

π_1 of smooth points of a log del Pezzo
surface is finite : II

By R. V. GURJAR and D.-Q. ZHANG

Abstract. Let S be a normal projective algebraic surface with at worst log terminal singularities (i.e., quotient singularities) and ample anti-canonical divisor $-K_S$. In this Part II, we shall give a structure theorem (Theorem 1.1) for S and complete the proof of the following result stated in the Part I: The smooth part of S has finite fundamental group.

Introduction

A normal projective surface S over \mathbf{C} is called a log del Pezzo surface if S has at most quotient singularities and $-K_S$ is ample, where K_S denotes the canonical divisor of S . In Part I (cf. [2]) of this paper we set out to prove the following :

MAIN THEOREM. *The fundamental group of the space of smooth points of a log del Pezzo surface is finite.*

In this part II, we will complete the proof of this result. We will use the notations and results from Part I freely. Recall from Part I that if \tilde{S} is a minimal resolution of singularities of S , then we can find a “minimal” (-1)-curve C on \tilde{S} (cf. Lemma 3.1 and Prop. 3.6 of Part I). In §3, §4, §5 of Part I, we reduced to consider the cases (II-3) and (II-4) there. As remarked in the Introduction of Part I, it suffices to consider the case (II-4) (the “2-component case”), to complete the proof of our Main Theorem. This will

1991 *Mathematics Subject Classification.* Primary 14J45; Secondary 14E20, 14J17, 14J26.

be done in this part **II** of our paper. As in Part **I**, our proof for the case (II-4) gives quite precise information about the configuration of $C + D$. After the results of parts **I** and **II** of our paper were announced in a conference in Kinoshita, Japan, A. Fujiki, R. Kobayashi and S. Lu have found another proof of our Main Theorem using differential geometric methods (cf. [1]). Their proof of the Main Theorem is short, but it does not seem to give as precise information about the singular locus of S as our proof.

Acknowledgements. The authors would like to thank the referee for very careful reading and valuable comments which make the paper much more readable.

1. The proof of the Main Theorem in the case (II-4)

In this section, we consider the case(II-4) in Remark 3.11 of Part **I**. We employ the notations there.

Recall that $f : \tilde{S} \rightarrow S$ is a minimal resolution of singularities of S and $D = f^{-1}(\text{Sing } S)$. We can also write

$$f^*K_S = K_{\tilde{S}} + D^*$$

where D^* is an effective \mathbf{Q} -divisor with support contained by D (cf. Lemma 1.1 of Part **I**).

The (-1) -curve C , used in the case(II-4) of Remark 3.11 in Part **I**, now meets exactly a (-2) -curve D_1 and a $(-n)$ -curve D_2 with $n \geq 3$. Let Δ_i be the connected component of D containing D_i . Let $C + T_i$ ($i = 1, 2$) be the maximal twig of $C + \Delta_i$ such that $T_i = 0$ if D_i is not a tip component of Δ_i and T_i is the maximal twig of Δ_i containing D_i otherwise.

Our aim is to prove the following Theorem 1.1, which will imply the Main Theorem in the case (II-4).

THEOREM 1.1. *Suppose that the case (II-4) in Remark 3.11 occurs. Then one of the following five cases occurs :*

(1) Δ_i is a linear chain with D_i as a tip for $i = 1$ or 2. Hence $\pi_1(S^o)$ is finite (cf. Lemma 1.2 below).

(2) There are irreducible components A_i ($i = 1, \dots, a$), B_j ($j = 1, \dots, b$) of $\Delta_1 + \Delta_2$ and there is a \mathbf{P}^1 -fibration $\varphi : \tilde{S} \rightarrow \mathbf{P}^1$ such that

(2-1) a singular fiber of φ has support equal to $\text{Supp}(C + \sum_i A_i)$,
 (2-2) every component of $D - \sum_j B_j$ is contained in a singular fiber of φ , and

(2-3) $F \cdot \sum_j B_j \leq 2$ for a general fiber F of φ .

In particular, there is a \mathbf{C}^* -fibration on S° and hence $\pi_1(S^\circ)$ is finite (cf. Lemma 2.2 of Part **I**).

(3) For $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$, the intersection matrix of $C + T_i + \Delta_j$ has a positive eigenvalue and hence $\kappa(\tilde{S}, C + T_i + \Delta_j) = 2$.

In particular, $\pi_1(S^\circ)$ is finite (cf. Lemma 1.12).

(4) $C + \Delta_1 + \Delta_2$ is described in Figure 1, 2, 3 or 4 below. Moreover, there is a \mathbf{P}^1 -fibration $\varphi : \tilde{S} \rightarrow \mathbf{P}^1$ such that $C + D$ and all singular fibers of φ are precisely described in the proof of Lemma 1.10. (We shall call them Case (4-1), (4-2), (4-3) or (4-4) of Theorem 1.1.)

Hence $\pi_1(S^\circ)$ is finite (cf. Lemma 1.13).

(5) $C + \Delta_1 + \Delta_2$ is described in Figure 5 or 6 below, where the divisor H in Figure 5 might be a zero divisor. (We shall call them Case (5-5) or (5-6) of Theorem 1.1.)

Hence $\pi_1(S^\circ)$ is finite (cf. Lemma 1.13).

Theorem 1.1 is a consequence of Lemmas 1.3, 1.8, 1.9, 1.10 and 1.11 below.

LEMMA 1.2. Suppose that Δ_i is a linear chain with D_i as a tip for $i = 1$ or 2. Then $\pi_1(S^\circ)$ is a finite group.

PROOF. Suppose Δ_1 is a linear chain with D_1 as a tip. As the Picard number $\rho(S) = 1$, we see that $C + \Delta_1 + \Delta_2$ supports a divisor with strictly positive self-intersection. By Lemma 1.10 of Part **I**, we have a surjection $\pi_1(U - \Delta_1 - \Delta_2) \rightarrow \pi_1(\tilde{S} - D)$, where U is a small tubular neighborhood of $C \cup \Delta_1 \cup \Delta_2$. We can write $U = U_1 \cup U_2$, where U_i is a small neighborhood of $C \cup \Delta_i$. It is easy to see that $U_i - \Delta_1 - \Delta_2$ contains a small neighborhood N_i of Δ_i as a strong deformation retract for $i = 1, 2$. By assumption, $\pi_1(N_i - \Delta_i)$ is finite for $i = 1, 2$ and by Mumford's presentation (cf. [3]), $\pi_1(N_1 - \Delta_1)$ is a cyclic group generated by "the" loop γ_1 in $C - \Delta_1 - \Delta_2$ around the point $C \cap \Delta_1$. Now an easy application of Van-Kampen's theorem for the covering $U_1 - \Delta_1 - \Delta_2$ and $U_2 - \Delta_1 - \Delta_2$ of $U - \Delta_1 - \Delta_2$ shows that $\pi_1(U - \Delta_1 - \Delta_2)$ is finite and hence so is $\pi_1(\tilde{S} - D)$. \square

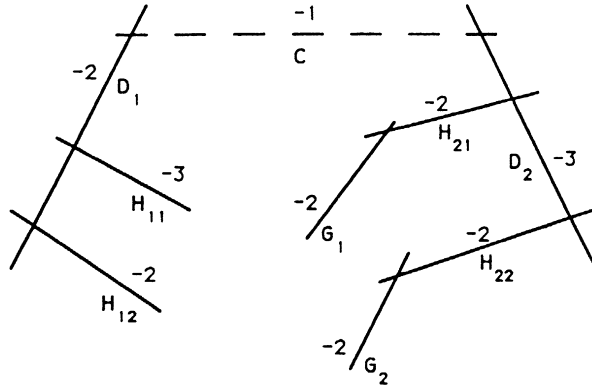


Fig. 1

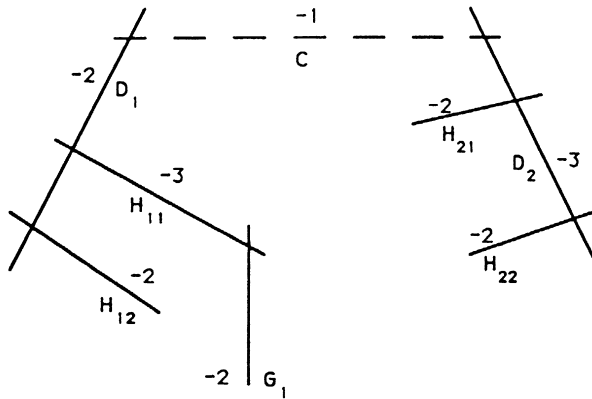


Fig. 2

LEMMA 1.3. (1) Suppose that Δ_1 contains G_i ($i = 1, \dots, s; s \geq 3$) such that $G_i^2 = -2$, $G_1 = D_1$, $G_j \cdot G_{j+1} = G_{s-2} \cdot G_s = 1$ ($j = 1, \dots, s - 2$). (This is the case if Δ_1 consists of only (-2) -curves but D_1 is not a tip of Δ_1 .) Then Theorem 1.1 (2) or (3) occurs.

(2) Suppose that Δ_1 is a fork with D_1 as its central component. Then Theorem 1.1 (3) occurs.

