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An Abel-Tauber theorem for Fourier sine transforms

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Abstract. We prove an Abel-Tauber theorem for Fourier sine transforms. It can be considered as the analogue of the Abel-Tauber theorem of Pitman in the boundary case. We apply it to Fourier sine series as well as to the tail behavior of a probability distribution.

1. Introduction and results

The aim of this paper is to prove an Abel-Tauber theorem for Fourier sine transforms. It characterizes, for example, the asymptotic behavior $f(t) \sim 1/t^2$ as $t \to \infty$ in terms of the Fourier sine transform of f, where f is a locally integrable, eventually non-increasing function on $[0, \infty)$ such that $\lim_{t\to\infty} f(t) = 0$. A similar result for Fourier sine series will be obtained as a corollary.

To state our results, we recall and introduce some notation. We denote by R_0 the whole class of slowly varying functions at infinity; that is, R_0 is the class of positive measurable l, defined on some neighborhood of infinity, satisfying

$$\forall \lambda > 0, \qquad \lim_{x\to\infty} l(\lambda x)/l(x) = 1.$$

For $l \in R_0$, the class Π_l is the class of measurable g satisfying

$$\forall \lambda \geq 1, \qquad \lim_{x\to\infty} \{g(\lambda x) - g(x)\}/l(x) = c \log \lambda$$

for some constant c called the *l-index* of g. It is useful to name the class of functions of which we define the Fourier sine transforms. The function $f : [0, \infty) \to \mathbb{R}$ belongs to $DL^1_{loc}[0, \infty)$ if it is locally integrable and eventually

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non-increasing on $[0,\infty)$, $\lim_{t\to\infty} f(t) = 0$. For $f \in DL^1_{loc}[0,\infty)$, we define its *Fourier sine transform* F_s by

$$F_s(\xi) = \int_0^{\infty-} f(t)\sin t\xi dt \qquad (0<\xi<\infty), \tag{1.1}$$

where we write $\int_0^{\infty-}$ to denote an improper integral obtained from $\int_0^M$ by letting $M \uparrow \infty$. Since the improper integral on the right converges uniformly on each (a,∞) with $a>0$, F_s is a continuous function on $(0,\infty)$. See the proof of Theorem 6 of Titchmarsh [6].

Here is the main theorem of this paper:

THEOREM 1.1. *Let $l \in R_0$ and $f \in DL^1_{loc}[0,\infty)$. Let F_s be the Fourier sine transform of f. Then*

$$f(t) \sim t^{-2} l(t) \qquad (t\to\infty) \tag{1.2}$$

if and only if

$$xF_s(1/x) \in \Pi_l \text{ in } x \text{ with } l\text{-index } 1. \tag{1.3}$$

The analogue for Fourier sine series is:

THEOREM 1.2. *Let $l \in R_0$. Suppose that the real sequence $\{b_n\}$ is eventually non-increasing, and tends to 0 as $n\to\infty$. We set*

$$g_s(\xi) = \sum_{n=1}^{\infty} b_n \sin n\xi \qquad (0<\xi<2\pi). \tag{1.4}$$

Then

$$b_n \sim n^{-2} l(n) \qquad (n\to\infty) \tag{1.5}$$

if and only if

$$xg_s(1/x) \in \Pi_l \text{ in } x \text{ with } l\text{-index } 1. \tag{1.6}$$

Now we recall the Abel-Tauber theorem of Pitman [4] which is closely related to Theorem 1.1. Let $l \in R_0$ and $0 < \alpha < 2$. Let $f \in DL^1_{loc}[0, \infty)$, and let F_s be the Fourier sine transform of f. Then, by Pitman [4],

$$f(t) \sim t^{-\alpha} l(t) \qquad (t \to \infty) \tag{1.7}$$

if and only if

$$F_s(\xi) \sim \xi^{\alpha-1} l(1/\xi) \frac{\pi}{2\Gamma(\alpha)\sin(\pi\alpha/2)} \qquad (\xi \to 0+). \tag{1.8}$$

From this and Theorem 1.1, we see that the behavior (1.2) is a critical case; we need Π-variation to characterize (1.2) in terms of the Fourier sine transform of f. For the related Abel-Tauber theorems for integral transforms, we refer to Chapter 4 of Bingham, Goldie and Teugels [1].

