0} u_n(z')z_0^{q_n})$ in S_1, where Ω' is a neighbourhood of $z=0$.*

Proof. Consider

$$L^v(w) = L(v+w) - L(v) = g(z) - L(v) = g_*(z) \in Asy^{0}_{\{\gamma_{i_*},\mathcal{L}^v\}}(\Omega_{S_*}). \tag{1.19}$$

Then it follows from Theorem 1.9 that there exists $w_{S_1}(z) \in Asy^{0}_{\{\gamma_{i_*},\mathcal{L}^v\}}(\Omega_{S_*})$ such that $L^v(w_{S_1}(z)) = g_*(z)$. Hence $u_{S_1}(z) = v(z) + w_{S_1}(z)$ is a desired solution. □

Now we cite a few results in Ōuchi [7] concerning the existence of a formal solution $\tilde{u}(z)$ of {Eq} with a Gevrey type estimate. For a given $q \in \boldsymbol{R}$, put

$$q^* = \min\{s_A q + e_A; A \in \mathcal{N}^M\}, \tag{1.20}$$

$$\Delta_L(q) = \{A \in \mathcal{N}^M; s_A q + e_A = q^*\}. \tag{1.21}$$

In Ōuchi [7], we define q^* and $\Delta_L(q)$ using quantities $d_{A,L} - |A|$ insted of e_A. But we will be able to easily notice that $e_A = d_{A,L} - |A|$ by representating $L(u)$ in the form (1.3).

Put for $A \in \mathcal{N}^M$

$$\mathfrak{L}_{0,A}(z', \mu, p) = b_{A,0}(z') \prod_{i=1}^{s_A} \mu(\mu-1)\dots(\mu - A_{i,0}+1) p_{A'_i} \tag{1.22}$$

and

$$\begin{aligned} &\mathfrak{L}_{1,A}(z', \lambda, \mu, p, \partial') \\ =&b_{A,0}(z')\{\sum_{i=1}^{s_A}(\prod_{h\neq i} \mu(\mu-1)\dots(\mu - A_{h,0}+1) p_{A'_h}) \\ &\times \lambda(\lambda-1)\dots(\lambda - A_{i,0}+1)\partial^{A'_i}\}, \end{aligned} \tag{1.23}$$

where $p = (p_{\alpha'};\ \alpha' \in \boldsymbol{N}^n)$ and λ, μ are parameters. $\mathfrak{L}_{1,A}(z', \lambda, \mu, p, \partial')$ is a linear partial differential operator with order $k'_A = \max\{|A'_i|; 1 \le i \le s_A\}$ and a polynomial of λ and ∂' with degree $k_A = \max\{|A_i|; 1 \le i \le s_A\}$. Define

$$\begin{cases} \mathfrak{L}_0(z', \mu, p) = \displaystyle\sum_{A \in \Delta_L(q)} \mathfrak{L}_{0,A}(z', \mu, p) \\ \mathfrak{L}_1(z', \lambda, \mu, p, \partial') = \displaystyle\sum_{A \in \Delta_L(q)} \mathfrak{L}_{1,A}(z', \lambda, \mu, p, \partial'). \end{cases} \tag{1.24}$$

$\mathfrak{L}_1(z', \lambda, \mu, p, \partial')$ is a linear partial differential operator with order $k'_L(q) = \max\{k'_A; A \in \Delta_L(q)\}$ and a polynomial of λ and ∂' with degree $k_L(q) = \max\{k_A; A \in \Delta_L(q)\}$.

Condition I. (1) *$\mathcal{S} \supset \{s_A q + e_A - q^*; A \in \mathcal{N}^M\}$ and $g(z) \sim z_0^{q_*}(\sum_{n=0}^{+\infty} g_n(z') z_0^{q_n})$, that is, $r = q^*$ in* {Eq}
(2) *There is a solution $u_0(z') \not\equiv 0$ of*

$$\mathfrak{L}_0(z', q, \partial^{\alpha'} u_0(z')) = g_0(z'), \tag{1.25}$$

which is holomorphic in a neighbourhood ω of $z' = 0$.

Suppose Condition I holds. Using $u_0(z')$ in Condition I, define

$$\mathfrak{L}_1(z', \lambda, \partial') = \mathfrak{L}_1(z', \lambda, q, \partial^{\alpha'} u_0(z'), \partial'). \tag{1.26}$$

Let $m_{\mathfrak{L}_1}$ be the order of $\mathfrak{L}_1(z', \lambda, \partial')$. Let $P.S.\mathfrak{L}_1(z', \lambda, \xi')$ be the principal symbol of $\mathfrak{L}_1(z', \lambda, \partial')$, which is homogeneous in ξ' with degree $m_{\mathfrak{L}_1}$, and $\overset{\circ}{k}_{\mathfrak{L}_1}$ be its degree as a polynomial of (λ, ξ'). We note that $m_{\mathfrak{L}_1} \le k'_L(q)$ and $\overset{\circ}{k}_{\mathfrak{L}_1} \le k_L(q)$.

Condition II. *$P.S.\mathfrak{L}_1(0, \lambda, \hat{\xi}')$, $\hat{\xi}' = (1, 0, \ldots, 0)$, is a polynomial of λ with degree $\overset{\circ}{k}_{\mathfrak{L}_1} - m_{\mathfrak{L}_1}$ and does not vanish for $\lambda = q + q_n$, $n = 1, 2, \ldots$.*

As for the existence of formal solutions with the formal valuation q, we have in Ōuchi [7]

Theorem 1.13. *Suppose that Conditions I and II hold. Then there exists a uniquely formal series $\tilde{u}(z) = z_0^q(\sum_{n=0}^{+\infty} u_n(z') z_0^{q_n})$ satisfing* {Eq} *formally and $\partial_1^h u_n(0, z'') = 0$ $(n \ge 1)$ for $0 \le h \le m_{\mathfrak{L}_1} - 1$.*

Condition I assures the existence of the nonzero initial term $u_0(z')$. We can determine $u_n(z')$ successively by Condition II.

Now we study the Gevrey estimate of the coefficients $u_n(z')$ $(n \ge 0)$ of $\tilde{u}(z)$ in Theorem 1.13. We try to find $0 < \gamma_* \le +\infty$ such that

$$|u_n(z')| \le AB^{q_n} \Gamma(\frac{q_n}{\gamma_*} + 1) \tag{Gev.γ_*}$$

for some constants A and B.

Condition III. *$P.S.\mathfrak{L}_1(0,\lambda,\hat{\xi}')$ is a polynomial of λ with degree $k_L(q)-m_{\mathfrak{L}_1}$.*

Condition III means $\overset{\circ}{k}_{\mathfrak{L}_1}=k_L(q)$, which is important in order to obtain an estimate such as $\{\mathrm{Gev}.\gamma_*\}$. We have in Ōuchi [7]

Theorem 1.14. *Put $\gamma_*=\gamma_{\min,L}(q)$. Suppose that Conditions I, II and III hold and $\gamma_*\le\min\{\gamma,\kappa\}$. Then the coefficients $u_n(z')$'s of formal solution $\tilde{u}(z)$ in Theorem 1.13 have the estimate $\{\mathrm{Gev}.\gamma_*\}$.*

We have given in Ōuchi [7] the Gevrey index γ_* more precisely. Let us explain it. We assume that Conditions I, II and III hold. For simpicity we assume $\kappa=+\infty$, that is, $b_A(z)\in Asy^{\mathcal{S}}_{\{+\infty\}}(\Omega_S)$. By using the coefficients $u_n(z')$ of $\tilde{u}(z)$ in Theorem 1.13, define

$$(1.27)\qquad u^*_l(z)=z_0^q\Big(\sum_{n=0}^{l}u_n(z')z_0^{q_n}\Big)\quad for\ \ l\ge 0,\quad u^*_{-1}(z)=0.$$

Let us consider operators $L^{u^*_l}(u)=L(u+u^*_l)-L(u^*_l)$ and $\mathcal{L}^{u^*_l}(z,\partial)$ for $l=-1,0,1,2,\dots$. We note $L^{u^*_{-1}}(u)=L(u)$. Then we have shown in Ōuchi [7]

Theorem 1.15. *Suppose that $\gamma_{\min,L^{u^*_{l-1}}}(q+q_l)\le\gamma$ for all $l\in\boldsymbol{N}$. Then for each $l\in\boldsymbol{N}$ the coefficients $u_n(z')$'s of formal solution $\tilde{u}(z)$ in Theorem 1.13 have the estimate $\{\mathrm{Gev}.\gamma_*\}$ for $\gamma_*=\gamma_{\min,L^{u^*_{l-1}}}(q+q_l)$.*

We have

Theorem 1.16. *Suppose that $L^{\tilde{u}}$ is linearly nondegenerate and $\gamma_{\min,\mathcal{L}^{\tilde{u}}}\le\gamma$. Then the coefficients $u_n(z')$'s of formal solution $\tilde{u}(z)$ in Theorem 1.13 have the estimate $\{\mathrm{Gev}.\gamma_*\}$ for $\gamma_*=\gamma_{\min,\mathcal{L}^{\tilde{u}}}$.*

Theorem 1.17. *Put $\gamma_*=\gamma_{\min,\mathcal{L}^{\tilde{u}}}$ and assume $\gamma_*\le\gamma$ in $\{\mathrm{Eq}\}$. Suppose that*
(1) $L^{\tilde{u}}$ is linearly nondegenerate, and
(2) $L^{\tilde{u}}(u)$ satisfies Condition 1-(i)'s for $1\le i\le p-1$.

Then for any sector $S_1 = S(\phi_{1,-}, \phi_{1,+})$ *with* $\phi_{1,+} - \phi_{1,-} < \pi/\gamma_{1,\mathcal{L}^{\tilde{u}}}$ *and* $S_1 \subset\subset S$, *there is a solution* $u_{S_1}(z) \in Asy^{\mathcal{S}}_{\{\gamma_*\}}(\Omega_{S_1})$ *of* {Eq} *with asymptotic expansion* $u_{S_1}(z) \sim \tilde{u}(z)$ *in* S_1.

The proofs of Theorems 1.16 and 1.17 are given in §5.

Remark 1.18. When $L(u)$ is a linear partial differential operator, say $L(\cdot) = L(z, \partial)$, the relation between solutions of formal power series and genuine solutions of $L(z, \partial)u = g(z)$ was investigated in Ōuchi [5]. The main result in [5], the existence of genuine solutions, follows from Theorem 1.17. The conditions in [5] to ensure it were given by the conditions on the vertices of the characteristic polygon, which are stronger than Conditions 1-(i)'s. So Theorem 1.17 is a generalization of the main result in [5] to not only nonlinear equations but also linear equations.