PROOF. (1) Let $S_0 = 2(C + G_1 + \dots + G_{s-2}) + G_{s-1} + G_s$ and let $\varphi :$

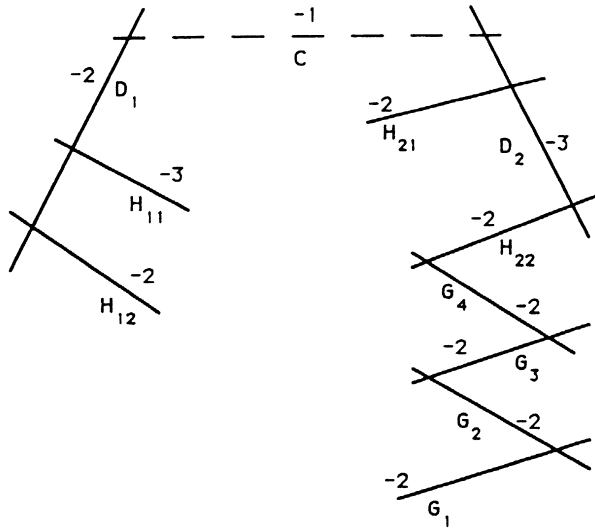


Fig. 3

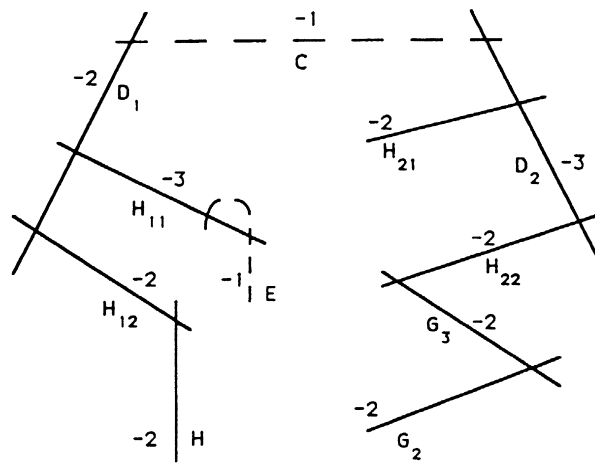


Fig. 4

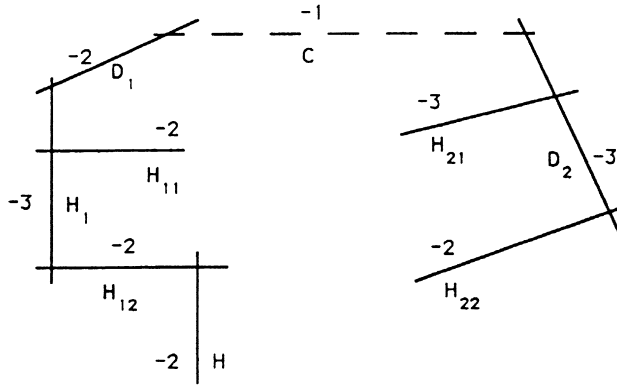


Fig. 5

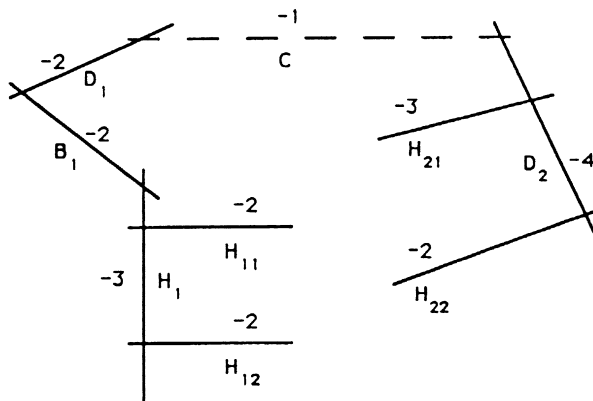


Fig. 6

$\tilde{S} \rightarrow \mathbf{P}^1$ be the \mathbf{P}^1 -fibration with S_0 as a singular fiber. If $\Delta_1 = \sum_i G_i$, then Theorem 1.1 (2) occurs with $\sum_i A_i = \sum G_i$, $\sum_i B_i = B_1 = D_2$. Otherwise, Theorem 1.1 (3) occurs. Indeed, the intersection matrix of $C + \Delta_1$ then has a positive eigenvalue.

(2) If the central component D_1 meets two (-2) -components of $\Delta_1 - D_1$, then we are reduced to the previous case. So we may assume that D_1 meets $\Delta_1 - D_1$ in one (-2) -component and two components of self intersections ≤ -3 . But then $D^* \geq 4/5D_1 + 1/3D_2$ (cf. Lemma 1.5 below) and $0 <$

$-C.(K_{\tilde{S}}+D^*) \leq 1-C.(4/5D_1+1/3D_2) = 1-4/5-1/3 < 0$, a contradiction (cf. Lemma 1.4 below).

This proves Lemma 1.3. \square

From now on till the end of the section, we shall assume the following hypothesis:

(*) *neither the case of Lemma 1.2 nor the cases of Lemma 1.3 occur.*

By the maximality of the twig $C + T_i$ and by the hypothesis(*), if D_i is a tip component of Δ_i (Δ_i is a fork in this case) then there are irreducible components H_i, H_{i1}, H_{i2} in $\Delta_i - T_i$ such that

$$T_i.(\Delta_i - T_i) = T_i.H_i = 1, \quad H_i.H_{i1} = H_i.H_{i2} = 1.$$

If D_i is not a tip component of Δ_i , then $T_i = 0$ and we let $H_i := D_i$ and H_{21}, H_{22} two components in $\Delta_i - D_i$ adjacent to D_i .

Let $\sigma : \tilde{S} \rightarrow \tilde{T}$ be the smooth blowing-down of curves in $C + T_1 + T_2$ such that

- (1) $\sigma(C + \Delta_1 + \Delta_2)$ consists of exactly one (-1) -curve \tilde{C} , with $\tilde{C} \leq \sigma(C + T_1 + T_2)$, and several $(-n_i)$ -curves with $n_i \geq 2$, and
- (2) the condition(1) will not be satisfied if σ is replaced by the composite of σ and the blowing-down of \tilde{C} .

Thus, $\sigma = id$ if and only if D_1 is not a tip of Δ_1 . If $\sigma \neq id$, then C is contracted by σ and $\sigma'(\tilde{C}) \leq D$.

Let $\tilde{D} = D$ (resp. $\tilde{\Delta}_i := \sigma(\Delta_i)$) if $\sigma = id$, and $\tilde{D} = \sigma(D) - \tilde{C}$ (resp. $\tilde{\Delta}_i := \sigma(\Delta_i)$ with \tilde{C} deleted if any) otherwise. Let $\tilde{H}_i = \sigma(H_i)$, $\tilde{H}_{ij} = \sigma(H_{ij})$, etc. By the definition of σ there is an irreducible component J_i in $T_i + H_i$ such that

$$\tilde{C}.\tilde{D} = \tilde{C}.(\tilde{J}_1 + \tilde{J}_2) = 2, \quad \tilde{C}.\tilde{J}_i = 1, \quad \text{where } \tilde{J}_i := \sigma(J_i).$$

The divisor \tilde{D} on \tilde{T} is contractible to quotient singularities with, say $g : \tilde{T} \rightarrow T$ the contraction morphism. T is again a log del Pezzo surface of

rank one with g as a minimal desingularization (cf. [4, Lemma 4.3]). So we can apply Lemma 1.1 of Part **I** for T . In particular, we have

$$g^*K_T = K_{\tilde{T}} + \tilde{D}^*, \quad -R.(K_{\tilde{T}} + \tilde{D}^*) > 0$$

for every curve R on \tilde{T} which is not contractible by g . Here \tilde{D}^* is an effective \mathbf{Q} -divisor with support in \tilde{D} .

Suppose that there are two smooth blowing-downs $\sigma_1 : \tilde{S} \rightarrow \tilde{S}_1$, $\sigma_2 : \tilde{S}_1 \rightarrow \tilde{T}$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Let E be the unique (-1) -curve in $\sigma_1(C + \Delta_1 + \Delta_2)$. Let $M := D$ if $\sigma_1 = id$ and $M := \sigma_1(D) - E$ otherwise. By [4, Lemma 4.3] and Lemma 1.1 in Part **I**, we have the following :

LEMMA 1.4. *M is contractible to quotient singularities with, say $f_1 : \tilde{S}_1 \rightarrow S_1$ the contraction morphism, and S_1 is again a log del Pezzo surface of rank one with f_1 as a minimal desingularization. In particular, we have*

$$f_1^*K_{S_1} = K_{\tilde{S}_1} + M^*, \quad -R.(K_{\tilde{S}_1} + M^*) > 0,$$

where M^* is an effective \mathbf{Q} -divisor with support contained in M and R is an arbitrary curve on \tilde{S}_1 not contractible by f_1 . (We can take $R = E$.)

Throughout the proof of Theorem 1.1, we shall frequently make use of Lemma 1.5 below to estimate the coefficients of the effective \mathbf{Q} -divisor M^* in Lemma 1.4. For instance, we often combine Lemmas 1.4 and 1.5 to rule out certain cases.

