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Some foliations on ruled surfaces II

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Abstract. In the previous paper [Sa5] entitled "Some foliations on ruled surfaces", we classified ruled surfaces with foliations on them leaving a curve invariant and having no singularities on the curve. There are three types of them. Some examples of such foliations were given.

In this paper, one of these types of such ruled surfaces and foliations on them are observed. It is proved that the examples given in [Sa5] are essential in the case that the genus of the base space is one.

§0. Introduction

In the previous papers [Sa4] and [Sa5], we classified ruled surfaces some foliations on which leave a curve invariant and have no singularities on it. (See Theorem 1.0 below.) We also gave examples of each case. In this paper, we assert that such a foliation on a decomposable ruled surface over a curve of genus one is non-singular and, if it is not the ruling, one of the examples in [Sa5]. We also investigate the case of genus $g > 1$. (Main Theorem 2.1). We will use the notations in [Sa5]. The author would like to thank Prof. K. Iwasaki and the referee for their helpful advice.

§1. Classification theorem and some properties of ruled surfaces

THEOREM 1.0. (Classification theorem —— [Sa5] Main Theorem 2.1.)
Let C be a closed Riemann surface of genus g, $X = \mathbf{P}(\mathcal{E}) \xrightarrow{\pi} C$ a ruled surface over C with the invariant e, where $\mathcal{E}$ is a normalized locally free $\mathcal{O}_C$-module, and C_0 a normalized section of $X \xrightarrow{\pi} C$. Assume that a foliation $\mathcal{F} \subset \Theta_X$ on X leaves an irreducible curve $C_1 \simeq_{num} aC_0 + bf$

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with $a > 0$ on X invariant and has no singularities on C_1. Then one of the following is the case.

I-i) $e = 0$, *$\mathcal{E}$ is decomposable and* $b = 0$.

I-ii) $e = 0$, *$\mathcal{E}$ is indecomposable and* $b = 0$.

II) $e < 0$, $a \geq 2$ *and* $b = \frac{1}{2}ea \in \mathbf{Z}$. *(In this case, $\mathcal{E}$ is indecomposable.)*

Here "$\simeq_{num}$" represents numerical equivalence of divisors on X.

In this paper, we are concerned with the case I-i). The following holds:

PROPOSITION 1.1.
For a ruled surface $X \xrightarrow{\pi} C$, there exists a normalized decomposable locally free $\mathcal{O}_C$-module $\mathcal{E}$ of rank two such that $X = \mathbf{P}(\mathcal{E})$ if and only if $X \xrightarrow{\pi} C$ has two sections with no intersection.

In what follows, we always assume that $X = \mathbf{P}(\mathcal{E}) \xrightarrow{\pi} C$ is a ruled surface over a closed Riemann surface C of genus $g \geq 1$ with

(1.2) $$\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L} \qquad \text{satisfying} \qquad deg\mathcal{L} = 0.$$

We put another assumption:

(1.3) $$\mathcal{L}^{\otimes n} \not\simeq \mathcal{O}_C \qquad \textit{for any} \qquad 0 \neq n \in \mathbf{Z}.$$

Let L be the line bundle over C whose *dual* bundle is the bundle the sheaf of germs of holomorphic sections of which is isomorphic to $\mathcal{L}$. We denote by 1_C the trivial line bundle over C and define a holomorphic vector bundle $E \xrightarrow{\pi_E} C$ of rank two by $E = 1_C \oplus L$ so that $\mathcal{E} = \mathcal{O}_C(E)^*$, the dual of $\mathcal{O}_C(E)$. Since $c(L) = 0$, L has a flat representative. Take a flat representative cocycle $(L_{\alpha\beta}) \in Z^1(\{U_\alpha\}, \mathbf{C}^\times)$ with respect to an open coordinate covering $\{(U_\alpha; z_\alpha)\}$ of C. Then the vector bundle E is locally trivialized with respect to this covering with fibre coordinates $(\lambda_\alpha, \mu_\alpha)$ on $\pi_E(U_\alpha) \simeq U_\alpha \times \mathbf{C}^2$, which satisfy the transition relations