We give examples. Let

$$L(u) = P_0(z', \partial')\partial_0 u + z_0^J \prod_{i=1}^{2} P_i(z', \partial')u, \tag{1.28}$$

where $J \in \boldsymbol{N}$, $P_i(z', \partial')$ is a linear partial differential operator of ∂' with order m_{P_i} and its principal symbol is denoted by $P.S.P_i(z', \xi')$. We assume

$$\begin{cases} m_{P_2} > m_{P_1} > m_{P_0} + 1 \geq 2 \\ P.S.P_i(0, \hat{\xi}') \neq 0, \ \hat{\xi}' = (1, 0, \cdots, 0), \quad for \ \ i = 0, 1, 2. \end{cases} \tag{1.29}$$

Let us consider $L(u) = g(z)$, where $g(z) = \sum_{n=0}^{+\infty} g_n(z')z_0^n$ is holomorphic in a neighbourhood of $z = 0$. We put $g_n(z') = 0$ for $n < 0$. We concern with a formal solution $\tilde{u}(z)$ with the formal valuation $q \in \boldsymbol{Z}$. So $\mathcal{S} = \boldsymbol{N}$ and $\tilde{u}(z) = z_0^q(\sum_{n=0}^{+\infty} u_n(z')z_0^n)$. If $\tilde{u}(z)$ exists, then

$$\begin{cases} L^{\tilde{u}}(u) = \mathcal{L}^{\tilde{u}}(z, \partial)u + M^{\tilde{u}}(u) \\ \mathcal{L}^{\tilde{u}}(z, \partial) = P_0(z', \partial')\partial_0 + z_0^J(P_2(z', \partial')\tilde{u})P_1(z', \partial') \\ \qquad\qquad + z_0^J(P_1(z', \partial')\tilde{u})P_2(z', \partial') \\ M^{\tilde{u}}(u) = z_0^J \prod_{i=1}^{2}(P_i(z', \partial')u). \end{cases} \tag{1.30}$$

Let $q > -J-1$. Then $u_n(z')$ $(n \geq 0)$ are determined by the following recursion formula,

$$q(P_0(z', \partial')u_0(z')) = g_{q-1}(z') \tag{1.31}$$

and for $n \geq 1$

$$\begin{aligned}&(n+q)P_0(z', \partial')u_n(z')\\ &+ \sum_{\{J+n_1+n_2+q+1=n\}} \prod_{i=1}^{2} P_i(z', \partial')u_{n_i}(z') = g_{n+q-1}(z').\end{aligned} \tag{1.32}$$

Since $P_0(z', \partial')$ is noncharacteristic with respect to $z_1 = 0$, we can find $u_0(z')$ such that $u_0(0) \neq 0$. If $q \geq 0$, $u_n(z')$ $(n \geq 1)$ are successively determined by (1.32), by imposing on (1.32) the initial conditions $\partial_1^h u(0, z'') = 0$ $(0 \leq h \leq m_{P_0} - 1)$. If $-J-1 < q < 0$, $u_{-q}(z')$ is not always determined. But here we assume

$$\sum_{\{J+n_1+n_2+q+1=-q\}} \prod_{i=1}^{2} P_i(z', \partial')u_{n_i}(z') = 0 \tag{1.33}$$

for a suitable choice of $\{u_n(z')\}$ $(0 \leq n < -q)$. Then $u_{-q}(z')$ is arbitrary and we can determine $u_n(z')$ $(n > -q)$ successively by (1.32), imposing on (1.32) the initial conditions $\partial_1^h u(0, z'') = 0$ $(0 \leq h \leq m_{P_0} - 1)$. Let j_i be the the formal valuation of $P_i(z', \partial')\tilde{u}$ $(i = 1, 2)$,

$$P_i(z', \partial')\tilde{u} = z_0^{j_i}(b_0^i(z') + O(z_0)) \tag{1.34}$$

and suppose $j_1 < +\infty$. Then $\mathcal{L}^{\tilde{u}}(z, \partial)$ is a linear operator with order m_{P_2} and $L^{\tilde{u}}(u)$ is linearly nondegenerate. Put $J_{m_{P_1}} = J + j_2$ and $J_{m_{P_2}} = J + j_1$. We have two cases:

(i) If $(J_{m_{P_2}} + 1)/(m_{P_2} - m_{P_0} - 1) > (J_{m_{P_1}} + 1)/(m_{P_1} - m_{P_0} - 1)$, then

$$\begin{aligned}0 = \gamma_{3,\mathcal{L}^{\tilde{u}}} < \gamma_{2,\mathcal{L}^{\tilde{u}}} &= \frac{J_{m_{P_1}} + 1}{m_{P_1} - m_{P_0} - 1}\\ < \gamma_{1,\mathcal{L}^{\tilde{u}}} &= \frac{J_{m_{P_2}} - J_{m_{P_1}}}{m_{P_2} - m_{P_1}} < \gamma_{0,\mathcal{L}^{\tilde{u}}} = +\infty.\end{aligned} \tag{1.35}$$

(ii) If $(J_{m_{P_2}}+1)/(m_{P_2}-m_{P_0}-1) \le (J_{m_{P_1}}+1)/(m_{P_1}-m_{P_0}-1)$, then

$$0=\gamma_{2,\mathcal{L}^{\tilde{u}}}<\gamma_{1,\mathcal{L}^{\tilde{u}}}=\frac{J_{m_{P_2}}+1}{m_{P_2}-m_{P_0}-1}<\gamma_{0,\mathcal{L}^{\tilde{u}}}=+\infty. \tag{1.36}$$

For the case (i) (the case (ii)) we have by Theorem 1.16

$$|u_n(z')| \le AB^n\Gamma(\frac{n}{\gamma_*}+1), \quad \gamma_*=\gamma_{\min,\mathcal{L}^{\tilde{u}}} \tag{1.37}$$

in a neigbourhood of $z'=0$, where $\gamma_{\min,\mathcal{L}^{\tilde{u}}}=\gamma_{2,\mathcal{L}^{\tilde{u}}}$ (resp. $\gamma_{1,\mathcal{L}^{\tilde{u}}}$). Moreover if $b_0^1(0)b_0^2(0)\neq 0$ (resp. $b_0^1(0)\neq 0$) in (1.34), the conditions in Theorem 1.17 are satisfied. Hence for any sector $S_1=S(\phi_{1,-},\phi_{1,+})$ with $\phi_{1,+}-\phi_{1,-}<\pi/\gamma_{1,\mathcal{L}^{\tilde{u}}}$, there exists a solution $u_{S_1}\in Asy_{\{\gamma_*\}}(\Omega_{S_1})$ of $L(u)=g(z)$ with $u_{S_1}(z)\sim\tilde{u}(z)$ in S_1.

Let $q\le -J-1$. Then $u_n(z')$ $(n\ge 0)$ are determined by the following recursion formula,

$$\begin{cases} \displaystyle\prod_{i=1}^{2}(P_i(z',\partial')u_0(z'))=0 \quad if \quad q<-J-1, \\ \displaystyle\prod_{i=1}^{2}(P_i(z',\partial')u_0(z'))+q(P_0(z',\partial')u_0(z'))=0 \quad if \quad q=-J-1, \end{cases} \tag{1.38}$$

and for $n\ge 1$

$$\begin{aligned} &(P_1(z',\partial')u_0(z'))P_2(z',\partial')u_n(z')+(P_2(z',\partial')u_0(z'))P_1(z',\partial')u_n(z') \\ &+\sum_{\left\{\substack{n_1+n_2=n \\ n_i\neq 0}\right\}}\prod_{i=1}^{2}P_i(z',\partial')u_{n_i}(z') \\ &+(n+2q+J+1)P_0(z',\partial')u_{n+q+J+1}(z')=g_{n+2q+J}(z'). \end{aligned} \tag{1.39}$$

Suppose that there is a solution $u_0(z')$ of (1.38) such that $(P_1(z,\partial')u_0(z'))|_{z'=0}\neq 0$. Since $P_2(z',\partial')$ is noncharacteristic with respect to $z_1=0$, $u_n(z')$ $(n\ge 1)$ are successively determined by (1.39), by imposing the initial conditions $\partial_1^h u(0,z'')=0$ $(0\le h\le m_{P_2}-1)$. Then it follows from Ishii [2] and Ōuchi [7] that $z_0^q(\sum_{n=0}^{+\infty}u_n(z')z_0^n)$ converges.

Secondly let

$$L(u) = z_0(\partial_1^2 u)(\partial_1^1 u) + (z_0\partial_0 - a_0 - a(z))u, \tag{1.40}$$

where $a(z) = \sum_{k=1}^{+\infty} a_k(z_1)z_0^k$ is holomorphic at $z = 0$ and $a_0 > -1$. We concern with a formal solution $\tilde{u}(z)$ of $L(u) = 0$ with formal valuation a_0. Let $\mathcal{S}$ be a semi-group generated by $\{1,\ a_0 + 1\}$ and try to find $\tilde{u}(z) = z_0^{a_0}(\sum_{n=0}^{+\infty} u_n(z_1)z_0^{q_n})$. We have

$$\begin{aligned} q_n u_n(z_1) - \sum_{\substack{k+q_{n_0}=q_n \\ k>0}} a_k(z_1)u_{n_0}(z') \\ + \sum_{q_{n_1}+q_{n_2}+a_0+1=q_n} \partial_1^2 u_{n_1}(z')\partial_1^1 u_{n_2}(z') = 0. \end{aligned} \tag{1.41}$$

So we can take $u_0(z')$ arbitrary and $u_n(z')$ $(n \geq 1)$ are successively determined. Then we have

$$\begin{cases} L^{\tilde{u}}(u) = \mathcal{L}^{\tilde{u}}(z,\partial)u + M^{\tilde{u}}(u), \\ \mathcal{L}^{\tilde{u}}(z,\partial) = z_0\partial_1^1\tilde{u}(z)\partial_1^2 + z_0\partial_1^2\tilde{u}(z)\partial_1^1 + z_0\partial_0 - (a_0 + a(z)), \\ M^{\tilde{u}}(u) = z_0(\partial_1^2 u)(\partial_1^1 u). \end{cases} \tag{1.42}$$

Suppose that there is an $N_0 \in \boldsymbol{N}$ such that $\partial_1 u_n(z') \equiv 0$ for all n with $0 \leq n \leq N_0 - 1$ and $\partial_1 u_{N_0}(z') \not\equiv 0$. Then $\mathcal{L}^{\tilde{u}}(z,\partial)$ is a linear operator with order 2 and $0 = \gamma_{2,\mathcal{L}^{\tilde{u}}} < \gamma_{1,\mathcal{L}^{\tilde{u}}} = q_{N_0} + a_0 + 1 < \gamma_{0,\mathcal{L}^{\tilde{u}}} = +\infty$. $L^{\tilde{u}}(u)$ has the strongly linear part with respect to the valuation $(a_0 + N_0)_+$. We have by Theorem 1.16

$$|u_n(z')| \leq AB^n\Gamma(\frac{n}{\gamma_*} + 1), \quad \gamma_* = q_{N_0} + a_0 + 1 \tag{1.43}$$

in a neigbourhood of $z' = 0$. Furthermore assume $\partial_1 u_{N_0}(0) \neq 0$. Then it follows from Theorem 1.17 that for any sector $S_1 = S(\phi_{1,-}, \phi_{1,+})$ with $\phi_{1,+} - \phi_{1,-} < \pi/\gamma_{1,\mathcal{L}^{\tilde{u}}}$, there exists a solution $u_{S_1}(z) \in Asy_{\{\gamma_*\}}(\Omega_{S_1})$ of $L(u) = 0$ with $u_{S_1}(z) \sim \tilde{u}(z)$ *in* S_1.

2. Majorant function

In order to show Theorems we need several estimates. So we make preparations for them. Firstly let us introduce a majorant function which is a modification of that in Lax [3] and Wagschal [8]. Put

$$\theta(t)=\sum_{n=0}^{+\infty}\frac{ct^n}{(n+1)^{m+2}}, \tag{2.1}$$

where $m \in \boldsymbol{N}$.

