The Andreotti Grauert vanishing theorem for dihedrons of $\mathbb{C}^n$

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Abstract. Let $X$ be a complex manifold, $\mathcal{O}_X$ the sheaf of analytic functions on $X$, $W$ an open set of $X$ with $C^2$-boundary $M = \partial W$ ($W$ locally on one side of $M$), $z_0$ a point of $M$, $p_0$ the exterior conormal to $W$ at $z_0$. If the number of negative eigenvalues for the Levi form of $M$ in a neighborhood of $p_0$ is $\geq s^-$ (resp. $\equiv s^-$), then vanishing of local cohomology groups of $\mathcal{O}_X$ over $W$ in degree $< s^-$ and $\neq 0$ (resp. $\neq 0$, $s^-$) is stated in [1], [2], [14] (resp. [10]). Let now $W$ be an open asymptotically convex dihedron with $C^2$ (transversal) faces $M_1$, $M_2$ and with “generic” edge $M_3$. If the numbers of negative eigenvalues for the Levi forms of $M_1$, $M_2$, and for the “microlocal” Levi form of $M_3$ are $\geq s^-$ along the exterior conormal cone to $W$ at $z_0$, then vanishing of cohomology $< s^-$ and $\neq 0$ is proved here in the line of [1]. Under the additional assumption that $M_3$ is real analytic and that $T^*_M X$ contains no germ of complex curve, the same result as by [10] is also stated. The content of this paper was exposed at the University Paris XIII, June 21 1991.

1. Notations

Let $X$ be a complex manifold, $M$ a $C^2$-submanifold, $\pi : T^*X \to X$ (resp. $\pi : T^*_M X \to M$, resp. $\tau : TM \to M$) the cotangent bundle to $X$ (resp. the conormal bundle to $M$, resp. the tangent bundle to $M$). Let $\text{or}_X$ (resp. $\text{or}_{M/X}$) be the orientation (resp. the relative orientation) sheaf. Let $z_0 \in M$ and $p \in \hat{T}^*_M X (= T^*_M X \setminus \{0\})$ with $\pi(p) = z_0$. We shall use the notations

$$e(p) = T_p T^* X$$

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\[ T_{z_0}^C M = T_{z_0} M \cap \sqrt{-1} T_{z_0} M \]
\[ \lambda_M(p) = T_p T^*_M X \]
\[ \lambda_o(p) = T_p \pi^{-1} \pi(p) \]
\[ \gamma_M(p) = \dim_C(\lambda_M(p) \cap \sqrt{-1} \lambda_M(p) \cap \lambda_o(p)) \]
\[ \nu(p) = \text{the complex line spanned by the radial vector field.} \]

We shall drop sometimes \( z_o \) and \( p \) in the above notations.

Let \( \alpha = \alpha^R + \sqrt{-1} \alpha^I \) be the canonical 1-form on \( T^*X \) and \( \sigma = \sigma^R + \sqrt{-1} \sigma^I \) (=\(d\alpha\)) the canonical 2-form. Let \( X^R \) (resp. \( (T^*X)^R \)) denote the real underlying manifold to \( X \) (resp. \( T^*X \)). We shall identify \( T^*(X^R) \) with \( T^*(X^R) \) by endowing \( T^*(X^R) \) with the 1-form \( \alpha^R \).

For an isotropic plane \( \mu \subset e \) and for a subspace \( \lambda \) of \( e \), we shall set \( \lambda^\mu = ((\lambda \cap \mu^\bot) + \mu)/\mu \). For \( \lambda \) (resp. \( \lambda^R \) (resp. \( \lambda^C \)) Lagrangian, we shall define \( s^\mu_+ , s^- \), \( s^0 \) to be the numbers of eigenvalues of \( L_\phi(z_o)|_{T_{z_o}^C M} \) which are respectively > 0, < 0, = 0; by (1.1) they do not depend on the choice of \( \phi \).

We shall denote by \( D^b(X) \) the derived category of the category of complexes of sheaves with bounded cohomology, and by \( D^b(X;p) \) the localized category with respect to the null-system \( \{ F \in D^b(X); p \notin \text{SS}(F) \} \). (Here \( \text{SS}(F) \) is the microsupport in the sense of [10] and [11], a closed conic subset of \( T^*X \).) We recall that for a complex \( F \) which verifies \( \text{SS}(F) \subset T^*_M X \) in a neighborhood of \( p, p \in T^*_M X \), one may find a complex \( M^\cdot \) of \( \mathbf{Z} \)–modules such that \( F \) is microlocally isomorphic to the constant complex \( M^\cdot \) at \( p \).
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(i.e. isomorphic in $D^b(X; p)$). This criterion, stated in [10, §6], for $M$ being a $C^2$-submanifold, easily extends to the case $C^1$ (see [6]).

