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The Andreotti Grauert vanishing theorem for dihedrons of $\mathbf{C}^n$

By Giuseppe ZAMPIERI

Abstract. Let X be a complex manifold, $\mathcal{O}_X$ the sheaf of analytic functions on X, W an open set of X with C^2-boundary $M = \partial W$ (W locally on one side of M), z_o a point of M, p_o the exterior conormal to W at z_o. If the number of negative eigenvalues for the Levi form of M in a neighborhood of p_o is $\geq s^-$ (resp. $\equiv s^-$), then vanishing of local cohomology groups of $\mathcal{O}_X$ over W in degree $< s^-$ and $\neq 0$ (resp. $\neq 0, s^-$) is stated in [1], [2], [14] (resp. [10]). Let now W be an open asymptotically convex dihedron with C^2 (transversal) faces M_1, M_2 and with "generic" edge M_3. If the numbers of negative eigenvalues for the Levi forms of M_1, M_2, and for the "microlocal" Levi form of M_3 are $\geq s^-$ along the exterior conormal cone to W at z_o, then vanishing of cohomology $< s^-$ and $\neq 0$ is proved here in the line of [1]. Under the additional assumption that M_3 is real analytic and that $T^*_{M_3}X$ contains no germ of complex curve, the same result as by [10] is also stated. The content of this paper was exposed at the University Paris XIII, June 21 1991.

1. Notations

Let X be a complex manifold, M a C^2- submanifold, $\pi : T^*X \to X$ (resp. $\pi : T^*_M X \to M$, resp. $\tau : TM \to M$) the cotangent bundle to X (resp. the conormal bundle to M, resp. the tangent bundle to M). Let or_X (resp. $\mathrm{or}_{M/X}$) be the orientation (resp. the relative orientation) sheaf. Let $z_o \in M$ and $p \in \dot{T}^*_M X$ $(= T^*_M X \setminus \{0\})$ with $\pi(p) = z_o$. We shall use the notations

$$e(p) = T_p T^* X$$

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$$\begin{aligned}
T^{\mathbf{C}}_{z_o}M =& T_{z_o}M \cap \sqrt{-1}T_{z_o}M \\
\lambda_M(p) =& T_pT^*_MX \\
\lambda_o(p) =& T_p\pi^{-1}\pi(p) \\
\gamma_M(p) =& \dim_{\mathbf{C}}(\lambda_M(p) \cap \sqrt{-1}\lambda_M(p) \cap \lambda_o(p)) \\
\nu(p) =& \text{the complex line spanned by the} \\
& \text{radial vector field.}
\end{aligned}$$

We shall drop sometimes z_o and p in the above notations.

Let $\alpha = \alpha^{\mathbf{R}} + \sqrt{-1}\alpha^{\mathbf{I}}$ be the canonical 1-form on T^*X and $\sigma = \sigma^{\mathbf{R}} + \sqrt{-1}\sigma^{\mathbf{I}}$ ($=d\alpha$) the canonical 2-form. Let $X^{\mathbf{R}}$ (resp. $(T^*X)^{\mathbf{R}}$) denote the real underlying manifold to X (resp. T^*X). We shall identify $T^*(X^{\mathbf{R}}) \simeq (T^*X)^{\mathbf{R}}$ by endowing $T^*(X^{\mathbf{R}})$ with the 1-form $\alpha^{\mathbf{R}}$.

For an isotropic plane $\mu \subset e$ and for a subspace λ of e, we shall set $\lambda^\mu = ((\lambda \cap \mu^\perp) + \mu)/\mu$. For λ (resp. l) $\mathbf{R}$- (resp. $\mathbf{C}$-) Lagrangian, we shall set $\mu = \lambda \cap \sqrt{-1}\lambda$ and define $\mathrm{L}_{l/\lambda}(u,v) \overset{\text{def.}}{=} \sigma^\mu(u,\bar{v})\ \forall u,v \in l^\mu$ where σ^μ is the form induced by σ on e^μ, and $\bar{v}$ is the conjugate to v, modulo μ, in the sum $(\lambda + \sqrt{-1}\lambda)/\mu$. Let ϕ be a C^2-function on $X^{\mathbf{R}}$ at z_o with $\phi\big|_M \equiv 0$ and $(z_o; d\phi(z_o)) = p$. We define the Levi form $L_\phi(z_o)$ in a local system of coordinates $z \in \mathbf{C}^n \simeq X$, as the Hermitian form with matrix $\left(\partial_{z_i}\partial_{\bar{z}_j}\phi(z_o)\right)_{1\le i,j\le n}$. According to [15], one may see that

$$L_\phi(z_o)\big|_{T^{\mathbf{C}}_{z_o}M} \sim L_{\lambda_o/\lambda_M} \tag{1.1}$$

where "$\sim$" means equivalence in signature and rank. (Cf. also [5] as for the case of higher codimension.) We shall also write $L_M(p)$ instead of $L_\phi(z_o)|_{T^{\mathbf{C}}_{z_o}M}$. We shall define $s^{+,-,0}_M(p)$ to be the numbers of eigenvalues of $L_\phi(z_o)\big|_{T^{\mathbf{C}}_{z_o}M}$ which are respectively > 0, < 0, $= 0$; by (1.1) they do not depend on the choice of ϕ.