To state Lemma 1.5 in a general setting, we need some preparation. Let X be a log del Pezzo surface. Let $h : \tilde{X} \rightarrow X$ be a minimal resolution of singularities of X and let $P = h^{-1}(\text{Sing } X)$. We decompose P into irreducible components : $P = \sum_{i=1}^n P_i$. By Lemma 1.1 in Part **I**, we can write

$$h^*K_X = K_{\tilde{X}} + P^*$$

where $P^* = \sum_{i=1}^n \alpha_i P_i$ for some non-negative rational number α_i . Let $\{Q_1, \dots, Q_r\}$ be a subset of $\{P_1, \dots, P_n\}$, say $Q_i = P_i$ for $1 \leq i \leq r$. We formally assign an integer Q_i^2 to Q_i so that $P_i^2 \leq Q_i^2 \leq -2$. Now we define rational numbers $\beta_i (1 \leq i \leq r)$ by the condition :

$$Q_j.(K_{\tilde{X}} + \sum_{i=1}^r \beta_i Q_i) = 0 \quad (j = 1, \dots, r),$$

where we set $Q_i.Q_j := P_i.P_j$ if $i \neq j$ and $Q_i.K_{\tilde{X}} := -2 - Q_i^2$. Then we have the following (cf. [4, Lemma 1.7]) :

LEMMA 1.5. *We have $\alpha_i \geq \beta_i$ for $1 \leq i \leq r$ and $\alpha_i \geq 1 + 2/P_i^2$ for $1 \leq i \leq n$.*

Let us continue the proof of Theorem 1.1. Suppose that for $a = 1$ or 2 , we have $J_a = H_a$ and $\tilde{H}_a^2 = -2$. Let $\tilde{G} \sim K_{\tilde{T}} + 2(\tilde{C} + \tilde{H}_a) + \tilde{H}_{a1} + \tilde{H}_{a2} + \tilde{J}_b$ where $\{a, b\} = \{1, 2\}$ as sets. Note that $H^2(\tilde{T}, \tilde{G}) \cong H^0(\tilde{T}, -(2(\tilde{C} + \tilde{H}_a) + \tilde{H}_{a1} + \tilde{H}_{a2} + \tilde{J}_b)) = 0$. Note also that $\tilde{G}.B = 0$ for $B = \tilde{C}, \tilde{H}_a, \tilde{H}_{a1}, \tilde{H}_{a2}, \tilde{J}_b$. Hence $\tilde{G}^2 = \tilde{G}.K_{\tilde{T}}$. Now the Riemann-Roch theorem implies that

$$h^0(\tilde{T}, \tilde{G}) \geq \frac{1}{2}\tilde{G}.(\tilde{G} - K_{\tilde{T}}) + 1 = 1.$$

So we may assume that $\tilde{G} \geq 0$.

LEMMA 1.6. *Assume the above conditions. Then we have :*

- (1) \tilde{G} is a nonzero effective divisor.
- (2) $\tilde{G} \cap (\tilde{C} + \tilde{H}_a + \tilde{H}_{a1} + \tilde{H}_{a2} + \tilde{J}_b) = \phi$. In particular, $\tilde{G}_1.\tilde{G} = \tilde{G}_1.K_{\tilde{T}}$ for every irreducible component \tilde{G}_1 of \tilde{G} .
- (3) We can decompose \tilde{G} into $\tilde{G} = \tilde{\Sigma} + \tilde{\Delta}$ such that $\text{Supp } \tilde{\Delta}$ is contained in $\text{Supp } \tilde{D}$ and $\tilde{\Sigma} = \sum_{i=1}^r \tilde{\Sigma}_i$ ($r \geq 1$) where $\tilde{\Sigma}_i$ is a (-1) -curve.
- (4) Write $\sigma^*\tilde{G} \sim \sigma^*(K_{\tilde{T}} + 2(\tilde{C} + \tilde{H}_a) + \tilde{H}_{a1} + \tilde{H}_{a2} + \tilde{J}_b) = K_{\tilde{S}} + sC +$ (an effective divisor with support in D). Then $r \leq s - 1$.
- (5) Let \tilde{B} be an irreducible component of $\tilde{D} - (\tilde{H}_a + \tilde{H}_{a1} + \tilde{H}_{a2} + \tilde{J}_b)$. Then $\tilde{B}.\tilde{G} > 0$ if and only if $\tilde{B}^2 \leq -3$ or $\tilde{B}.\tilde{G} > 0$.
- (6) If $\tilde{\Sigma}$ is a reduced divisor, then $\tilde{G} = \tilde{\Sigma}$ and $\tilde{\Sigma}$ is a disjoint union of $\tilde{\Sigma}_i$'s.

PROOF. From the definition of \tilde{G} , one can calculate that :

CLAIM(1). $\tilde{G}.\tilde{B} = 0$ if \tilde{B} is one of $\tilde{C}, \tilde{H}_a, \tilde{H}_{a1}, \tilde{H}_{a2}$ and \tilde{J}_b . Moreover, $\tilde{G}.\tilde{B} \geq 0$ for every irreducible component B of \tilde{D} .

By the fact that $|K_{\tilde{S}} + C + D| = \phi$ and the definition of σ , we get :

CLAIM(2). $|K_{\tilde{T}} + \tilde{C} + \tilde{D}| = \phi$.

(1) By the hypothesis(*) which is stated after Lemma 1.3, \tilde{J}_b meets an irreducible component \tilde{B} of $\tilde{\Delta}_b$. So, $\tilde{G} \cdot \tilde{B} = (K_{\tilde{T}} + \tilde{J}_b) \cdot \tilde{B} \geq 1$. Hence $\tilde{G} > 0$.

(2) Suppose $\tilde{G} \cap \tilde{C} \neq \phi$. Then $\tilde{C} \leq \tilde{G}$ by Claim(1). Now, $\tilde{H}_a \leq \tilde{G} - \tilde{C}$ because $\tilde{H}_a \cdot (\tilde{G} - \tilde{C}) = -\tilde{H}_a \cdot \tilde{C} = -1 < 0$. This leads to $0 \leq \tilde{G} - \tilde{C} - \tilde{H}_a \in |K_{\tilde{T}} + \tilde{C} + \tilde{H}_a + \tilde{H}_{a1} + \tilde{H}_{a2} + \tilde{J}_b| \subseteq |K_{\tilde{T}} + \tilde{C} + \tilde{D}|$, a contradiction to Claim(2). So, $\tilde{G} \cap \tilde{C} = \phi$. One iterates this argument and can prove (2).

(3) Decompose \tilde{G} into $\tilde{G} = \tilde{\Sigma} + \tilde{\Delta}$ where $\text{Supp } \tilde{\Delta} \subseteq \text{Supp } \tilde{D}$ and $\tilde{\Sigma}$ contains no irreducible components of \tilde{D} . First, by Claim(1), we have $\tilde{G} \cdot \tilde{\Delta}_i \geq 0$ for every irreducible component $\tilde{\Delta}_i$ of $\tilde{\Delta}$. Hence $0 \leq \tilde{G} \cdot \tilde{\Delta} = \tilde{\Sigma} \cdot \tilde{\Delta} + \tilde{\Delta}^2 < \tilde{\Sigma} \cdot \tilde{\Delta}$ when $\tilde{\Delta} \neq 0$, because $\text{Supp } \tilde{\Delta} \subseteq \text{Supp } \tilde{D}$ and \tilde{D} is negative definite. This proves that $\tilde{\Sigma} \neq 0$.

Let $\tilde{\Sigma}_i$ be an irreducible component of $\tilde{\Sigma}$. Note that $\tilde{\Sigma}_i \cdot K_{\tilde{T}} \leq \tilde{\Sigma}_i \cdot (K_{\tilde{T}} + \tilde{D}^*) < 0$ (cf. Lemma 1.4). So, if $\tilde{\Sigma}_i^2 < 0$, then $\tilde{\Sigma}_i$ is a (-1) -curve. Suppose that $\tilde{\Sigma}_i^2 \geq 0$. Then, by (2), $\tilde{\Sigma}_i^2 \leq \tilde{\Sigma}_i \cdot \tilde{G} = \tilde{\Sigma}_i \cdot K_{\tilde{T}} < 0$. We reach a contradiction. This proves (3).

(4) By (2), $\sigma^* \tilde{\Sigma}_i$ is again a (-1) -curve and $\sigma^*(\tilde{\Delta}) \subseteq D$. Write $f(C) \equiv c(-K_S)$, $f(\sigma^* \tilde{\Sigma}_i) \equiv e_i(-K_S)$, where $c > 0, e_i > 0$. Then $(sc - 1)(-K_S) \equiv f(\sigma^* \tilde{G}) \equiv \sum_{i=1}^r e_i(-K_S)$. Since $K_S^2 > 0$, we have

$$sc - 1 = \sum_i e_i \geq rc$$

by the minimality of $-C \cdot (K_{\tilde{S}} + D^*) = c(K_{\tilde{S}} + D^*)^2 = c(K_S)^2$ (cf. the choice of C in Part I). Hence $(s - r)c \geq 1 > 0$. (4) then follows.

(5) follows from the equality : $\tilde{B} \cdot \tilde{G} = \tilde{B} \cdot (K_{\tilde{T}} + \tilde{H}_{a1} + \tilde{H}_{a2} + \tilde{J}_b)$.