We shall denote by $O_X$ the sheaf of holomorphic functions on $X$ and by $Z_W$ ($W \subset X$ locally closed) the sheaf which is 0 on $X \setminus W$ and the constant sheaf with stalk $Z$ on $W$. We shall deal with the complex $\mu_W(O_X) \overset{\text{def.}}{=} \mu_{hom}(Z_W, O_X)$ of microfunctions along $W$ (in particular with the complex $\mu_M(O_X)$), where $\mu_{hom}(\cdot, \cdot)$ is the bifunctor of [10].

Finally we shall denote by $N(W)$ (resp. $N(W)^{oa}$) the normal cone to $W$ (resp. the polar antipodal cone to $N(W)$); this is an open cone of $TX$ (resp. a closed cone of $T^*X$).

2. Statement of the Vanishing Theorems

Let $X$ be a complex manifold, $W$ an open set of $X$ with $C^2$-boundary $M = \partial W$ ($W$ locally on one side of $M$), $z_o$ a point of $M$, $p_o$ the exterior conormal to $W$ at $z_o$.

**Theorem A.** (i) ([1]) In the above situation we have

\[
\lim_{\to B} H^j(W \cap B, O_X) = 0 \quad 0 < j < s^-_M(p_o),
\]

for $B$ describing a system of neighborhoods of $z_o$.

(ii) ([10]) If moreover $s^-_M(p) \equiv \text{const} \forall p \in T^*_M X$ near $p_o$, then (2.1) holds $\forall j \neq 0, s^-_M(p_o)$.

Refinements of (2.1) are given by several authors ([14], [2], [10]); in particular vanishing of (2.1) for $j > s^-_M + s^+_M$ is also stated. As for Th. A (ii) the result was already known before [10] under the stronger assumption of constant rank for $L_M(p)$ (cf. e.g. [14]).

The purpose of the present note is to generalize Theorem A to the case of a dihedron. Let $M_1, M_2$ be $C^2$-hypersurfaces of $X^R$ through $z_o$, $M_i^+, i = 1, 2$ open half-spaces with boundary $M_i$, and let $W$ be the dihedron defined by $W = M_1^+ \cap M_2^+$. We put $M_3 = M_1 \cap M_2$, and assume that $M_3$ is a manifold which verify the “genericity” condition:

\[
T^*_{M_3} X_{z_o} \cap \sqrt{-1} T^*_{M_3} X_{z_o} = \{0\}.
\]
which is equivalent to $\gamma_{M_3}(p) = 0$ for $p \in \pi^{-1}(z_0)$. We denote $p_i$, $i = 1, 2$ the exterior conormal to $M_i$, and denote $s^-$ the minimum between $s_{M_1}^-(p_1)$, $s_{M_2}^-(p_2)$ and $s_{M_3}^-(p) \forall p \in N_{z_0}(W)^{\alpha}$.

**Theorem 2.1.** Let $M_1, M_2$ intersect transversally, and let the intersection $M_3$ be generic. Then

$$(2.4) \quad \lim_{B} H^j(W \cap B, O_X) = 0 \quad 0 < \forall j < s^-$$

(and

$$(2.5) \quad \lim_{B} \Gamma(B, O_X) \sim \lim_{B} \Gamma(W \cap B, O_X) \quad \text{if} \quad s^- \geq 1).$$

In [13, Prop. 11 p. 148] vanishing of (2.4) for $j < s^- - 1, j \neq 0$ is proved (in a different frame); in particular the case $j = s^- - 1$ seems not to be treated.

**Theorem 2.2.** Let $M_1, M_2$ intersect transversally, let (2.2) be satisfied, and suppose moreover $M_3$ real analytic with $T^*_M X$ containing no germ of complex curve. Assume that for a constant $s^-$ and for an open set $B_0$:

$$(2.6) \quad s_{M_i}^-(p) \equiv s^- \quad \forall p \in T^*_M X \cap N(W)^{\alpha} \cap \pi^{-1}(B_0), \quad \text{and} \quad \forall i = 1, 2, 3.$$

Then (2.4) holds $\forall j \neq 0, s^-$ (and (2.5) is still fulfilled when $s^- \geq 1$).