We shall denote by $\mathrm{D}^b(X)$ the derived category of the category of complexes of sheaves with bounded cohomology, and by $\mathrm{D}^b(X;p)$ the localized category with respect to the null-system $\{\mathcal{F} \in \mathrm{D}^b(X); p \notin \mathrm{SS}(\mathcal{F})\}$. (Here $\mathrm{SS}(\mathcal{F})$ is the microsupport in the sense of [10] and [11], a closed conic subset of T^*X.) We recall that for a complex $\mathcal{F}$ which verifies $\mathrm{SS}(\mathcal{F}) \subset T^*_MX$ in a neighborhood of p, $p \in T^*_MX$, one may find a complex $M^{\cdot}$ of $\mathbf{Z}$-modules such that $\mathcal{F}$ is microlocally isomorphic to the constant complex $M^{\cdot}_M$ at p

(i.e. isomorphic in $\mathrm{D}^b(X;p)$). This criterion, stated in [10, §6], for M being a C^2-submanifold, easily extends to the case C^1 (see [6]).

We shall denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X and by $\mathbf{Z}_W$ ($W \subset X$ locally closed) the sheaf which is 0 on $X \setminus W$ and the constant sheaf with stalk $\mathbf{Z}$ on W. We shall deal with the complex $\mu_W(\mathcal{O}_X) \overset{\text{def.}}{=} \mu\mathrm{hom}(\mathbf{Z}_W, \mathcal{O}_X)$ of microfunctions along W (in particular with the complex $\mu_M(\mathcal{O}_X)$), where $\mu\mathrm{hom}(\cdot,\cdot)$ is the bifunctor of [10].

Finally we shall denote by $N(W)$ (resp. $N(W)^{oa}$) the normal cone to W (resp. the polar antipodal cone to $N(W)$); this is an open cone of TX (resp. a closed cone of T^*X).

2. Statement of the Vanishing Theorems

Let X be a complex manifold, W an open set of X with C^2-boundary $M = \partial W$ (W locally on one side of M), z_o a point of M, p_o the exterior conormal to W at z_o.

Theorem A. (i) ([1]) *In the above situation we have*

$$\varinjlim_{B} H^j(W \cap B, \mathcal{O}_X) = 0 \quad 0 < {}^\forall j < s_M^-(p_o), \tag{2.1}$$

for B describing a system of neighborhoods of z_o.

(ii) *([10]) If moreover $s_M^-(p) \equiv$ const $\forall p \in T_M^*X$ near p_o, then (2.1) holds $\forall j \neq 0, s_M^-(p_o)$.*

Refinements of (2.1) are given by several authors ([14], [2], [10]); in particular vanishing of (2.1) for $j > s_M^- + s_M^o$ is also stated. As for Th. A (ii) the result was already known before [10] under the stronger assumption of constant rank for $L_M(p)$ (cf. e.g. [14]).

The purpose of the present note is to generalize Theorem A to the case of a dihedron. Let M_1, M_2 be C^2-hypersurfaces of $X^{\mathbf{R}}$ through z_o, $M_i^+, i = 1,2$ open half-spaces with boundary M_i, and let W be the dihedron defined by $W = M_1^+ \cap M_2^+$. We put $M_3 = M_1 \cap M_2$, and assume that M_3 is a manifold which verify the "genericity" condition:

$$T_{M_3}^*X_{z_o} \cap \sqrt{-1}T_{M_3}^*X_{z_o} = \{0\}. \tag{2.2}$$

which is equivalent to $\gamma_{M_3}(p) = 0$ for $p \in \pi^{-1}(z_o)$. We denote p_i, $i = 1, 2$ the exterior conormal to M_i, and denote s^- the minimum between $s^-_{M_1}(p_1)$, $s^-_{M_2}(p_2)$ and $s^-_{M_3}(p)\,\forall p \in N_{z_o}(W)^{o\,a}$.

THEOREM 2.1. *Let M_1, M_2 intersect transversally, and let the intersection M_3 be generic. Then*

$$\varinjlim_{B} \mathrm{H}^j(W \cap B, \mathcal{O}_X) = 0 \quad 0 < {}^\forall j < s^- \tag{2.4}$$

(and

$$\varinjlim_{B} \Gamma(B, \mathcal{O}_X) \xrightarrow{\sim} \varinjlim_{B} \Gamma(W \cap B, \mathcal{O}_X) \textit{ if } s^- \geq 1). \tag{2.5}$$

In [13, Prop. 11 p. 148] vanishing of (2.4) for $j < s^- - 1, j \neq 0$ is proved (in a different frame); in particular the case $j = s^- - 1$ seems not to be treated.

