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Existence of axially symmetric H-surfaces of annulus type

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Abstract. In this paper, we solve Plateau's problem for surfaces of constant mean curvature spanning two circular (axially symmetric) contours. To prove the existence of such surfaces, we use the Leray-Schauder degree theory.

1. Introduction

A surface of prescribed mean curvature H is called an *H-surface* for short. As a natural generalization of the classical Plateau problem, we can propose the problem of whether there exist H-surfaces spanning a Jordan curve in $\mathbb{R}^3$ (Plateau's problem is this problem for $H = 0$). This problem has been studied by many mathematicians (see Remark 1.7 and Struwe [32]).

However, little has been known about the existence of H-surfaces spanning two or more Jordan curves in $\mathbb{R}^3$, except for the case of $H = 0$ (see [2], [5], [8], [17], [24], [31], [33]).

In this paper, we show the existence of H-surfaces spanning two Jordan curves in $\mathbb{R}^3$ which have the same topological structure as an annulus. Such a surface is called an *annulus-type* H-surface for short.

Before stating our results, we introduce two notations: For real numbers a and b, let

$$\Gamma(a, b) := \{(a\cos\theta, a\sin\theta, b) \in \mathbb{R}^3 \mid 0 \leqq \theta < 2\pi\}.$$

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For a real number $\rho < 1$, let

$$A_\rho := \{w = (u,v) \in \mathbb{R}^2 \mid \rho < |w| < 1\}.$$

For a given real constant H and $\Gamma(a,b)$, we show the existence of annulus-type H-surfaces spanning $\Gamma(a,b)$ and $\Gamma(1,0)$ under some conditions on a, b and H.

To put it more precisely, we look for a real number $\rho \in (0,1)$ and a surface

$$X = (X^1, X^2, X^3) : \bar{A}_\rho \longrightarrow \mathbb{R}^3$$

such that

$$X \in C^0(\bar{A}_\rho; \mathbb{R}^3) \cap C^2(A_\rho; \mathbb{R}^3), \tag{1.1}$$

$$\Delta X = 2H X_u \wedge X_v \quad \textit{in } A_\rho, \tag{1.2}$$

$$|X_u|^2 - |X_v|^2 = 0 \quad \textit{in } A_\rho, \tag{1.3}$$

$$X_u \cdot X_v = 0 \quad \textit{in } A_\rho, \tag{1.4}$$

and

$$X\big|_{\partial B_1} : \partial B_1 \longrightarrow \Gamma(1,0), X\big|_{\partial B_\rho} : \partial B_\rho \longrightarrow \Gamma(a,b) \tag{1.5}$$

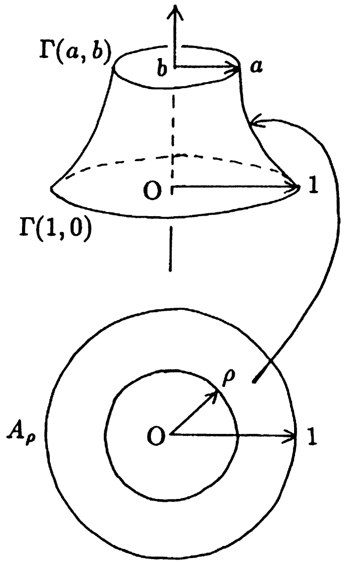
and they are weakly monotone oriented parametrizations of $\Gamma(1,0)$ *and* $\Gamma(a,b)$ *respectively.*

Here and hereafter, for a real number $r > 0$, let $B_r = \{w = (u,v) \in \mathbb{R}^2 \mid |w| < r\}$, let $X_u = \frac{\partial X}{\partial u}$, etc. For $p = (p^1, p^2, p^3) \in \mathbb{R}^3$ and $q = (q^1, q^2, q^3) \in \mathbb{R}^3$, $p \wedge q = (p^2q^3 - p^3q^2, p^3q^1 - p^1q^3, p^1q^2 - p^2q^1)$ denotes their exterior product, and $p \cdot q = p^1q^1 + p^2q^2 + p^3q^3$ denotes their scalar product.

Since $\Gamma(1,0)$ and $\Gamma(a,b)$ are axially symmetric, it is natural to expect the existence of such a surface with axial symmetry (see [Fig. 1]).

Here we present two examples of axially symmetric H-surfaces of annulus type. Sphere (see [Fig. 2]) is most typical. In 1841 Delaunay [4] constructed H-surfaces by rolling a conic section on a straight line in a plane and rotating the trace of its focus around that line (see [Fig. 3], Remark 1.4 and Eells [6]).

Now, we state our main results.



[Fig. 1]

THEOREM 1.1. *Let $a, b, H \in \mathbb{R}$ satisfy*

$$\frac{3}{4} < a < 1, \tag{1.6}$$

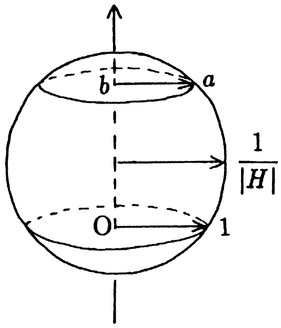
$$|H| < \frac{(1-a)(2a-1)-|b|}{4a(1-a)^2}, \tag{1.7}$$

$$|H| \leqq \frac{4(1-a)\sqrt{(a-\frac{1}{4})(a-\frac{3}{4})}-|b|}{8a(1-a)^2}. \tag{1.8}$$

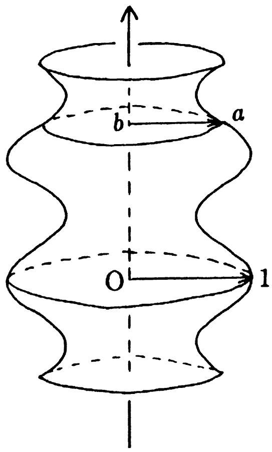
Then there exist a real number $\rho \in (0,1)$ and a solution $X \in C^\infty(\bar{A}_\rho;\mathbb{R}^3)$ to (1.2)–(1.5) of the form

$$X(r\cos\theta, r\sin\theta) = (f(r)\cos\theta, f(r)\sin\theta, g(r)) \quad (\rho \leqq r \leqq 1, 0 \leqq \theta < 2\pi), \tag{1.9}$$

where $f, g \in C^\infty([\rho, 1])$.



[Fig. 2] Sphere.



[Fig. 3] Delaunay's surface.

THEOREM 1.2. *Let $a, b, H \in \mathbb{R}$ satisfy*

$$1 < a < \frac{3}{2}, \tag{1.10}$$

$$|H| < \frac{(a-1)-(2a-1)|b|}{4a(a-1)^2}, \tag{1.11}$$

$$|H| \leqq \frac{2(a-1)\sqrt{(a+\frac{1}{2})(\frac{3}{2}-a)}-(2a-1)|b|}{8a(a-1)^2}. \tag{1.12}$$

Then there exist a real number $\rho \in (0,1)$ and a solution $X \in C^\infty(\bar{A}_\rho;\mathbb{R}^3)$ to (1.2)–(1.5) of the form (1.9) with some $f, g \in C^\infty([\rho,1])$.

Note that we do need some assumptions on a, b and H, if we look for a solution to (1.1)–(1.5) of the form (1.9) (cf. Heinz [11]; see Remark 1.7).

In fact, using a result of Kenmotsu [19], we can get the following:

THEOREM 1.3. *Suppose*

$$|H| > 1, |H(1-a)| > 1. \tag{1.13}$$

Then there exists no solution to (1.1)–(1.5) of the form (1.9), regardless of b.

REMARK 1.4. In 1980 Kenmotsu [19] generalized the result of Delaunay [4] (see also Hsiang-Yu [14]) and proved as a corollary of his main theorem that any complete surface of revolution with constant mean curvature is a sphere, a catenoid or an H-surface whose generating curve is given by

$$\begin{aligned} &X(s;H,B) \\ &= \left(\int_0^s \frac{1+B\sin 2Ht}{\sqrt{1+B^2+2B\sin 2Ht}}dt, \frac{1}{2|H|}\sqrt{1+B^2+2B\sin 2Hs}\right) \\ &=: (x(s),y(s)) \quad for\ s\in\mathbb{R} \end{aligned} \tag{1.14}$$

with some constant $B \in \mathbb{R}$.

PROOF OF THEOREM 1.3. Take $X(s;H,B), x(s)$ and $y(s)$ as in (1.14). Then we may assume that $B \geqq 0$.