(By (2.2), if $M_3$ contains no germ of complex curve, the same is also true for $T^*_M X$.)

3. Proofs

**Proof of Theorem 2.1**

We shall use the notations:

$$\Lambda_i = T^*_M X \cap N(W)^{\alpha}, \quad N_i = \pi(\Lambda_i),$$

$$\hat{\Lambda}_i = \Lambda_i \setminus (\Lambda_j \cup \Lambda_k), \quad \hat{N}_i = N_i \setminus (N_j \cup N_k), \quad j, k \neq i.$$
We first deal with a neighborhood of the conormal $p_1$ to $M_1$; since $p_1 \in (\Lambda_1 \cap \Lambda_3) \setminus \Lambda_2$, we shall forget $\Lambda_2$ for a while.

**Lemma 3.1.** We may find a (complex) contact transformation $\chi$ between neighborhoods of $p_1$ and $q_1 \overset{\text{def.}}{=} \chi(p_1)$ which interchanges:

\[
\begin{cases}
T_{M_3}^*X \to T_{\tilde{M}_3}^*X, & \text{cod } \tilde{M}_3 = 1, s_{\tilde{M}_3}(q_1) = 0 \\
T_{M_1}^*X \to T_{\tilde{M}_1}^*X, & \text{cod } \tilde{M}_1 = 1.
\end{cases}
\]

**Proof.** We shall deal in the linear space $e' = e^\nu$ (due to $\dim_{\mathbb{R}}(\lambda_{M_i} \cap \nu) = 1$). According to [17], to solve (3.1) is equivalent as to find a $\mathbb{C}$-Lagrangian plane $l'_o \subset e'$ such that (with $\lambda'_i \overset{\text{def.}}{=} \lambda^\nu_{M_i}$):

\[
\begin{align}
l'_o \cap \lambda'_3 &= \{0\}, \\
L_{l'_o/\lambda'_3} &\geq 0, \\
l'_o \cap \lambda'_1 &= \{0\}.
\end{align}
\]

Now the complex planes $l'_o$ which verify (3.4) are an open dense set in the Lagrangian Grassmannian of $e'$ (obvious). The ones which verify (3.2), (3.3) are an open non-empty set. In fact this is clearly non-empty (cf. e.g. the proof of [10, Prop. 11.3.1]); moreover the rank of $L_{l'_o/\lambda'_3}$ is constant ($= n - 1 - \dim(\lambda'_3 \cap \sqrt{-1}\lambda'_3)$) for $l'_o$ varying in the open set defined by $l'_o \cap \lambda'_3 = \{0\}$ and hence the signature is constant in each component of the above set. Thus the problem of finding $l'_o$ as in (3.2)–(3.4) can be solved. □

(This was already remarked in [7].)

**Remark 3.2.** In the same way as before one may construct a contact transformation satisfying the requirements (3.1) but with $s_{\tilde{M}_1}(q_1) = 0$ instead of $s_{\tilde{M}_3}(q_1) = 0$.

We shall put in what follows:

$$
\tilde{\Lambda}_i = \chi(\Lambda_i), \quad \hat{\Lambda}_i = \chi(\tilde{\Lambda}_i), \quad \tilde{N}_i = \pi(\hat{\Lambda}_i), \quad \hat{N}_i = \pi(\tilde{\Lambda}_i).
$$
(b) We recall from [5] that

\[ s_{M_3}^{-}(p_1) \leq s_{M_1}^{-}(p_1) \leq s_{M_3}^{-}(p_1) + 1. \]

We also choose a quantization \( \Phi_K \) of \( \chi \) by a kernel \( K \) according to [10, ch. 11].