THEOREM 2.2. *Let M_1, M_2 intersect transversally, let (2.2) be satisfied, and suppose moreover M_3 real analytic with $T^*_{M_3}X$ containing no germ of complex curve. Assume that for a constant s^- and for an open set B_o:*

$$s^-_{M_i}(p) \equiv s^- \quad \forall p \in T^*_{M_i}X \cap N(W)^{oa} \cap \pi^{-1}(B_o), \text{ and } \forall i = 1, 2, 3. \tag{2.6}$$

Then (2.4) holds $\forall j \neq 0, s^-$ (and (2.5) is still fulfilled when $s^- \geq 1$).

(By (2.2), if M_3 contains no germ of complex curve, the same is also true for $T^*_{M_3}X$.)

3. Proofs

Proof of Theorem 2.1

We shall use the notations:

$$\Lambda_i = T^*_{M_i}X \cap N(W)^{o\,a}, \quad N_i = \pi(\Lambda_i),$$
$$\overset{o}{\Lambda}_i = \Lambda_i \setminus (\Lambda_j \cup \Lambda_k), \quad \overset{o}{N}_i = N_i \setminus (N_j \cup N_k),\ j, k \neq i.$$

(a) We first deal with a neighborhood of the conormal p_1 to M_1; since $p_1 \in (\Lambda_1 \cap \Lambda_3) \setminus \Lambda_2$, we shall forget Λ_2 for a while.

LEMMA 3.1. *We may find a (complex) contact transformation χ between neighborhoods of p_1 and $q_1 \overset{\text{def.}}{=} \chi(p_1)$ which interchanges:*

$$\begin{cases} T^*_{M_3}X \to T^*_{\tilde{M}_3}X, & cod\,\tilde{M}_3 = 1, s^-_{\tilde{M}_3}(q_1) = 0 \\ T^*_{M_1}X \to T^*_{\tilde{M}_1}X, & cod\,\tilde{M}_1 = 1. \end{cases} \tag{3.1}$$

PROOF. We shall deal in the linear space $e' = e^\nu$ (due to $\dim_{\mathbf{R}}(\lambda_{M_i} \cap \nu) = 1$). According to [17], to solve (3.1) is equivalent as to find a $\mathbf{C}$-Lagrangian plane $l'_o \subset e'$such that (with $\lambda'_i \overset{\text{def.}}{=} \lambda^\nu_{M_i}$):

$$l'_o \cap \lambda'_3 = \{0\}, \tag{3.2}$$

$$L_{l'_o/\lambda'_3} \geq 0, \tag{3.3}$$

$$l'_o \cap \lambda'_1 = \{0\}. \tag{3.4}$$

Now the complex planes l'_o which verify (3.4) are an open dense set in the Lagrangian Grassmannian of e' (obvious). The ones which verify (3.2), (3.3) are an open non-empty set. In fact this is clearly non-empty (cf. e.g. the proof of [10, Prop. 11.3.1]); moreover the rank of $L_{l'_o/\lambda'_3}$ is constant $(= n - 1 - \dim(\lambda'_3 \cap \sqrt{-1}\lambda'_3))$ for l'_o varying in the open set defined by $l'_o \cap \lambda'_3 = \{0\}$ and hence the signature is constant in each component of the above set. Thus the problem of finding l'_o as in (3.2)–(3.4) can be solved. □

(This was already remarked in [7].)

REMARK 3.2. In the same way as before one may construct a contact transformation satisfying the requirements (3.1) but with $s^-_{\tilde{M}_1}(q_1) = 0$ instead of $s^-_{\tilde{M}_3}(q_1) = 0$.

We shall put in what follows:

$$\tilde{\Lambda}_i = \chi(\Lambda_i),\ \overset{o}{\tilde{\Lambda}}_i = \chi(\overset{o}{\tilde{\Lambda}}_i),\ \tilde{N}_i = \pi(\tilde{\Lambda}_i),\ \overset{o}{\tilde{N}}_i = \pi(\overset{o}{\tilde{\Lambda}}_i).$$

(b) We recall from [5] that

$$s^-_{M_3}(p_1) \leq s^-_{M_1}(p_1) \leq s^-_{M_3}(p_1) + 1.$$

We also choose a quantization Φ_K of χ by a kernel K according to [10, ch. 11].