$$\lambda_\alpha = L_{\alpha\beta}\lambda_\beta \qquad \text{and} \qquad \mu_\alpha = \mu_\beta,$$

and $X = \mathbf{P}(\mathcal{E})$ is obtained by patching $U_\alpha \times \mathbf{P}^1$'s together identifying

$$\begin{array}{lccc} & U_\alpha \times \mathbf{P}^1 & & U_\beta \times \mathbf{P}^1 \\ & \bigcup & & \bigcup \\ (1.4) & (U_\alpha \cap U_\beta) \times \mathbf{P}^1 & = & (U_\alpha \cap U_\beta) \times \mathbf{P}^1 \\ & (z, \zeta_\alpha) & = & (z, L_{\alpha\beta}\zeta_\beta), \end{array}$$

where $\zeta_\alpha = \frac{\lambda_\alpha}{\mu_\alpha}$ is the inhomogeneous coordinate on $U_\alpha \times \mathbf{P}^1$.

Under the local trivialization (1.4), local equations $\zeta_\alpha = 0$ and $\zeta_\alpha = \infty$ on $U_\alpha \times \mathbf{P}^1$ define global curves with the properties of a normalized section. We denote them by C_0 and C_∞, respectively. Thus

$$C_0 \simeq_{num} C_\infty \qquad \text{and} \qquad {C_0}^2 = C_0 \cdot C_\infty = {C_\infty}^2 = 0.$$

The following lemma is important.

Lemma 1.5.

Assume that $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ with $deg\mathcal{L} = 0$ and that no $0 \neq n \in \mathbf{Z}$ satisfies $\mathcal{L}^{\otimes n} \simeq \mathcal{O}_C$. Let $C_1 \simeq_{num} aC_0$ be an irreducible curve in $X = \mathbf{P}(\mathcal{E})$. Then C_1 is either C_0 or C_∞.

Proof.

It follows from $C_1 \simeq_{num} aC_0$ that $C_1 \cdot C_\infty = 0$. Suppose that C_1 were neither C_0 nor C_∞. A local equation φ_α of C_1 on $U_\alpha \times \mathbf{P}^1$ of the local trivialization (1.4) would be written as follows:

$$\varphi_\alpha = \sum_{j=0}^{a} p_{j,\alpha}{\zeta_\alpha}^j \qquad \text{with} \qquad p_{j,\alpha} \in \mathcal{O}_C(U_\alpha).$$

Noting $C_1 \cdot C_0 = 0$, φ_α could be taken so that $p_{\alpha,0} = 1 \in \mathcal{O}_C(U_\alpha)$. Considering the transition relation $\zeta_\alpha = L_{\alpha\beta}\zeta_\beta$, these φ_α's would patch together and $p_{j,\alpha}$'s would define global holomorphic sections $p_j \in \Gamma(C, \mathcal{L}^{\otimes j})$ for $j = 1,\ldots,a$. Since $deg\mathcal{L}^{\otimes j} = 0$ and $\mathcal{L}^{\otimes j} \not\simeq \mathcal{O}_C$, $p_j = 0$, which is a contradiction. $\square$

Tensoring an invertible $\mathcal{O}_C$-module to $\mathcal{E}$ if necessary, we may assume $C_1 = C_0$.

The differential map of the projection $X \xrightarrow{\pi} C$ defines the following exact sequence of holomorphic vector bundles over X.

$$0 \longrightarrow Tf \longrightarrow TX \xrightarrow{D\pi} \pi^*TC \longrightarrow 0 \tag{1.6}$$

Here Tf is the kernel of $TX \xrightarrow{D\pi} \pi^*TC$, which consists of vector fields tangent to fibres of $X \xrightarrow{\pi} C$.