In the proof of Theorem 1.1, we will find that it is enough to prove the theorem for $f \in DL^1_{loc}[0, \infty)$ finite and non-increasing on $[0, \infty)$. For such f, we have the inversion formula which represents f by the Fourier sine transform of f (see Theorem 7 of Titchmarsh [6]). However it is difficult to use to prove the Tauberian implication (1.3) ⇒ (1.2). The difficulty is in that the problem we have to deal with involves both Π-variation and improper integrals. The key to the proof is to reduce the problem to the analogous result of Inoue [2], [3] for Fourier cosine transforms.

The proofs of Theorems 1.1 and 1.2 will be given in section 2. In section 3, we apply Theorem 1.1 to the tail behavior of a probability distribution.

2. Proofs of Theorems 1.1 and 1.2

For $f \in DL^1_{loc}[0, \infty)$, we define its *Fourier cosine transform* F_c by

$$F_c(\xi) = \int_0^{\infty-} f(t) \cos t\xi dt \qquad (0 < \xi < \infty). \tag{2.1}$$

We have the following Abel-Tauber theorem for Fourier cosine transforms:

Theorem 2.1 (Pitman [4] and Inoue [2], [3]). *Let $l \in R_0$ and $f \in DL^1_{loc}[0, \infty)$. Let F_c be the Fourier cosine transform of f. Then*

$$f(t) \sim t^{-1} l(t) \qquad (t \to \infty) \tag{2.2}$$

if and only if

$$F_c(1/\cdot) \in \Pi_l \text{ with } l\text{-index } 1. \tag{2.3}$$

The Abelian implication (2.2) $\Rightarrow$ (2.3) is essentially Theorem 7 (iii) of Pitman [4], while the Tauberian assertion (2.3) $\Rightarrow$ (2.2) is due to Inoue [2], [3].

Using Theorem 2.1, we shall prove Theorem 1.1.

PROOF OF THEOREM 1.1. Choose M so large that f is positive, finite and non-increasing on $[M, \infty)$. We set

$$g(t) = \begin{cases} f(M) & (0 \le t < M), \\ f(t) & (M \le t < \infty). \end{cases}$$

Let G_s be the Fourier sine transform of g. We set

$$a(x) = x^{-1} \sin x \qquad (0 < x < \infty). \tag{2.4}$$

Then by the mean-value theorem,

$$|a(x) - a(y)| \le \text{const.} |x - y| \qquad (0 < x, y < 1), \tag{2.5}$$

so for any $\lambda > 1$,

$$\begin{aligned} &|\lambda x F_s(1/(\lambda x)) - x F_s(1/x) - \lambda x G_s(1/(\lambda x)) + x G_s(1/x)|/l(x) \\ &= \frac{1}{l(x)} \left| \int_0^M t \{f(t) - f(M)\} \{a(t/(\lambda x)) - a(t/x)\} dt \right| \\ &\le \text{const.} \frac{(1 - \lambda^{-1})}{x l(x)} \int_0^M t^2 |f(t) - f(M)| dt \to 0 \qquad (x \to \infty), \end{aligned}$$

whence (1.3) holds if and only if $x G_s(1/x) \in \Pi_l$ in x with l-index 1. Therefore we may assume that f is positive, finite and non-increasing on $[0, \infty)$.

First we assume (1.2). Then f is integrable over $[0, \infty)$, and so, by integration by parts,

$$F_s(\xi)/\xi = \int_0^{\infty -} h(t) \cos t\xi dt \qquad (0 < \xi < \infty) \tag{2.6}$$

with

$$h(t) = \int_t^\infty f(s)ds \qquad (0 \le t < \infty). \tag{2.7}$$

Since (1.2) is equivalent to $h(t) \sim t^{-1}l(t)$ as $t \to \infty$ (see page 39 of [1]), we immediately obtain (1.3) by the Abelian implication of Theorem 2.1.