Lemma 3.3. *Let $s^-_{M_1}(p_1) = s^-_{M_3}(p_1)$. Then, if χ is a contact transformation which satisfies (3.1), we have:*

$$\tag{3.5} \begin{cases} s^-_{\tilde{M}_1}(q_1) = 0, \\ \tilde{\Lambda}_1 \cup \tilde{\Lambda}_3 = T^*_Y X \ \text{ for a } C^1\text{-hypersurface } Y, \\ \Phi_K(\mathbf{Z}_{\overset{o}{N}_1}) \simeq \mathbf{Z}_Y[s^-_{M_1}(p_1)] \quad \text{in } \mathrm{D}^b(X; q_1). \end{cases}$$

Proof. We remark that $T^*_{\tilde{M}_1}X \cap T^*_{\tilde{M}_3}X$ is clean of codimension 1, whence $\tilde{M}_1, \tilde{M}_3$ intersect at the order 2 along a submanifold $\tilde{R}$ of codimension 2. In particular $T\tilde{M}_1\big|_{\tilde{R}} = T\tilde{M}_3\big|_{\tilde{R}}$ and hence either $\tilde{N}_1 \cup \tilde{N}_3$ or $\tilde{N}_1 \cup (\tilde{M}_3 \setminus \tilde{N}_3)$ is a C^1-hypersurface Y. To make the right choice we exploit Φ_K and remark that

$$\mathrm{SS}(\Phi_K(\mathbf{Z}_{\overset{o}{N}_1})) \subset \tilde{\Lambda}_1 \cup \tilde{\Lambda}_3 \subset \pi^{-1}(\tilde{N}_1 \cup \tilde{N}_3),$$
$$\mathrm{SS}(\Phi_K(\mathbf{Z}_{N_1})) \subset \tilde{\Lambda}_1 \cup (T^*_{\tilde{M}_3}X \setminus \tilde{\Lambda}_3) \subset \pi^{-1}(\tilde{N}_1 \cup (\tilde{M}_3 \setminus \tilde{N}_3)).$$

Hence one of the above complexes has its microsupport contained in $T^*_Y X$; and this complex must then be constant due to [10] and [6]. But we have

$$\tag{3.6} \Phi_K(\mathbf{Z}_{\overset{o}{N}_1}) = \begin{cases} \mathbf{Z}_{\tilde{M}_1}[s^-_{M_1}(p_1) - s^-_{\tilde{M}_1}(q_1)] & \text{in } \overset{o}{\tilde{\Lambda}}_1, \\ \mathbf{Z}_{\tilde{M}_3}[s^-_{M_3}(p_1)] & \text{in } \overset{o}{\tilde{\Lambda}}_3, \end{cases}$$
$$\Phi_K(\mathbf{Z}_{N_1}) = \begin{cases} \mathbf{Z}_{\tilde{M}_1}[s^-_{M_1}(p_1) - s^-_{\tilde{M}_1}(q_1)] & \text{in } \overset{o}{\tilde{\Lambda}}_1, \\ \mathbf{Z}_{\tilde{M}_3}[s^-_{M_3}(p_1) + 1] & \text{in } T^*_{\tilde{M}_3}X \setminus \tilde{\Lambda}_3. \end{cases}$$

Recall that $s^-_{M_3}(p_1) = s^-_{M_1}(p_1)$; hence constancy may hold only for $\Phi_K(\mathbf{Z}_{\overset{o}{N}_1})$ (which implies $\tilde{N}_1 \cup \tilde{N}_3 = Y \in C^1$) and under the additional condition $s^-_{\tilde{M}_1}(p_1) = 0$.

We also observe that $\Phi_K(\mathbf{Z}_{\overset{o}{N}_1})$ turns out to be a constant sheaf along Y and by (3.6) this is simple with shift $\frac{1}{2} + s^-_{M_1}(p_1)$ along $\overset{o}{\tilde{\Lambda}}_1$ and $\overset{o}{\tilde{\Lambda}}_3$. This completes the proof of (3.5). □

Denote by $\Sigma_{\tilde{M}_i}, i = 1, 2$ the closed half–spaces with boundary $\tilde{M}_i$ and inward conormal q_1.

Lemma 3.4. *Let $s^-_{M_1}(p_1) = s^-_{M_3}(p_1) + 1$. Then by the contact transformation of Remark 3.2, we have*

$$\text{(3.5')} \qquad \begin{cases} s^-_{\tilde{M}_3}(q_1) = 0,\ \Sigma_{\tilde{M}_1} \supset \Sigma_{\tilde{M}_3}, \\ \pi(\tilde{\Lambda}_1 \cup \tilde{\Lambda}_3) = \partial(\Sigma_{\tilde{M}_1} \setminus \Sigma_{\tilde{M}_3})^+, \\ \Phi_K(\mathbf{Z}_{\overset{o}{N}_1}) \simeq \mathbf{Z}_{(\Sigma_{\tilde{M}_1} \setminus \Sigma_{\tilde{M}_3})^+}, \end{cases}$$

where $(\cdot)^+$ is a component of $\cdot$.