Let Θ_X be the sheaf of germs of holomorphic vector fields over X. Since $\frac{\partial \zeta_\beta}{\partial z_\alpha} = 0$, local vector fields $\frac{\partial}{\partial z_\alpha} \in \Gamma(\pi^{-1}(U_\alpha), \Theta_X)$ define a well-defined invertible subsheaf of Θ_X, which is mapped isomorphically onto $\mathcal{O}_X(\pi^*TC)$ by $TX \xrightarrow{D\pi} \pi^*TC$.

PROPOSITION 1.7.
Let $X = \mathbf{P}(\mathcal{E}) \xrightarrow{\pi} C$ be a ruled surface with the invariant $e = 0$ and $\mathcal{E}$ decomposable. Then the exact sequence (1.6) splits.

In what follows, we identify $\mathcal{O}_X(\pi^*TC)$ with the invertible subsheaf of Θ_X described above. Thus we have

$$TX = Tf \oplus \pi^*TC \tag{1.8}$$

Take a local trivialization of $X \xrightarrow{\pi} C$ as in (1.4) and set $\rho_\alpha = \frac{\mu_\alpha}{\lambda_\alpha} = {\zeta_\alpha}^{-1}$. Note that

$$\frac{\partial}{\partial \rho_\alpha} = -{\zeta_\alpha}^2 \frac{\partial}{\partial \zeta_\alpha} \in \Gamma(\pi^{-1}(U_\alpha), \Theta_X).$$

Holomorphic vector fields $\zeta_\alpha \frac{\partial}{\partial \zeta_\alpha} = -\rho_\alpha \frac{\partial}{\partial \rho_\alpha} \in \Gamma(\pi^{-1}(U_\alpha), \Theta_X)$ are patched together into a global holomorphic vector field

$$\begin{aligned} &\eta \in \Gamma(X, \mathcal{O}_X(Tf)) \subset \Gamma(X, \Theta_X) \qquad \text{satisfying} \\ &\eta|_{\pi^{-1}(U_\alpha)} = \zeta_\alpha \frac{\partial}{\partial \zeta_\alpha}, \end{aligned} \tag{1.9}$$

which is logarithmic with respect to $C_0 = C_1$ (and C_∞).

Under the identification (1.8), vector fields belonging to $\mathcal{O}_X(\pi^*TC)$ are also logarithmic with respect to C_0.

LEMMA 1.10.
Let X, C_0 and η be as above. Then

$$\mathrm{Der}_X(\log C_0)\,|_{\pi^{-1}(U_\alpha)} = \mathcal{O}_X(\pi^*TC)\,|_{\pi^{-1}(U_\alpha)} \\ \oplus (\mathcal{O}_X\eta\,|_{\pi^{-1}(U_\alpha)} + \mathcal{O}_{\pi^{-1}(U_\alpha)}\frac{\partial}{\partial\rho_\alpha}),$$

where $\mathrm{Der}_X(\log C_0)$ *is the sheaf of germs of logarithmic vector fields with respect to* C_0.

Generally, a non-zero global meromorphic vector field on a complex manifold M determines a foliation of dimension one on M. Note that a foliation $\mathcal{F} \subset \Theta_M$ of dimension one on M defines, by taking local generators of $\mathcal{F}$, a morphism $\tau \xrightarrow{\varphi} TM$ of holomorphic vector bundles over M of a holomorphic line bundle τ into the holomorphic tangent bundle TM, whose zero loci $\{\varphi = 0\}$ are of codimension ≥ 2. Conversely, a morphism $\tau \xrightarrow{\varphi} TM$ with zero loci of codimension ≥ 2 defines a foliation of dimension one on M. (cf. [GM2] and [Sa1].)

Since every holomorphic line bundle over a ruled surface X is meromorphically trivial, $\tau \xrightarrow{\varphi} TX$ defines a global meromorphic vector field on X upto multiplication of global meromorphic functions. Thus we may consider a foliation on a ruled surface as a global meromorphic vector field upto multiplication of global meromorphic functions.