**Lemma 3.3.** Let \( s_{M_1}^{-}(p_1) = s_{M_3}^{-}(p_1) \). Then, if \( \chi \) is a contact transformation which satisfies (3.1), we have:

\[
\begin{cases}
  s_{M_1}^{-}(q_1) = 0, \\
  \tilde{\Lambda}_1 \cup \tilde{\Lambda}_3 = T^*_Y X \text{ for a } C^1 \text{-hypersurface } Y, \\
  \Phi_K(Z_{\overset{o}{N}_1}) \simeq Z_Y[s_{M_1}^{-}(p_1)] \text{ in } D^b(X; q_1).
\end{cases}
\]

**Proof.** We remark that \( T^*_M X \cap T^*_M X \) is clean of codimension 1, whence \( \tilde{M}_1, \tilde{M}_3 \) intersect at the order 2 along a submanifold \( \tilde{R} \) of codimension 2. In particular \( T\tilde{M}_1|_{\tilde{R}} = T\tilde{M}_3|_{\tilde{R}} \) and hence either \( \tilde{N}_1 \cup \tilde{N}_3 \) or \( \tilde{N}_1 \cup (\tilde{M}_3 \setminus \tilde{N}_3) \) is a \( C^1 \)-hypersurface \( Y \). To make the right choice we exploit \( \Phi_K \) and remark that

\[
\text{SS}(\Phi_K(Z_{\overset{o}{N}_1})) \subset \tilde{\Lambda}_1 \cup \tilde{\Lambda}_3 \subset \pi^{-1}(\tilde{N}_1 \cup \tilde{N}_3),
\]

\[
\text{SS}(\Phi_K(Z_{N_1})) \subset \tilde{\Lambda}_1 \cup (T^*_M X \setminus \tilde{\Lambda}_3) \subset \pi^{-1}(\tilde{N}_1 \cup (\tilde{M}_3 \setminus \tilde{N}_3)).
\]

Hence one of the above complexes has its microsupport contained in \( T^*_Y X \); and this complex must then be constant due to [10] and [6]. But we have

\[
\begin{align*}
\Phi_K(Z_{\overset{o}{N}_1}) &= \begin{cases}
  Z_{\overset{o}{M}_1}[s_{M_1}^{-}(p_1) - s_{M_1}^{-}(q_1)] & \text{in } \overset{o}{\tilde{\Lambda}_1}, \\
  Z_{\overset{o}{M}_3}[s_{M_3}^{-}(p_1)] & \text{in } \overset{o}{\tilde{\Lambda}_3},
\end{cases} \\
\Phi_K(Z_{N_1}) &= \begin{cases}
  Z_{\overset{o}{M}_1}[s_{M_1}^{-}(p_1) - s_{M_1}^{-}(q_1)] & \text{in } \overset{o}{\tilde{\Lambda}_1}, \\
  Z_{\overset{o}{M}_3}[s_{M_3}^{-}(p_1) + 1] & \text{in } T^*_M X \setminus \tilde{\Lambda}_3.
\end{cases}
\end{align*}
\]
Recall that $s_{M_3}(p_1) = s_{M_1}(p_1)$; hence constancy may hold only for $\Phi_K(Z_{\tilde{N}_1})$ (which implies $\tilde{N}_1 \cup \tilde{N}_3 = Y \in C^1$) and under the additional condition $s_{\tilde{M}_1}(p_1) = 0$.

We also observe that $\Phi_K(Z_{\tilde{N}_1})$ turns out to be a constant sheaf along $Y$ and by (3.6) this is simple with shift $\frac{1}{2} + s_{\tilde{M}_1}(p_1)$ along $\tilde{A}_1$ and $\tilde{A}_3$. This completes the proof of (3.5). □

Denote by $\Sigma_{\tilde{M}_i}, i = 1, 2$ the closed half–spaces with boundary $\tilde{M}_i$ and inward conormal $q_1$.

**Lemma 3.4.** Let $s_{\tilde{M}_1}(p_1) = s_{\tilde{M}_3}(p_1) + 1$. Then by the contact transformation of Remark 3.2, we have

$$(3.5') \begin{cases} 
  s_{\tilde{M}_3}(q_1) = 0, \Sigma_{\tilde{M}_1} \supset \Sigma_{\tilde{M}_3}, \\
  \pi(\tilde{A}_1 \cup \tilde{A}_3) = \partial(\Sigma_{\tilde{M}_1} \setminus \Sigma_{\tilde{M}_3})^+, \\
  \Phi_K(Z_{\tilde{N}_1}) \cong Z_{(\Sigma_{\tilde{M}_1} \setminus \Sigma_{\tilde{M}_3})^+},
\end{cases}$$

where $(\cdot)^+$ is a component of $\cdot$.