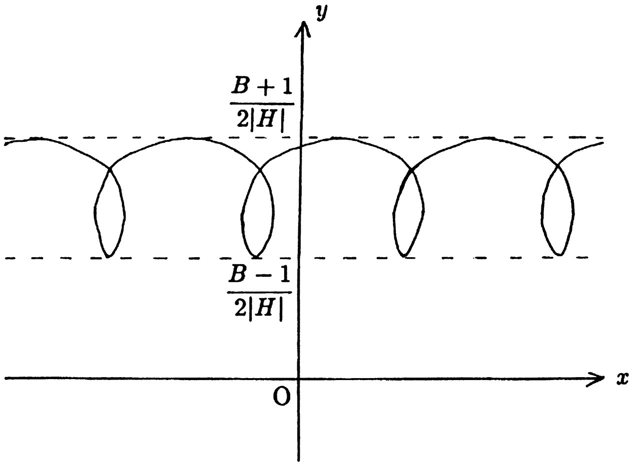
Note that

$$y(s) \leqq \frac{1+B}{2|H|} < 1 \quad in\ \mathbb{R} \tag{1.15}$$

if $|H| > 1$ and $0 \leqq B \leqq 1$.

On the other hand, if $|H| > 1$ and $B > 1$, the generating curve $X(s;H,B)$ is as in [Fig. 4] and

$$\frac{B-1}{2|H|} \leqq y(s) \leqq \frac{B+1}{2|H|} \quad in\ \mathbb{R} \tag{1.16}$$



[Fig. 4]

(see [19]).

Therefore, if there exists a solution to (1.1)–(1.5) of the form (1.9) for $|H| > 1$, we have

$$\frac{B-1}{2|H|} \leqq \min\{1, a\} \leqq \max\{1, a\} \leqq \frac{B+1}{2|H|}$$

for some $B > 1$. Hence we have

$$\frac{B+1}{2|H|} - \frac{B-1}{2|H|} \geqq |1-a|,$$

that is to say

$$\frac{1}{|H|} \geqq |1-a|. \tag{1.17}$$

However, (1.17) contradicts to (1.13). Hence we have proved the theorem. □

Now, to show the nonuniqueness of the pair (ρ, X) satisfying (1.1)–(1.5), we prepare the following:

PROPOSITION 1.5. **(1)** *Let $a, b, H \in \mathbb{R}$ satisfy*

$$0 < |H| \leqq 1, \ a > 0, \ bH < 0, \tag{1.18}$$

$$a^2 + \left(b + \sqrt{\frac{1-H^2}{H^2}}\right)^2 = \frac{1}{H^2}. \tag{1.19}$$

Let

$$\rho = \frac{|H|}{1 + \frac{|H|}{H}\sqrt{1-H^2}} \sqrt{\frac{1 + \frac{|H|}{H}\sqrt{1-H^2} + bH}{1 - \frac{|H|}{H}\sqrt{1-H^2} - bH}}. \tag{1.20}$$

Then we have

$$\rho \in (0,1), \tag{1.21}$$

and for this ρ, there exists a solution X to (1.1)–(1.5) of the form (1.9) with

$$f(r) = \frac{2(1+\sqrt{1-H^2})r^{\frac{|H|}{H}}}{(1+\sqrt{1-H^2})^2 r^{\frac{2|H|}{H}} + H^2}, \tag{1.22}$$

$$g(r) = \frac{1}{|H|} - \sqrt{\frac{1-H^2}{H^2}} - \frac{2|H|}{(1+\sqrt{1-H^2})^2 r^{\frac{2|H|}{H}} + H^2}. \tag{1.23}$$

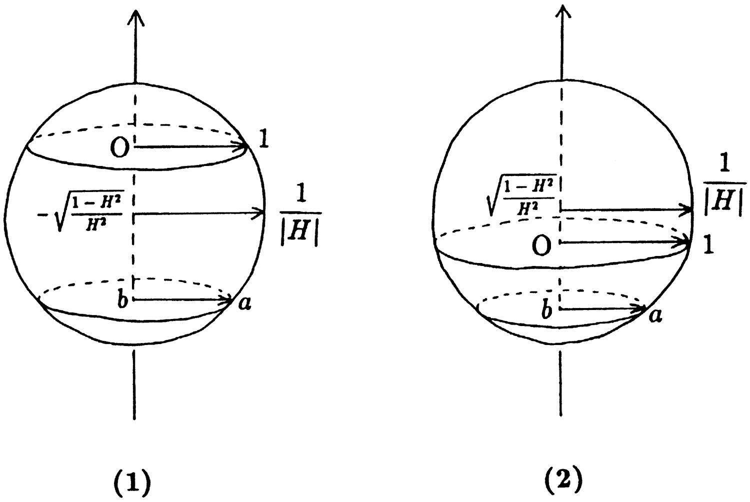
Moreover, then we have

$$X(A_\rho) = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + \left(z + \sqrt{\frac{1-H^2}{H^2}}\right)^2 = \frac{1}{H^2}, \right. \\ \left. \min\{b, 0\} < z < \max\{b, 0\} \right\} \tag{1.24}$$

(see [Fig. 5 (1)] *).*

(2) *Let $a, b, H \in \mathbb{R}$ satisfy*

$$0 < |H| \leqq 1, \ a > 0, \ bH < 0 \tag{1.25}$$



[Fig. 5]

and

$$(1.26)\qquad a^2 + \left(b - \sqrt{\frac{1-H^2}{H^2}}\right)^2 = \frac{1}{H^2}.$$

Let

$$(1.27)\qquad \rho = \frac{|H|}{1 - \frac{|H|}{H}\sqrt{1-H^2}} \sqrt{\frac{1 - \frac{|H|}{H}\sqrt{1-H^2} + bH}{1 + \frac{|H|}{H}\sqrt{1-H^2} - bH}}.$$

Then we have

$$(1.28)\qquad \rho \in (0,1),$$

and for this ρ*, there exists a solution* X *to (1.1)–(1.5) of the form (1.9) with*

$$(1.29)\qquad f(r) = \frac{2(1+\sqrt{1-H^2})r^{\frac{|H|}{H}}}{(1+\sqrt{1-H^2})^2 + H^2 r^{\frac{2|H|}{H}}},$$

$$(1.30)\qquad g(r) = -\frac{1}{|H|} + \sqrt{\frac{1-H^2}{H^2}} + \frac{2|H|r^{\frac{2|H|}{H}}}{(1+\sqrt{1-H^2})^2 + H^2 r^{\frac{2|H|}{H}}}.$$

Moreover, then we have

$$(1.31)\quad X(A_\rho) = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + \left(z - \sqrt{\frac{1-H^2}{H^2}}\right)^2 = \frac{1}{H^2},$$
$$\min\{b,0\} < z < \max\{b,0\}\ \}$$

(see [Fig. 5 (2)] *).*

Proposition 1.6. *For some $a, b, H \in \mathbb{R}$, there are two or more different pairs of $\rho \in (0,1)$ and X which satisfy (1.1)–(1.5).*

Proof. For example, let $H = \frac{1}{2}, a = 1$ and $b = -2\sqrt{3}$. Let $\rho_1 = e^{-2\sqrt{3}}$ and $\rho_2 = (2-\sqrt{3})^2$. For $i = 1, 2$, let

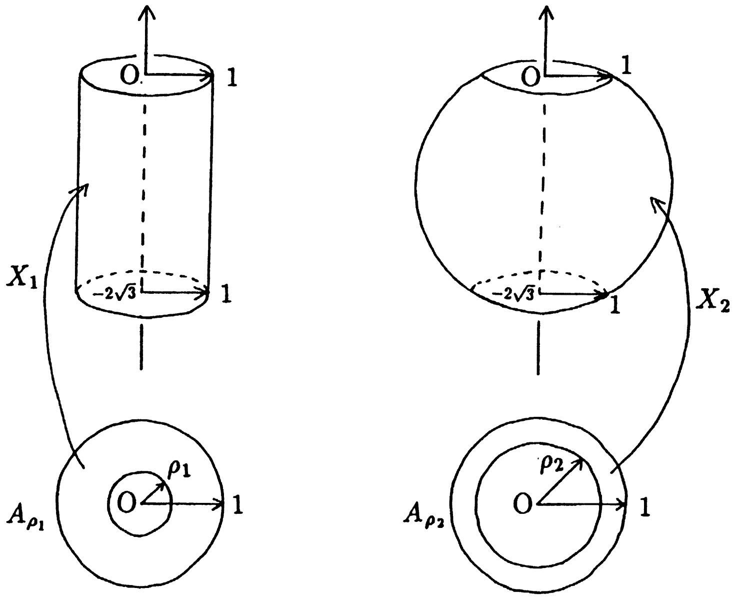
$$X_i(w) := \left(f_i(|w|)\frac{u}{|w|}, f_i(|w|)\frac{v}{|w|}, g_i(|w|)\right) \quad for\ w = (u,v) \in \bar{A}_{\rho_i},$$

where

$$f_1(r) = 1, \qquad f_2(r) = \frac{4r}{(2+\sqrt{3})r^2 + (2-\sqrt{3})},$$
$$g_1(r) = \log r, \qquad g_2(r) = \frac{r^2 - 1}{(2+\sqrt{3})r^2 + (2-\sqrt{3})}.$$

Then $(f, g) = (f_1, g_1)$ clearly satisfies (2.1)–(2.4) below for $\rho = \rho_1$. Hence (1.1)–(1.5) hold for $(\rho, X) = (\rho_1, X_1)$. On the other hand, by Proposition 1.5.(1), (1.1)–(1.5) hold for $(\rho, X) = (\rho_2, X_2)$ (see [Fig. 6] and Remark 1.7 below). □