§2. Main theorem

There are meromorphic sections $\in \Gamma(X, \mathcal{M}_X(\pi^*TC))$ with no zero, which we consider as meromorphic vector fields $\in \Gamma(X, \mathcal{M}_X(\pi^*TC)) \subset \Gamma(X, \mathcal{M}_X(TX))$. Explicitly as follows: Take a local trivialization of $X \xrightarrow{\pi} C$ and a coordinate covering $\{(U_\alpha; z_\alpha)\}$ of C as (1.4). There is a global holomorphic 1-form $0 \neq u = (u_\alpha dz_\alpha) \in \Gamma(C, \mathcal{O}_C(T^*C))$. It defines a global 1-form $\in \Gamma(X, \mathcal{O}_X(T^*X))$. Then, since $\frac{\partial\zeta_\beta}{\partial z_\alpha} = 0$,

$$(\frac{1}{u_\alpha}\frac{\partial}{\partial z_\alpha}) \in \Gamma(X, \mathcal{M}_X(\pi^*TC))$$

is a well-defined global meromorphic vector field on X with no zero, which we will denote by $\pi^*(\frac{1}{u})$. Note that every line bundle over X is meromorphically trivial. Thus we have, fixing a $0 \neq u \in \Gamma(C, \mathcal{O}_C(T^*C))$ arbitrarily,

$$\mathcal{M}_X \pi^*(\frac{1}{u}) = \mathcal{M}_X(\pi^*TC) \subset \mathcal{M}_X(TX). \tag{2.0}$$

If C is of genus one, then π^*TC is trivial and we have a global holomorphic vector field $v \in \Gamma(X, \mathcal{O}_X(\pi^*TC)) \subset \Gamma(X, \Theta_X)$. Namely, in the construction (1.4), we can take a coordinate covering $\{(U_\alpha; z_\alpha)\}$ of C so that $dz_\alpha = dz_\beta$. Let $u \in \Gamma(C, \mathcal{O}_C(T^*C))$ be the holomorphic 1-form on C defined by $u|_{U_\alpha} = dz_\beta$. Then $\Gamma(C, \mathcal{O}_C(T^*C)) = \mathbf{C}u \simeq \mathbf{C}$. Vector fields $\frac{\partial}{\partial z_\alpha} \in \Gamma(\pi^{-1}(U_\alpha), \mathcal{O}_X(\pi^*TC)) \subset \Gamma(\pi^{-1}(U_\alpha), \Theta_X)$ patch together to define a well-defined global holomorphic vector field $v = \pi^*(\frac{1}{u}) \in \Gamma(X, \mathcal{O}_X(\pi^*TC)) \subset \Gamma(X, \Theta_X)$. Note that $\mathcal{O}_X(\pi^*TC) = \mathcal{O}_X v$ and that, for $0 \neq w = ku \in \Gamma(C, \mathcal{O}_C(T^*C))$ with $0 \neq k \in \mathbf{C}$,

$$\pi^*(\frac{1}{w}) = \frac{1}{k} v \in \Gamma(X, \mathcal{O}_X(\pi^*TC)) = \mathbf{C}v \subset \Gamma(X, \Theta). \tag{2.0.1}$$

MAIN THEOREM 2.1.

· The case $g = 1$.

Let $X = \mathbf{P}(\mathcal{E}) \xrightarrow{\pi} C$ be a ruled surface over an elliptic curve C with $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ satisfying $deg\mathcal{L} = 0$ and $\mathcal{L}^{\otimes n} \not\simeq \mathcal{O}_C$ for any $0 \neq n \in \mathbf{Z}$. Let $v \in \Gamma(X, \Theta_X)$ be as above and $\eta \in \Gamma(X, \Theta_X)$ as (1.9). Assume that a foliation $\mathcal{F} \subset \Theta_X$ leaves an irreducible curve C_1 on X invariant and that $\mathcal{F}$ has no singularities on C_1. Then $\mathcal{F}$ is a non-singular foliation. Moreover, if $\mathcal{F}$ is not the ruling, then $\mathcal{F}$ is generated by a global holomorphic vector field

$$v + t\eta \in \Gamma(X, \Theta_X) \qquad \textit{with} \qquad t \in \mathbf{C}.$$