**Proof.** The proof is the same as in Lemma 3.3. □

We should prove Lemma 3.4 (and similarly Lemma 3.3) also by another argument. We recall that $\tilde{M}_1, \tilde{M}_3$ intersect at the order 2; thus either $\Sigma_{\tilde{M}_1} \supset \Sigma_{\tilde{M}_3}$ or $\Sigma_{\tilde{M}_1} \subset \Sigma_{\tilde{M}_3}$. But

$$\text{Hom}_{D^b(X;p_1)}(Z_{M_1}, Z_{M_3}) \cong \text{Hom}_{D^b(X;q_1)}(Z_{\Sigma_{\tilde{M}_1}}, Z_{\Sigma_{\tilde{M}_3}}[-s_{\tilde{M}_3}(q_1)])$$

$$\cong H^{-s_{\tilde{M}_3}(q_1)}R\Gamma_{\Sigma_{\tilde{M}_1}}(Z_{\Sigma_{\tilde{M}_3}})z'_0, (z'_0 = \pi(q_1)).$$

This gives the first of (3.5'). The proof of the remaining statements is then easy.

(c) We have

$$(3.7) \quad Z_W \sim Z_{\tilde{N}_1}[-1] \otimes \text{or}_{M_1/X} \quad \text{in} \quad D^b(X;p_1).$$
In fact one uses the distinguished triangle

\[ Z_{\bar{W}\setminus N_1} \to Z_{\bar{W}} \to Z_{N_1} \xrightarrow{+1}, \]

applies the functor \( \cdot^* \overset{\text{def.}}{=} \mathcal{R}\text{Hom}(\cdot, Z_X) \), and gets

\[ Z_{\frac{N_1}{-1}} \otimes_{\mathcal{O}_{M_1/X}} Z_W \to Z_{\frac{\bar{W}
abla N_1}{+1}}. \]

But

\[ \text{SS}(Z_{\bar{W}\setminus N_1})^*_{z_0} = \text{SS}(Z_{\bar{W}\setminus N_1})^g_{z_0} \]

is the convex cone of the plane \( T_{M_3}^* X_{z_0} \)
bounded by \( \mathbb{R}^+ p_2, \mathbb{R}^- p_1 \).

Thus \( p_1 \notin \text{SS}(Z_{\bar{W}\setminus N_1})^*_{z_0} \) and (3.7) follows. We remark that along with (3.7) we have

(3.8) \[ Z_W \overset{\sim}{\leftarrow} Z_{M_3}[-2] \otimes_{\mathcal{O}_{M_3/X}} \text{in } D^b(X; \mathcal{O}_3) \]

(3.9) \[ Z_W \overset{\sim}{\leftarrow} Z_{M_1}[-1] \otimes_{\mathcal{O}_{M_1/X}} \text{in } D^b(X; \mathcal{O}_1). \]

(d) We first assume \( s_{M_1}^{-}(p_1) = s_{M_3}^{-}(p_1) \), put \( s_{1}^{-} \overset{\text{def.}}{=} s_{M_1}^{-}(p_1) \), and denote by \( \bar{W} \) the open subset of \( X \) with boundary \( Y \) and exterior conormal \( q_1 \) (cf. Lemma 3.2). By (3.5), (3.7) and by the analogous for \( \bar{W} \), we get

(3.10) \[ \Phi_K(Z_W) \simeq \Phi_K(Z_{\frac{N_1}{-1}}) \otimes_{\mathcal{O}_{M_1/X}} \]
\[ \simeq Z_Y[s_1^{-} - 1] \otimes_{\mathcal{O}_{Y/X}} \simeq Z_{\bar{W}}[s_1^{-}] \]

Let \( \mu_{\bar{W}}(\cdot) = \mu_{\text{hom}}(Z_{\bar{W}}, \cdot) \) be the functor defined in [K-S 1,ch. 5], and recall the triangle

(3.11) \[ (\mathcal{O}_{X}|_{\bar{W}})_{z_o} \to \mathcal{R}\Gamma_{\bar{W}}(\mathcal{O}_X)_{z_o'} \to \mu_{\bar{W}}(\mathcal{O}_X)_{q_o} \xrightarrow{+1} (z_o' = \pi(q_1)). \]
Since \( O_X z_o \hookrightarrow H^0_W(O_X) z_o \) is injective by analytic continuation, then:
\[ H^j \mu_W(O_X)_{q_1} = 0 \forall j < 0. \]
By (3.10), by the fact that \( \Phi_K(O_X) = O_X \), and by [10, Th. 7.4.1], one concludes

\[
(3.12) \quad H^j \mu_W(O_X)_{p_1} = 0 \quad \forall j < s_1^-.
\]