LEMMA 2.1. *There is a constant $c>0$ in (2.1) such that for $0 \le k' \le k \le m$*

$$\theta^{(k)}(t)\theta^{(k')}(t) \ll \theta^{(k)}(t). \tag{2.2}$$

PROOF. We have

$$\theta^{(k)}(t)=\sum_{n=0}^{+\infty} c\frac{(n+k)(n+k-1)\dots(n+1)}{(n+k+1)^{m+2}}t^n \tag{2.3}$$

and

$$\begin{aligned}&\theta^{(k)}(t)\theta^{(k')}(t)\\ &\quad = c^2\sum_{n=0}^{+\infty}\{\sum_{n_1+n_2=n}\frac{(n_1+k)\dots(n_1+1)}{(n_1+k+1)^{m+2}}\times\frac{(n_2+k')\dots(n_2+1)}{(n_2+k'+1)^{m+2}}\}t^n.\end{aligned}$$

It holds that

$$\frac{B}{(n+1)^{m-k+2}} \le \frac{(n+k)\dots(n+1)}{(n+k+1)^{m+2}} \le \frac{A}{(n+1)^{m-k+2}}, \tag{2.4}$$

where $A, B > 0$ depend on m. We have from (2.4)

$$\begin{aligned}&\sum_{n_1+n_2=n}\frac{(n_1+k)\dots(n_1+1)}{(n_1+k+1)^{m+2}}\times\frac{(n_2+k')\dots(n_2+1)}{(n_2+k'+1)^{m+2}}\\ &\le \sum_{n_1+n_2=n}\frac{A^2}{(n_1+1)^{m-k+2}(n_2+1)^{m-k+2}} \le \frac{C}{(n+1)^{m-k+2}}.\end{aligned}$$

So if we choose $c > 0$ so that $c^2 C \le cB$, (2.2) follows from (2.4). $\square$

We fix $c > 0$ so that (2.2) holds. We have

LEMMA 2.2. *(1)* $\theta^{(k)}(t) \ll \frac{C}{k+1}\theta^{(k+1)}(t)$.
(2) Let $0 \le k' \le k \le m$. *Then*

$$\sum_{i=0}^{n} \binom{n}{i} \theta^{(k+p+n-i)}(t)\theta^{(k'+i+p')}(t) \ll \frac{p!p'!}{(p+p')!}\theta^{(k+n+p+p')}(t). \tag{2.5}$$

PROOF. Let us return to (2.3). We have

$$\frac{(n+k)\dots(n+1)}{(n+k+1)^{m+2}}\frac{(n+k+2)^{m+2}}{(n+k+1)\dots(n+1)} \le \frac{(n+k+2)^{m+2}}{(n+k+1)^{m+3}} \le \frac{C}{n+k+1},$$

which means (1). Differentiate (2.2) $p+p'$ times. Then we have

$$\sum_{i=0}^{p+p'} \binom{p+p'}{i} \theta^{(k+p+p'-i)}(t)\theta^{(k'+i)}(t) \ll \theta^{(k+p+p')}(t)$$

and in particular

$$\frac{(p+p')!}{p!p'!}\theta^{(k+p)}(t)\theta^{(k'+p')}(t) \ll \theta^{(k+p+p')}(t). \tag{2.6}$$

By differentiate (2.6) n times, we have (2.5). $\square$

Put $\varphi_R(t) = \theta(t/R)$, where $0 < R < 1$. We have from Lemma 2.2

PROPOSITION 2.3. *(1)* $\varphi_R^{(k)}(t) \ll \frac{C}{k+1}\varphi_R^{(k+1)}(t)$.
(2) Let $0 \le k' \le k \le m$. *Then*

$$\sum_{i=0}^{n} \binom{n}{i} \varphi_R^{(k+n-i+p)}(t)\varphi_R^{(k'+i+p')}(t) \ll R^{-k'}\frac{p!p'!}{(p+p')!}\varphi_R^{(k+n+p+p')}(t). \tag{2.7}$$

Let us introduce a function space $X^{\gamma}_{p,q,c}(S)$, where $p \in \boldsymbol{N}$, $q, c, \gamma \geq 0$ and $\zeta = (\zeta_0, \zeta') \in (\boldsymbol{R}_+)^{n+1}$.

Definition 2.4. Let $S = \{z_0; |\arg z_0| < \phi, 0 < |z_0| \leq r\}$ and $\omega = \{z' \in \boldsymbol{C}^n;\ \sum_{i=1}^n \zeta_i |z_i| < R\}$. $X^{\gamma}_{p,q,c}(S)$ is the totality of $u(z) \in \mathcal{O}(S \times \omega)$ with the following bounds: There is a constant C such that for all $n \in \boldsymbol{N}$

$$(z_0\partial_0)^n u(z) \zeta_0^n |z_0|^q \exp(-c|z_0|^{-\gamma}) \varphi_R^{(n+p)}(\zeta' \cdot z') \tag{2.8}$$

as a holomorphic function of z'. The norm of $u(z)$ is defined by the infimum of C satisfying the above bounds and denoted by $\|u\|_{p,q,c,\gamma}$.

It is obvious that $X^{\gamma}_{p,q,c}(S)$ is a Banach space and $z_0^r \in X^{\gamma}_{0,r,0}(S)$ ($r \in \boldsymbol{R}$, $\zeta_0 > |r|$). We can define for $u(z) \in X^{\gamma}_{p,q,c}(S)$

$$((z_0\partial_0)^{-1}u)(z) = \int_0^{z_0} \frac{u(t, z')}{t} dt, \tag{2.9}$$

which is also in $\mathcal{O}(S \times \omega)$ and

$$(z_0\partial_0)(z_0\partial_0)^{-1} = (z_0\partial_0)^{-1}(z_0\partial_0) = Id. \tag{2.10}$$

We have from the definition

Proposition 2.5. *(1).* $X^{\gamma}_{p,q,r} \subset X^{\gamma}_{p+1,q,r}$ *and* $\|u\|_{p+1,q,c,\gamma} \leq \frac{C}{p+1}\|u\|_{p,q,c,\gamma}$.
(2). Let $u(z) \in X^{\gamma}_{p,q,c}(S)$. *Then* $(z_0\partial_0)^{\alpha_0}\partial'^{\alpha'} u(z) \in X_{p+|\alpha|,q,c,\gamma}(S)$ *and*

$$\|(z_0\partial_0)^{\alpha_0}\partial'^{\alpha'} u\|_{p+|\alpha|,q,c,\gamma} \leq \zeta^{\alpha}\|u\|_{p,q,c,\gamma}. \tag{2.11}$$

Proposition 2.6. *Let* S^* *be a sector in* $\boldsymbol{C}$ *such that* $S \subset\subset S^*$ *and* ω^* *be a neighbourhood of* $z' = 0$ *such that* $\omega \subset\subset \omega^*$. *Let* $a(z) \in \mathcal{O}(S^* \times \omega^*)$. *If* $a(z)$ *is bounded on* $S^* \times \omega^*$, *then there exists a* $\zeta = \{\zeta_0, \zeta'\} \in \boldsymbol{R}_+^{n+1}$ *such that* $a(z) \in X^{\gamma}_{0,0,0}(S)$.

Proof. Put $S^* = \{z_0;\ |\arg z_0| < \phi^*, 0 < |z_0| \le r^*\}$, where $\phi^* > \phi$ and $r^* > r$, r and ϕ being those in the definition of S. We have by Cauchy's formula

$$\partial_0^n a(z) = \frac{n!}{2\pi i}\int_{\mathcal{C}} \frac{a(t,z')}{(t-z_0)^{n+1}}dt.$$

We choose the circle $|t - z_0| = \delta$ as the integration path $\mathcal{C}$, where $\delta = \min\{|z_0|\sin((\phi^* - \phi)/2), (r^* - r)/2\}$. So we have $|z_0^n \partial_0^n a(z)| \le MC^n n!$ for all $n \in \boldsymbol{N}$, $M = \sup_{z\in S^*\times\omega^*} |a(z)|$. Hence from these estimates, we have $|(z_0\partial_0)^n a(z)| \le MC_1^n n!$ and there exists a $\zeta = \{\zeta_0, \zeta'\} \in \boldsymbol{R}_+^{n+1}$ such that $a(z) \in X_{0,0,0}^{\gamma}(S)$. □

We choose $\zeta \in \boldsymbol{R}_+^{n+1}$ in the following so that Propsition 2.6 holds, if necessary.

Proposition 2.7. *Let $u(z) \in X_{p+k,q,c}^{\gamma}(S)$ and $v(z) \in X_{p'+k',q',c'}^{\gamma}(S)$, where $0 \le k' \le k \le m$. Then $u(z)v(z) \in X_{p+p'+k,q+q',c+c'}^{\gamma}(S)$ and*

$$\|uv\|_{p+p'+k,q+q',c+c',\gamma} \le \frac{p!p'!}{R^{k'}(p+p')!}\|u\|_{p+k,q,c,\gamma}\|v\|_{p'+k',q',c',\gamma}. \tag{2.12}$$

Proof. We have

$$(z_0\partial_0)^n uv = \sum_{i=0}^{n}\binom{n}{i}(z_0\partial_0)^{n-i}u\cdot(z_0\partial_0)^i v.$$

From the definition of $X_{p,q,c}^{\gamma}(S)$ and Proposition 2.3

$$(z_0\partial_0)^n uv \ll R^{-k'}\frac{p!p'!}{(p+p')!}\|u\|_{p+k,q,c,\gamma}\|v\|_{p'+k',q',c',\gamma}$$
$$\times\zeta_0^n|z_0|^{q+q'}\exp(-(c+c')|z_0|^{-\gamma})\varphi_R^{(k+n+p+p')}(\zeta'\cdot z'),$$

which means the statement. □

Corollary 2.8. *(1). Let $\zeta_0 > |r|$. Then*

$$\|z_0^r u\|_{p,q+r,c,\gamma} \le \|z_0^r\|_{0,r,0,\gamma}\|u\|_{p,q,c,\gamma}. \tag{2.13}$$

(2). Suppose that $u(z) \in X^{\gamma}_{p,q,c}(S)$ and $a(z) \in \mathcal{O}(S^* \times \omega^*)$ is bounded, where $S \subset\subset S^*$ and $\omega \subset\subset \omega^*$. Then $a(z)u(z) \in X^{\gamma}_{p,q,c}(S)$ and

$$\|au\|_{p,q,c,\gamma} \le \|a\|_{0,0,0,\gamma}\|u\|_{p,q,c,\gamma} \tag{2.14}$$

The statement (1) is obvious and (2) follows from Proposition 2.6. We have

Proposition 2.9. *Let $u_i(z) \in X^{\gamma}_{p_i,q_i,c_i}(S)$ $(1 \le i \le s)$ and $A \in \mathcal{N}^s$ with $k_A \le m$. Then $\prod_{i=1}^s (z_0\partial_0)^{A_{i,0}} \partial'^{A'_i} u_i(z) \in X^{\gamma}_{p_1+\cdots+p_s+k_A, q_1+\cdots+q_s,c}(S)$ and*

$$\begin{aligned}(p_1+p_2+\cdots+p_s)!\|\prod_{i=1}^s (z_0\partial_0)^{A_{i,0}}\partial'^{A'_i}u_i\|_{p_1+\cdots+p_s+k_A,q_1+\cdots+q_s,c,\gamma} \\ \le C^s R^{-k_A(s-1)}\zeta^A \prod_{i=1}^s p_i!\|u_i\|_{p_i,q_i,c,\gamma}.\end{aligned} \tag{2.15}$$

Proof. We have from Propositions 2.5 and 2.7

$$\begin{aligned}&(p_1+p_2+\cdots+p_s)!\|\prod_{i=1}^s (z_0\partial_0)^{A_{i,0}}\partial'^{A'_i}u_i(z)\|_{p_1+\cdots+p_s+k_A,q_1+\cdots+q_s,c,\gamma} \\ &\quad\le R^{-k_A(s-1)}\prod_{i=1}^s p_i!\|(z_0\partial_0)^{A_{i,0}}\partial'^{A'_i}u_i\|_{p_i+k_A,q_i,c,\gamma} \\ &\quad\le C^s R^{-k_A(s-1)}\zeta^A\prod_{i=1}^s p_i!\|u_i\|_{p_i,q_i,c,\gamma}. \ \square\end{aligned}$$

Proposition 2.10. *(1) Let $u(z) \in X^{\gamma}_{p,q,c}(S)$ $(q > 0)$. Then $(z_0\partial_0)^{-1}u(z) \in X^{\gamma}_{p,q,c}(S)$ and*

$$\|(z_0\partial_0)^{-1}u\|_{p,q,c,\gamma} \le q^{-1}\|u\|_{p,q,c,\gamma} \tag{2.16}$$

(2) Let $u(z) \in X^{\gamma}_{p,q,c}(S)$ $(c,\gamma > 0)$. Then $z_0^{-\gamma}(z_0\partial_0)^{-1}u(z) \in X^{\gamma}_{p,q,c}(S)$ and

$$\|z_0^{-\gamma}(z_0\partial_0)^{-1}u\|_{p,q,c,\gamma} \le \frac{C}{c\gamma}\|u\|_{p,q,c,\gamma}. \tag{2.17}$$

(3) Let $u(z) \in X^{\gamma}_{p,q,c}(S)$ $(c, \gamma > 0)$. *Then* $z_0^{-\gamma}u(z) \in X^{\gamma}_{p+1,q,c}(S)$ *and*

$$\|z_0^{-\gamma}u\|_{p+1,q,c,\gamma} \leq \frac{C}{c\gamma}\|u\|_{p,q,c,\gamma}. \tag{2.18}$$