(6) By the condition, $\tilde{\Sigma}_i \neq \tilde{\Sigma}_j$ if $i \neq j$. So,

$$-1 = \tilde{\Sigma}_i^2 = \tilde{\Sigma}_i \cdot \tilde{G} - \tilde{\Sigma}_i \cdot (\tilde{\Delta} + \sum_{j \neq i} \tilde{\Sigma}_j) \leq \tilde{\Sigma}_i \cdot \tilde{G} = \tilde{\Sigma}_i \cdot K_{\tilde{T}} = -1.$$

Thus, $\tilde{\Sigma}_i \cdot (\tilde{\Delta} + \sum_{j \neq i} \tilde{\Sigma}_j) = 0$ for every i . So, $\tilde{\Sigma}$ is a disjoint union of $\tilde{\Sigma}_i$'s and $\tilde{\Sigma} \cap \tilde{\Delta} = \phi$. In particular, $\tilde{G} \cdot \tilde{\Delta} = \tilde{\Delta}^2$. By Claim(1), we have $\tilde{G} \cdot \tilde{\Delta} \geq 0$. So, $\tilde{\Delta}^2 \geq 0$. Since $\tilde{\Delta}$ is contained in \tilde{D} and \tilde{D} is negative definite, we have $\tilde{\Delta} = 0$. This proves (6) and Lemma 1.6 is proved. \square

COROLLARY 1.7. *Assume that σ is a contraction of curves in $C + T_1$. Assume further that $J_1 = H_1$ and $\tilde{H}_1^2 = -2$ (hence $J_2 = D_2$ and the hypothesis in Lemma 1.6 is satisfied with $a = 1$). Then $K_{\tilde{T}} + 2(\tilde{C} + \tilde{H}_1) + \tilde{H}_{11} + \tilde{H}_{12} + \tilde{J}_2 \sim \tilde{G} = \tilde{\Sigma} = \tilde{\Sigma}_1$, i.e., \tilde{G} is reduced and a (-1) -curve.*

PROOF. We apply Lemma 1.6 to $\tilde{G} \sim K_{\tilde{T}} + 2(\tilde{C} + \tilde{H}_1) + \tilde{H}_{11} + \tilde{H}_{12} + \tilde{J}_2$. By the hypothesis, $\sigma^*\tilde{G} \sim K_{\tilde{S}} + 2C +$ (an effective divisor with support in D). Then Corollary 1.7 follows from Lemma 1.6. \square

LEMMA 1.8. *Suppose the case (II-4) in Remark 3.11 of Part I occurs. Then one of the following two cases occurs :*

- (1) *Theorem 1.1, (2) or (3) occurs.*
- (2) *$(\tilde{J}_a^2, \tilde{J}_b^2) = (-2, -2), (-2, -3)$ or $(-2, -4)$ where $\{a, b\} = \{1, 2\}$ as sets. If $\tilde{J}_k^2 = -2$ (this is the case if $k = a$), then $J_k = H_k$ and $H_{kj}^2 \leq -3$ for $j = 1$ or 2 . Moreover, Δ_2 is not a fork with D_2 as its central component.*

PROOF. By [4, Lemma 4.4], $\tilde{J}_a^2 = -2$ for $a = 1$ or 2 . Let $\{a, b\} = \{1, 2\}$ as sets.

Case(1) $\tilde{J}_b^2 = -2$. If \tilde{J}_s is a tip of $\tilde{\Delta}_s$ for $s = a$ or b , say $s = b$, then $J_b \neq H_b$ and Theorem 1.1 (3) occurs. Indeed, the intersection matrix of $\tilde{C} + \tilde{J}_b + \tilde{\Delta}_a$ has a positive eigenvalue and so does $C + T_b + \Delta_a$. Thus we may assume $J_a = H_a, J_b = H_b$.

Suppose $H_{s_1}^2 = H_{s_2}^2 = -2$ for $s = a$ or b , say $s = a$. Let $S_0 := 2(\tilde{C} + \tilde{H}_a) + \tilde{H}_{a1} + \tilde{H}_{a2}$ and let $\psi : \tilde{S} \rightarrow \mathbf{P}^1$ be the \mathbf{P}^1 -fibration with S_0 as a singular fiber. If $\tilde{\Delta}_a = \tilde{H}_a + \tilde{H}_{a1} + \tilde{H}_{a2}$, then Theorem 1.1 (2) occurs with $\varphi = \psi \cdot \sigma$, $\sum_i B_i = B_1 = H_b$. If $\tilde{\Delta}_a > \tilde{H}_a + \tilde{H}_{a1} + \tilde{H}_{a2}$, Theorem 1.1 (3) occurs. Indeed, the intersection matrix of $\tilde{C} + \tilde{\Delta}_a$ then has a positive eigenvalue and so does $C + T_b + \Delta_a$. Thus we may assume that $H_{aj}^2 \leq -3$ for $j = 1$ or 2 . The same argument works for $s = b$.

To finish the proof of Lemma 1.8 in this case, we have to consider the case where $\tilde{\Delta}_2$ is a fork with \tilde{D}_2 as its central component. By the previous arguments, now we have $J_1 = H_1, J_2 = H_2 = D_2, H_{11}^2 \leq -3$ say, and \tilde{H}_2 meets $\tilde{\Delta}_2 - \tilde{H}_2$ in one (-2) -component \tilde{H}_{23} and two components $\tilde{H}_{21}, \tilde{H}_{22}$ of self intersections ≤ -3 .

Let $\sigma_2 : \tilde{S}_1 \rightarrow \tilde{T}$ be the blowing-up of the point $\tilde{C} \cap \tilde{D}_2$ and let L be

the exceptional curve of σ_2 . Note that $\sigma \neq id$ because $\tilde{D}_2^2 = -2$ while $D_2^2 \leq -3$. So we have a smooth blowing-down $\sigma_1 : \tilde{S} \rightarrow \tilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we get $-L.(K_{\tilde{S}_1} + M^*) > 0$, where $M^* \geq 1/2\sigma'_2\tilde{H}_{11} + 1/2\sigma'_2\tilde{H}_1 + 1/4\sigma'_2\tilde{H}_{12} + 1/4\sigma'_2\tilde{C} + 5/11\sigma'_2\tilde{H}_{23} + 10/11\sigma'_2\tilde{D}_2 + 7/11\sigma'_2\tilde{H}_{22} + 7/11\sigma'_2\tilde{H}_{21}$ (cf. Lemma 1.5). This leads to $-L.(K_{\tilde{S}_1} + M^*) \leq 1 - L.(1/4\sigma'_2\tilde{C} + 10/11\sigma'_2\tilde{D}_2) = 1 - 1/4 - 10/11 < 0$. We reach a contradiction. So it is impossible that $\tilde{\Delta}_2$ is a fork with \tilde{D}_2 as its central component. Lemma 1.8 is proved in the present case.

Case(2) $\tilde{J}_b^2 \leq -3$. Then by the definition of σ (cf. the second condition), $J_a = H_a$, i.e., \tilde{J}_a is not a tip of $\tilde{\Delta}_a$. If $H_{a1}^2 = H_{a2}^2 = -2$, then by the arguments in the above paragraph, Theorem 1.1, (2) or (3) occurs. So we may assume that $H_{aj}^2 \leq -3$ for $j = 1$ or 2 , say $j = 1$.

We now prove that $d := -\tilde{J}_b^2 \leq 4$. Since it is impossible that $\tilde{\Delta}_b$ is a linear chain with \tilde{J}_b as a tip, we have $\tilde{D}^* \geq (d-2)/(d-1)\tilde{J}_b + 3/7\tilde{H}_{a1} + 2/7\tilde{H}_a + 1/7\tilde{H}_{a1}$ (cf. Lemma 1.5). By Lemma 1.4, we have $0 < -\tilde{C}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{C}((d-2)/(d-1)\tilde{J}_b + 2/7\tilde{H}_a) = 1/(d-1) - 2/7$. Hence $d \leq 4$.

To finish the proof of Lemma 1.8 in this case, we still have to consider the case where $\tilde{\Delta}_2$ is a fork with \tilde{D}_2 as its central component. Now $J_2 = H_2 = D_2$ and $J_a = H_a$. If $\tilde{D}_2^2 = -2$, i.e., if $a = 2, b = 1$, then $\tilde{J}_1^2 \leq -3$, and by the previous argument, \tilde{D}_2 meets $\tilde{\Delta}_2 - \tilde{D}_2$ in one (-2) -component and two components of self intersections ≤ -3 . This will lead to a contradiction to $0 < -\tilde{C}.(K_{\tilde{T}} + \tilde{D}^*)$ as in Lemma 1.3 (2). So, we have $\tilde{D}_2^2 \leq -3, H_{11}^2 \leq -3$ and $\tilde{H}_1^2 = -2$, i.e., $a = 1, b = 2$.

If $\sigma \neq id$, then a contradiction is derived as in the case(1) above. If $\sigma = id$, then $J_1 = H_1 = D_1, K_{\tilde{S}} + 2(C + D_1) + H_{11} + H_{12} + D_2 \sim \tilde{G} = \tilde{\Sigma}$, where $\tilde{\Sigma}$ is a (-1) -curve (cf. (4) and (6) of Lemma 1.6). We have also $\tilde{\Sigma}.\tilde{H}_{2j} > 0$ for $j = 1, 2$ and 3 , where \tilde{H}_{2j} are irreducible components of D adjacent to D_2 (cf. Lemma 1.6 (5)). Now applying Lemma 1.5, we get $D^* \geq 2/3D_2 + 1/3H_{21} + 1/3H_{22} + 1/3H_{23}$. Hence $-\tilde{\Sigma}.(K_{\tilde{S}} + D^*) \leq 1 - \tilde{\Sigma}.(1/3H_{21} + 1/3H_{22} + 1/3H_{23}) \leq 0$, a contradiction to Lemma 1.4.