Remark 1.7. Here we restrict ourselves to surfaces of constant mean curvature. Under the normalization that a Jordan curve Γ in $\mathbb{R}^3$ is contained in a closed unit ball about the origin of $\mathbb{R}^3$, Hildebrandt [12] succeeded in proving the existence of disk-type H-surfaces spanning Γ for $|H| \leqq 1$, improving earlier results of Heinz [10] and Werner [35]. This is the best possible result because Heinz [11] proved that there exists no solution for $|H| > 1$ if Γ is a circle of radius 1. In addition, under the same normalization for Γ, even nonuniqueness of H-surfaces spanning Γ ("Rellich's



[Fig. 6]

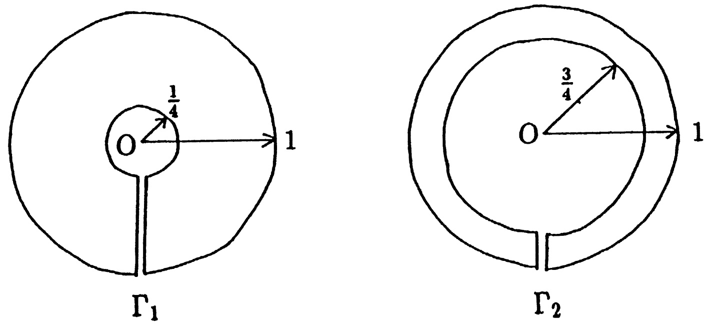
conjecture") was proved for small $|H|$ by Struwe [30] and Steffen [27], and for $0 < |H| < 1$ by Struwe [29] and Brezis-Coron [1]. On the other hand, improving earlier results of Wente [34], Steffen [25] proved the existence of disc-type H-surfaces spanning a Jordan curve Γ in $\mathbb{R}^3$, under the condition that Γ is rectifiable and

$$|H| < \sqrt{\frac{2\pi}{3}\frac{1}{A_\Gamma}}, \tag{1.32}$$

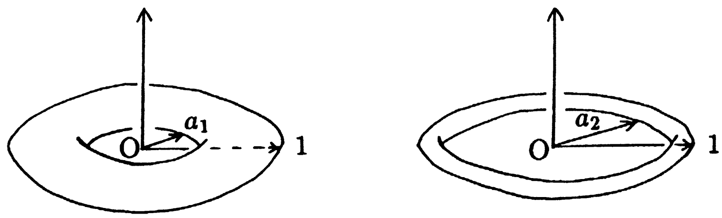
where A_Γ is the infimum of the area functional on

$$\begin{aligned} C(\Gamma) := \{X \in H^{1,2}(B_1;\mathbb{R}^3) \mid \quad & X\big|_{\partial B_1} \in C^0(\partial B_1;\mathbb{R}^3), \\ & X\big|_{\partial B_1} : \partial B_1 \longrightarrow \Gamma \ \textit{ is a weakly monotone} \\ & \qquad \textit{oriented parametrization of } \Gamma.\}. \end{aligned}$$

Note that the constant $\frac{2\pi}{3}$ in (1.32) is not optimal but π is conjectured to be the best possible constant (see Steffen [25; p.122] or Struwe [32; Theorem



[Fig. 7]



[Fig. 8] ($\frac{3}{4} < a_1 < a_2 < 1$).

III.3.4. and remarks after that]). For example, $A_{\Gamma_1} > A_{\Gamma_2}$ in [Fig. 7]. Therefore, by (1.32), we expect that we can span Γ_2 with H-surfaces with larger $|H|$ than any absolute values of the mean curvatures of H-surfaces spanning Γ_1.

REMARK 1.8. The conditions for $|H|$ in our existence theorems for annulus-type H-surfaces correspond to (1.32) in Steffen's existence theorem for disk-type H-surfaces. For example, let $b = 0$ and $\frac{3}{4} < a_1 < a_2 < 1$ (or $1 < a_2 < a_1 < \frac{3}{2}$). Then the right-hand sides of (1.7) and (1.8) (or (1.11) and (1.12)) tend to infinity as a_2 approaches 1. Hence we expect that we can span $\Gamma(1, 0)$ and $\Gamma(a_2, 0)$ with H-surfaces with larger $|H|$ than any absolute values of the mean curvatures of H-surfaces spanning $\Gamma(1, 0)$ and $\Gamma(a_1, 0)$ (see [Fig. 8] and compare it with [Fig. 7]).

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2. Proof of the main results

In this section, we prove Theorem 1.1, Theorem 1.2 and Proposition 1.5. Before proving them, we introduce some notations: For $\rho \in (0,1)$, let

$$I_\rho := (\rho, 1), \quad J_\rho := (\log \rho, 0).$$

Now suppose there exist a real number $\rho \in (0,1)$ and a solution $X \in C^\infty(\bar{A}_\rho; \mathbb{R}^3)$ to (1.2)–(1.5) of the form

$$X(r\cos\theta, r\sin\theta) = (f(r)\cos\theta, f(r)\sin\theta, g(r)) \qquad (\rho \leqq r \leqq 1, 0 \leqq \theta < 2\pi), \tag{1.9}$$

for some $f,\ g \in C^\infty(\bar{I}_\rho)$. Then, introducing polar coordinates, (1.2)–(1.5) can be reduced to a system of ordinary differential equations with boundary conditions:

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{1}{r^2} f = -\frac{2H}{r} f \frac{dg}{dr} \quad in\ I_\rho, \tag{2.1}$$

$$\frac{d^2 g}{dr^2} + \frac{1}{r}\frac{dg}{dr} = \frac{2H}{r} f \frac{df}{dr} \quad in\ I_\rho, \tag{2.2}$$

$$\left(\frac{df}{dr}\right)^2 + \left(\frac{dg}{dr}\right)^2 = \frac{1}{r^2} f^2 \quad in\ I_\rho, \tag{2.3}$$

$$f(1) = 1,\ f(\rho) = a,\ g(1) = 0,\ g(\rho) = b. \tag{2.4}$$

Moreover, let

$$\tilde{f}(t) := f(e^t), \quad \tilde{g}(t) := g(e^t). \tag{2.5}$$

Then (2.1)–(2.4) can be reduced to another system:

$$\frac{d^2\tilde{f}}{dt^2} - \tilde{f} = -2H\tilde{f}\frac{d\tilde{g}}{dt} \quad \text{in } J_\rho, \tag{2.6}$$

$$\frac{d^2\tilde{g}}{dt^2} = 2H\tilde{f}\frac{d\tilde{f}}{dt} \quad \text{in } J_\rho, \tag{2.7}$$

$$\left(\frac{d\tilde{f}}{dt}\right)^2 + \left(\frac{d\tilde{g}}{dt}\right)^2 = \tilde{f}^2 \quad \text{in } J_\rho, \tag{2.8}$$

$$\tilde{f}(0) = 1,\ \tilde{f}(\log\rho) = a,\ \tilde{g}(0) = 0,\ \tilde{g}(\log\rho) = b. \tag{2.9}$$

Conversely, if there exist $\tilde{f},\ \tilde{g} \in C^\infty(\bar{J}_\rho)$ satisfying (2.6)–(2.9) for some $\rho \in (0,1)$, then $X \in C^\infty(\bar{A}_\rho; \mathbb{R}^3)$ of the form (1.9) for f and g determined by (2.5) clearly satisfies (1.2)–(1.5).

Therefore, to prove Theorem 1.1 and Theorem 1.2, it is sufficient to prove that there exist $\rho \in (0,1)$, $\tilde{f} \in C^2(\bar{J}_\rho)$ and $\tilde{g} \in C^2(\bar{J}_\rho)$ satisfying (2.6)–(2.9), since it is clear that $\tilde{f},\ \tilde{g} \in C^\infty(\bar{J}_\rho)$ by (2.6) and (2.7).