Proof. The proof is the same as in Lemma 3.3. □

We should prove Lemma 3.4 (and similarly Lemma 3.3) also by another argument. We recall that $\tilde{M}_1, \tilde{M}_3$ intersect at the order 2; thus either $\Sigma_{\tilde{M}_1} \supset \Sigma_{\tilde{M}_3}$or $\Sigma_{\tilde{M}_1} \subset \Sigma_{\tilde{M}_3}$. But

$$\begin{aligned} \mathrm{Hom}_{D^b(X;p_1)}(\mathbf{Z}_{M_1}, \mathbf{Z}_{M_3}) &\overset{\Phi_K}{\simeq} \mathrm{Hom}_{D^b(X;q_1)}(\mathbf{Z}_{\Sigma_{\tilde{M}_1}}, \mathbf{Z}_{\Sigma_{\tilde{M}_3}}[-s^-_{\tilde{M}_3}(q_1)]) \\ &\cong \mathrm{H}^{-s^-_{\tilde{M}_3}(q_1)} \mathbf{R}\Gamma_{\Sigma_{\tilde{M}_1}}(\mathbf{Z}_{\Sigma_{\tilde{M}_3}})_{z'_0},\ (z'_0 = \pi(q_1)). \end{aligned}$$

This gives the first of (3.5'). The proof of the remaining statements is then easy.

(c) We have

$$\text{(3.7)} \qquad \mathbf{Z}_W \overset{\sim}{\leftarrow} \mathbf{Z}_{\overset{o}{N}_1}[-1] \otimes \mathrm{or}_{M_1/X} \quad \text{in } \mathrm{D}^b(X;p_1).$$

In fact one uses the distinguished triangle

$$\mathbf{Z}_{\bar{W}\setminus N_1} \to \mathbf{Z}_{\bar{W}} \to \mathbf{Z}_{N_1} \overset{+1}{\to},$$

applies the functor $\cdot^* \overset{\text{def.}}{=} \mathbf{R}\mathcal{H}\mathrm{om}(\cdot, \mathbf{Z}_X)$, and gets

$$\mathbf{Z}_{\overset{o}{N}_1}[-1] \otimes \mathrm{or}_{M_1/X} \to \mathbf{Z}_W \to \mathbf{Z}^*_{\bar{W}\setminus N_1} \overset{+1}{\to} .$$

But

$$\begin{aligned} \mathrm{SS}(\mathbf{Z}_{\bar{W}\setminus N_1})^*_{z_o} &= \mathrm{SS}(\mathbf{Z}_{\bar{W}\setminus N_1})^a_{z_o} \\ &= \text{the convex cone of the plane } T^*_{M_3}X_{\ z_o} \\ &\quad\ \text{bounded by } \mathbf{R}^+p_2,\ \mathbf{R}^-p_1. \end{aligned}$$

Thus $p_1 \notin \mathrm{SS}(\mathbf{Z}_{\bar{W}\setminus N_1})^*_{z_o}$ and (3.7) follows. We remark that along with (3.7) we have

$$\mathbf{Z}_W \overset{\sim}{\leftarrow} \mathbf{Z}_{M_3}[-2] \otimes \mathrm{or}_{M_3/X} \text{ in } \mathrm{D}^b(X; \overset{o}{\Lambda}_3) \tag{3.8}$$

$$\mathbf{Z}_W \overset{\sim}{\leftarrow} \mathbf{Z}_{M_1}[-1] \otimes \mathrm{or}_{M_1/X} \text{ in } \mathrm{D}^b(X; \overset{o}{\Lambda}_1). \tag{3.9}$$

(d) We first assume $s^-_{M_1}(p_1) = s^-_{M_3}(p_1)$, put $s^-_1 \overset{\text{def.}}{=} s^-_{M_1}(p_1)$, and denote by $\tilde{W}$ the open subset of X with boundary Y and exterior conormal q_1 (cf. Lemma 3.2). By (3.5), (3.7) and by the analogous for $\tilde{W}$, we get

$$\begin{aligned} \Phi_K(\mathbf{Z}_W) &\simeq \Phi_K(\mathbf{Z}_{\overset{o}{N}_1}[-1] \otimes \mathrm{or}_{M_1/X}) \\ &\simeq \mathbf{Z}_Y[s^-_1 - 1] \otimes \mathrm{or}_{Y/X} \simeq \mathbf{Z}_{\tilde{W}}[s^-_1] \end{aligned} \tag{3.10}$$

Let $\mu_{\tilde{W}}(\cdot) = \mu\mathrm{hom}(\mathbf{Z}_{\tilde{W}}, \cdot)$ be the functor defined in [K-S 1,ch. 5], and recall the triangle

$$\left(\mathcal{O}_X|_{\overline{\tilde{W}}}\right)_{z'_o} \to \mathbf{R}\Gamma_{\tilde{W}}(\mathcal{O}_X)_{z'_o} \to \mu_{\tilde{W}}(\mathcal{O}_X)_{q_o} \overset{+1}{\to} (z'_o = \pi(q_1)). \tag{3.11}$$