Proof. Put $v(z) = ((z_0\partial_0)^{-1}u)(z)$. Then

$$(z_0\partial_{z_0})^n v(z) = (z_0\partial_{z_0})^{-1}(z_0\partial_{z_0})^n u(z) = \int_0^{z_0} t^{-1}(t\partial_t)^n u(t, z')dt$$
$$\ll \|u\|_{p,q,c,\gamma}\ \zeta_0^n\ \varphi_R^{(p+n)}(\zeta' \cdot z') \int_0^{|z_0|} |t|^{q-1}\exp(-\frac{c}{|t|^\gamma})dt.$$

Since

$$\int_0^{|z_0|} |t|^{q-1}\exp(-\frac{c}{|t|^\gamma})dt \leq q^{-1}|z_0|^q\exp(-\frac{c}{|z_0|^\gamma}),$$

we have (2.16). Let us show (2). Put $v(z) = z_0^{-\gamma}((z_0\partial_0)^{-1}u)(z)$. Then

$$(z_0\partial_0)^n v(z) = \sum_{i=0}^{n}\binom{n}{i}(-\gamma)^{n-i}z_0^{-\gamma}(z_0\partial_{z_0})^{i-1}u(z).$$

We have

$$z_0^{-\gamma}(z_0\partial_{z_0})^{i-1}u(z) = z_0^{-\gamma}(z_0\partial_{z_0})^{-1}(z_0\partial_{z_0})^i u(z)$$
$$= z_0^{-\gamma}\int_0^{z_0} t^{-1}(t\partial_t)^i u(t, z')dt$$
$$\ll \|u\|_{p,q,c,\gamma}\ \zeta_0^i\ |z_0|^{-\gamma}\ \varphi_R^{(p+i)}(\zeta' \cdot z')$$
$$\cdot \int_0^{|z_0|} |t|^{q-1}\exp(-\frac{c}{|t|^\gamma})dt.$$

Since

$$|z_0|^{-\gamma}\int_0^{|z_0|} \frac{t^{q+\gamma}}{t^{1+\gamma}}\exp(-\frac{c}{t^\gamma})dt \leq |z_0|^q\int_0^{|z_0|}\frac{1}{t^{1+\gamma}}\exp(-\frac{c}{t^\gamma})dt$$
$$= |z_0|^q\frac{1}{c\gamma}\exp(-\frac{c}{|z_0|^\gamma})$$

and $\gamma^{n-i} \ll C\varphi_R^{(n-i)}$, we have (2.17) by Proposition 2.3. We show (3). We have

$$\|z_0^{-\gamma}(z_0\partial_0)^{-1}z_0\partial_0 u\|_{p+1,q,c,\gamma} \leq \frac{C'}{c\gamma}\|z_0\partial_0 u\|_{p+1,q,c,\gamma} \leq \frac{C}{c\gamma}\|u\|_{p,q,c,\gamma},$$

which implies (3). □

3. Construction of solutions with exponential decay I

Now we proceed to prove Theorem 1.8. So we assume that Condition 0 and Condition-1-{i} and construct $u(z)$ with exponential decay satisfying (1.12) or $\{Eq^0\}$. For the simplicity we use the following notations:

$$\left\{\begin{aligned} \gamma = \gamma_{i,\mathcal{L}},\ \gamma^* = \gamma_{i-1,\mathcal{L}}, \\ k^* = k_{i-1},\ L = l_{k_{i-1},\mathcal{L}}, \\ e_{k^*} = e_{k_{i-1},\mathcal{L}},\ e_k = e_{k,\mathcal{L}}. \end{aligned}\right. \tag{3.1}$$

It follows from Proposition 1.6 that $L(u)$ has the strongly linear part with respect to valuation r and we may assume $r = 0$. Hence

$$\begin{cases} e_A - e_{k^*} = \gamma^*(k_A - k^*) + J_A^+ & (J_A^+ \geq 0) \quad for\ k_A > k^* \\ e_A - e_{k^*} = -\gamma(k^* - k_A) + J_A^- & (J_A^- \geq 0) \quad for\ k_A \leq k^*. \end{cases} \tag{3.2}$$

Recall $\mathcal{L}_i(z,\partial)$ introduced in §1. Put

$$\mathfrak{L}(z,\partial) = z_0^{-e_{k^*}}\mathcal{L}_i(z,\partial), \quad \mathfrak{M}(z,\partial^\alpha u) = z_0^{-e_{k^*}}(L(u) - \mathcal{L}_i(z,\partial)u). \tag{3.3}$$

Then

$$\mathfrak{L}(z,\partial) = \sum_{\left\{\substack{(k,l)\ ;\ e(k,l)=e_k \\ e_{k^*}-e_k=\gamma(k^*-k)}\right\}} z_0^{e_k - e_{k^*}} b_{k,l}(z,\partial)(z_0\partial_0)^{k-l}. \tag{3.4}$$

Condition-1-{i} means that $b_{k^*,L}(0,\xi') \not\equiv 0$ and $L > l$ in (3.4). We have

$$\mathfrak{M}(z,\partial^\alpha u) = \mathfrak{M}^+(z,\partial^\alpha u) + \mathfrak{M}^-(z,\partial^\alpha u), \tag{3.5}$$

where

$$\left\{\begin{aligned} \mathfrak{M}^+(z,\partial^\alpha u) &= \sum_{\{A;\ k_A > k^*\}} z_0^{\gamma^*(k_A-k^*)+J_A^+} b_A(z) \prod_{i=1}^{s_A} \partial^{A_i'}(z_0\partial_0)^{A_{i,0}} u \\ & \qquad\qquad J_A^+ \geq 0, \\ \mathfrak{M}^-(z,\partial^\alpha u) &= \sum_{\{A;\ k_A \leq k^*\}} z_0^{-\gamma(k^*-k_A)+J_A^-} b_A(z) \prod_{i=1}^{s_A} \partial^{A_i'}(z_0\partial_0)^{A_{i,0}} u \\ & \qquad\qquad J_A^- > 0, \end{aligned}\right. \tag{3.6}$$

where $J_A^- > 0$ for A appearing in $\mathfrak{M}^-(z, \partial^\alpha u)$ follows from the assumption that $L(u)$ has the strongly linear part with respect to valuation 0. We fix a positive constant δ_-:

$$(3.7) \qquad 0 < \delta_- < \min\{J_A^-; A \textit{ in } \mathfrak{M}^-(z, \partial^\alpha u)\} \quad \textit{with} \quad \gamma^*/\delta_- \in \boldsymbol{N}.$$

If $i = 1$, $\mathfrak{M}_+(z, \partial^\alpha u)$ does not appear, $\gamma^* = +\infty$ $(1/\gamma^* = 0)$ and $0 < \delta_- < \min\{J_A^-; A \textit{ in } \mathfrak{M}^-(z, \partial^\alpha u)\}$. As for sectors in z_0 space we may assume that they are symmetric in the positive real axis. So $S = \{z_0; |\arg z_0| < \theta,\ 0 < |z_0| \le r\}$ and $S' = \{z_0; |\arg z_0| < \theta',\ 0 < |z_0| \le r'\}$, where $0 < \theta' < \theta$ and $0 < r' < r$. Let ω and ω' be neighbourhood of $z' = 0$ such that $\omega' \subset\subset \omega$. Recall $\Omega_S = S \times \omega$. In the sequel $r' > 0$ and ω' are chosen so small, if necessary. Further we may assume that if $i > 1$, $0 < \theta' < \pi/2\gamma_{i-1,\mathcal{L}}$ and $0 < \theta < \pi/2\gamma_{i,\mathcal{L}}$, and if $i = 1$, $0 < \theta' < \theta < \pi/2\gamma_{1,\mathcal{L}}$.

Multiplying $\{\mathrm{Eq}^0\}$ by $z_0^{-e_{k^*}}$ and denoting $z_0^{-e_{k^*}} g(z)$ by $g(z)$, we may consider

$$\{\mathrm{Eq}^*\} \quad \begin{cases} \mathfrak{L}(z, \partial)u + \mathfrak{M}(z, \partial^\alpha u) \equiv g(z) \mod Asy^0_{\{\gamma_{i-1,\mathcal{L}}\}}(\Omega'_{S'}) & \textit{for } i > 1 \\ \mathfrak{L}(z, \partial)u + \mathfrak{M}(z, \partial^\alpha u) = g(z) \quad \textit{for } i = 1 \end{cases}$$

where $g(z) \in Asy^0_{\{\gamma_{i,\mathcal{L}}\}}(\Omega_S)$, so $|g(z)| \le C_0 \exp(-c_0|z_0|^{-\gamma_{i,\mathcal{L}}})$ $(c_0 > 0)$. Thus Theorem 1.8 becomes the following:

THEOREM 3.1. *There exists a solution* $u(z) \in Asy^0_{\{\gamma_{i,\mathcal{L}}\}}(\Omega'_{S'})$ *of* $\{\mathrm{Eq}^*\}$.

The proof of Theorem 3.1 is divided into 4 steps. We try to construct $u(z) = \sum_{n=0}^{+\infty} u_n(z)$, which does not always converge but formally satisfies $\{\mathrm{Eq}^*\}$. In this section we give a system of equations to determine $u_n(z)$, which is the first step. Substituting $u(z)$ into the left hand side of $\{\mathrm{Eq}^*\}$, we have

$$(3.8) \qquad \begin{aligned} &\mathfrak{L}(z, \partial)\Big(\sum_{n=0}^{+\infty} u_n(z)\Big) + \mathfrak{M}\Big(z, \partial^\alpha \sum_{n=0}^{+\infty} u_n(z)\Big) \\ &= \sum_{n=0}^{+\infty} \mathfrak{L}(z, \partial)u_n(z) + \sum_{n=1}^{+\infty} \mathfrak{M}_n(z, u_{n'}(z); n' < n), \end{aligned}$$

where

$$\mathfrak{M}_n(z, u_{n'}(z); n' < n) = \mathfrak{M}_n^+(z, u_{n'}(z); n' < n) + \mathfrak{M}_n^-(z, u_{n'}(z); n' < n)$$

and

$$(3.9)\quad \begin{cases} \mathfrak{M}_n^+(z, u_{n'}(z); n' < n) = \displaystyle\sum_{\{A;\ k_A > k^*\}} {z_0}^{\gamma^*(k_A - k^*) + J_A^+} b_A(z) \\ \qquad \times \Big(\displaystyle\sum_{\left\{ \substack{n_1, \cdots, n_s \\ n_1 + \cdots + n_s + \gamma^*(k_A - k^*)/\delta_- = n} \right\}} \prod_{i=1}^{s} \partial^{A_i'} (z_0 \partial_0)^{A_{i,0}} u_{n_i}(z) \Big) \\ \mathfrak{M}_n^-(z, u_{n'}(z); n' < n) = \displaystyle\sum_{\{A;\ k_A \le k^*\}} {z_0}^{-\gamma(k^* - k_A) + J_A^-} b_A(z) \\ \qquad \times \Big(\displaystyle\sum_{\left\{ \substack{n_1, \cdots, n_s \\ n_1 + \cdots + n_s = n-1} \right\}} \prod_{i=1}^{s} \partial^{A_i'} (z_0 \partial_0)^{A_{i,0}} u_{n_i}(z) \Big). \end{cases}$$

So we try to determine $u_n(z)$ with the following recursion formula:

$$(3.10)\quad \begin{cases} \mathfrak{L}(z, \partial) u_0(z) = g(z) \\ \mathfrak{L}(z, \partial) u_n(z) + \mathfrak{M}_n(z, u_{n'}(z); n' < n) = 0. \end{cases}$$

So the second step is to solve (3.10). Firstly we show the solvability of $\mathfrak{L}(z, \partial) v(z) = g(z)$. Here

$$(3.11)\quad \begin{aligned} \mathfrak{L}(z, \partial) = & \sum_{\left\{ \substack{(k,l)\ ;\ e(k,l) = e_k \\ e_{k^*} - e_k = \gamma(k^* - k)} \right\}} z_0^{e_k - e_{k^*}} b_{k,l}(z, \partial)(z_0 \partial_0)^{k-l} \\ & = b_{k^*,L}(z_0 \partial_0)^{k^* - L} + \Sigma' z_0^{e_k - e_{k^*}} b_{k,l}(z, \partial)(z_0 \partial_0)^{k-l}, \end{aligned}$$

where Σ' means the sum of (k, l) in $\{(k, l);\ (k, l) \neq (k^*, L), e(k, l) = e_k, e_{k^*} - e_k = \gamma(k^* - k)\}$. In this section we use Condition-1-{i}. So it follows from it that $L := l_{k_{i-1}, \mathcal{L}} > l$ in $\mathfrak{L}(z, \partial)$ and we may assume $b_{k^*,L}(0, \hat{\xi}') \neq 0,\ \hat{\xi}' = (1, 0, \ldots, 0)$, that is, the coefficient $b(z)$ of ${\partial_1}^L$ in $b_{k^*,L}(0, \partial')$ does not vanish in a neighbourhood of $z' = 0$. So, multiplying (3.11) by $b(z)^{-1}$, we assume $b(z) = 1$. Let $\zeta' = (\zeta_1, \zeta'')$, where $\zeta_1 > \zeta_i$ for $i > 2$.