So it is impossible that $\tilde{\Delta}_2$ is a fork with \tilde{D}_2 as its central component. Lemma 1.8 is proved in the present case. \square

LEMMA 1.9. *Suppose the case(2) in Lemma 1.8 occurs. Then it is*

impossible that $\tilde{J}_1^2 = \tilde{J}_2^2 = -2$.

PROOF. We consider the case where $\tilde{J}_1^2 = \tilde{J}_2^2 = -2$. By the hypothesis, we have $J_i = H_i$, $\tilde{H}_i^2 = -2$ for $i = 1, 2$ and we may assume that $H_{11}^2 \leq -3$, $H_{21}^2 \leq -3$.

Case(1) σ is a contraction of curves contained in $C + T_1$.

Then the conditions of Corollary 1.7 are satisfied. Hence $K_{\tilde{T}} + 2(\tilde{C} + \tilde{H}_1) + \tilde{H}_{11} + \tilde{H}_{12} + \tilde{H}_2 \sim \tilde{G} = \tilde{\Sigma}$ where $\tilde{\Sigma}$ is a (-1) -curve. Note that $\tilde{\Sigma}.\tilde{H}_{21} = \tilde{G}.\tilde{H}_{21} = (K_{\tilde{T}} + \tilde{H}_2).\tilde{H}_{21} \geq 1 + 1$ (cf. Lemma 1.6 (2)). Let $\Sigma := \sigma^*(\tilde{\Sigma})$. Then Σ is again a (-1) -curve with $\Sigma.H_{21} \geq 2$ (cf. Lemma 1.6 (2)). On the other hand, $D^* \geq 1/2D_2 + 1/2H_{21}$ because $D_2^2 \leq -3$, $H_{21}^2 \leq -3$ (cf. Lemma 1.5). This leads to $-\Sigma.(K_{\tilde{\Sigma}} + D^*) \leq 1 - 1/2\Sigma.H_{21} \leq 0$, a contradiction to Lemma 1.4. So Case(1) is impossible.

Case(2) σ contracts at least one irreducible component of the maximal twig T_2 of Δ_2 .

By noting that $D_1^2 = -2$, $D_2^2 \leq -3$, there are two smooth blowing-downs $\sigma_1 : \tilde{S} \rightarrow \tilde{S}_1, \sigma_2 : \tilde{S}_1 \rightarrow \tilde{T}$ such that $\sigma = \sigma_2 \cdot \sigma_1$ and that :

- (1) $\sigma_1(T_1 + C + T_2) = T'_1 + E + T'_2$ where E is a (-1) -curve and $T'_i \leq \sigma_1(T_i)$,
- (2) $T'_1 + \sigma_1(H_1) = \sum_{i=1}^s L_i$, $E.L_1 = L_i.L_{i+1} = 1$ ($i = 1, \dots, s-1$; $s \geq 2$), $L_s = \sigma_1(H_1)$, $L_1^2 = -2$, $L_2^2 = -(t+1)$, $L_j^2 = -2$ ($2 < j < s$), and
- (3) $T'_2 + \sigma_1(H_2) = \sum_{i=1}^t M_i$, $E.M_1 = M_i.M_{i+1} = 1$ ($i = 1, \dots, t-1$; $t \geq 2$), $M_t = \sigma_1(H_2)$, $M_1^2 = -3$, $M_t^2 = -s$, $M_j^2 = -2$ ($2 \leq j < t, j \neq t$).

Since $\sigma_1(\Delta_1 + \Delta_2) - E$ is contractible to quotient singularities, we have $(s, t) = (2, 2), (2, 3)$ or $(3, 2)$. By Lemma 1.4, we have $1 - E.M^* = -E.(K_{\tilde{S}_1} + M^*) > 0$. We can also get lower bounds for coefficients of M^* by applying Lemma 1.5 to $X = S_1$, $\sum_{i=1}^{s+t+4} Q_i = H_{11} + H_{21} + H_{12} + H_{22} + \sum_i L_i + \sum_j M_j$, $Q_1^2 = Q_2^2 = -3$, $Q_3^2 = Q_4^2 = -2$. Now the inequality $1 - E.M^* > 0$, together with these lower bounds, will deduce an inequality $(2t-1)(s-1) < 3$. This is impossible because $s \geq 2$ and $t \geq 2$.

This proves Lemma 1.9. \square

In the proof of the following Lemmas 1.10 and 1.11, to rule out most of the cases, we shall frequently use Lemma 1.5 to get an estimate on the coefficients of M^* and then deduce a contradiction to Lemma 1.4.

LEMMA 1.10. *Suppose that the case in Corollary 1.7 occurs. Suppose further that the case(2) in Lemma 1.8 occurs with $(\tilde{J}_a^2, \tilde{J}_b^2) = (-2, -3)$ or $(-2, -4)$ (hence $a = 1, b = 2, J_1 = H_1, J_2 = D_2$). Then Theorem 1.1, (3) or (4) occurs.*

PROOF. By the hypothesis in the case(2) of Lemma 1.8, we may assume that $H_{11}^2 \leq -3$. By Corollary 1.7, $K_{\tilde{T}} + 2(\tilde{C} + \tilde{H}_1) + \tilde{H}_{11} + \tilde{H}_{12} + \tilde{J}_2 \sim \tilde{G} = \tilde{\Sigma}$ where $\tilde{\Sigma}$ is a (-1) -curve.

CLAIM(1). (1) $\tilde{D}^* \geq 3/7\tilde{H}_{11} + 2/7\tilde{H}_1 + 1/7\tilde{H}_{12} + (a-2)/(a-1)\tilde{J}_2$. Here $a := -\tilde{J}_2^2 \geq 3$ and hence $(a-2)/(a-1) \geq 1/2$.

(2) $\tilde{\Delta}_1$ is a linear chain.

(3) Either $\tilde{\Delta}_2$ is a linear chain with $\tilde{J}_2 = \tilde{H}_2$, or $\tilde{\Delta}_2$ is a fork with \tilde{J}_2 as a tip.

(4) $\tilde{\Delta}_1 - \tilde{H}_{11}$ consists of (-2) -curves.

(5) $\tilde{\Delta}_2 - \tilde{J}_2$ consists of (-2) -curves.

Since $\tilde{H}_{11}^2 \leq -3$ and since it is impossible that $\tilde{\Delta}_2$ is a linear chain with \tilde{J}_2 as a tip (cf. the hypothesis(*) after Lemma 1.3), (1) follows.

If $\tilde{\Delta}_1$ is not a linear chain, then $\tilde{D}^* \geq 1/2\tilde{H}_1 + 1/2\tilde{H}_{11}$ (cf. Lemma 1.5). This leads to $-\tilde{C} \cdot (K_{\tilde{S}} + \tilde{D}^*) \leq 1 - \tilde{C} \cdot (1/2\tilde{H}_1 + 1/2\tilde{J}_2) = 0$, a contradiction to Lemma 1.4. So, (2) of Claim(1) is true.

Suppose (3) of Claim(1) is false. Then $\tilde{\Delta}_2$ contains $L_i (i = 1, \dots, s; s \geq 4)$ such that $L_2 = \tilde{J}_2, L_i \cdot L_{i+1} = L_{s-2} \cdot L_s = 1 (i = 1, \dots, s-2)$. So we have $\tilde{D}^* \geq 1/3L_1 + 2/3 \sum_{i=2}^{s-2} L_i + 1/3L_{s-1} + 1/3L_s$ (cf. Lemma 1.5). On the other hand, for $i = 1, 3$ (and also for $i = 4$ if $s = 4$), we have $L_i \cdot \tilde{\Sigma} = L_i \cdot (K_{\tilde{T}} + \tilde{D}_2) \geq 1$ (cf. Lemma 1.6). This leads to $-\tilde{\Sigma} \cdot (K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{\Sigma} \cdot (1/3L_1 + 2/3 \sum_{i=2}^{s-2} L_i + 1/3L_{s-1} + 1/3L_s) \leq 0$. We reach a contradiction to Lemma 1.4. Thus, (3) of Claim(1) is true.

Suppose $\tilde{\Delta}_1 - \tilde{H}_{11}$ contains a $(-n)$ -curve B with $n \geq 3$. If B and \tilde{H}_{12} are in the same connected component of $\tilde{\Delta}_1 - \tilde{H}_1$, then $\tilde{D}^* \geq 1/2\tilde{H}_1 + 1/2\tilde{J}_2$ (cf. Lemma 1.5) and hence $-\tilde{C} \cdot (K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{C} \cdot (1/2\tilde{H}_1 + 1/2\tilde{J}_2) = 0$, a contradiction to Lemma 1.4. If B and \tilde{H}_{11} are in the same connected component of $\tilde{\Delta}_1 - \tilde{H}_1$, we let $L_1 + \dots + L_s$ be a linear chain in $\tilde{\Delta}_1$ such that $L_1 = \tilde{H}_{11}, L_s = B, L_i \cdot L_{i+1} = 1 (i = 1, \dots, s-1)$. Then one has $D^* > 1/2 \sum_i L_i$ (cf. Lemma 1.5). Moreover, $L_i \cdot \tilde{\Sigma} = L_i \cdot (K_{\tilde{T}} + \tilde{H}_{11}) \geq 1$ for $i = 2, s$

and $L_2.\tilde{\Sigma} \geq 2$ if $s = 2$. This leads to $-\tilde{\Sigma}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{\Sigma}.1/2 \sum_i L_i \leq 0$, a contradiction to Lemma 1.4. Therefore, (4) of Claim(1) is true.