Lemma 2.1. *Let $H = 0$. Suppose $a > 0$ and $(a,b) \neq (1,0)$. Then the necessary and sufficient condition for $\rho \in (0,1)$, $\tilde{f} \in C^2(\bar{J}_\rho)$ and $\tilde{g} \in C^2(\bar{J}_\rho)$ satisfying (2.6)–(2.9) to exist is that a and b satisfy the inequality*

$$b^2 \leqq \sup\left\{\frac{4(r^{-1}-a)(a-r)(\log r)^2}{(r^{-1}-r)^2} \;\middle|\; 0 < r \leqq \min\{a, a^{-1}\}\right\} =: m(a). \tag{2.10}$$

Proof. Since $H = 0$, by (2.6), (2.7) and (2.9) we have

$$\tilde{f}(t) = \frac{(\rho^{-1}-a)e^t + (a-\rho)e^{-t}}{\rho^{-1}-\rho}, \qquad \tilde{g}(t) = \frac{b}{\log\rho}t.$$

Hence, by (2.8) we have

$$\frac{\{(\rho^{-1}-a)e^t - (a-\rho)e^{-t}\}^2}{(\rho^{-1}-\rho)^2} + \left(\frac{b}{\log\rho}\right)^2 = \frac{\{(\rho^{-1}-a)e^t + (a-\rho)e^{-t}\}^2}{(\rho^{-1}-\rho)^2}.$$

Hence we have

$$b^2 = \frac{4(\rho^{-1} - a)(a - \rho)(\log \rho)^2}{(\rho^{-1} - \rho)^2} =: h(\rho).$$

Note that

$$\begin{aligned} h(\rho) &= \frac{16(1 - a\rho)(a - \rho)(\sqrt{\rho}\log\sqrt{\rho})^2}{(1 - \rho^2)^2} \\ &= \frac{16(1 - a\rho)(a - \rho)}{(1 - \rho^2)^2} \left\{ \frac{(-\log\sqrt{\rho})}{e^{-\log\sqrt{\rho}}} \right\}^2 \\ &\longrightarrow 0 \quad (\ as\ \rho \longrightarrow +0). \end{aligned}$$

On the other hand,

$$h(\rho) \longrightarrow 0 \quad (\ as\ \rho \longrightarrow \min\{a, a^{-1}\}).$$

Hence there exists

$$\rho_0 \in \big(0, \min\{a, a^{-1}\}\big) \subset (0, 1)$$

such that

$$h(\rho_0) = m(a).$$

Therefore, the conclusion of the lemma holds. □

Lemma 2.2. *Let $H = 0$. Suppose a and b satisfy the assumptions of Theorem 1.1 or Theorem 1.2. Then (2.10) holds.*

Proof. We define $m(a)$ as in (2.10). Let

$$\varphi(r) := \frac{2\log r\sqrt{(r^{-1} - a)(a - r)}}{r - r^{-1}}.$$

Since $\frac{3}{4} < a < \frac{3}{2}$, we have

$$\sqrt{m(a)} \geqq \varphi\left(\frac{1}{2}\right) = \frac{4}{3}\log 2\sqrt{(2 - a)\left(a - \frac{1}{2}\right)}. \tag{2.11}$$

Now suppose a and b satisfy (1.6)–(1.8) for $H = 0$. Then, by (1.6) and (1.7) we have

$$|b| < (1-a)(2a-1) = -2\left(a - \frac{3}{4}\right)^2 + \frac{1}{8} < \frac{1}{8}.$$

But, by (2.11) we have

$$\sqrt{m(a)} > \frac{4}{3} \cdot \frac{1}{2}\sqrt{(2-1)\left(\frac{3}{4} - \frac{1}{2}\right)} = \frac{1}{3}.$$

Hence we have

$$\sqrt{m(a)} > |b|.$$

Therefore, (2.10) holds.

On the other hand, suppose a and b satisfy (1.10)–(1.12) for $H = 0$. Then by (1.10) and (1.11) we have

$$|b| < \frac{a-1}{2a-1} = \frac{1}{2} - \frac{1}{4a-2} < \frac{1}{4}.$$

But, by (2.11) we have

$$\sqrt{m(a)} > \frac{4}{3} \cdot \frac{1}{2}\sqrt{\left(2 - \frac{3}{2}\right)\left(1 - \frac{1}{2}\right)} = \frac{1}{3}.$$

Hence we have

$$\sqrt{m(a)} > |b|.$$

Therefore, (2.10) holds. □

Now, to prove Theorem 1.1, we prepare the following:

Lemma 2.3. *Let $a, b, H \in \mathbb{R}$ satisfy (1.6)–(1.8). Suppose $H \neq 0$. Let*

$$a_1 = 2a - 1,\ \rho_1 = e^{-(1-a_1)},\ \rho_2 = e^{-\frac{1-a_1}{2}},\ \rho_0 = \frac{1}{2}(\rho_1 + \rho_2)$$

and let

$$a_2 = \sqrt{\frac{1}{2}\left({a_1}^2 + \frac{1}{|H|}a_1 + \frac{|b|}{|H|\log \rho_2} + 1\right)}.$$

Then,

$$0 < \rho_1 < \rho_0 < \rho_2 < 1, \tag{2.12}$$

$$0 < a_1 < 1 < a_2, \tag{2.13}$$

$${a_1}^2 > H^2 \left({a_1}^2 - {a_2}^2 + \frac{|b|}{|H| \log \rho_2} \right)^2, \tag{2.14}$$

$$\frac{1 - a_1}{2} + \log \rho_1 \sqrt{{a_1}^2 - H^2 \left({a_1}^2 - 1 + \frac{|b|}{|H| \log \rho_1} \right)^2} \leqq 0. \tag{2.15}$$

Proof. By (1.6) we have

$$\frac{1}{2} < a_1 < 1.$$

Hence we have

$$1 - a_1 > 0.$$

Hence (2.12) holds.

By (1.7) we have

$$|H| < \frac{a_1(1 - a_1) - 2|b|}{(1 - {a_1}^2)(1 - a_1)}.$$

Since $H \neq 0$ and $1 - {a_1}^2 > 0$, we have

$$1 - {a_1}^2 < \frac{1}{|H|} a_1 + \frac{|b|}{|H| \log \rho_2}.$$

Hence we have $1 < {a_2}^2$ and so (2.13) holds. Moreover, since

$$\begin{aligned} {a_2}^2 = & \left({a_1}^2 + \frac{1}{|H|} a_1 + \frac{|b|}{|H| \log \rho_2} \right) \\ & - \frac{1}{2} \left({a_1}^2 + \frac{1}{|H|} a_1 + \frac{|b|}{|H| \log \rho_2} - 1 \right), \end{aligned}$$

we have

$$\left(- \frac{1}{|H|} a_1 \right)^2$$

$$= \left\{ \left({a_1}^2 - {a_2}^2 + \frac{|b|}{|H| \log \rho_2} \right) - \frac{1}{2} \left({a_1}^2 + \frac{1}{|H|} a_1 + \frac{|b|}{|H| \log \rho_2} - 1 \right) \right\}^2$$
$$> \left({a_1}^2 - {a_2}^2 + \frac{|b|}{|H| \log \rho_2} \right)^2 .$$

Hence (2.14) holds.

By (1.8) we have

$$|H| \leqq \frac{(1-a_1)\sqrt{{a_1}^2 - \frac{1}{4}} - |b|}{(1-{a_1}^2)(1-a_1)}.$$

Then, we have

$$\left\{ |H| \left(1 - {a_1}^2 + \frac{|b|}{|H|(1-a_1)} \right) \right\}^2 \leqq \left(\sqrt{{a_1}^2 - \frac{1}{4}} \right)^2 .$$

Hence we have

$$\sqrt{\frac{1}{4}} \leqq \sqrt{{a_1}^2 - H^2 \left({a_1}^2 - 1 + \frac{|b|}{|H| \log \rho_1} \right)^2 }.$$

Since $1 - a_1 > 0$, (2.15) holds. □

Proof of Theorem 1.1. We prove this theorem, using the Leray-Schauder degree theory. To define the Leray-Schauder degree, we introduce some notations first. By Lemma 2.1 and Lemma 2.2, it is sufficient to prove only for the case of $H \neq 0$. Then, take a_1, a_2, ρ_1, ρ_2 and ρ_0 as in Lemma 2.3 and let

$$L := \frac{a_1 - 1}{2 \log \rho_1} + 2,$$
$$M := \left\{ v \in C^1(\bar{J}_{\rho_1}) \;\middle|\; a_1 < v < a_2,\ 0 < \frac{dv}{dt} < L \right\},$$
$$\Omega := (\rho_1, \rho_2) \times M.$$

For $(\rho, v) \in \bar{\Omega}$, let

$$h(\rho, v) := \frac{1}{\log \rho} \left\{ \int_0^{\log \rho} v(s)^2 ds - \frac{b}{H} \right\}.$$

Then, on $\bar{J}_{\rho_1}$, by (2.14) we have

$$\begin{aligned} &v^2 - H^2 \left\{ v^2 - h(\rho, v) \right\}^2 \\ &> {a_1}^2 - H^2 \left\{ {a_1}^2 - \left({a_2}^2 + \left| \frac{b}{H \log \rho} \right| \right) \right\}^2 \\ &\geqq {a_1}^2 - H^2 \left({a_1}^2 - {a_2}^2 + \frac{|b|}{|H| \log \rho_2} \right)^2 \\ &> 0. \end{aligned} \tag{2.16}$$

Hence, for $(\rho, v) \in \bar{\Omega}$ and $\lambda \in [0, 1]$ we can define $\varphi_\lambda : \bar{\Omega} \longrightarrow C^1(\bar{J}_{\rho_1})$ such that

$$\begin{aligned} \varphi_\lambda(\rho, v) = (1 - \lambda) &\left(1 + \frac{a_1 - 1}{2 \log \rho_1} t \right) \\ &+ \lambda \left(1 + \int_0^t \sqrt{v(s)^2 - H^2 (v(s)^2 - h(\rho, v))^2} ds \right). \end{aligned}$$

Moreover, we put

$$\begin{aligned} &\zeta_\lambda(\rho, v) = (1 - \lambda)\rho_0 + \lambda(\rho + a - v(\log \rho)), \\ &I(\rho, v) = (\rho, v), \\ &G_\lambda(\rho, v) = (\zeta_\lambda(\rho, v), \varphi_\lambda(\rho, v)), \\ &F_\lambda = I - G_\lambda. \end{aligned}$$

Now we prepare the following:

LEMMA 2.4. *Under the conditions as above, assume that* $F_1(\rho, v) = 0$ *for some* $(\rho, v) \in \Omega$. *For such* (ρ, v), *let*

$$\tilde{f}(t) = v(t) \quad in\ \bar{J}_{\rho_1}, \tag{2.17}$$

$$\tilde{g}(t) = \int_0^t H(v(s)^2 - h(\rho, v)) ds \quad in\ \bar{J}_{\rho_1}. \tag{2.18}$$

Then, for this $\rho \in (0,1)$, $\tilde f$ and $\tilde g$ are smooth solutions to (2.6)–(2.9).