Lemma 3.2. *Let* $g(z) \in X^{\gamma}_{p+k^*,\rho,c}(S')$ $(\rho > 0)$. *Then there is a unique solution* $v(z) \in X^{\gamma}_{p+k^*-L,\rho,c}(S')$ *of*

$$(3.12) \qquad \begin{cases} \partial_1^L v(z) = g(z) \\ \partial_1^h v(z)|_{z_1=0} = 0 \quad for \quad 0 \le h \le L-1 \end{cases}$$

such that

$$(3.13) \qquad \|v\|_{p+k^*-L,\rho,c,\gamma} \le {\zeta_1}^{-L} \|g\|_{p+k^*,\rho,c,\gamma}.$$

Proof. Since

$$\partial_1^L \varphi_R^{(n+p+k^*-L)}(\zeta' \cdot z') = \zeta_1^L \varphi_R^{(n+p+k^*)}(\zeta' \cdot z'),$$

we have the estimate. □

Lemma 3.3. *Let* $g(z) \in X^{\gamma}_{p+k^*,\rho,c}(S')$ $(\rho > 0)$. *Then there is a* $\zeta_1 > 1$ *such that*

$$(3.14) \qquad \{b_{k^*,L}(z,\partial') + \Sigma' z_0^{-\gamma(k^*-k)} b_{k,l}(z,\partial')(z_0\partial_0)^{k-l-k^*+L}\} v(z) = g(z)$$

has a solution $v(z) \in X^{\gamma}_{p+k^*-L,\rho,c}(S')$ *with*

$$(3.15) \qquad \|v\|_{p+k^*-L,\rho,c,\gamma} \le C \|g\|_{p+k^*,\rho,c,\gamma}.$$

Proof. Put

$$b'_{k^*,L}(z,\partial') = b_{k^*,L}(z,\partial') - \partial_1^L.$$

Define $v_n(z)$ $(n \ge 0)$:

$$(3.16) \qquad \begin{cases} \partial_1^L v_0(z) = g(z), \\ \partial_1^L v_n(z) = - b'_{k^*,L}(z,\partial') v_{n-1}(z) \\ \qquad\qquad - \Sigma' z_0^{-\gamma(k^*-k)} b_{k,l}(z,\partial')(z_0\partial_0)^{k-l-k^*+L} v_{n-1}(z) \end{cases}$$

with $\partial_1^h v_n(0, z'') = 0$ for $0 \le h \le L-1$. We have by Lemma 3.2

$$\|v_0\|_{p+k^*-L,\rho,c,\gamma} \le \zeta_1^{-L} \|g\|_{p+k^*,\rho,c,\gamma}. \tag{3.17}$$

We show

$$\|v_n\|_{p+k^*-L,\rho,c,\gamma} \le (\frac{C'}{\zeta_1})^{n+1} \|g\|_{p+k^*,\rho,c,\gamma} \tag{3.18}$$

by induction. Then by Propositions 2.10, 2.5 and the inductive hypothesis

$$\begin{aligned} &\|z_0^{-\gamma(k^*-k)} b_{k,l}(z,\partial')(z_0\partial_0)^{k-l-k^*+L} v_{n-1}\|_{p+k^*,\rho,c,\gamma} \\ \le& C_1 \|b_{k,l}(z,\partial')(z_0\partial_0)^{L-l} v_{n-1}\|_{p+k^*,\rho,c,\gamma} \\ \le& C_1 {\zeta_0}^{L-l} {\zeta_1}^{l} \|v_{n-1}\|_{p+k^*-L,\rho,c,\gamma} \le C_2 {C'}^n {\zeta_1}^{-n+L-1} \|g\|_{p+k^*,\rho,c,\gamma}. \end{aligned}$$

Similarily we have

$$\begin{aligned} \|b'_{k^*,L}(z,\partial') v_{n-1}(z)\|_{p+k^*,\rho,c,\gamma} &\le C_3 {\zeta_1}^{L-1} \|v_{n-1}(z)\|_{p+k^*-L,\rho,c,\gamma} \\ &\le C_3 {C'}^n {\zeta_1}^{-n+L-1} \|g\|_{p+k^*,\rho,c,\gamma}. \end{aligned}$$

Hence by Lemma 3.2 we have (3.18) for some $C' > 0$. We take the constant $\zeta_1 > 1$ with $C'\zeta_1^{-1} < 1/2$. So $v(z) = \sum_{n=0}^{+\infty} v_n(z)$ converges and is a desired solution. □

We choose $\zeta_1 > 1$ so that Lemma 3.3 holds.

Lemma 3.4. *Let $v(z) \in X^\gamma_{p,\rho,c}(S')$ $\rho > 0$. Then there is a $u(z) \in X^\gamma_{p,\rho,c}(S')$ such that $(z_0\partial_0)^{k^*-L} u(z) = v(z)$ and*

$$\|u\|_{p,\rho,c,\gamma} \le C\rho^{-k^*+L} \|v\|_{p,\rho,c,\gamma}\ , \tag{3.19}$$

where $C > 0$ is independent of ρ.

Proof. Let $u(z) = (z_0\partial_0)^{-(k^*-L)} v(z)$. Then we have by Propostion 2.10-(1)

$$\|u\|_{p,\rho,c,\gamma} \le \frac{C'}{\rho^{k^*-L}} \|v\|_{p,\rho,c,\gamma}. \ \square$$

By Lemmas 3.3 and 3.4 we have

Proposition 3.5. *Let $g(z) \in X_{p+k^*,\delta n,c,\gamma}(S')$ $(\delta > 0)$. Then there is a $u(z) \in X_{p+k^*-L,\delta n,c,\gamma}(S')$ such that $\mathfrak{L}(z,\partial)u(z) = g(z)$ and*

$$\|u\|_{p+k^*-L,\delta n,c,\gamma} \leq Cn^{-k^*+L}\|g\|_{p+k^*,\delta n,c,\gamma}. \tag{3.20}$$

Thus we can find $u_n(z')$ satisfying (3.10) by Propostion 3.5. Let us obtain the estimate of $u_n(z)$, which is the third step. Recall the definition of δ_- (see (3.7)).

Proposition 3.6. *The follwing estimate holds:*

$$\|u_n\|_{p_n,q_n,c,\gamma} \leq \frac{CB^n}{(1+n^2)((k^*-L)n)!}, \tag{3.21}$$

where $p_n = [(n\delta_-)/\gamma^*] + (k^* - L)n$ and $q_n = n\delta_-$.

Before the proof we note that if $i = 1$, $\gamma^* = +\infty$ and $p_n = (k^* - L)n$.

Proof. Firstly we note that $u_n(z')$ $(n \geq 0)$ are determined by (3.10). Let $0 < c < c_0$. Then inequality (3.21) holds for $n = 0$. We assume $u_r(z) \in X^{\gamma}_{p_r,q_r,c}(S')$ and $\|u_r\|_{p_r,q_r,c,\gamma} \leq AB^r/(1+r^2)((k^*-L)r)!$ for all $r < n$. Put

$$U(z) = b_A(z)\prod_{i=1}^{s_A}\partial^{A'_i}(z_0\partial_0)^{A_{i,0}}u_{n_i}(z), \tag{3.22}$$

$n' = \sum_{i=1}^{s_A} n_i$ $p' = \sum_{i=1}^{s_A} p_{n_i}$ and $q' = \sum_{i=1}^{s_A} q_{n_i}$. Consider $\mathfrak{M}^-_n(z,\ u_r(z);\ r < n)$. So the terms appearing in it are corresponding to $A \in \mathcal{N}^s$ with $k_A \leq k^*$. Since $n' = n-1$, we have $q_n = q' + \delta_-$ and

$$p_n + k_A - k * + L = [\frac{n\delta_-}{\gamma^*}] + (k^* - L)(n-1) + k_A \geq \sum_{i=1}^{s} p_i + k_A = p' + k_A.$$

So it follows from Corollary 2.8 and Propsitions 2.5, 2.9 and 2.10 that

$$\begin{aligned} p'!\|{z_0}^{-\gamma(k^*-k_A)+J_A^-}U(z)\|_{p_n+L,\ q_n,\ c,\gamma} &\le C'p'!\|U(z)\|_{p_n+k_A-k^*+L,\ q',c,\gamma} \\ \le C'p'!\|U(z)\|_{p'+k_A,\ q',c,\gamma} &\le C_1^{s_A+1}\prod_{i=1}^{s_A} p_{n_i}!\|u_{n_i}\|_{p_{n_i},\ q_{n_i},\ c,\gamma}. \end{aligned}$$

It follows from the inductive hypothesis and $\prod_{i=1}^{s_A} p_{n_i}!((k^*-L)n_i)!^{-1} \le p'!((k^*-L)n')!^{-1}$ that

$$\begin{aligned} &\|{z_0}^{-\gamma(k^*-k_A)+J_A^-}U(z)\|_{p_n+L,\ q_n,\ c,\gamma} \\ \le & C_1^{s_A+1}C^{s_A}B^{n'}(p'!)^{-1}\prod_{i=1}^{s_A}\frac{1}{(1+n_i)^2}\prod_{i=1}^{s_A}\frac{p_{n_i}!}{((k^*-L)n_i)!} \\ \le & C_1^{s_A+1}C^{s_A}B^{n'}\frac{1}{((k^*-L)n')!}\prod_{i=1}^{s_A}\frac{1}{(1+n_i)^2}. \end{aligned}$$

Hence

$$\begin{aligned} &\|\mathfrak{M}_n^-(z,u_{n'}(z);n'<n)\|_{p_n+L,\ q_n,\ c,\gamma} \\ \le & \sum_A \sum_{\left\{\substack{n_1,\cdots,n_{s_A} \\ n_1+n_2+\cdots+n_{s_A}=n-1}\right\}} C_1^{s_A+1}C^{s_A}B^{n'}\frac{1}{((k^*-L)n')!}\prod_{i=1}^{s_A}\frac{1}{1+n_i^2} \\ \le & \frac{K_1C^MB^{n-1}}{(1+n^2)((k^*-L)(n-1))!}. \ \square \end{aligned}$$

Now consider $\mathfrak{M}_n^+(z,u_{n'}(z);n'<n)$. So the terms appearing in it are corresponding to $A\in\mathcal{N}^s$ with $k_A>k^*$. Since $n'=n-\gamma^*(k_A-k^*)/\delta_-$, we have $q_n=n\delta_-\le q'+\gamma^*(k_A-k^*)$ and

$$\begin{aligned} p_n+L-(p'+k_A) =& [\frac{n\delta_-}{\gamma^*}]+(k^*-L)n+L-p'-k_A \\ \ge& [\frac{n'\delta_-}{\gamma^*}]+(k_A-k^*)+(k^*-L)n+L-p'-k_A \\ \ge& [\frac{n'\delta_-}{\gamma^*}]+(k^*-L)(n-1)-p' \ge (k^*-L)(n-n'-1). \end{aligned}$$