Suppose that $\tilde{\Delta}_2 - \tilde{J}_2$ contains a $(-n)$ -curve B with $n \geq 3$. Let $L_1 + \dots + L_s$ be a linear chain contained in $\tilde{\Delta}_2$ such that $L_1 = \tilde{J}_2$, $L_s = B$, $L_i.L_{i+1} = 1$ ($i = 1, \dots, s - 1$). Then we have $\tilde{D}^* \geq 1/2 \sum_i L_i$ (cf. Lemma 1.5). Note that for $i = 2, s$, we have $L_i.\tilde{\Sigma} = L_i.(K_{\tilde{T}} + \tilde{J}_2) \geq 1$. Moreover, $L_2.\tilde{\Sigma} \geq 2$ if $s = 2$. This leads to $-\tilde{\Sigma}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{\Sigma}.(1/2 \sum_i L_i) \leq 0$. We reach a contradiction to Lemma 1.4. Therefore, (5) of Claim(1) is true.

This proves Claim(1).

CLAIM(2). Suppose that $\tilde{J}_2^2 = -4$. Then Theorem 1.1 (3) occurs.

We consider the case $\tilde{J}_2^2 = -4$. Then $\tilde{D}^* \geq 2/3\tilde{J}_2$ by Claim(1). If \tilde{H}_{11} is not a tip of $\tilde{\Delta}_1$ (resp. \tilde{H}_{12} is not a tip, or $H_{11}^2 \leq -4$), then by Lemma 1.5, $D^* \geq 6/11\tilde{H}_{11} + 4/11\tilde{H}_1 + 2/11\tilde{H}_{12}$ (resp. $D^* \geq 4/9\tilde{H}_{11} + 3/9\tilde{H}_1 + 2/9\tilde{H}_{12}$, or $D^* \geq 3/5\tilde{H}_{11} + 2/5\tilde{H}_1 + 1/5\tilde{H}_{12}$). Any of the three cases implies that $-\tilde{C}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{C}.(1/3\tilde{H}_1 + 2/3\tilde{J}_2) = 0$, a contradiction to Lemma 1.4.

Thus, $\tilde{\Delta}_1 = \tilde{H}_1 + \tilde{H}_{11} + \tilde{H}_{12}$ and $\tilde{H}_1^2 = -2, \tilde{H}_{11}^2 = -3, \tilde{H}_{12}^2 = -2$ (cf. Claim(1)). If \tilde{J}_2 is a tip of $\tilde{\Delta}_2$, i.e., if $J_2 \neq H_2$, then Theorem 1.1 (3) occurs since the intersection matrix of $\tilde{C} + \tilde{J}_2 + \tilde{\Delta}_1$ and hence that of $C + T_2 + \Delta_1$ have a positive eigenvalue.

We may now assume that $J_2 = H_2$. Then $\tilde{D}^* \geq 2/3\tilde{H}_2 + 1/3\tilde{H}_{21} + 1/3\tilde{H}_{22}$ (cf. Claim(1)). We shall show that this leads to a contradiction. By Claim(1), $\tilde{\Delta}_2$ is now a linear chain. If H_{2j} is not tip of $\tilde{\Delta}_2$ for $j = 1$ and 2 , then $D^* \geq 2/4\tilde{H}_{21} + 3/4\tilde{H}_2 + 2/4\tilde{H}_{22}$. This leads to $0 < -\tilde{C}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{C}.(2/7\tilde{H}_1 + 3/4\tilde{H}_2) = 1 - 2/7 - 3/4 < 0$, a contradiction.

So we may assume that H_{21} is a tip of $\tilde{\Delta}_2$. If $\tilde{\Delta}_2$ has more than four irreducible components, then $D^* \geq 4/11\tilde{H}_{21} + 8/11\tilde{H}_2 + 6/11\tilde{H}_{22}$. This leads to $0 < -\tilde{C}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{C}.(2/7\tilde{H}_1 + 8/11\tilde{H}_2) = 1 - 2/7 - 8/11 < 0$, a contradiction. Therefore, $H := \tilde{\Delta}_2 - (\tilde{H}_{21} + \tilde{H}_2 + \tilde{H}_{22})$ is zero or a (-2) -curve adjacent to \tilde{H}_{22} (cf. Claim(1)).

Note that $\tilde{\Sigma}.\tilde{H}_{2j} = (K_{\tilde{T}} + \tilde{H}_2).\tilde{H}_{2j} = 1$ for $j = 1$ and 2 (cf. Lemma 1.6). If $B.\tilde{\Sigma} > 0$ for some irreducible component B of $\tilde{D} - (\tilde{H}_{21} + \tilde{H}_{22})$, then B is not contained in $\tilde{\Delta}_1$ nor $\tilde{\Delta}_2$, $B^2 \leq -3$ and $B.\tilde{\Sigma} = B.K_{\tilde{T}}$ (cf. Lemma 1.6

(5)). Hence $\tilde{D}^* \geq 1/3B$. This leads to $0 < -\tilde{\Sigma} \cdot (K_{\tilde{T}} + D^*) \leq 1 - \tilde{\Sigma} \cdot (1/3B + 1/3\tilde{H}_{21} + 1/3\tilde{H}_{22}) = 0$, a contradiction. So, $\tilde{\Sigma}$ meets transversally only \tilde{H}_{21} and \tilde{H}_{22} in \tilde{D} .

Let $S'_0 := 2\tilde{\Sigma} + \tilde{H}_{21} + \tilde{H}_{22}$ and let $\psi : \tilde{T} \rightarrow \mathbf{P}^1$ be the \mathbf{P}^1 -fibration with S'_0 as a singular fiber. Let S'_1 be the singular fiber containing $\tilde{C} + \tilde{\Delta}_1$. Then there is a (-1) -curve E such that $E \cdot \tilde{H}_{11} = 1$ and $S'_1 = 2(\tilde{C} + \tilde{H}_1) + \tilde{H}_{11} + \tilde{H}_{12} + E$. Since $\rho(\tilde{T}) = 1$ and since every irreducible component of $\tilde{D} - (H + \tilde{H}_2)$ is contained in singular fibers of ψ , every singular fiber S'_2 other than S'_1 consists of one (-1) -curve and several irreducible components of \tilde{D} (cf. Lemma 1.1 (4) of Part I). Here we set $H := \tilde{\Delta}_2 - (\tilde{H}_{21} + \tilde{H}_2 + \tilde{H}_{22})$. Moreover, $H \neq 0$ because $\rho(T) = 1$. So, H is a (-2) -curve adjacent to \tilde{H}_{22} . Since H is a cross-section, $H \cdot E = 1$ and S'_0, S'_1 are the only singular fibers of ψ for otherwise H would meet a (-1) -curve F in some singular fiber S'_2 and F has multiplicity at least two.

Let $\tau : \tilde{T} \rightarrow \Sigma_2$ be the smooth blowing-down of curves in singular fibers of ψ such that $\tau(H)^2 = -2$. On the one hand, \tilde{H}_2 is a 2-section with $\tilde{H}_2 \cap H = \phi$ and hence $\tau(\tilde{H}_2)^2 = 8$. On the other hand, a calculation shows that $\tau(\tilde{H}_2)^2 = \tilde{H}_2^2 + 1 + 4 = 1$. We reach a contradiction.

This proves Claim(2).

In view of Claim(2), we may assume that $\tilde{J}_2^2 = -3$. If \tilde{J}_2 is a tip of $\tilde{\Delta}_2$, i.e., if $J_2 \neq H_2$, then Theorem 1.1 (3) occurs. Indeed, the intersection matrix of $\tilde{C} + \tilde{J}_2 + \tilde{\Delta}_1$ and hence that of $C + T_2 + \Delta_1$ then have a positive eigenvalue.

Thus we may assume that $J_2 = H_2$. Then $\tilde{\Delta}_2$ is a linear chain (cf. Claim(1)). We have also

$$\tilde{D}^* \geq 3/7\tilde{H}_{11} + 2/7\tilde{H}_1 + 1/7\tilde{H}_{12} + 1/4\tilde{H}_{21} + 2/4\tilde{H}_2 + 1/4\tilde{H}_{22}.$$

Note that $H \cdot \tilde{\Sigma} = H \cdot (K_{\tilde{T}} + \tilde{H}_{11} + \tilde{H}_{12} + \tilde{H}_2) = 1$ (cf. Lemma 1.6) if H is an irreducible component of $\tilde{D} - \tilde{H}_1$ adjacent to one of $\tilde{H}_{11}, \tilde{H}_{12}, \tilde{H}_2$. In particular, $\tilde{\Sigma} \cdot \tilde{H}_{21} = \tilde{\Sigma} \cdot \tilde{H}_{22} = 1$.

CLAIM(3). $\tilde{D} - (\tilde{H}_{11} + \tilde{H}_2)$ consists of (-2) -curves.