Let now \( s^{-}_{M_3}(p_1) = s^{-}_{M_1}(p_1) + 1 \); in this case we have

\[
\Phi_K(Z_W) \simeq Z(\Sigma_{M_1} \setminus \Sigma_{M_3})^+ [s_1^- - 1].
\]

We also remark that

\[
H^1_{\Sigma_{M_3}}(O_X) z_o \hookrightarrow H^1_{\Sigma_{M_1}}(O_X) z_o
\]
is injective. Then (3.12) is still fulfilled in this case

\[ (e) \]
One may repeat the same argument as in (a)–(d) and get (3.12) with \( p_1, s_1^- \) replaced by \( p_2, s_2^- \). One may also use (3.10) and get in similar (and simpler) way:

\[
(3.13) \quad H^j \mu_W(O_X)_p = 0 \quad \forall p \in \Lambda_3, \forall j < s^{-}_{M_3}(p)
\]

Finally by the triangle

\[
(3.14) \quad (O_X)_{\bar{W}} \to R\Gamma_W(O_X) \to R\pi_* \mu_W(O_X)^{+1},
\]
we get \( H^j R\Gamma_W(O_X)_{z_o} = 0 \) if \( j \neq 0 \), and \( j < s_1^-, s_2^-, \inf_{p \in \Lambda_3} s^{-}_{M_3}(p) \).

From (3.14), one also gets \( O_{X z_o} \simeq H^0_W(O_X)_{z_o} \) when the infimum of the above \( s^- \)'s is \( \geq 1 \).
Proof of Theorem 2.2

(a) We first focus our attention in a neighborhood of \( p_1 \). Observing that \( s_{M_1}(p_1) = s_{M_3}(p_1) \) in the present case, then by the proof of Theorem 2.1 we get \( \Phi_K(Z_W) = Z_{\tilde{W}}[s^-, (s^- = s_T^-)] \) at \( q_1 \) for \( \tilde{W} \) open with \( C^1 \)-boundary. According to [10, Th. 11.2.8] \( s_{M_1}(q) - s_{M_1}(p) \equiv \text{const} \); but \( s_{M_i}(p) \equiv \text{const}, s_{M_i}(q_1) = 0 \) whence \( s_{M_i}(q) \equiv 0 \). We also observe that in the present situation \( \Sigma_{\tilde{M}_3} \supset \Sigma_{\tilde{M}_1} \).

Lemma 3.5. \( \tilde{W} \) is pseudoconvex in a neighborhood of \( z^{(\text{def.})}_o (p_1) = \pi(q_1) \).

Proof. We identify \( X \simeq \mathbb{C}^n \) and recall from [9, ch. 2, 4] that \( \tilde{W} \) is pseudoconvex at \( z' = \pi(q_1) \) if and only if \( -\log \delta \) is plurisubharmonic (with \( \delta(z) = \text{dist}(z, X \setminus \tilde{W}) \)). We remark that \( \forall z \in \tilde{W} \) there exists an unique \( z^* \in \partial \tilde{W} \) such that \( \delta(z) = |z - z^*| \) (due to \( \tilde{W} \supset X \setminus \Sigma_{\tilde{M}_3} \)). We then let \( S = \{z \in \tilde{W}; z^* \in \tilde{M}_1 \cap \tilde{M}_3\} \) (which is a \( C^2 \)-hypersurface), and denote by \( \tilde{W}_h, h = 1, 2 \) the two components of \( \tilde{W} \setminus S \). We also define \( \delta_h \) to be the distance to \( \tilde{M}_h \); clearly \( \forall z \in \tilde{W} \setminus S \) we have \( \delta = \delta_1 \) or \( \delta_3 \) at \( z \) whence \( \delta \in C^2(\tilde{W} \setminus S) \cap C^1(\tilde{W}) \). It is also easy to see (by adapting the proof by [9, Th. 2.6.12]) that \( -\log \delta \) is plurisubharmonic if and only if \( s^-_{M_h}(q') \equiv 0 \) \( \forall q' \in \tilde{N}_h \times \tilde{M}_h T^*_{\tilde{M}_h} X \). Thus \( \forall w \in \mathbb{C}^n \) and \( \forall g \in C^\infty_c(\tilde{W}) \), \( g \geq 0 \):

\[
\int_{\tilde{W}} \sum_i \sum_j \partial_i \bar{\partial}_j (-\log \delta) w_i \bar{w}_j g \, d\lambda = \sum_{h=1,2} \int_{\tilde{W}_h} \sum_i \partial_i \bar{\partial}_j (-\log \delta) w_i \bar{w}_j g \, d\lambda \geq 0
\]