Hence it follows from Corollary 2.8 and Propsitions 2.5, 2.9 and 2.10 that

$$
\begin{aligned}
&(\sum_{i=1}^{s_A}[n_i\delta_*/\gamma^*] + (k^*-L)(n-1))!\ \|{z_0}^{\gamma^*(k_A-k^*)+J_A^+}U(z)\|_{p_n+L,q_n,\ c,\gamma} \\
\le & C(\sum_{i=1}^{s_A}[n_i\delta_*/\gamma^*] + ((k^*-L)n')!\ \|{z_0}^{\gamma^*(k_A-k^*)+J_A^+}U(z) \\
&\cdot \|_{p'+k_A,\ q'+\gamma^*(k_A-k^*),\ c,\gamma} \\
\le & C'p'!\ \|U(z)\|_{p'+k_A,\ q',c,\gamma} \le C_1^{s_A+1}\prod_{i=1}^{s_A} p_{n_i}!\|u_{n_i}\|_{p_{n_i},\ q_{n_i},\ c,\gamma}
\end{aligned}
$$

and by the similar method to the preceding

$$
\|\mathfrak{M}_n^+(z,u_{n'}(z);n'<n)\|_{p_n+L,\ q_n,c,\gamma} \le \frac{K_1 C^M B^{n-1}}{(1+n^2)((k^*-L)(n-1))!}.
$$

Thus we have

$$
\|\mathfrak{M}_n(z,u_{n'}(z);n'<n)\|_{p_n+L,\ q_n,c,\gamma} \le \frac{K' B^{n-1}}{(1+n^2)((k^*-L)(n-1))!}.
$$

Hence by (3.10) and Proposition 3.5 we have for a large $B>0$

$$
\begin{aligned}
\|u_n\|_{p_n,\ q_n,\ c,\gamma} \le & C'n^{-k^*+L}\|\mathfrak{M}_n(z,u_{n'}(z);n'<n)\|_{p_n+L,\ q_n,\ c,\gamma} \\
\le & \frac{KC^M B^{n-1}}{(1+n^2)((k^*-L)(n-1))!n^{k^*-L}} \le \frac{CB^n}{(1+n^2)((k^*-L)n)!}.
\end{aligned}
$$

By Proposition 3.6 if $i=1$,

$$
u_n(z) \ll \|u_n\|_{p_n,\ q_n,\ c,\gamma}|z_0|^{n\delta_-}\exp(-c|z_0|^{-\gamma})\varphi_R^{(k^*-L)n}(\zeta'\cdot z'), \tag{3.23}
$$

which means

$$
|u_n(z)| \le CB^n|z_0|^{n\delta_-}\exp(-c|z_0|^{-\gamma}) \tag{3.24}
$$

in a neighbourhood ω' of $z'=0$. So $\sum_{n=0}^{+\infty} u_n(z)$ converges in $S'\times\omega'$, where $S'=\{z_0; 0<|z_0|\le r',\ |\arg z_0|<\theta'\}$ for small $r'>0$.

If $i \geq 2$, by Proposition 3.6,

$$|u_n(z)| \leq CB^n |z_0|^{n\delta_-} \exp(-c|z_0|^{-\gamma})\Gamma(n\delta_-/\gamma^* + 1) \tag{3.25}$$

Remark 3.7. We assume Condition-1-(i) in order that there is a solution $v(z)$ of

$$\mathfrak{L}(z, \partial)v(z) = g(z), \tag{3.26}$$

which has a good estimate (Propositions 3.5 and 3.6). So if we put another condition to assure the existence of a solution of (3.26) with a good estimate, we will be able to obtain results similar to Theorems 1.8 and 1.9.

4. Construction of solutions with exponential decay II

Now we proceed to the fourth step and complete the proof of Theorem 3.1 for $i > 1$. So suppose $\gamma^* < +\infty$. For the proof we need lemmas concering fuctions with zero asymtotic exapnsion. In the following of this sections functions $\{u_n(z)\}_{n \in \boldsymbol{N}}$ means that constructed in §3. As stated in §1 we denote different constants by the same notations, if confusions will not occur. ω is a neighbourhood of $z' = 0$ and $\Omega_S = S \times \omega$. Recall $S' = \{z_0; |\arg z_0| \leq \theta',\ 0 < |z_0| \leq r'\}$ be a sector such that $S' \subset\subset S$, where $0 < \theta' < \pi/2\gamma^*$ and $r' > 0$ is chosen small, if necessary.

Let us define

$$\begin{cases} \hat{u}_n(z, \xi) = (\dfrac{u_n(z)}{{z_0}^{n\delta_- + \gamma^*}}) \dfrac{\xi^{n\delta_-/\gamma^*}}{\Gamma(n\delta_-/\gamma^* + 1)}, \\ \hat{u}_N(z, \xi) = \displaystyle\sum_{n=N+1}^{+\infty} \hat{u}_n(z, \xi), \\ \hat{u}(z, \xi) = \displaystyle\sum_{n=0}^{+\infty} \hat{u}_n(z, \xi). \end{cases} \tag{4.1}$$

It follows from the estimate (3.25) that $\hat{u}(z, \xi)$ and $\hat{u}_N(z, \xi)$ converges on $\Omega_S \times \{|\xi| \leq \hat{\xi}_0\}$ for some $\hat{\xi}_0 > 0$. We have

Lemma 4.1. *There exist* $\hat{\xi}$, $\hat{\xi}_0 > \hat{\xi} > 0$, B *and* B_1 *such that*

$$|\hat{u}_N(z, \xi)| \leq AB^{N+1} |z_0|^{-\gamma^*} \exp(-c|z_0|^{-\gamma}) |\xi|^{(N+1)\delta_-/\gamma^*} \tag{4.2}$$

on $\Omega_S \times \{|\xi| \le \hat{\xi}\}$ *and*

$$\sum_{n=0}^{N} |\hat{u}_n(z,\xi)| \le AB_1^{N+1}|z_0|^{-\gamma^*} \exp(-c|z_0|^{-\gamma})|\xi|^{(N+1)\delta_-/\gamma^*} \tag{4.3}$$

on $\Omega_S \times \{|\xi| \ge \hat{\xi}\}$.

Proof. We have easily the first inequality. By (3.25) there are $\hat{\xi}$ and $B_1 > B$ such that if $|\xi| \ge \hat{\xi}$

$$\begin{aligned}\sum_{n=0}^{N} |\hat{u}_n(z,\xi)| &\le A|z_0|^{-\gamma^*} \exp(-c|z_0|^{-\gamma})(B_1|\xi|^{\delta_-/\gamma^*})^N \sum_{n=0}^{N} (B_1|\xi|^{\delta_-/\gamma^*})^{n-N} \\ &\le A|z_0|^{-\gamma^*} \exp(-c|z_0|^{-\gamma})(B_1|\xi|^{\delta_-/\gamma^*})^{N+1}. \ \square\end{aligned}$$

Proposition 4.2. *There is a* $u_{S'}(z) \in Asy_{\{\gamma^*\}}(\Omega_{S'})$ *such that*

$$\begin{aligned}&|u_{S'}(z) - \sum_{n=0}^{N} u_n(z)| \\ &\le AB^{N+1}\Gamma(\frac{(N+1)\delta_-}{\gamma^*} + 1)|z_0|^{(N+1)\delta_-} \exp(-c|z_0|^{-\gamma})\end{aligned} \tag{4.4}$$

in $\Omega_{S'}$.

Proof. Define

$$u_{S'}(z) = \int_0^{\hat{\xi}} \exp(-\xi z_0^{-\gamma^*})\hat{u}(z,\xi)d\xi. \tag{4.5}$$

We have

$$\begin{aligned}&u_{S'}(z) - \sum_{n=0}^{N} u_n(z) \\ &= \int_0^{\hat{\xi}} \exp(-\xi z_0^{-\gamma^*})\hat{u}(z,\xi)d\xi - \int_0^{+\infty} \exp(-\xi z_0^{-\gamma^*}) \sum_{n=0}^{N} \hat{u}_n(z,\xi)d\xi \\ &= \int_0^{\hat{\xi}} \exp(-\xi z_0^{-\gamma^*})\hat{u}_N(z,\xi)d\xi - \int_{\hat{\xi}}^{+\infty} \exp(-\xi z_0^{-\gamma^*}) \sum_{n=0}^{N} \hat{u}_n(z,\xi)d\xi \\ &= I_{1,N} + I_{2,N}.\end{aligned}$$

From lemma 4.1 if $z_0 \in S'$,

$$
\begin{aligned}
|I_{1,N}| \le & AB^{N+1}|z_0|^{-\gamma^*}\exp(-c|z_0|^{-\gamma})\int_0^{\hat{\xi}} \exp(-c'\xi|z_0|^{-\gamma^*})\xi^{(N+1)\delta_-/\gamma^*}d\xi \\
\le & AB_2^N\Gamma((N+1)\delta_-/\gamma^*+1)\exp(-c|z_0|^{-\gamma})|z_0|^{(N+1)\delta_-/\gamma^*}
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{2,N}| \le & AB_1^{N+1}|z_0|^{-\gamma^*}\exp(-c|z_0|^{-\gamma})\int_{\hat{\xi}}^{+\infty} \exp(-c'\xi|z_0|^{-\gamma^*})\xi^{(N+1)\delta_-/\gamma^*}d\xi \\
\le & AB_2^{N+1}\Gamma((N+1)\delta_-/\gamma^*+1)\exp(-c|z_0|^{-\gamma})|z_0|^{(N+1)\delta_-/\gamma^*}.
\end{aligned}
$$

This completes the proof. □

We have

Proposition 4.3. *Let $u_{S'}(z)$ be a function defined by* (4.5) *and $g_{S'}(z) = L(u_{S'}) - g(z)$. Then $g_{S'}(z) \in Asy^0_{\{\gamma^*\}}(\Omega_{S'})$.*

Put $v_N(z) = \sum_{n=0}^N u_n(z)$ and $w_N(z) = u_{S'}(z) - v_N(z)$. Then we have

$$
\begin{aligned}
g_{S'}(z) = & L(u_{S'}) - g(z) \\
= & \{\mathfrak{L}(z,\partial)(v_N + w_N) + \mathfrak{M}(z,\partial^\alpha v_N) - g(z)\} \\
& + \{\mathfrak{M}(z,\partial^\alpha(v_N+w_N)) - \mathfrak{M}(z,\partial^\alpha v_N)\} \\
= & \{\mathfrak{M}(z,\partial^\alpha v_N) - \sum_{n=1}^N \mathfrak{M}_n(z,u_{n'};n'<n)\} \\
& + \{\mathfrak{L}(z,\partial)w_N + \mathfrak{M}(z,\partial^\alpha(v_N+w_N)) - \mathfrak{M}(z,\partial^\alpha v_N)\} \\
= & J_{1,N} + J_{2,N}.
\end{aligned} \tag{4.6}
$$

So in order to show Proposition 4.3 we estimate $J_{1,N}$ and $J_{2,N}$. Put $v_{N,n}(z) = u_n(z)$ $(0 \le n \le N)$ and $v_{N,n}(z) = 0$ $(n \ge N+1)$. Then $v_N(z) = \sum_{n=0}^{+\infty} v_{N,n}(z)$ and $J_{1,N} = \sum_{n\ge N+1}\mathfrak{M}_n(z, v_{N,n'}; n'<n)$. We have

Lemma 4.4. *There are c_1, $c_2 > 0$ such that if $c_1/(N+1) \le |z_0|^{\gamma^*} \le c_1/N$,*

$$
|J_{1,N}|,\ |J_{2,N}| \le AC^M \exp(-c_2/|z_0|^{\gamma^*}). \tag{4.7}
$$

PROOF. Firstly we show the inequality for $J_{1,N}$. Put

$$V^N(z) = b_A(z)\prod_{i=1}^{s_A}\partial^{A_i}(z_0\partial_0)^{A_{i,0}}v_{n_i}^N(z).$$

Note $p_n = [n\delta_-/\gamma^*] + (k-L)n$ and $n' = \sum_{i=1}^{s_A} n_i$. Then we have, by the same way as in the proof of Proposition 3.6 for $k_A \le k^*$

$$\begin{aligned}&\|z_0^{e_A-e_{k^*}}V^N\|_{p_n+L,\ q_n,c,\gamma}\\ \le& C_1^{s_A+1}B^{s_A}C^{n'}(p'!)^{-1}\prod_{i=1}^{s_A}\frac{1}{(1+n_i^2)}\prod_{i=1}^{s_A}\frac{p_{n_i}!}{((k^*-L)n_i)!}\end{aligned}$$

Thus we have

$$|z_0^{e_k-e_{k^*}}V^N(z)| \le C_1^{s_A+1}B^{s_A}\tilde{C}^{n'}|z_0|^{n\delta_-}e^{-c|z_0|^{-\gamma}}\prod_{i=1}^{s_A}\frac{1}{(1+n_i^2)}\prod_{i=1}^{s_A}(n_i\delta_-/\gamma^*)!.$$