Since $\mathcal{O}_{X\, z_o'} \hookrightarrow \mathcal{H}^0_{\tilde{W}}(\mathcal{O}_X)_{\, z_o'}$ is injective by analytic continuation, then: $\mathrm{H}^j\mu_{\tilde{W}}(\mathcal{O}_X)_{q_1} = 0\, \forall j < 0$. By (3.10), by the fact that $\Phi_K(\mathcal{O}_X) = \mathcal{O}_X$, and by [10, Th. 7.4.1], one concludes

$$\mathrm{H}^j\mu_W(\mathcal{O}_X)_{p_1} = 0 \quad \forall j < s_1^-. \tag{3.12}$$

Let now $s^-_{M_3}(p_1) = s^-_{M_1}(p_1) + 1$; in this case we have

$$\Phi_K(\mathbf{Z}_W) \simeq \mathbf{Z}_{(\Sigma_{\tilde{M}_1} \setminus \Sigma_{\tilde{M}_3})^+}[s_1^- - 1].$$

We also remark that

$$\mathcal{H}^1_{\Sigma_{\tilde{M}_3}}(\mathcal{O}_X)_{z_0'} \hookrightarrow \mathcal{H}^1_{\Sigma_{\tilde{M}_1}}(\mathcal{O}_X)_{z_0'}$$

is injective. Then (3.12) is still fulfilled in this case

(e) One may repeat the same argument as in (a)–(d) and get (3.12) with p_1, s_1^- replaced by p_2, s_2^-. One may also use (3.10) and get in similar (and simpler) way:

$$\mathrm{H}^j\mu_W(\mathcal{O}_X)_p = 0 \quad \forall p \in \overset{o}{\Lambda}_3,\, \forall j < s^-_{M_3}(p) \tag{3.13}$$

Finally by the triangle

$$(\mathcal{O}_X)_{\bar{W}} \to \mathbf{R}\Gamma_W(\mathcal{O}_X) \to \mathbf{R}\dot{\pi}_*\mu_W(\mathcal{O}_X) \overset{+1}{\to}, \tag{3.14}$$

we get $\mathrm{H}^j\mathbf{R}\Gamma_W(\mathcal{O}_X)_{z_o} = 0$ if $j \neq 0$, and $j < s_1^-,\ s_2^-,\ \inf\limits_{p \in \overset{o}{\Lambda}_3} s^-_{M_3}(p)$.

From (3.14), one also gets $\mathcal{O}_{X\, z_o} \simeq \mathcal{H}^0_W(\mathcal{O}_X)_{z_o}$ when the infimum of the above s^-'s is ≥ 1.

Proof of Theorem 2.2

(a) We first focus our attention in a neighborhood of p_1. Observing that $s^-_{M_1}(p_1) = s^-_{M_3}(p_1)$ in the present case, then by the proof of Theorem 2.1 we get $\Phi_K(\mathbf{Z}_W) = \mathbf{Z}_{\tilde{W}}[s^-]$, $(s^- = s^-_1)$ at q_1 for $\tilde{W}$ open with C^1-boundary. According to [10, Th. 11.2.8] $s^-_{\tilde{M}_1}(q) - s^-_{M_1}(p) \equiv$ const; but $s^-_{M_i}(p) \equiv$ const, $s^-_{\tilde{M}_i}(q_1) = 0$ whence $s^-_{\tilde{M}_i}(q) \equiv 0$. We also observe that in the present situation $\Sigma_{\tilde{M}_3} \supset \Sigma_{\tilde{M}_1}$.

Lemma 3.5. *$\tilde{W}$ is pseudoconvex in a neighborhood of $z'_o(\overset{\text{def.}}{=} \pi(q_1))$.*

Proof. We identify $X \simeq \mathbf{C}^n$ and recall from [9, ch. 2, 4] that $\tilde{W}$ is pseudoconvex at z'_o if and only if $-\log\delta$, is plurisubharmonic (with $\delta(z) = \operatorname{dist}(z, X\backslash\tilde{W})$. We remark that $\forall z \in \tilde{W}$ there exists an unique $z^* \in \partial\tilde{W}$ such that $\delta(z) = |z - z^*|$ (due to $\tilde{W} \supset X \setminus \Sigma_{\tilde{M}_3}$). We then let $S = \{z \in \tilde{W}; z^* \in \tilde{M}_1 \cap \tilde{M}_3\}$ (which is a C^2–hypersurface), and denote by $\tilde{W}_h$, $h = 1, 2$ the two components of $\tilde{W} \setminus S$. We also define δ_h to be the distance to $\tilde{M}_h$; clearly $\forall z \in \tilde{W} \setminus S$ we have $\delta = \delta_1$ or δ_3 at z whence $\delta \in C^2(\tilde{W} \setminus S) \cap C^1(\tilde{W})$. It is also easy to see (by adapting the proof by [9, Th. 2.6.12]) that $-\log\delta\big|_{\tilde{W}\setminus S}$ is plurisubharmonic if and only if $s^-_{\tilde{M}_h}(q') \equiv 0 \, \forall q' \in \tilde{N}_h \times_{\tilde{M}_h} T^*_{\tilde{M}_h}X$. Thus $\forall w \in \mathbf{C}^n$ and $\forall g \in C^\infty_c(\tilde{W})$, $g \geq 0$:

$$\int_{\tilde{W}} \sum (-\log\delta) w_i \bar{w}_j \partial_i \bar{\partial}_j g \mathrm{d}\lambda = \sum_{h=1,2} \int_{\tilde{W}_h} \sum \partial_i \bar{\partial}_j (-\log\delta) w_i \bar{w}_j g \mathrm{d}\lambda \geq 0$$

(which holds, by Stokes formula, because $\delta \in C^2(\tilde{W} \setminus S) \cap C^1(\tilde{W})$ with bounded second derivatives). The lemma immediately follows. $\square$

By the above lemma we have $\mathrm{H}^j\mathbf{R}\Gamma_{\tilde{W}}(\mathcal{O}_X)_{z'} = 0 \, \forall j \neq 0$, $(z' = \pi(q))$ due to [9, Corollary 7.4.2]. By the triangle (3.11) and by (3.10), we then conclude

$$\mathrm{H}^j \mu_W(\mathcal{O}_X)_{p_1} = 0 \quad \forall j \neq s^-. \tag{3.15}$$

(b) The same as (3.15) also holds for p_2 and $\forall p \in \overset{o}{\Lambda}_3$. We need now:

Lemma 3.6. *We have*

$$\mathrm{H}^j \mathbf{R}\dot{\pi}_* \mu_W(\mathcal{O}_X)_{z_o} = 0 \quad \forall j \neq s^-. \tag{3.16}$$

Proof. We set $\mathcal{F} = \mathrm{H}^{s^-}\mu_W(\mathcal{O}_X)$, $Z = \dot{N}_{z_o}(W)^{o\,a} = \mathbf{R}^+ p_1 \cup \mathbf{R}^+ p_2 \cup \operatorname{int} N_{z_o}(W)^{o\,a}$ and denote B_i, $i = 1, 2$ conic neighborhoods of p_i in Z. Recall that M_3 is real analytic and that $T^*_{M_3}X$ contains no germ of complex curve. Then [16] applies and gives that $\mathcal{F}\big|_{\overset{o}{\Lambda}_3} = \mathrm{H}^{s^-+2}\mu_{M_3}(\mathcal{O}_X)\big|_{\overset{o}{\Lambda}_3}$ is conically flabby. In particular

$$\mathrm{H}^j(\overset{o}{\Lambda}_3, \mathcal{F}) = 0\, \forall j \neq 0; \quad \Gamma(\overset{o}{\Lambda}_3, \mathcal{F}) \twoheadrightarrow \Gamma(\overset{o}{\Lambda}_3 \cap B_i, \mathcal{F}),\ i = 1, 2 \tag{3.17}$$

(where“$\twoheadrightarrow$” stands for surjectivity). By an elementary application of the Mayer-Vietoris long exact sequence (and by taking the B_i’s small enough) one gets

$$\mathrm{H}^j(B_1 \cup B_2 \cup \overset{o}{\Lambda}_3, \mathcal{F}) = 0 \quad \forall j \neq 0,$$

which implies immediately (3.16). $\square$

By the triangle (3.14) and by Lemma 3.6 we get at once the conclusion of the proof of Theorem 2.2.

4. Remarks and Examples

Remark 4.1. Let M_1, M_2 be C^2-hypersurfaces, M_1^+, M_2^+ open half-spaces with boundary M_1, M_2, and set $M_3 = M_1 \cap M_2$. We do not assume that M_1, M_2 are transversal but still suppose that M_3 is a manifold and that there is an inclusion e.g. $M_2^+ \supset M_1^+$. We denote by W a domain whose boundary is the union of one component of $M_1 \setminus M_3$ and one of $M_2 \setminus M_3$ and which has proper conormal cone. We also assume that $\partial W \in C^1$ (or equivalently that $T^*_{M_1}X \cap T^*_{M_2}X = M_3 \times_{M_1} T^*_{M_i}X$, $i = 1, 2$). Then we get (2.4) $0 < {}^{\forall}j < s^-$ (resp. $\forall j \neq 0, s^-$ if $s^-_{M_i}(p) \equiv s^- \forall p \in \Lambda_i \overset{\text{def.}}{:=} T^*_{M_i}X \cap N(W)^{o\,a}, i = 1, 2$); we also get (2.5) if $s^- \geq 1$. In fact we first remark that $s^-_{M_1}(p_1) \leq s^-_{M_2}(p_1)$ due to $M_2^+ \supset M_1^+$. Assume first $s^-_{M_1}(p_1) = s^-_{M_2}(p_1)$.