Proof. By (2.17) and (2.18) we have

$$\tilde g(t) = \int_0^t H(\tilde f(s)^2 - h(\rho, \tilde f))ds. \tag{2.19}$$

Hence (2.7) holds.

Since $F_1(\rho, \tilde f) = 0$, by (2.19) we have

$$\tilde f(\log\rho) = a, \tag{2.20}$$

$$\tilde f(t) = 1 + \int_0^t \sqrt{\tilde f(s)^2 - \left(\frac{d\tilde g}{dt}(s)\right)^2} ds. \tag{2.21}$$

By (2.21), (2.8) holds.

Since $(\rho, \tilde f) \in \Omega$, by (2.21) we have

$$0 < \frac{d\tilde f}{dt} = \sqrt{\tilde f^2 - \left(\frac{d\tilde g}{dt}\right)^2}.$$

Hence, by (2.7) we have

$$\begin{aligned}\frac{d^2\tilde f}{dt^2} &= \frac{2\tilde f\frac{d\tilde f}{dt} - 2\frac{d\tilde g}{dt}\frac{d^2\tilde g}{dt^2}}{2\frac{d\tilde f}{dt}} \\ &= \tilde f - 2H\tilde f\frac{d\tilde g}{dt}.\end{aligned}$$

Hence (2.6) holds.

By (2.19) we have

$$\begin{aligned}\tilde g(\log\rho) &= H\int_0^{\log\rho} \tilde f(s)^2 ds - H\left\{\int_0^{\log\rho} \tilde f(s)^2 ds - \frac{b}{H}\right\} \\ &= b.\end{aligned}$$

Hence, by (2.19)–(2.21), (2.9) holds.

By (2.6) and (2.7), $\tilde f$ and $\tilde g$ are clearly smooth. □

By (2.16) and the Ascoli-Arzelà theorem,

$$G_\lambda : \bar{\Omega} \longrightarrow \mathbb{R} \times C^1(\bar{J}_{\rho_1})$$

is a compact continuous map for any $\lambda \in [0,1]$. Hence, to define the Leray-Schauder degree $\deg(F_\lambda, 0, \Omega)$, it is sufficient to show the following:

LEMMA 2.5. *Under the conditions as above, $F_\lambda \neq 0$ on $\partial\Omega$ for any $\lambda \in [0,1]$.*

PROOF Since

$$F_0(\rho, v) = 0$$

if and only if

$$(\rho, v) = \left(\rho_0, 1 + \frac{a_1 - 1}{2\log\rho_1} t\right) \in \Omega,$$

we have

$$F_0 \neq 0 \quad \text{on } \partial\Omega.$$

Now we assume that $F_\lambda(\rho, v) = 0$, $(\rho, v) \in \bar{\Omega}$ and $0 < \lambda \leqq 1$. Then we have

$$\begin{aligned} v(t) =& (1-\lambda)\left(1 + \frac{a_1 - 1}{2\log\rho_1} t\right) \\ &+ \lambda\left(1 + \int_0^t \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho, v))^2} ds\right), \end{aligned} \tag{2.22}$$

$$(1-\lambda)(\rho - \rho_0) = \lambda(a - v(\log\rho)). \tag{2.23}$$

Since $v \in \bar{M}$, we have

$$\frac{dv}{dt} \geqq 0.$$

Hence we have

$$v(t) \leqq v(0) = 1 < a_2 \quad \textit{for any } t \in \bar{J}_{\rho_1}.$$

On the other hand, by (2.16) and (2.22) we have

$v(t)$

$$\begin{aligned}
&\geqq v(\log\rho_1)\\
&= (1-\lambda)\frac{a_1+1}{2} + \lambda\left(1 + \int_0^{\log\rho_1} \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho,v))^2}ds\right)\\
&= \frac{a_1+1}{2} + \lambda\left(\frac{1-a_1}{2} + \int_0^{\log\rho_1} \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho,v))^2}ds\right)\\
&> \frac{a_1+1}{2} + \lambda\left(\frac{1-a_1}{2} + \log\rho_1\right)\\
&= \frac{a_1+1}{2} - \lambda\frac{1-a_1}{2}\\
&\geqq a_1 \quad for\ any\ t \in \bar{J}_{\rho_1}.
\end{aligned}$$

Hence we have

$$v(t) > a_1 \quad for\ any\ t \in \bar{J}_{\rho_1}.$$

By (2.22) we have

$$\frac{dv}{dt} = (1-\lambda)\frac{a_1-1}{2\log\rho_1} + \lambda\sqrt{v^2 - H^2(v^2 - h(\rho,v))^2}.$$

Hence, by (2.16) we have

$$\frac{dv}{dt} > 0.$$

On the other hand, we have

$$\frac{dv}{dt} \leqq \frac{a_1-1}{2\log\rho_1} + 1 < L.$$

Now we assume $\rho = \rho_1$. Then, by (2.23) we have

$$0 \geqq (1-\lambda)(\rho_1 - \rho_0) = \lambda(a - v(\log\rho_1)).$$

Hence we have

$$v(\log\rho_1) \geqq a.$$

But, by (2.15) and (2.22) we have

$$v(\log\rho_1)$$

$$
\begin{aligned}
&= (1-\lambda)\frac{a_1+1}{2} + \lambda\left(1 + \int_0^{\log\rho_1} \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho_1, v))^2} ds\right) \\
&= a + \lambda\left(\frac{1-a_1}{2} + \int_0^{\log\rho_1} \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho_1, v))^2} ds\right) \\
&\leqq a + \lambda\left[\frac{1-a_1}{2} + (\log\rho_1)\min\left\{\sqrt{v(0)^2 - H^2(v(0)^2 - h(\rho_1, v))^2}\ ,\right.\right. \\
&\left.\left.\sqrt{v(\log\rho_1)^2 - H^2(v(\log\rho_1)^2 - h(\rho_1, v))^2}\right\}\right] \\
&\leqq a + \lambda\left[\frac{1-a_1}{2}\right. \\
&+ (\log\rho_1)\min\left\{\sqrt{v(0)^2 - H^2\left\{v(0)^2 - \left(v(\log\rho_1)^2 - \left|\frac{b}{H\log\rho_1}\right|\right)\right\}^2},\right. \\
&\left.\left.\sqrt{v(\log\rho_1)^2 - H^2\left\{v(\log\rho_1)^2 - \left(v(0)^2 + \left|\frac{b}{H\log\rho_1}\right|\right)\right\}^2}\right\}\right] \\
&= a + \lambda\left(\frac{1-a_1}{2}\right. \\
&\left.+ \log\rho_1\sqrt{v(\log\rho_1)^2 - H^2\left\{v(\log\rho_1)^2 - \left(1 + \left|\frac{b}{H\log\rho_1}\right|\right)\right\}^2}\right) \\
&< a + \lambda\left(\frac{1-a_1}{2} + \log\rho_1\sqrt{{a_1}^2 - H^2\left({a_1}^2 - 1 + \frac{|b|}{|H|\log\rho_1}\right)^2}\right) \\
&\leqq a.
\end{aligned}
$$

This contradiction implies that the assumptipon of $\rho = \rho_1$ is false, and we see $\rho \neq \rho_1$. On the other hand, we assume $\rho = \rho_2$. Then, by (2.23) we have

$$0 \leqq (1-\lambda)(\rho_2 - \rho_0) = \lambda(a - v(\log\rho_2)).$$

Hence we have

$$v(\log\rho_2) \leqq a.$$

But, by (2.22) we have

$$v(\log\rho_2)$$

$$
\begin{aligned}
&= (1-\lambda)\left(1+\frac{(a_1-1)\log\rho_2}{2\log\rho_1}\right)\\
&\qquad\qquad + \lambda\left(1+\int_0^{\log\rho_2}\sqrt{v(s)^2-H^2(v(s)^2-h(\rho_2,v))^2}ds\right)\\
&\geqq (1-\lambda)\left(1+\frac{a_1-1}{2}\right)\\
&\qquad\qquad + \lambda\left(1+\int_0^{\log\rho_2}\sqrt{v(s)^2-H^2(v(s)^2-h(\rho_2,v))^2}ds\right)\\
&= \frac{a_1+1}{2}+\lambda\left(\frac{1-a_1}{2}+\int_0^{\log\rho_2}\sqrt{v(s)^2-H^2(v(s)^2-h(\rho_2,v))^2}ds\right)\\
&> a+\lambda\left(\frac{1-a_1}{2}+\log\rho_2\right)\\
&= a.
\end{aligned}
$$

This contradiction implies that the assumption of $\rho = \rho_2$ is false, and we see $\rho \neq \rho_2$.