By Stiring's formula we have

$$(\frac{n_i\delta_-}{\gamma^*})! \le C'(\frac{n_i\delta_-}{\gamma^*})^{n_i\delta_-/\gamma^*+1/2}e^{-(n_i\delta_-)/\gamma^*}$$

and

$$\prod_{i=1}^{s_A}(\frac{n_i\delta_-}{\gamma^*})! \le C'^{s_A}e^{-n'\delta_-/\gamma^*}\prod_{i=1}^{s_A}(\frac{n_i\delta_-}{\gamma^*})^{n_i\delta_-/\gamma^*+1/2}.$$

Hence

$$\begin{aligned}|z_0^{e_A-e_{k^*}}V^N(z)| \le& C_1^{s_A+1}B^{s_A}C_2^{n'}\exp(-c|z_0|^{-\gamma}-n'\delta_-/\gamma^*)\\ &\times|z_0|^{n\delta_-}\prod_{i=1}^{s_A}\frac{1}{1+n_i^2}\prod_{i=1}^{s_A}(\frac{n_i\delta_-}{\gamma^*})^{n_i\delta_-/\gamma^*+1/2}\end{aligned} \tag{4.8}$$

Let $c_1/(N+1) \le |z_0|^{\gamma^*} \le c_1/N$, where $c_1 > 0$ is chosen small later. Let $0 < \epsilon < 1$. If $n_i \le N$, we have for small $c_1 > 0$

$$|z_0|^{n\delta_-}\prod_{i=1}^{s_A}(\frac{n_i\delta_-}{\gamma^*})^{n_i\delta_-/\gamma^*+1/2} \le (\frac{c_1}{N})^{n\delta_-/\gamma^*}\prod_{i=1}^{s_A}(\frac{n_i\delta_-}{\gamma^*})^{n_i\delta_-/\gamma^*+1/2} \le A\epsilon^n.$$

Since $v_{N,n}(z') = 0$ for $n \geq N+1$, if $c_1/(N+1) \leq |z_0|^{\gamma^*} \leq c_1/N$, we have

$$(4.9) \quad |z_0^{e_A - e_{k^*}} V^N(z)| \leq B_1^{s_A+1} C_2^{n'} \epsilon^n \exp(-c|z_0|^{-\gamma} - n'\delta_-/\gamma^*) \prod_{i=1}^{s_A} \frac{1}{1+n_i^2}$$

and by the same method as in the proof of Proposition 3.6

$$(4.10) \quad |\mathfrak{M}_n^-(z, v_{N,n'}; n' < n)| \leq B_1^{s_A+1} C_2^{n-1} \epsilon^n \exp(-c|z_0|^{-\gamma} - n'\delta_-/\gamma^*).$$

We also have an estimate for $\mathfrak{M}_n^+(z, v_{N,n'}; n' < n)$ similar to (4.10). Thus if $c_1/(N+1) \leq |z_0|^{\gamma^*} \leq c_1/N < 1$,

$$(4.11) \qquad |\mathfrak{M}_n(z, v_{N,n'}; n' < n)| \leq K C_2^{n-1} \epsilon^n \exp(-c|z_0|^{-\gamma} - n'\delta_-/\gamma^*).$$

We choose $c_1 > 0$ so small that $\epsilon(\exp(-\delta_-/\gamma^*)C_2)^n < 1/2$. Hence if $c_1/(N+1) \leq |z_0|^{\gamma^*} \leq c_1/N$,

$$\begin{aligned}
|J_{1,N}| &\leq \sum_{n=N+1}^{+\infty} |\mathfrak{M}_n(z, v_{N,n'}; n' < n)| \\
&\leq K \exp(-c|z_0|^{-\gamma^*}) \sum_{n=N+1}^{+\infty} \exp(-n\delta_-/\gamma) C_2^{n-1} \epsilon^n \\
&\leq K' 2^{-N-1} \exp(-c|z_0|^{-\gamma}) \\
&\leq K' \exp(-c_2|z_0|^{-\gamma^*} - c|z_0|^{-\gamma}).
\end{aligned}$$

Secondly we show the inequality for $J_{2,N}$. We have

$$(4.12) \qquad J_{2,N} = \mathfrak{L}(z, \partial) w_N + \sum_{|\alpha| \leq m} \partial^\alpha w_N \int_0^1 \partial_{p_\alpha} \mathfrak{M}(z, \partial^\alpha(v_N + \theta w_N)) d\theta.$$

It follows from Proposition 4.2 that for $0 < c' < c$, c being that in Proposition 4.2,

$$(4.13) \qquad |J_{2,N}| \leq A' B^{N+1} \Gamma(\frac{(N+1)\delta_-}{\gamma^*} + 1) |z_0|^{(N+1)\delta_-} \exp(-c'|z_0|^{-\gamma}).$$

If $c_1/(N+1) \le |z_0|^{\gamma^*} \le c_1/N$, where $c_1 > 0$ is chosen so small, we have

$$\begin{aligned}|J_{2,N}| \le & A'B^{N+1}\Gamma(\frac{(N+1)\delta_-}{\gamma^*}+1)(c_1/N)^{(N+1)\delta_-/\gamma^*}\exp(-c'|z_0|^{-\gamma}) \\ \le & A'B^{N+1}(\frac{(N+1)\delta_-}{\gamma^*})^{N\delta_-/\gamma^*+1/2}e^{-N\delta_-/\gamma^*} \\ & \cdot (c_1/N)^{(N+1)\delta_-/\gamma^*}\exp(-c'|z_0|^{-\gamma}) \\ \le & A'2^{-N-1}(\frac{N+1}{N})^{(N+1)\delta_-/\gamma^*}\exp(-c'|z_0|^{-\gamma}) \\ \le & A'\exp(-c_2|z_0|^{-\gamma^*}-c'|z_0|^{-\gamma}).\end{aligned}$$

Thus we have (4.7). □

By Lemma 4.4 for $c_1/(N+1) \le |z_0|^{\gamma^*} \le c_1/N$

$$|g_{S'}(z)| \le A\exp(-c_2|z_0|^{-\gamma^*}), \tag{4.14}$$

where c_1 and A are independent of N. So we have $g_{S'}(z) \in Asy^{\,0}_{\{\gamma^*\}}(\Omega'_S)$ and this completes the proof of Proposition 4.3.

Thus we have Theorem 3.1 for $i > 1$ from Proposition 4.3 and as mentioned before Theorem 1.8 follows from Theorem 3.1.

Proof of Theorem 1.9. Let $S_i = \{z_0;\ |\arg z_0| \le \theta_i,\ 0 < |z_0| \le r_i\}$ $(1 \le i \le i^*)$, where $\theta_1 < \ldots < \theta_{i^*}$ and $0 < \theta_i < \pi/2\gamma_i$. If $i^* = 1$, Theorem 1.9 follows from Theorem 1.8-(2). Suppose that $i^* \ge 2$. It follows from Theorem 1.8-(1) that there exist $u_{i^*}(z) \in Asy^{\,0}_{\{\gamma_{i^*}\}}(\Omega_{S_{i^*-1}})$ and $g_{i^*-1}(z) \in Asy^{\,0}_{\{\gamma_{i^*-1}\}}(\Omega_{S_{i^*-1}})$ such that

$$L(u_{i^*}) = g(z) - g_{i^*-1}(z). \tag{4.15}$$

Put $L^{<i^*>}(u) = L(u)$ and $g_{i^*}(z) = g(z)$. Suppose that there exist $u_i(z) \in Asy^{\,0}_{\{\gamma_i\}}(\Omega_{S_{i-1}})$, $g_{i-1}(z) \in Asy^{\,0}_{\{\gamma_{i-1}\}}(\Omega_{S_{i-1}})$ and nonlinear operators $L^{<i>}(u)$ for $k+1 \le i \le i^*$ such that

$$L^{<i>}(u_i) = g_i(z) - g_{i-1}(z). \tag{4.16}$$

Define

$$L^{<k>}(u) = L^{<k+1>}(u+u_{k+1}) - L^{<k+1>}(u_{k+1}).$$

Consider the equation

$$L^{<k>}(u) = g_k(z). \tag{4.17}$$

It follows again from Theorem 1.8 that there exist $u_k(z) \in Asy^{0}_{\{\gamma_k\}}(\Omega_{S_{k-1}})$ and $g_{k-1}(z) \in Asy^{0}_{\{\gamma_{k-1}\}}(\Omega_{S_{k-1}})$ such that

$$L^{<k>}(u_k) = g_k(z) - g_{k-1}(z). \tag{4.18}$$

By repeating the method, we arrive at the following equation,

$$L^{<1>}(u) = g_1(z), \tag{4.19}$$

where $g_1(z) \in Asy^{0}_{\{\gamma_1\}}(\Omega_{S_1})$. The equation (4.19) has a solution $u_1(z) \in Asy^{0}_{\{\gamma_1\}}(\Omega_{S_1})$ by Theorem 1.8-(2). Hence we have

$$\sum_{i=1}^{i^*} L^{<i>}(u_i) = \sum_{i=1}^{i^*}(g_i(z) - g_{i-1}(z)) = g(z) \quad (g_0(z) = 0).$$

Since $\sum_{i=1}^{i^*} L^{<i>}(u_i) = L(\sum_{i=1}^{i^*} u_i(z)) = g(z)$, $u_{S'} = \sum_{i=1}^{i^*} u_i(z)$ satisfies $L(u_{S'}) = g(z)$. Thus we have Theorem 1.9. □

5. Proofs of Propositions and Theorems

In §5 we give the proofs of Propositions 1.6, 1.7, 1.11, Theorems 1.16 and 1.17.

Proof of Proposition 1.6. Put $R(u) = L(z_0^r u)$. Then

$$R(u) = \sum_{s=1}^{M} \sum_{\{A;\ s_A = s\}} z_0^{e_A + s_A r} b_A(z) \prod_{i=1}^{s} \partial'^{A'_i}(z_0\partial_0 + r)^{A_{i,0}} u, \tag{5.1}$$

from which we have $\Sigma^*_L(r'+r) = \Sigma^*_R(r')$, $\Sigma^*_{\mathcal{R}} = \Sigma^*_{\mathcal{L}} + (0, r)$, $\mathcal{R}_i(z, z_0\partial_0, \partial') = z_0^r \mathcal{L}_i(z, z_0\partial_0, \partial')$ for $1 \le i \le p-1$ and $\mathcal{R}_p(z, z_0\partial_0, \partial') = z_0^r \mathcal{L}_p(z, z_0\partial_0 + r, \partial')$.