We then perform a contact transformation χ similar as in §3 but satisfying in the present case:

$$\chi(T^*_{M_1}X) = T^*_{\tilde{M}_1}X, \text{ codim } \tilde{M}_1 = 1,\ s^-_{\tilde{M}_1}(q_1) = 0;$$
$$\chi(T^*_{M_2}X) = T^*_{\tilde{M}_2}X, \text{ codim } \tilde{M}_2 = 1.$$

By quantization we get accordingly (with $s_i^- = s^-_{M_i}(p_1)$ and $\tilde{s}_i^- = s^-_{\tilde{M}_i}(q_1)$):

$$\Phi_K(\mathbf{Z}_{\partial W}) = \begin{cases} \mathbf{Z}_{\tilde{M}_1}[s_1^-] & \text{in } \overset{o}{\tilde{\Lambda}}_1 \\ \mathbf{Z}_{\tilde{M}_2}[s_2^- - \tilde{s}_2^-] & \text{in } \overset{o}{\tilde{\Lambda}}_2 \end{cases}$$
$$\Phi_K(\mathbf{Z}_{M_2^+ \setminus M_1^+}) = \begin{cases} \mathbf{Z}_{\tilde{M}_1}[s_1^-] & \text{in } \overset{o}{\tilde{\Lambda}}_1 \\ \mathbf{Z}_{\tilde{M}_2}[s_2^- - \tilde{s}_2^- - 1] & \text{in } T^*_{\tilde{M}_2}X \setminus \overset{o}{\tilde{\Lambda}}_2. \end{cases}$$

Thus constancy may hold only for the first sheaf and provided that $\tilde{s}_2^- = 0$. The proof then goes in the same line as in §3. The case $s_2^- = s_1^- + 1$ is similar.

REMARK 4.2. It seems that we may extend Theorem 2.2 to polyhedrons. Let $M_i^+, i = 1, ..., m$ be half-spaces with C^2 transversal boundaries and let $W = \underset{i}{\cap} M_i^+$. Let $\beta = (\beta_1, \ldots, \beta_m), \beta_i \in \{0, 1\}$, define $M_\beta = \bigcap_{\{i:\beta_i=1\}} M_i$ and assume that $M_\beta, |\beta| = m$ is generic at z_o, $z_o \in M_\beta$ (i.e. $\gamma_{M_\beta}(p) = 0, p \in \pi^{-1}(z_o)$). We remark that if $\gamma_{M_\beta}(p) = 0$ holds for $|\beta| = m$, then it also holds $\forall |\beta| \leq m$. If we then suppose that

$$s^-_{M_\beta}(p) \equiv s^- \ \forall p \in \dot{T}^*_{M_\beta}X \cap N(W)^{o\,a} \cap \pi^{-1}(B_o),\ \forall \beta,$$

we get (2.4) $\forall j \neq 0, s^-$ (and (2.5) when $s^- \geq 1$). We shall prove it in our forthcoming papers. It seems that the above conclusions also hold for polyhedrons with non-transversal faces (but with C^1-boundary) in the frame of Remark 4.1.

Example 4.3. (Dihedron with transversal faces.) Let $X = \mathbf{C}^n$ with coordinates $z = (z_1, z_2, z'), z = x + \sqrt{-1}y$, let $M_1 = \{z \in X; y_1 = y_2 +$

$|y'|^2\}, M_2 = \{y_1 = -y_2 - |y'|^2\}, M_3 = \{y_1 = 0, y_2 = -|y'|^2\}, W = \{y_2 > -|y'|^2, -(y_2 + |y'|^2) < y_1 < y_2 + |y'|^2\}$. In this case $s^-_{M_i}(p) \equiv n - 2\, \forall p \in T^*_{M_i}X \cap N(W)^{o\,a}$, $i = 1, 2, 3$. It follows (2.4) $\forall j \neq 0, n-2$ (and (2.5) when $n > 2$).

Example 4.4. (Dihedron with C^1-boundary.) Let $W = \{z \in X; y_1 > -|y'|^2 + c_1 y_2^2 \text{ for } y_2 \geq 0;\ y_1 > -|y'|^2 + c_2 y_2^2 \text{ for } y_2 \leq 0\}$ (c_1 and $c_2 \geq 0$, $c_1 \neq c_2$). We still have (2.4) $\forall j \neq 0, n-2$ (and (2.5) for $n > 2$).

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Dipartimento di Matematica
Pura ed Applicata
Università Degli Studi di Padova
v. Belzoni 7, 35131 Padova
Italy