Therefore, we have proved that $(\rho, v) \notin \partial\Omega$. Hence we have

$$F_\lambda \neq 0 \ \ on\ \partial\Omega\ for\ any\ \lambda \in (0,1]. \ \square$$

By Lemma 2.5, we can define the Leray-Schauder degree $\deg(F_\lambda, 0, \Omega)$ for any $\lambda \in [0,1]$, and by its homotopy invariance we have

$$\deg(F_1, 0, \Omega) = \deg(F_0, 0, \Omega).$$

Moreover, since

$$F_0 = I - \left(\rho_0, 1 + \frac{a_1-1}{2\log\rho_1}t\right),$$

we have

$$\deg(F_0, 0, \Omega) = 1.$$

Hence we have

$$\deg(F_1, 0, \Omega) \neq 0.$$

Hence there exists $(\rho, v) \in \Omega$ such that $F_1(\rho, v) = 0$. Therefore, by Lemma 2.4 we can conclude the proof of Theorem 1.1 for the case of $H \neq 0$. $\square$

We next prove Theorem 1.2. For that purpose, we prepare the following:

Lemma 2.6. *Let $a, b, H \in \mathbb{R}$ satisfy (1.10)–(1.12). Suppose $H \neq 0$. Let*

$$a_2 = 2a - 1,\ \rho_1 = e^{-\frac{a_2-1}{a_2}},\ \rho_2 = e^{-\frac{a_2-1}{2a_2}},\ \rho_0 = \frac{1}{2}(\rho_1 + \rho_2)$$

and let

$$a_1 = \frac{1}{2|H|}\left\{-1 + \sqrt{\frac{1}{2}\left\{(2|H|+1)^2 + 1 + 4|H|\left(|H|{a_2}^2 - \frac{|b|}{\log \rho_2}\right)\right\}}\right\}.$$

Then,

$$0 < \rho_1 < \rho_0 < \rho_2 < 1, \tag{2.24}$$

$$0 < a_1 < 1 < a_2, \tag{2.25}$$

$${a_1}^2 > H^2\left({a_1}^2 - {a_2}^2 + \frac{|b|}{|H|\log \rho_2}\right)^2, \tag{2.26}$$

$$\frac{a_2 - 1}{2} + \log \rho_1 \sqrt{1 - H^2\left(1 - {a_2}^2 + \frac{|b|}{|H|\log \rho_1}\right)^2} \leqq 0. \tag{2.27}$$

Proof. By (1.10) we have

$$1 < a_2 < 2.$$

Hence we have

$$a_2 - 1 > 0.$$

Hence (2.24) holds.

Since $H \neq 0$, we have

$$a_1 > 0.$$

By (1.11) we have

$$|H| < \frac{(a_2 - 1) - 2a_2|b|}{(a_2 + 1)(a_2 - 1)^2}.$$

Then, we have

$$|H|({a_2}^2 - 1) < 1 - \frac{2a_2|b|}{a_2 - 1}.$$

Hence we have

$$(2|H|+1)^2 > 1 + 4|H|\left(|H|{a_2}^2 + \frac{2a_2|b|}{a_2-1}\right).$$

Hence we have

$$\begin{aligned} a_1 &< \frac{1}{2|H|}\left\{-1+\sqrt{(2|H|+1)^2}\right\} \\ &= 1, \end{aligned}$$

which shows (2.25).

Moreover, since

$$(2|H|a_1+1)^2 > 1 + 4|H|\left(|H|{a_2}^2 - \frac{|b|}{\log \rho_2}\right),$$

we have

$$4|H|(|H|{a_1}^2 + a_1) > 4|H|\left(|H|{a_2}^2 - \frac{|b|}{\log \rho_2}\right).$$

Dividing both sides by $4|H| > 0$, we have

$$|H|{a_1}^2 + a_1 > |H|{a_2}^2 - \frac{|b|}{\log \rho_2}.$$

Hence we have

$$\begin{aligned} a_1 &> -|H|\left({a_1}^2 - {a_2}^2 + \frac{|b|}{|H|\log \rho_2}\right) \\ &> 0, \end{aligned}$$

which shows (2.26).

By (1.12) we have

$$|H| \leqq \frac{(a_2-1)\sqrt{1-\frac{{a_2}^2}{4}} - a_2|b|}{({a_2}^2-1)(a_2-1)}.$$

Then, we have

$$\left\{|H|\left(a_2{}^2-1+\frac{a_2|b|}{|H|(a_2-1)}\right)\right\}^2 \leqq \left(\sqrt{1-\frac{a_2{}^2}{4}}\right)^2.$$

Hence we have

$$\sqrt{\frac{a_2{}^2}{4}} \leqq \sqrt{1-H^2\left(a_2{}^2-1-\frac{|b|}{|H|\log\rho_1}\right)^2}.$$

Since $a_2-1>0$, (2.27) holds. □

Proof of Theorem 1.2. We proceed in the same way as in the proof of Theorem 1.1. By Lemma 2.1 and Lemma 2.2, it is sufficient to prove only for the case of $H \neq 0$. Then, take a_1, a_2, ρ_1, ρ_2 and ρ_0 as in Lemma 2.8 and let

$$\begin{aligned}
L &:= -\frac{a_2-1}{2\log\rho_1}+a_2+1,\\
M &:= \left\{v \in C^1(\bar{J}_{\rho_1}) \;\middle|\; a_1 < v < a_2, -L < \frac{dv}{dt} < 0\right\},\\
\Omega &:= (\rho_1,\rho_2)\times M.
\end{aligned}$$

For $(\rho, v) \in \bar{\Omega}$, let

$$h(\rho,v) := \frac{1}{\log\rho}\left\{\int_0^{\log\rho} v(s)^2 ds - \frac{b}{H}\right\}.$$

Then, on $\bar{J}_{\rho_1}$, by (2.26) we have

$$\begin{aligned}
(2.28)\qquad & v^2 - H^2\left\{v^2-h(\rho,v)\right\}^2\\
&> a_1{}^2 - H^2\left\{a_1{}^2-\left(a_2{}^2+\left|\frac{b}{H\log\rho}\right|\right)\right\}^2\\
&\geqq a_1{}^2 - H^2\left(a_1{}^2-a_2{}^2+\frac{|b|}{|H|\log\rho_2}\right)^2\\
&> 0.
\end{aligned}$$

Hence, for $(\rho, v) \in \bar{\Omega}$ and $\lambda \in [0,1]$ we can define $\varphi_\lambda : \bar{\Omega} \longrightarrow C^1(\bar{J}_{\rho_1})$ such that

$$\varphi_\lambda(\rho, v) = (1-\lambda)\left(1 + \frac{a_2 - 1}{2\log\rho_1} t\right) + \lambda\left(1 - \int_0^t \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho, v))^2} ds\right).$$

Moreover, we define

$$\begin{aligned} &\zeta_\lambda(\rho, v) = (1-\lambda)\rho_0 + \lambda(\rho + v(\log\rho) - a), \\ &I(\rho, v) = (\rho, v), \\ &G_\lambda(\rho, v) = (\zeta_\lambda(\rho, v), \varphi_\lambda(\rho, v)), \\ &F_\lambda = I - G_\lambda. \end{aligned}$$

Now we prepare the following:

LEMMA 2.7. *Under the conditions as avove, assume that $F_1(\rho, v) = 0$ for some $(\rho, v) \in \Omega$. For such (ρ, v), let*

$$\tilde{f}(t) = v(t) \quad in\ \bar{J}_{\rho_1}, \tag{2.29}$$

$$\tilde{g}(t) = \int_0^t H(v(s)^2 - h(\rho, v)) ds \quad in\ \bar{J}_{\rho_1}. \tag{2.30}$$

Then, for this $\rho \in (0,1)$, $\tilde{f}$ and $\tilde{g}$ are smooth solutions to (2.6)–(2.9).