Let $\mathcal{F}o\ell$ be the set of foliations with the properties described above. There is a one-to-one correspondence from $\Gamma(C, \mathcal{O}_C(T^*C)) \sqcup \{\infty\} \simeq \mathbf{P}^1$ onto $\mathcal{F}o\ell$. Namely,

$$\begin{array}{ccccl}
\mathbf{P}^1 & \simeq & \Gamma(C, \mathcal{O}_C(T^*C)) \sqcup \{\infty\} & \longrightarrow & \mathcal{F}o\ell \\
 & & ku & \mapsto & \mathcal{O}_X(\frac{1}{k}v + \eta) \quad (0 \neq k \neq \infty) \\
 & & 0 & \mapsto & \mathcal{O}_X(\pi^*TC) \\
 & & \infty & \mapsto & \textit{the ruling.}
\end{array}$$

· The case $g > 1$.

Let $X = \mathbf{P}(\mathcal{E}) \xrightarrow{\pi} C$ be a ruled surface over a closed Riemann surface C of genus $g > 1$ with $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ satisfying $deg\mathcal{L} = 0$ and $\mathcal{L}^{\otimes n} \not\simeq \mathcal{O}_C$ for any $0 \neq n \in \mathbf{Z}$. Let $\eta \in \Gamma(X, \Theta_X)$ be as (1.9). Assume that a foliation $\mathcal{F} \subset \Theta_X$ leaves an irreducible curve C_1 on X invariant and that $\mathcal{F}$ has no singularities on C_1. Then $\mathcal{F}$ is defined by a global meromorphic vector field

$$\theta = h\pi^*(\frac{1}{u}) + \eta \in \Gamma(X, \mathcal{M}_X(TX)),$$

where $h \in \mathcal{M}_X(X)$ is a global meromorphic function $\neq 0$ on X defined as follows: Take a holomorphic line bundle $\xi \in \mathrm{H}^1(C, \mathcal{O}_C^*)$, a holomorphic section

$$s \in \Gamma(C, \mathcal{O}_C(\xi)) \subset \Gamma(X, \mathcal{O}_X(\pi^*\xi))$$

satisfying $(u) - (s) \geq 0$ on C, a non-negative integer $m \in \mathbf{Z}$ and $m+1$ holomorphic sections

$$q_j \in \Gamma(C, \mathcal{O}_C(\xi \otimes L^{\otimes(-j)})) \subset \Gamma(X, \mathcal{O}_X(\pi^*(\xi \otimes L^{\otimes(-j)}))) \qquad (j = 0, \cdots, m),$$

not all of which are zero-sections. h is defined on $\pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbf{P}^1$ of the local trivialization (1.4),

$$h|_{\pi^{-1}(U_\alpha)} = \frac{s_\alpha(z_\alpha)}{\sum_{j=0}^m q_{j,\alpha}(z_\alpha)\zeta_\alpha{}^j}.$$

Here $s_\alpha(z_\alpha)$ and $q_{j,\alpha}(z_\alpha) \in \mathcal{O}_C(U_\alpha) \subset \mathcal{O}_X(\pi^{-1}(U_\alpha))$ represnt the sections $s \in \Gamma(X, \mathcal{O}_X(\pi^*\xi))$ and $q_j \in \Gamma(X, \mathcal{O}_X(\pi^*(\xi \otimes L^{\otimes(-j)})))$ on $\pi^{-1}(U_\alpha)$, respectively.

Proof of Main Theorem 2.1.