Suppose to the contrary that Claim(3) is false. Then $\tilde{D} - (\tilde{\Delta}_1 + \tilde{\Delta}_2)$ contains a $(-n)$ -curve B with $n \geq 3$ (cf. Claim(1)). By Lemma 1.6, we have

$B.\tilde{\Sigma} = B.K_{\tilde{T}} = n-2$. Note that $\tilde{D}^* \geq (n-2)/nB$ and $0 < -\tilde{\Sigma}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{\Sigma}.(n-2)/nB = 1 - (n-2)^2/n$. So, $n = 3$ and $B.\tilde{\Sigma} = 1$.

If $\tilde{D} - \tilde{H}_1$ has an irreducible component H adjacent to \tilde{H}_{11} , then $\tilde{D}^* \geq 3/11H + 6/11\tilde{H}_{11} + 4/11\tilde{H}_1 + 2/11\tilde{H}_{12}$. This leads to $0 < -\tilde{\Sigma}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{\Sigma}.(1/3B + 3/11H + 1/4\tilde{H}_{21} + 1/4\tilde{H}_{22}) = 1 - 1/3 - 3/11 - 1/4 - 1/4 < 0$. We reach a contradiction. So, \tilde{H}_{11} is a tip of $\tilde{\Delta}_1$.

If $\tilde{D} - \tilde{H}_1$ has an irreducible component H adjacent to \tilde{H}_{12} but H is not a tip of $\tilde{\Delta}_1$, then $\tilde{D}^* \geq 2/11H + 3/11\tilde{H}_{12} + 4/11\tilde{H}_1 + 5/11\tilde{H}_{11}$. This leads to $0 < -\tilde{\Sigma}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{\Sigma}.(1/3B + 2/11H + 1/4\tilde{H}_{21} + 1/4\tilde{H}_{22}) = 1 - 1/3 - 2/11 - 1/4 - 1/4 < 0$. We reach again a contradiction. Thus, $H := \tilde{\Delta}_1 - (\tilde{H}_{11} + \tilde{H}_1 + \tilde{H}_{12})$ is zero or a (-2) -curve adjacent to \tilde{H}_{12} (cf. Claim(1)).

Let $S'_0 := 2\tilde{\Sigma} + \tilde{H}_{21} + \tilde{H}_{22}$ and let $\psi : \tilde{T} \rightarrow \mathbf{P}^1$ be the \mathbf{P}^1 -fibration with S'_0 as a singular fiber. Let S'_1 be the singular fiber containing $\tilde{C} + \tilde{H}_1 + \tilde{H}_{11} + \tilde{H}_{12}$.

Suppose $H_{11}^2 = -3$. Then there is a (-1) -curve E such that $E.\tilde{H}_{11} = 1$ and $S'_1 = 2(\tilde{C} + \tilde{H}_1) + \tilde{H}_{12} + \tilde{H}_{11} + E$. Since B is a 2-section, we have $B.E = 2$. This leads to $0 < -E.(K_{\tilde{T}} + D^*) \leq 1 - E.(1/3B + 3/7\tilde{H}_{11}) = 1 - (1/3) \cdot 2 - 3/7 < 0$, a contradiction. So, $H_{11}^2 \leq -4$.

Suppose $\sigma \neq id$. Let $\sigma_2 : \tilde{S}_1 \rightarrow \tilde{T}$ be the blowing-up of the point $P_2 := \tilde{C} \cap \tilde{H}_2$ and set $L := \sigma_2^{-1}(P_2)$. Then by the hypothesis in Corollary 1.7, there is a smooth blowing-down $\sigma_1 : \tilde{S} \rightarrow \tilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we have $-L.(K_{\tilde{S}_1} + M^*) > 0$, where $M^* \geq 2/3\sigma'_2\tilde{H}_{11} + 2/3\sigma'_2\tilde{H}_1 + 1/3\sigma'_2\tilde{H}_{12} + 1/3\sigma'_2\tilde{C} + 1/3\sigma'_2\tilde{H}_{21} + 2/3\sigma'_2\tilde{H}_2 + 1/3\sigma'_2\tilde{H}_{22}$. This leads to $0 < -L.(K_{\tilde{S}_1} + M^*) \leq 1 - L.(1/3\sigma'_2\tilde{C} + 2/3\sigma'_2\tilde{H}_2) = 0$, a contradiction. So, $\sigma = id$. Hence $\tilde{T} = \tilde{S}$, $H_i = D_i (i = 1, 2)$.

Let $S_0 := 3C + 2D_1 + H_{12} + D_2$ and let $\varphi : \tilde{S} \rightarrow \mathbf{P}^1$ be the \mathbf{P}^1 -fibration with S_0 as a singular fiber. Then $\tilde{\Sigma}$ and the (-3) -curve B are contained in the same singular fiber of φ , say S_1 . By the minimality of $-C.(K_{\tilde{S}} + D^*)$ and by noting that C has multiplicity 3 in S_0 and the summation of the multiplicities of (-1) -curves in S_1 is at least 3 (cf. [4, Lemma 1.6]), every (-1) -curve F in S_1 , especially $\tilde{\Sigma}$, satisfies $-F.(K_{\tilde{S}} + D^*) = -C.(K_{\tilde{S}} + D^*)$. So, every singular fiber of the previous fibration ψ defined by $|2\tilde{\Sigma} + \tilde{H}_{21} + \tilde{H}_{22}|$ has one of two types in Lemma 6.11 of Part I. However, S'_1 above contains a curve H_{11} with $H_{11}^2 \leq -4$. We reach a contradiction.

This proves Claim(3).

Let

$$S_0 := 3\tilde{C} + 2\tilde{H}_1 + \tilde{H}_{12} + \tilde{H}_2$$

and let $\varphi : \tilde{T} \rightarrow \mathbf{P}^1$ be the \mathbf{P}^1 -fibration with S_0 as a singular fiber. $\tilde{H}_{21}, \tilde{H}_{22}$ (resp. \tilde{H}_{11}) is a cross-section (resp. 2-section). Denote by S_1 the singular fiber containing $\tilde{\Sigma}$. Let

$$S_i \quad (i = 0, 1, \dots, r)$$

be all singular fibers of φ . By Claim(3), every singular fiber S_i ($i \geq 1$) consists of only (-1) or (-2) -curves. So, S_i has one of two types in Lemma 6.11 of Part **I**.

CLAIM(4). Suppose that S_k has the first type in Lemma 6.11 of Part **I** for some $k \geq 1$. Then Case(4-1) of Theorem 1.1 occurs.

Suppose S_1 has the first type in Lemma 6.11 of Part **I**. Namely, $\tilde{\Sigma}$ is the unique (-1) -curve in S_1 . Then the 2-section \tilde{H}_{11} meets two multiplicity-one or one multiplicity-two irreducible component(s) other than $\tilde{\Sigma}$ in S_1 . This implies that $\tilde{\Delta}_1$ is a fork (cf. Lemma 1.1 (4) of Part **I**), a contradiction to Claim(1). So, S_1 consists of two (-1) -curves $\tilde{\Sigma}, E$ and several (-2) -curves.

Suppose that S_k has the first type in Lemma 6.11 of Part **I** for some $k \geq 2$, say $k = 2$. Namely, there is a unique (-1) -curve F in S_2 . Since $\tilde{H}_{2j} \cdot S_2 = 1$ ($j = 1, 2$), there are two (-2) -curves G_j ($j = 1, 2$) such that $F \cdot G_j = 1, \tilde{H}_{2j} \cdot G_j = \tilde{H}_{11} \cdot F = 1$ and

$$S_2 = 2F + G_1 + G_2.$$

Now we have (cf. Claim(1)) :

$$\tilde{\Delta}_2 = G_1 + \tilde{H}_{21} + \tilde{H}_2 + \tilde{H}_{22} + G_2.$$

We have also $\tilde{D}^* \geq 1/5G_1 + 2/5\tilde{H}_{21} + 3/5\tilde{H}_2 + 2/5\tilde{H}_{22} + 1/5G_2$.

If H is an irreducible component of $\tilde{\Delta}_1 - \tilde{H}_1$ adjacent to \tilde{H}_{12} , then H is a cross-section and $H \cdot G_j = 1$ for $j = 1$ or 2 . This leads to $\tilde{\Delta}_1 = \tilde{\Delta}_2$, a contradiction. So, \tilde{H}_{12} is a tip of $\tilde{\Delta}_1$.

If H is an irreducible component of $\tilde{\Delta}_1 - \tilde{H}_1$ adjacent to \tilde{H}_{11} , then $\tilde{D}^* \geq 3/11H + 6/11\tilde{H}_{11} + 4/11\tilde{H}_1 + 2/11\tilde{H}_{12}$. This leads to $0 < -\tilde{\Sigma} \cdot (K_{\tilde{T}} +$

$D^*) \leq 1 - \tilde{\Sigma} \cdot (3/11H + 2/5\tilde{H}_{21} + 2/5\tilde{H}_{22}) = 1 - 3/11 - 2/5 - 2/5 < 0$, a contradiction. So, \tilde{H}_{11} is tip of \tilde{H}_1 .

Therefore,

$$\tilde{\Delta}_1 = \tilde{H}_1 + \tilde{H}_{11} + \tilde{H}_{12}.$$

In particular, $\tilde{\Sigma}$ meets only $\tilde{H}_{2j} (j = 1, 2)$ in \tilde{D} (cf. Lemma 1.6 and Claim(3)). So,

$$S_1 = \tilde{\Sigma} + E$$

with $\tilde{\Sigma} \cdot E = 1$ and $\tilde{H}_{11} \cdot E = 2$.