Hence the following (i) and (ii) are equivalent: (i) $L(u)$ has the strongly linear part with respect to valuation r, (ii) $R(u)$ has the stronly linear part with respect to valuation 0. We show (2). Let $L(u)$ is an operator with order m. Hence $e_A \neq +\infty$ for some $A \in \mathcal{N}^M$ with $k_A = m$. So if $L(u)$ has the strongly linear part with respect to valuation ρ_+, we have $e_m \neq +\infty$. Conversely supppose that $L(u)$ is linearly nondegenerate, that is, $e_m \neq +\infty$. Then if $s_A \geq 2$, $s_A r + e_A > r + e_m$ for large r and this implies $\Sigma_M^*(r) \subset\subset \Sigma_{\mathcal{L}}^*(r)$ and ρ equals to the infimum of such r. □

Let us proceed to show Proposition 1.7. For $v(z) \in \mathcal{F}$ and $v(z) = \sum_{n=0}^{+\infty} v_n(z')z_0^{r_n}$ we have defined an formal operator $L^v(u) := L(u+v) - L(v)$ and

$$v_{-1}^*(z) = 0, \quad v_l^*(z) = \sum_{n=0}^{l} v_n(z')z_0^{r_n} \quad for \;\; l \in \boldsymbol{N}. \tag{5.2}$$

Let us give the forms of $L^v(u)$ and $\mathcal{L}^v(z,\partial)$ more concterely. Let $A = (A_1, A_2, \ldots, A_{s_A}) \in \mathcal{N}^M$, $\mathcal{I}$ be a subset of $\{1, 2, \ldots, s_A\}$ and $|\mathcal{I}|$ be the cardinal number of the set $|I|$. By putting

$$\begin{cases} L_A^v(u) = z_0^{e_A} b_A(z) \\ \qquad \cdot \{ \displaystyle\sum_{\{\mathcal{I};|\mathcal{I}|\geq 1\}} (\prod_{h\notin\mathcal{I}} \partial'^{A'_h}(z_0\partial_0)^{A_{h,0}} v)(\prod_{i\in\mathcal{I}} \partial'^{A'_i}(z_0\partial_0)^{A_{i,0}} u)\} \\ \mathcal{L}_A^v(z,\partial) = z_0^{e_A} b_A(z) \{ \displaystyle\sum_{i=1}^{s_A} (\prod_{h\neq i} \partial'^{A'_h}(z_0\partial_0)^{A_{h,0}} v) \partial'^{A'_i}(z_0\partial_0)^{A_{i,0}} \} \\ M_A^v(u) = L_A^v(u) - \mathcal{L}_A^v(z,\partial)u, \end{cases} \tag{5.3}$$

we have

$$\begin{aligned} &L^v(u) = \sum_A L_A^v(u), \quad \mathcal{L}^v(z,\partial) = \sum_A \mathcal{L}_A^v(z,\partial), \\ &M^v(u) = \sum_A M_A^v(u). \end{aligned} \tag{5.4}$$

Proof of Proposition 1.7. We have from (5.3)

$$\begin{cases} L^v(u) = \displaystyle\sum_A c_A(z, v(z)) \prod_{i=1}^{s_A} \partial'^{A'_i}(z_0\partial_0)^{A_{i,0}} u \\ \mathcal{L}^v(z,\partial) = \displaystyle\sum_{\{|\alpha|\leq m\}} l_\alpha(z, v(z)) \partial^{\alpha'}(z_0\partial_0)^{\alpha_0}. \end{cases} \tag{5.5}$$

Let $e(l_\alpha(v))$ and $e(c_A(v))$ be the formal valuation of $l_\alpha(z,v(z))$ and that of $c_A(z,v(z))$ respectively. From the assumption $e_m := \min_{\{\alpha;\ |\alpha|=m\}} e(l_\alpha(v)) < +\infty$. Let us consider only α with $e(l_\alpha(v)) < +\infty$ and $A \in \mathcal{N}^M$ with $e(c_A(v)) < +\infty$. For such α and A there is an N_0 such that for $N \geq N_0$

$$e(c_A(v)) = e(c_A(v_N^*)), \quad e(l_\alpha(v)) = e(l_\alpha(v_N^*)). \tag{5.6}$$

The first assertion follows from (5.6). Since $L^v(u)$ is linearly nondegenerate, by Proposition 1.6 there is a ρ such that $\Sigma^*_{L^v}(r) = \Sigma^*_{\mathcal{L}^v}(r)$ for any $r > \rho$. Thus we have the second assertion. □

Proof of Proposition 1.11. We may assume $S = S(\pi/2\kappa)$. Put

$$\hat{f}(z',\xi) = \sum_{n=0}^{+\infty} \frac{f_n(z')\xi^{q_n/\kappa}}{\Gamma(q_n/\kappa+1)}, \tag{5.7}$$

which converges in $\{\xi; 0 < |\xi| \leq \hat{\xi}_0\}$ for some $\hat{\xi}_0 > 0$. Define for $0 < \hat{\xi} < \hat{\xi}_0$

$$f(z) = z_0^{-\kappa} \int_0^{\hat{\xi}} \exp(-\xi z_0^{-\kappa}) \hat{f}(z',\xi) d\xi. \tag{5.8}$$

It is obvious that $f(z) \in \mathcal{O}(\Omega_S)$ and we can show $f(z) \in Asy^{\mathcal{S}}_{\{\kappa\}}(\Omega_S)$ with asymptotic expasion $f(z) \sim \sum_{n=0}^{+\infty} f_n(z') z_0^{q_n}$ by the way similar to the proof of Proposition 4.2. □

Before showing Theorem 1.16, we give

Proposition 5.1. *Let $\tilde{u}(z)$ be that in Theorem 1.13. Suppose that $L^{\tilde{u}}(u)$ is linearly nondegenerate. Then there is an $N \in \boldsymbol{N}$ such that $\gamma_{\min,\mathcal{L}^{\tilde{u}}} = \gamma_{\min,\mathcal{L}^{u_N^*}} = \gamma_{\min,L^{u_{N-1}^*}}(q+q_N)$.*

Proof. Let N_0 and ρ be those in Proposition 1.7. We choose N such that $N-1 \geq N_0$ and $q+q_N > \rho$. Then it follows from Proposition 1.7 that

$$\Sigma^*_{L^{\tilde{u}}}(q+q_N) = \Sigma^*_{\mathcal{L}^{\tilde{u}}}(q+q_N) = \Sigma^*_{L^{u^*_{N-1}}}(q+q_N) = \Sigma^*_{\mathcal{L}^{u^*_{N-1}}}(q+q_N). \tag{5.9}$$

This means $\gamma_{\min,\mathcal{L}^{\tilde{u}}} = \gamma_{\min,\mathcal{L}^{u^*_N}} = \gamma_{\min,L^{u^*_{N-1}}}(q + q_N)$. □

PROOF OF THEOREM 1.16. By combinig Theorem 1.15 with Proposition 5.1, we have Theorem 1.16. □

We proceed to show Theorem 1.17. Before doing so, we give

PROPOSITION 5.2. *Let* $f(z), g(z) \in Asy^{\mathcal{S}}_{\{\gamma\}}(\Omega_S)$. *Then* $f(z)g(z)$ *and* $(z_0\partial_0)^{\alpha_0}\partial'^{\alpha'}f(z)$ *are functions in* $Asy^{\mathcal{S}}_{\{\gamma\}}(\Omega_S)$. *Further if* $f(z) \sim 0$, *for any* $S' \subset\subset S$ *there is a* $c' > 0$ *such that*

$$|f(z)| \le C\exp(-c'|z_0|^{-\gamma}) \quad in\ S'.$$

PROOF. Let $S' \subset\subset S$, $f(z) = \sum_{n=0}^{N-1} f_n(z')z_0^{q_n} + f^N(z)$ and $g(z) = \sum_{n=0}^{N-1} g_n(z')z_0^{q_n} + g^N(z)$ where

$$|f^N(z)|,\ |g^N(z)| \le AB^{q_N}\Gamma(\frac{q_N}{\gamma}+1)|z_0|^{q_N} \quad in\ S'. \tag{5.10}$$

Put $h(z) = f(z)g(z)$. Then we have

$$\begin{aligned} h(z) &= \sum_{n_1=0}^{N-1} f_{n_1}(z')z_0^{q_{n_1}}g(z) + f^N(z)g(z) \\ &= \sum_{n_1=0}^{N-1} f_{n_1}(z')z_0^{q_{n_1}}(\sum_{n_2=0}^{N(n_1)-1} g_{n_2}(z')z_0^{q_{n_2}} + g^{N(n_1)}(z)) + f^N(z)g(z), \end{aligned}$$

where $q_{n_1} + q_{N(n_1)-1} \le q_{N-1}$ and $q_{n_1} + q_{N(n_1)} \ge q_N$. We have

$$h(z) = \sum_{n=0}^{N-1} h_n(z')z_0^{q_n} + h^N(z),$$

where

$$\begin{cases} h_n(z') = \displaystyle\sum_{q_{n_1}+q_{n_2}=q_n} f_{n_1}(z')g_{n_2}(z') \quad for\ 0 \le n \le N-1 \\ h^N(z) = \displaystyle\sum_{n_1=0}^{N-1} f_{n_1}(z')g^{N(n_1)}(z) + f^N(z)g(z) \end{cases}$$

If $q_{n_1}+q_{n_2}=q_n$, $|f_{n_1}(z')g_{n_2}(z')|\le A'B^{q_n}\Gamma(q_n/\gamma+1)$. Since $\mathcal{S}$ is finitely generated, we have $|h_n(z')|\le A'B'^{q_n}\Gamma(q_n/\gamma+1)$. For $h^N(z)$ we have by the similar method $|h^N(z)|\le A'B'^{q_N}\Gamma(q_N/\gamma+1)|z_0|^{q_N}$. So $f(z)g(z)\in Asy^{\mathcal{S}}_{\{\gamma\}}(\Omega_S)$. Let us show the second assertion. It is not difficult to show $\partial'^{\alpha'}f(z)\in Asy^{\mathcal{S}}_{\{\gamma\}}(\Omega_S)$. So let us show $z_0\partial_0 f(z)\in Asy^{\mathcal{S}}_{\{\gamma\}}(\Omega_S)$. We have

$$z_0\partial_0 f(z)=\sum_{n=0}^{N-1}q_nf_n(z')z_0^{q_n}+((q_N+z_0\partial_0)f^N(z))z_0^{q_N}. \tag{5.11}$$

By Cauchy's formula

$$z_0\partial_0 f^N(z)=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{z_0f^N(t,z')}{(t-z_0)^2}dt.$$

As the proof of Proposition 2.6, we choose the circle $|t-z_0|=c|z_0|$ as the integration path $\mathcal{C}$, where c is a small constant depending on S'. We have for $z_0\in S'$

$$\begin{aligned}|z_0\partial_0 f^N(z)|\le&\frac{1}{2\pi}\int_0^{2\pi}|\frac{f^N(z_0+c|z_0|e^{i\theta},z')}{c}|d\theta\\ \le& A'B^{q_N}\Gamma(q_N/\gamma+1)|(1+c)z_0|^{q_N}\le A'B'^{q_N}\Gamma(q_N/\gamma+1)|z_0|^{q_N},\end{aligned}$$

which implies $z_0\partial_0 f(z)\in Asy^{\mathcal{S}}_{\{\gamma\}}(\Omega_S)$. Suppose $f(z)\sim 0$. Then for all $q_N\in\mathcal{S}$

$$|f(z)|\le AB^{q_N}\Gamma(q_N/\gamma+1)|z_0|^{q_N}. \tag{5.12}$$

Let $d>0$ be a generator of $\mathcal{S}$. Then we have, by Stiring's formula, for all $n\in\boldsymbol{N}$.

$$|f(z)|\le AB^{nd}\Gamma(nd/\gamma+1)|z_0|^{nd}\le A'B'^{nd}|z_0|^{nd}(nd/\gamma)^{nd/\gamma}\exp(-nd/\gamma).$$

For $b(nd/\gamma)^{-1/\gamma}/2\le|z_0|\le b(nd/\gamma)^{-1/\gamma}$ we have

$$|f(z)|\le A'B'^{nd}b^{nd}\exp(-nd/\gamma)\le A'B'^{nd}b^{nd}\exp(-(\frac{b}{2})^{\gamma}|z|^{-\gamma}).$$

Choose b with $B'b \leq 1$. Then $|f(z)| \leq A' \exp(-c|z|^{-\gamma})$, which implies $f(z) \in Asy^{0}_{\{\gamma\}}(\Omega_S)$. $\square$

PROOF OF THEOREM 1.17. We apply Theorem 1.12 to the proof. Assumptions 1 and 2 are obviously satisfied and Assumption 3 is satisfied for $i_* = p-1$. Conditions for $L^{\tilde{u}}$ are assumed. So we only check Assumption 4. Put $\gamma_* = \gamma_{p-1,\mathcal{L}^{\tilde{u}}}$ and $S_* = S(\phi_-, \phi_+)$ be a sector with $\phi_+ - \phi_- < \pi/\gamma_*$. Let $v(z) \in z_0^q\, Asy^{\mathcal{S}}_{\{\gamma_*\}}(\Omega_{S_*})$ with $v(z) \sim \tilde{u}(z)$. Then it follows from Proposition 5.2 that $L(v) \in Asy^{\mathcal{S}}_{\{\gamma_*\}}(\Omega_{S_*})$. Hence $g_*(z) := g(z) - L(v) \in Asy^{\mathcal{S}}_{\{\gamma_*\}}(\Omega_{S_*})$. Since $g_*(z) \sim 0$, $g_*(z) \in Asy^{0}_{\{\gamma_*\}}(\Omega_{S_*})$ by Proposition 5.2.

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