PROOF. The proof is the same as that of Lemma 2.4. □

By (2.28) and the Ascoli-Arzelà theorem,

$$G_\lambda : \bar{\Omega} \longrightarrow \mathbb{R} \times C^1(\bar{J}_{\rho_1})$$

is a compact continuous map for any $\lambda \in [0,1]$. Hence, to define the Leray-Schauder degree $\deg(F_\lambda, \Omega, 0)$, it is sufficient to show the following:

LEMMA 2.8. *Under the conditions as above, $F_\lambda \neq 0$ on $\partial\Omega$ for any $\lambda \in [0,1]$.*

PROOF. Since

$$F_0(\rho, v) = 0$$

if and only if

$$(\rho, v) = \left(\rho_0, 1 + \frac{a_2 - 1}{2\log\rho_1} t\right) \in \Omega,$$

we have

$$F_0 \neq 0 \quad on\ \partial\Omega.$$

Now we assume that $F_\lambda(\rho, v) = 0,\ (\rho, v) \in \bar{\Omega}$ and $0 < \lambda \leqq 1$. Then we have

$$\begin{aligned}(2.31)\quad v(t) = (1-\lambda)&\left(1 + \frac{a_2 - 1}{2\log\rho_1} t\right) \\ &+ \lambda\left(1 - \int_0^t \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho, v))^2} ds\right),\end{aligned}$$

$$(2.32) \qquad (1-\lambda)(\rho - \rho_0) = \lambda(v(\log\rho) - a).$$

Since $v \in \bar{M}$, we have

$$\frac{dv}{dt} \leqq 0.$$

Hence we have

$$v(t) \geqq v(0) = 1 > a_1 \quad for\ any\ t \in \bar{J}_{\rho_1}.$$

On the other hand, by (2.28) and (2.31) we have

$$\begin{aligned}&v(t) \\ &\leqq v(\log\rho_1) \\ &= (1-\lambda)\frac{a_2 + 1}{2} + \lambda\left(1 - \int_0^{\log\rho_1} \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho, v))^2} ds\right) \\ &= \frac{a_2 + 1}{2} + \lambda\left(-\frac{a_2 - 1}{2} - \int_0^{\log\rho_1} \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho, v))^2} ds\right) \\ &< \frac{a_2 + 1}{2} + \lambda\left(-\frac{a_2 - 1}{2} - a_2\log\rho_1\right)\end{aligned}$$

$$= \frac{a_2+1}{2} + \lambda \frac{a_2-1}{2}$$
$$\leqq a_2 \quad for\ any\ t \in \bar{J}_{\rho_1}.$$

Hence we have

$$v(t) < a_2 \quad for\ any\ t \in \bar{J}_{\rho_1}.$$

By (2.31) we have

$$\frac{dv}{dt} = (1-\lambda)\frac{a_2-1}{2\log\rho_1} - \lambda\sqrt{v^2 - H^2(v^2 - h(\rho, v))^2}.$$

Hence, by (2.28) we have

$$\frac{dv}{dt} < 0.$$

On the other hand, we have

$$\frac{dv}{dt} \geqq \frac{a_2-1}{2\log\rho_1} - a_2 > -L.$$

Now we assume $\rho = \rho_1$. Then, by (2.32) we have

$$0 \geqq (1-\lambda)(\rho_1 - \rho_0) = \lambda(v(\log\rho_1) - a).$$

Hence we have

$$v(\log\rho_1) \leqq a.$$

But, by (2.27) and (2.31) we have

$$\begin{aligned}
& v(\log\rho_1) \\
&= (1-\lambda)\frac{a_2+1}{2} + \lambda\left(1 - \int_0^{\log\rho_1} \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho_1, v))^2} ds\right) \\
&= \frac{a_2+1}{2} - \lambda\left(\frac{a_2-1}{2} + \int_0^{\log\rho_1} \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho_1, v))^2} ds\right) \\
&> a - \lambda\left(\frac{a_2-1}{2} + \log\rho_1\sqrt{1 - H^2\left(1 - {a_2}^2 + \frac{|b|}{|H|\log\rho_1}\right)^2}\right) \\
&\geqq a.
\end{aligned}$$

This contradiction implies that the assumption of $\rho = \rho_1$ is false, and we see $\rho \neq \rho_1$. On the other hand, we assume $\rho = \rho_2$. Then, by (2.32) we have

$$0 \leqq (1-\lambda)(\rho_2 - \rho_0) = \lambda(v(\log \rho_2) - a).$$

Hence we have

$$v(\log \rho_2) \geqq a.$$

But, by (2.31) we have

$$\begin{aligned}
&v(\log \rho_2) \\
&= (1-\lambda)\left(1 + \frac{(a_2 - 1)\log \rho_2}{2 \log \rho_1}\right) \\
&\qquad + \lambda\left(1 - \int_0^{\log \rho_2} \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho_2, v))^2} ds\right) \\
&\leqq (1-\lambda)\left(1 + \frac{a_2 - 1}{2}\right) \\
&\qquad + \lambda\left(1 - \int_0^{\log \rho_2} \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho_2, v))^2} ds\right) \\
&= \frac{a_2 + 1}{2} - \lambda\left(\frac{a_2 - 1}{2} + \int_0^{\log \rho_2} \sqrt{v(s)^2 - H^2(v(s)^2 - h(\rho_2, v))^2} ds\right) \\
&< a - \lambda\left(\frac{a_2 - 1}{2} + a_2 \log \rho_2\right) \\
&= a.
\end{aligned}$$

This contradiction implies that the assumption of $\rho = \rho_2$ is false, and we see $\rho \neq \rho_2$.

Therefore, we have proved that $(\rho, v) \notin \partial\Omega$. Hence, we have

$$F_\lambda \neq 0 \ \ \textit{on } \partial\Omega \textit{ for any } \lambda \in (0, 1]. \ \square$$

By Lemma 2.8, we can define the Leray-Schauder degree $\deg(F_\lambda, 0, \Omega)$ for any $\lambda \in [0, 1]$, and by its homotopy invariance we have

$$\deg(F_1, 0, \Omega) = \deg(F_0, 0, \Omega).$$

Moreover, since

$$F_0 = I - \left(\rho_0, 1 + \frac{a_2 - 1}{2\log\rho_1} t\right),$$

we have

$$\deg(F_0, 0, \Omega) = 1.$$

Hence we have

$$\deg(F_1, 0, \Omega) \neq 0.$$

Hence, there exists $(\rho, v) \in \Omega$ such that $F_1(\rho, v) = 0$. Therefore, by Lemma 2.7 we can conclude the proof of Theorem 1.2 for the case of $H \neq 0$. □

Finally we state the proof of Proposition 1.5.

PROOF OF PROPOSITION 1.5. **(1)** Suppose X is a solution to (1.1)–(1.5) of the form (1.9). Then f and g satisfy (2.1)–(2.4). On the other hand, X satisfies (1.24) if and only if f and g satisfy the equation

$$f^2 + \left(g + \sqrt{\frac{1 - H^2}{H^2}}\right)^2 = \frac{1}{H^2}. \tag{2.33}$$

Now we look for f and g satisfying (2.1)–(2.4) and (2.33). By (2.2) we have

$$\frac{d}{dr}\left(r\frac{dg}{dr}\right) = \frac{d}{dr}\left(Hf^2\right) \quad in\ I_\rho. \tag{2.34}$$

Hence we have

$$r\frac{dg}{dr} = Hf^2 + \left(\alpha - \frac{1}{H}\right), \tag{2.35}$$

where α is some real constant. By (2.33) and (2.35) we have

$$r\frac{dg}{dr} = -H\left(g + \sqrt{\frac{1 - H^2}{H^2}}\right)^2 + \alpha. \tag{2.36}$$

On the other hand, by differentiating both sides of (2.33), we have

$$f\frac{df}{dr} = -\left(g + \sqrt{\frac{1-H^2}{H^2}}\right)\frac{dg}{dr}. \tag{2.37}$$

Hence, by (2.3) we have

$$\left\{-\left(g + \sqrt{\frac{1-H^2}{H^2}}\right)\frac{dg}{dr}\right\}^2 + f^2\left(\frac{dg}{dr}\right)^2 = \frac{1}{r^2}f^4. \tag{2.38}$$

Hence, by (2.33) and (2.35) we have

$$\frac{1}{H^2}\left(\frac{dg}{dr}\right)^2 = \frac{1}{r^2}\left\{\frac{1}{H}\left(r\frac{dg}{dr} + \frac{1}{H} - \alpha\right)\right\}^2. \tag{2.39}$$

Hence we have

$$\frac{2}{r}\left(\frac{1}{H} - \alpha\right)\left\{\frac{dg}{dr} - \frac{1}{2r}\left(\alpha - \frac{1}{H}\right)\right\} = 0. \tag{2.40}$$

Now suppose $\alpha \neq \frac{1}{H}$. Then, by (2.40) we have

$$\frac{dg}{dr} = \frac{1}{2r}\left(\alpha - \frac{1}{H}\right). \tag{2.41}$$

Hence, by (2.4) we have

$$g(r) = \frac{1}{2}\left(\alpha - \frac{1}{H}\right)\log r. \tag{2.42}$$

Then, by (2.36), (2.41) and (2.42) we have

$$\frac{1}{2}\left(\alpha - \frac{1}{H}\right) = -H\left\{\frac{1}{2}\left(\alpha - \frac{1}{H}\right)\log r + \sqrt{\frac{1-H^2}{H^2}}\right\}^2 + \alpha. \tag{2.43}$$