Next, we prove that (1.3) implies (1.2). By the second mean-value theorem for integrals,

$$F_s(\xi) = \lim_{U\to\infty} \int_0^U f(t)\sin t\xi dt = \lim_{U\to\infty} f(0+) \int_0^\zeta \sin t\xi dt$$

for some $\zeta \in (0, U)$, and so $\xi F_s(\xi)$ is bounded on $(0, \infty)$. In particular, for any $x > 0$, $F_s(\xi)(1 - \cos x\xi)/\xi$ is integrable over $(0, \infty)$. Then by applying Theorem 38 of Titchmarsh [6] to the pair of $\mathrm{I}_{[0,x]}$ and f, we obtain

$$\int_0^x f(t)dt = \frac{2}{\pi} \int_0^\infty F_s(\xi) \frac{1 - \cos x\xi}{\xi} d\xi \qquad (0 < x < \infty).$$

By Theorem 3.7.4 of Bingham, Goldie and Teugels [1], (1.3) implies $|xF_s(1/x)| \in R_0$ in x, so $F_s(\xi)/\xi$ is integrable on $(0, 1)$ whence on $(0, \infty)$. Thus for any $x > 0$,

$$\int_0^x f(t)dt \le \frac{4}{\pi} \int_0^\infty \frac{|F_s(\xi)|}{\xi} d\xi < \infty,$$

whence f is integrable over $(0, \infty)$. Again we arrive at (2.6) with (2.7), and similarly we obtain (1.2) from (1.3) by the Tauberian implication of Theorem 2.1. □

Following the method of Soni and Soni [5], we shall prove Theorem 1.2 as a corollary of Theorem 1.1.

Proof of Theorem 1.2. We set

$$f(t) = \begin{cases} 0 & (0 \le t < 1/2), \\ b_n & (n - 1/2 \le t < n + 1/2, \quad n = 1, 2, \cdots). \end{cases}$$

Then f is in $DL^1_{loc}[0,\infty)$, and (1.5) is equivalent to

$$f(t) \sim t^{-2}l(t) \qquad (t \to \infty). \tag{2.8}$$

Let F_s be the Fourier sine transform of f. Then by a simple calculation,

$$F_s(\xi) = a(\xi/2)g_s(\xi) \qquad (0 < \xi < 2\pi),$$

where a is defined by (2.4). So we have for any $\lambda > 1$ and $x > 0$,

(2.9)
$$\begin{aligned}&\{\lambda x F_s(1/(\lambda x)) - xF_s(1/x)\}/l(x)\\ &\quad = a(1/(2\lambda x))\{\lambda x g_s(1/(\lambda x)) - xg_s(1/x)\}/l(x)\\ &\qquad\qquad + xg_s(1/x)\{a(1/(2\lambda x)) - a(1/(2x))\}/l(x).\end{aligned}$$

By Theorem 3.7.4 of Bingham, Goldie and Teugels [1], (1.6) implies $|xg_s(1/x)| \in R_0$ in x, so by (2.5) the second term on the right of (2.9) tends to 0 as $x \to \infty$. Therefore (1.6) implies $xF_s(1/x) \in \Pi_l$ in x with l-index 1, which, by the Tauberian implication of Theorem 1.1, implies (2.8) whence (1.5).

Conversely, we set $c(x) = 1/a(x)$ for $x > 0$. Then we also have

$$|c(x) - c(y)| \le \text{const.}|x - y| \qquad (0 < x, y < 1).$$

Arguing similarly, we obtain (1.6) from (1.5), which completes the proof. □

3. Application to the tail behavior

In this section, we apply Theorem 1.1 to the tail behavior of a probability distribution. For the related results, we refer to Pitman [4] as well as pp. 336-337 of Bingham, Goldie and Teugels [1], and Inoue [3].

Let X be a real random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. The *tail-sum* of X is the function T defined by

$$T(x) = P(X \le -x) + P(X > x) \qquad (0 \le x < \infty).$$

Let U be the real part of the characteristic function of X:

$$U(\xi) = E[\cos \xi X] \qquad (\xi \in \mathbb{R}).$$

Then we have

$$\{1 - U(\xi)\}/\xi = \int_0^{\infty -} T(x) \sin \xi x dx \qquad (0 < \xi < \infty).$$

Since T is finite and non-increasing on $[0, \infty)$, $\lim_{x\to\infty} T(x) = 0$, by Theorem 1.1 we immediately obtain

THEOREM 3.1. *Let $l \in R_0$. Then $T(x) \sim x^{-2}l(x)$ as $x \to \infty$ if and only if $x^2\{1 - U(1/x)\} \in \Pi_l$ in x with l-index 1.*

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