At first, we assume simply $g \geq 1$. Suppose that $\mathcal{F}$ is neither the ruling nor $\mathcal{O}_X(\pi^*TC)$. Then we may assume that $\mathcal{F}$ leaves C_0 invariant and has no singularities on C_0. It follows from Lemma 1.10 that $\mathcal{F}$ is determined by a global meromorphic vector field $\eta + h\pi^*(\frac{1}{u})$, where $0 \neq h \in \mathcal{M}_X(X)$. Consider the divisors $(h) = (h)_+ - (h)_-$ and $D = (h\pi^*(\frac{1}{u})) = D_+ - D_-$, where $(h)_+$ and D_+ are the effective divisors defined by the zero loci of h

and $h\pi^*(\frac{1}{u})$ and $(h)_-$ and D_- are the effective divisors defined by the pole loci of h and $h\pi^*(\frac{1}{u})$, respectively.

LEMMA 2.3.

0) *The divisor $(h)_+ - D_+$ is effective and there is an effective divisor δ on C such that*

$$(h)_+ - D_+ = \pi^*\delta.$$

1) *$(u) - \delta$ is an effective divisor on C.*
2) *The divisor $D_+ - C_0$ (on X) is not effective.*
3) *For some $0 \le m \in \mathbf{Z}$,*

$$D_+ \simeq_{num} mC_0.$$

PROOF OF LEMMA 2.3.
0) and 1). It follows from $0 \neq u \in \Gamma(C, \mathcal{O}_C(T^*C))$ and $D = (h) - \pi^*(u)$.
2) and 3). Note that $\mathcal{F}$ leaves C_0 invariant. Thus $D_+ - C_0 \not\geq 0$. Let $D_+ \simeq_{num} mC_0 + nf$. Then $C_0 \cdot D_+ = n$. Since $\mathcal{F}$ has no singularities on C_0 and D_+ is effective, $m \ge 0$ and $n = 0$. □

PROOF OF MAIN THEOREM 2.1. (continued)
Let $\xi \in \mathbf{H}^1(C, \mathcal{O}_C{}^*)$ be a holomorphic line bundle over C such that $\mathcal{O}_C(\xi) \simeq \mathcal{L}(\delta)$. Fix a holomorphic section $s \in \Gamma(C, \mathcal{O}_C(\xi))$ with $(s) = \delta$. s defines a holomorphic section of $\mathcal{L}(\pi^*\delta)$ on X defining the divisor $\pi^*\delta = (h)_+ - D_+$, which we also denote by s: $s \in \Gamma(X, \mathcal{L}(\pi^*\delta))$. Assume that, with respect to the local trivialization (1.4), ξ is represented by a 1-cocycle $(\xi_{\alpha\beta}) \in \mathrm{Z}^1(\{U_\alpha\}, \mathcal{O}_C{}^*)$ and $s = (s_\alpha)$ with $s_\alpha \in \mathcal{O}_C(U_\alpha) \subset \mathcal{O}_X(\pi^{-1}(U_\alpha))$. h can be written with respect to the local trivialization (1.4) as follows: Let λ_α and μ_α be the fibre coordinates of the vector bundle E on $\pi_E{}^{-1}(U_\alpha) \simeq U_\alpha \times \mathbf{C}^2$. It follows from Lemma 2.3 that we can take holomorphic functions $P_\alpha(z_\alpha, \lambda_\alpha, \mu_\alpha)$ and $Q_\alpha(z_\alpha, \lambda_\alpha, \mu_\alpha)$ on $U_\alpha \times \mathbf{C}^2$, which are homogeneous polynomials of degree m with respect to λ_α and μ_α, so that $s_\alpha P_\alpha$ and Q_α define the zero and pole loci of $h|_{\pi^{-1}(U_\alpha)}$, respectively, and that, substituting $\zeta_\alpha = \frac{\lambda_\alpha}{\mu_\alpha}$,

$$h(z_\alpha, \zeta_\alpha) = \frac{s_\alpha(z_\alpha) P_\alpha(z_\alpha, \lambda_\alpha, \mu_\alpha)}{Q_\alpha(z_\alpha, \lambda_\alpha, \mu_\alpha)}. \tag{2.4}$$