(which holds, by Stokes formula, because \( \delta \in C^2(\tilde{W} \setminus S) \cap C^1(\tilde{W}) \) with bounded second derivatives). The lemma immediately follows. \( \square \)

By the above lemma we have \( H^j R_{\bar{\partial}^*} (O_X) z' = 0 \) \( \forall j \neq 0 \), \( (z' = \pi(q)) \) due to [9, Corollary 7.4.2]. By the triangle (3.11) and by (3.10), we then conclude

(3.15) \( H^j \mu_W (O_X)_{p_1} = 0 \) \( \forall j \neq s^- \).

(b) The same as (3.15) also holds for \( p_2 \) and \( \forall p \in \overset{\circ}{\Lambda}_3 \). We need now:
**Lemma 3.6.** We have

\[ (3.16) \quad H^j R\hat{\cdot} \mu_W (\mathcal{O}_X)_{z_0} = 0 \quad \forall j \neq s^- . \]

**Proof.** We set \( F = H^{s^-} \mu_W (\mathcal{O}_X), Z = \hat{\cdot} N_{z_0} (W)^{o_a} = R^+ p_1 \cup R^+ p_2 \cup \text{int} N_{z_0} (W)^{o_a} \) and denote \( B_i, i = 1, 2 \) conic neighborhoods of \( p_i \) in \( Z \). Recall that \( M_3 \) is real analytic and that \( T^*_{M_3} X \) contains no germ of complex curve. Then [16] applies and gives that \( F|_{\hat{\cdot} \Lambda_3} = H^{s^-+2} \mu_{M_3} (\mathcal{O}_X)|_{\hat{\cdot} \Lambda_3} \) is conically flabby. In particular

\[ (3.17) \quad H^j (\hat{\cdot} \Lambda_3, F) = 0 \quad \forall j \neq 0; \quad \Gamma (\hat{\cdot} \Lambda_3, F) \twoheadrightarrow \Gamma (\hat{\cdot} \Lambda_3 \cap B_i, F), i = 1, 2 \]

(where \( \twoheadrightarrow \) stands for surjectivity). By an elementary application of the Mayer-Vietoris long exact sequence (and by taking the \( B_i \)'s small enough) one gets

\[ H^j (B_1 \cup B_2 \cup \hat{\cdot} \Lambda_3, F) = 0 \quad \forall j \neq 0, \]

which implies immediately (3.16). \( \square \)

By the triangle (3.14) and by Lemma 3.6 we get at once the conclusion of the proof of Theorem 2.2.

### 4. Remarks and Examples

**Remark 4.1.** Let \( M_1, M_2 \) be \( C^2 \)-hypersurfaces, \( M_1^+, M_2^+ \) open half-spaces with boundary \( M_1, M_2 \), and set \( M_3 = M_1 \cap M_2 \). We do not assume that \( M_1, M_2 \) are transversal but still suppose that \( M_3 \) is a manifold and that there is an inclusion e.g. \( M_2^+ \supset M_1^+ \). We denote by \( W \) a domain whose boundary is the union of one component of \( M_1 \setminus M_3 \) and one of \( M_2 \setminus M_3 \) and which has proper conormal cone. We also assume that \( \partial W \in C^1 \) (or equivalently that \( T^*_{M_1} X \cap T^*_{M_2} X = M_3 \times_{M_1} T^*_{M_1} X, i = 1, 2 \)). Then we get (2.4) \( 0 < \forall j < s^- \) (resp. \( \forall j \neq 0, s^- \) if \( s^-_{M_i} (p) \equiv s^- \forall p \in \Lambda_i \) : \( = T^*_{M_i} X \cap N (W)^{o.a}, i = 1, 2 \)); we also get (2.5) if \( s^- \geq 1 \). In fact we first remark that \( s^-_{M_1} (p_1) \leq s^-_{M_2} (p_1) \) due to \( M_2^+ \supset M_1^+ \). Assume first \( s^-_{M_1} (p_1) = s^-_{M_2} (p_1) \).
We then perform a contact transformation $\chi$ similar as in §3 but satisfying in the present case:

\[
\chi(T_{M_1}^* X) = T_{\tilde{M}_1}^* X, \text{ codim } \tilde{M}_1 = 1, s_{\tilde{M}_1}^-(q_1) = 0;
\]

\[
\chi(T_{M_2}^* X) = T_{\tilde{M}_2}^* X, \text{ codim } \tilde{M}_2 = 1.
\]