However, since $\alpha \neq \frac{1}{H}$, (2.43) can not hold for all r. This contradiction implies that our assumption is false, and we have

$$\alpha = \frac{1}{H}. \tag{2.44}$$

Hence, by (2.36) and (2.44) we have

$$r\frac{dg}{dr} = -H\left(g + \sqrt{\frac{1-H^2}{H^2}}\right)^2 + \frac{1}{H}. \tag{2.45}$$

Now, let

$$z(r) := g(r) + \sqrt{\frac{1-H^2}{H^2}} - \frac{1}{|H|}. \tag{2.46}$$

Then, by (2.45) we have

$$r\frac{dz}{dr} = -Hz^2 - 2\frac{H}{|H|}z. \tag{2.47}$$

Moreover, let

$$u(r) := z(r)r^{\frac{2|H|}{H}}. \tag{2.48}$$

Then, by (2.47) and (2.48) we have

$$\frac{du}{dr}(r) = -Hu(r)^2 r^{-1-\frac{2|H|}{H}}. \tag{2.49}$$

Still more, let

$$v := \frac{1}{u}. \tag{2.50}$$

Then, by (2.49) and (2.50) we have

$$-\frac{1}{v^2}\frac{dv}{dr} = -\frac{H}{v^2}r^{-1-\frac{2|H|}{H}}. \tag{2.51}$$

Hence we have

$$\frac{dv}{dr} = Hr^{-1-\frac{2|H|}{H}}. \tag{2.52}$$

Equation (2.52) gives us

$$v(r) = -\frac{|H|}{2} r^{-\frac{2|H|}{H}} + \beta, \tag{2.53}$$

where β is some real constant. Then, by (2.46), (2.48), (2.50) and (2.53) we have

$$g(r) = \frac{1}{|H|} - \sqrt{\frac{1-H^2}{H^2}} + \frac{2}{2\beta r^{\frac{2|H|}{H}} - |H|}. \tag{2.54}$$

Hence we have

$$g(1) = \frac{1}{|H|} - \sqrt{\frac{1-H^2}{H^2}} + \frac{2}{2\beta - |H|}. \tag{2.55}$$

On the other hand, by (2.4) we have $g(1) = 0$. Hence we have

$$\frac{1}{|H|} - \sqrt{\frac{1-H^2}{H^2}} + \frac{2}{2\beta - |H|} = 0. \tag{2.56}$$

Equation (2.56) yields

$$\beta = -\frac{(1+\sqrt{1-H^2})^2}{2|H|}. \tag{2.57}$$

By (2.54) and (2.57), (1.23) holds.

On the other hand, by (2.33), let

$$f = \sqrt{\frac{1}{H^2} - \left(g + \sqrt{\frac{1-H^2}{H^2}}\right)^2}. \tag{2.58}$$

Then, by (1.23) and (2.58), (1.22) holds.

Now, by (2.4) we have $g(\rho) = b$. Hence, by (1.23) we have

$$\frac{1}{|H|} - \sqrt{\frac{1-H^2}{H^2}} - \frac{2|H|}{(1+\sqrt{1-H^2})^2 \rho^{\frac{2|H|}{H}} + H^2} = b. \tag{2.59}$$

Hence we have

$$(2.60)\qquad (1+\sqrt{1-H^2})^2\rho^{\frac{2|H|}{H}}+H^2=\frac{2|H|}{\frac{1}{|H|}-\sqrt{\frac{1-H^2}{H^2}}-b}.$$

This leads to

$$(2.61)\qquad (1+\sqrt{1-H^2})^2\rho^{\frac{2|H|}{H}}=\frac{(1+\sqrt{1-H^2}+b|H|)H^2}{1-\sqrt{1-H^2}-b|H|}.$$

Therefore, a simple calculation yields (1.20).

If ρ satisfies (1.20), we have

$$(2.62)\qquad \rho=\frac{|H|}{1+\frac{|H|}{H}\sqrt{1-H^2}}\sqrt{-1+\frac{2}{1-\frac{|H|}{H}\sqrt{1-H^2}-bH}}.$$

Hence (1.21) holds, because $bH<0$ by (1.18).

Conversely, we easily see that f and g given by (1.22) and (1.23) respectively satisfy (2.1)–(2.3) and (2.58). Moreover, we also see that $f(1)=1$, $g(1)=0$ and $g(\rho)=b$ for ρ given by (1.20). Hence (1.24) holds. On the other hand, by (1.19) and (2.58) we have $f(\rho)=a$. Hence (2.4) holds and we conclude that there exists a solution X to (1.1)–(1.5) of the form (1.9) for these f and g.

(2) Suppose X is a solution to (1.1)–(1.5) of the form (1.9). Then f and g satisfy (2.1)–(2.4). On the other hand, X satisfies (1.31) if and only if f and g satisfy the equation

$$(2.63)\qquad f^2+\left(g-\sqrt{\frac{1-H^2}{H^2}}\right)^2=\frac{1}{H^2}.$$

Now we look for f and g satisfying (2.1)–(2.4) and (2.63). In the same manner as in the proof of (1), we get

$$(2.64)\qquad r\frac{dg}{dr}=-H\left(g-\sqrt{\frac{1-H^2}{H^2}}\right)^2+\frac{1}{H}.$$

Now, let

$$z(r) := g(r) - \sqrt{\frac{1-H^2}{H^2}} + \frac{1}{|H|}. \tag{2.65}$$

Then, by (2.64) we have

$$r\frac{dz}{dr} = -Hz^2 + 2\frac{H}{|H|}z. \tag{2.66}$$

Moreover, let

$$u(r) := z(r)r^{-\frac{2|H|}{H}}. \tag{2.67}$$

Then, by (2.66) and (2.67) we have

$$\frac{du}{dr}(r) = -Hu(r)^2 r^{-1+\frac{2|H|}{H}}. \tag{2.68}$$

Still more, let

$$v := \frac{1}{u}. \tag{2.69}$$

Then, by (2.68) and (2.69) we have

$$-\frac{1}{v^2}\frac{dv}{dr} = -\frac{H}{v^2}r^{-1+\frac{2|H|}{H}}. \tag{2.70}$$

Hence we have

$$\frac{dv}{dr} = Hr^{-1+\frac{2|H|}{H}}. \tag{2.71}$$

Equation (2.71) gives us

$$v(r) = \frac{|H|}{2}r^{\frac{2|H|}{H}} + \beta, \tag{2.72}$$

where β is some real constant. Then, by (2.65), (2.67), (2.69) and (2.72) we have

$$g(r) = -\frac{1}{|H|} + \sqrt{\frac{1-H^2}{H^2}} + \frac{2r^{\frac{2|H|}{H}}}{2\beta + |H| r^{\frac{2|H|}{H}}}. \tag{2.73}$$

Hence we have

$$g(1) = -\frac{1}{|H|} + \sqrt{\frac{1-H^2}{H^2}} + \frac{2}{2\beta + |H|}. \tag{2.74}$$

On the other hand, by (2.4) we have $g(1) = 0$. Hence we have

$$-\frac{1}{|H|} + \sqrt{\frac{1-H^2}{H^2}} + \frac{2}{2\beta + |H|} = 0. \tag{2.75}$$

Equation (2.75) yields

$$\beta = \frac{(1+\sqrt{1-H^2})^2}{2|H|}. \tag{2.76}$$

By (2.73) and (2.76), (1.30) holds.

On the other hand, by (2.63), let

$$f = \sqrt{\frac{1}{H^2} - \left(g - \sqrt{\frac{1-H^2}{H^2}}\right)^2}. \tag{2.77}$$

Then, by (1.30) and (2.77), (1.29) holds.

Now, by (2.4) we have $g(\rho) = b$. Hence, by (1.30) we have

$$-\frac{1}{|H|} + \sqrt{\frac{1-H^2}{H^2}} + \frac{2|H|\rho^{\frac{2|H|}{H}}}{(1+\sqrt{1-H^2})^2 + H^2 \rho^{\frac{2|H|}{H}}} = b. \tag{2.78}$$

This leads to

$$(1+\sqrt{1-H^2})^2 \rho^{-\frac{2|H|}{H}} + H^2 = \frac{2|H|}{\frac{1}{|H|} - \sqrt{\frac{1-H^2}{H^2}} + b}. \tag{2.79}$$

Hence we have

$$(2.80)\qquad (1+\sqrt{1-H^2})^2\rho^{-\frac{2|H|}{H}} = \frac{(1+\sqrt{1-H^2}-b|H|)H^2}{1-\sqrt{1-H^2}+b|H|}.$$

Therefore, a simple calculation yields (1.27).

If ρ satisfies (1.27), we have

$$(2.81)\qquad \rho = \frac{|H|}{1-\frac{|H|}{H}\sqrt{1-H^2}}\sqrt{-1+\frac{2}{1+\frac{|H|}{H}\sqrt{1-H^2}-bH}}.$$

Hence (1.28) holds, because $bH < 0$ by (1.25).

We can show the converse in the same manner as in the proof of (1). □

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