We can choose P_α, which defines on $\pi^{-1}(U_\alpha)$ the divisor D_+, such as

$$P_\alpha(z_\alpha,\lambda_\alpha,\mu_\alpha)={\mu_\alpha}^m+\sum_{j=1}^m p_{j,\alpha}(z_\alpha){\lambda_\alpha}^j{\mu_\alpha}^{m-j}. \tag{2.5}$$

Here

$$p_{j,\alpha}(z_\alpha)\in\mathcal{O}_C(U_\alpha)\subset\mathcal{O}_X(\pi_E^{-1}(U_\alpha)).$$

Similarly,

$$Q_\alpha(z_\alpha,\lambda_\alpha,\mu_\alpha)=\sum_{j=0}^m q_{j,\alpha}(z_\alpha){\lambda_\alpha}^j{\mu_\alpha}^{m-j}$$

with

$$q_{j,\alpha}(z_\alpha)\in\mathcal{O}_C(U_\alpha)\subset\mathcal{O}_X(\pi_E^{-1}(U_\alpha)).$$

Recall the following transition relations:

$$\lambda_\alpha=L_{\alpha\beta}\lambda_\beta\qquad\text{and}\qquad\mu_\alpha=\mu_\beta \tag{2.6}$$

with $L_{\alpha\beta}\in\mathbf{C}^\times$.

LEMMA 2.7.
On ${\pi_E}^{-1}(U_\alpha\cap U_\beta)\simeq(U_\alpha\cap U_\beta)\times\mathbf{C}^2$,

$$P_\alpha(z_\alpha,\lambda_\alpha,\mu_\alpha)=P_\beta(z_\beta,\lambda_\beta,\mu_\beta).$$

PROOF OF LEMMA 2.7.
Since the zero loci of these holomorphic functions in $(U_\alpha\cap U_\beta)\times\mathbf{C}^2$ coincide with each other, there exists a non-vanishing holomorphic function $\psi_{\alpha\beta}\in\mathcal{O}^*((U_\alpha\cap U_\beta)\times\mathbf{C}^2)$ such that

$$P_\alpha=\psi_{\alpha\beta}P_\beta.$$

Using the transition relations (2.6) and $z_\alpha=z_\alpha(z_\beta)$, we regard P_α as a function of z_β, λ_β and μ_β. Both P_α and P_β are, with respect to λ_β and μ_β, homogeneous polynomials of degree m. Thus $\psi_{\alpha\beta}$ depends only on z_β.

Recall (2.5). Since the coefficients of ${\mu_\beta}^m$ of both functions are 1, we have $\psi_{\alpha\beta} = 1$ and $P_\alpha = P_\beta$. □

PROOF OF MAIN THEOREM 2.1. (continued)
Thus the coefficients $p_{j,\alpha}$'s define

$$p_j \in \Gamma(C, \mathcal{O}_C(L^{\otimes(-j)})).$$

Since $\Gamma(C, \mathcal{O}_C(L^{\otimes(-j)})) = 0$ for $j \neq 0$, we have $P_\alpha = \mu_\alpha^m$. It follows from the transition relations (2.6) that $q_{j,\alpha}$'s define

$$q_j \in \Gamma(C, \mathcal{O}_C(\xi \otimes L^{\otimes(-j)})).$$

Thus we have the case $g > 1$.
· *The case* $g = 1$.

In this case, the divisor $(u) = 0$ and the line bundle ξ must be trivial. We have $h|_{\pi^{-1}(U_\alpha)} = \frac{P_\alpha}{Q_\alpha}$. It follows that $Q_\alpha = Q_\beta$ and that $q_{j,\alpha}$'s define $q_j \in \Gamma(C, \mathcal{O}_C(L^{\otimes(-j)}))$'s. Thus $Q_\alpha = q_0 \mu_\alpha^m$ and $h = \frac{1}{q_0}$.

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