By quantization we get accordingly (with $s_i^- = s_{M_i}(p_1)$ and $\tilde{s}_i^- = s_{\tilde{M}_i}(q_1)$):

\[
\Phi_K(Z_{\partial W}) = \begin{cases} Z_{\tilde{M}_1}[s^{-}_1] & \text{in } \tilde{\Lambda}_1 \\
Z_{\tilde{M}_2}[s^{-}_2 - \tilde{s}^{-}_2] & \text{in } \tilde{\Lambda}_2 \end{cases}
\]

\[
\Phi_K(Z_{M_2^+ \setminus M_1^+}) = \begin{cases} Z_{\tilde{M}_1}[s^{-}_1] & \text{in } \tilde{\Lambda}_1 \\
Z_{\tilde{M}_2}[s^{-}_2 - \tilde{s}^{-}_2 - 1] & \text{in } T_{\tilde{M}_2}^* X \setminus \tilde{\Lambda}_2. \end{cases}
\]

Thus constancy may hold only for the first sheaf and provided that $\tilde{s}^{-}_2 = 0$. The proof then goes in the same line as in §3. The case $s^{-}_2 = s^{-}_1 + 1$ is similar.

**Remark 4.2.** It seems that we may extend Theorem 2.2 to polyhedrons. Let $M_i^+, i = 1, ..., m$ be half-spaces with $C^2$ transversal boundaries and let $W = \cap_i M_i^+$. Let $\beta = (\beta_1, \ldots, \beta_m), \beta_i \in \{0, 1\}$, define $M_\beta = \bigcap_{\{i; \beta_i = 1\}} M_i$ and assume that $M_\beta, |\beta| = m$ is generic at $z_0, z_\in M_\beta$ (i.e. $\gamma_{M_\beta}(p) = 0, p \in \pi^{-1}(z_0)$). We remark that if $\gamma_{M_\beta}(p) = 0$ holds for $|\beta| = m$, then it also holds $\forall |\beta| \leq m$. If we then suppose that

\[
s_{M_\beta}^-(p) \equiv s^- \forall p \in \hat{T}_{M_\beta}^* X \cap N(W)^{\alpha a} \cap \pi^{-1}(B_0), \forall \beta,
\]

we get (2.4) $\forall j \neq 0, s^-$ (and (2.5) when $s^- \geq 1$). We shall prove it in our forthcoming papers. It seems that the above conclusions also hold for polyhedrons with non-transversal faces (but with $C^1$-boundary) in the frame of Remark 4.1.

**Example 4.3.** (Dihedron with transversal faces.) Let $X = C^n$ with coordinates $z = (z_1, z_2, z')$, $z = x + \sqrt{-1} y$, let $M_1 = \{z \in X; y_1 = y_2 +$
\[ |y'|^2, \{ y_1 = -y_2 - |y'|^2 \}, M_2 = \{ y_1 = 0, y_2 = -|y'|^2 \}, W = \{ y_2 > -|y'|^2, -(y_2 + |y'|^2) < y_1 < y_2 + |y'|^2 \}. \]
In this case \( s_{M_i}^*(p) \equiv n - 2 \forall p \in T_{M_i}^* X \cap N(W)^a, i = 1, 2, 3. \) It follows (2.4) \( \forall j \neq 0, n - 2 \) (and (2.5) when \( n > 2 \)).

**Example 4.4.** (Dihedron with \( C^1 \)-boundary.) Let \( W = \{ z \in X; y_1 > -|y'|^2 + c_1 y_2^2 \text{ for } y_2 \geq 0; y_1 > -|y'|^2 + c_2 y_2^2 \text{ for } y_2 \leq 0 \} \) \( (c_1 \text{ and } c_2 \geq 0, c_1 \neq c_2) \). We still have (2.4) \( \forall j \neq 0, n - 2 \) (and (2.5) for \( n > 2 \)).

**References**


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