If $\tilde{H}_{11}^2 \leq -4$, then $D^* > 1/2\tilde{H}_{11}$ and $0 < -E \cdot (K_{\tilde{T}} + \tilde{D}^*) \leq 1 - E \cdot 1/2\tilde{H}_{11} = 0$, a contradiction. So, $\tilde{H}_{11}^2 = -3$.

For every $i \geq 3$, since \tilde{H}_{21} meets a (-1) -curve of multiplicity one in S_i , the fiber S_i has the second type in Lemma 6.11 of part I. Since $\tilde{D} - (\tilde{H}_{21} + \tilde{H}_{22} + \tilde{H}_{11})$ are contained in singular fibers of φ and since $\rho(T) = 1$, we see that $r = 3$ and

$$S_i \ (i = 0, 1, 2, 3)$$

are all singular fibers of φ (cf. [4, Lemma 1.5 (1)]). Let $E_j (j = 1, 2)$ be the two (-1) -curves in S_3 .

Let $\tau : \tilde{T} \rightarrow \Sigma_2$ be the smooth blowing-down of curves in singular fibers such that $\tau(\tilde{H}_{21})^2 = -2$. Then $\tau(\tilde{H}_{22}) \sim \tau(\tilde{H}_{21}) + 2\tau(S_0)$ and $\tau(\tilde{H}_{11}) \sim 2\tau(\tilde{H}_{21}) + 4\tau(S_0)$. In particular, $\tau(\tilde{H}_{22})^2 = 2$ and $\tau(\tilde{H}_{11})^2 = 8$. So we may assume that $\tilde{H}_{2j} \cdot E_j = \tilde{H}_{11} \cdot E_j = 1 \ (j = 1, 2)$. Moreover,

$$S_3 = E_1 + G_3 + G_4 + E_2$$

where $G_3 + G_4$ is a connected component of \tilde{D} with two (-2) -curves (cf. Lemma 1.1 (4) of Part I) and with $E_j \cdot G_{j+2} = 1$.

Now $\tilde{H}_{11}^2 = -3$, and

$$\tilde{\Delta}_1, \tilde{\Delta}_2, G_3 + G_4$$

are all connected components of \tilde{D} (cf. Lemma 1.1, (4) of Part I). To show that Case(4-1) of Theorem 1.1 occurs, it suffices to show that $\sigma = id$. Let $\sigma_2 : \tilde{S}_1 \rightarrow \tilde{T}$ be the blowing-up of the point $P_2 := \tilde{C} \cap \tilde{H}_2$ and let $L := \sigma_2^{-1}(P_2)$. Suppose to the contrary that $\sigma \neq id$. Then by the hypothesis in Corollary 1.7, there is a smooth blowing-down $\sigma_1 : \tilde{S} \rightarrow \tilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Now applying Lemma 1.4, we get $-L \cdot (K_{\tilde{S}_1} + M^*) > 0$, where $M^* =$

$1/2\sigma'_2\tilde{H}_{11}+1/2\sigma'_2\tilde{H}_1+1/4\sigma'_2\tilde{H}_{12}+1/4\sigma'_2\tilde{C}+1/4\sigma'_2G_1+2/4\sigma'_2\tilde{H}_{21}+3/4\sigma'_2\tilde{H}_2+2/4\sigma'_2\tilde{H}_{22}+1/4\sigma'_2G_2$. This leads to $-L.(K_{\tilde{S}_1}+M^*)=1-L.(1/4\sigma'_2\tilde{C}+3/4\sigma'_2\tilde{H}_2)=0$. We reach a contradiction. So, $\sigma=id$ and Case(4-1) of Theorem 1.1 occurs.

This proves Claim(4).

In view of Claim(4), we may assume that each singular fiber S_i ($i=1, \dots, r$) has the second type in Lemma 6.11 of Part I. Then the number of singular fibers containing two (-1) -curves is one less than the number of sectional-components of \tilde{D} because $\rho(T)=1$. So, $r=2$ and S_0, S_1, S_2 are all singular fibers if \tilde{H}_{12} is a tip of $\tilde{\Delta}_1$, or $r=3$ and S_0, S_1, S_2, S_3 are all singular fibers otherwise. Let

$$\mu : \tilde{T} \rightarrow \Sigma_2$$

be the smooth blowing-down of curves in singular fibers of φ such that $\mu(\tilde{H}_{21})^2=-2$. Write $\mu(\tilde{H}_{ij})=\bar{H}_{ij}$, $\mu(S_i)=\bar{S}_i$, etc. Then $\bar{H}_{22} \sim \bar{H}_{21}+2\bar{S}_0$ and $\bar{H}_{11} \sim 2\bar{H}_{21}+4\bar{S}_0$. In particular, $\bar{H}_{22}^2=2$, $\bar{H}_{11}^2=8$, $\bar{H}_{22}.\bar{H}_{11}=4$.

CLAIM(5). Suppose that \tilde{H}_{11} is not a tip. Then Case(4-2) of Theorem 1.1 occurs.

One can see that \tilde{H}_{11} is a (-3) -curve, as in the proof of Claim (4) above. Note that $r \geq 2$ and we can write

$$S_1 = \tilde{\Sigma} + \sum_{i=1}^s G_i + E$$

such that $E^2=-1$, $G_i^2=-2$, $\tilde{H}_{11}.G_1 = \tilde{\Sigma}.G_1 = G_j.G_{j+1} = G_s.E = 1$ ($j=1, \dots, s-1$) (cf. Lemma 1.6), and

$$S_2 = E_1 + \sum_{i=s+1}^{s+t} G_i + E_2$$

such that $E_i^2=-1$, $G_j^2=-2$, $E_1.G_{s+1} = G_j.G_{j+1} = G_{s+t}.E_2 = 1$ ($j \leq s+t-1$). Note that $\tilde{H}_{11}.E = 1$ for $\tilde{H}_{11}.S_1 = 2$ and $\tilde{H}_{11}.G_1 = 1$.

Note that $\tilde{D}^* \geq 2/11\tilde{H}_{12} + 4/11\tilde{H}_1 + 6/11\tilde{H}_{11} + 3/11G_1$. If $F.\tilde{H}_{11} \geq 2$ for some (-1) -curve F , then $0 < -F.(K_{\tilde{T}} + D^*) \leq 1 - 6/11F.\tilde{H}_{11} \leq 1 - 2 \cdot (6/11) < 0$, a contradiction. So, $F.\tilde{H}_{11} \leq 1$ for every (-1) -curve F and the equality holds if F is in S_i ($i \geq 2$) because $\tilde{H}_{11}.S_i = 2$ (cf. (2) of Claim(1)).

Case(5.1) \tilde{H}_{12} is a tip of $\tilde{\Delta}_1$, while \tilde{H}_{2j} is not a tip of $\tilde{\Delta}_2$ for $j = 1$ or 2 , say $j = 1$. Then $r = 2$. We may assume $\tilde{H}_{21}.G_{s+1} = 1$. Since $\overline{H}_{22}^2 = 2$, one gets $\tilde{H}_{22}.E_2 = 1$ and $t = 4$. This leads to $\tilde{D}^* \geq 1/10G_{s+4} + 2/10G_{s+3} + 3/10G_{s+2} + 4/10G_{s+1} + 5/10\tilde{H}_{21} + 6/10\tilde{H}_2 + 3/10\tilde{H}_{22}$ and $0 < -\tilde{\Sigma}.(K_{\tilde{T}} + D^*) \leq 1 - \tilde{\Sigma}.(5/10\tilde{H}_{21} + 3/10\tilde{H}_{22} + 3/11G_1) = 1 - 5/10 - 3/10 - 3/11 < 0$, a contradiction. So, Case(5.1) is impossible.

Case(5.2). \tilde{H}_{12} is a tip of $\tilde{\Delta}_1$ and both \tilde{H}_{21} and \tilde{H}_{22} are tips of $\tilde{\Delta}_2$. Then $r = 2$, i.e.,

$$S_i \quad (i = 0, 1, 2)$$

are all singular fibers of φ , and

$$\tilde{\Delta}_1 = \tilde{H}_{12} + \tilde{H}_1 + \tilde{H}_{11} + \sum_{i=1}^s G_i, \quad \tilde{\Delta}_2 = \tilde{H}_{21} + \tilde{H}_2 + \tilde{H}_{22},$$

because $\tilde{\Delta}_i$'s are linear chains. Moreover,

$$\tilde{\Delta}_1, \quad \tilde{\Delta}_2, \quad \sum_{i=s+1}^{s+t} G_i$$

are all connected components of \tilde{D} (cf. Lemma 1.1 (4) of Part **I**). We shall show that Case(4-2) of Theorem 1.1 occurs. We may assume that $\tilde{H}_{21}.E_1 = 1$. By the same reasoning as in the previous case, we have $\tilde{H}_{22}.E_2 = 1$ and $t = 3$. Then $8 = \overline{H}_{11}^2 = \tilde{H}_{11}^2 + 2 + (s + 4) + 4$. Hence $s = -\tilde{H}_{11}^2 - 2$. If $s \geq 2$, then $\tilde{H}_{11}^2 \leq -4$ and $\tilde{D}^* \geq 1/4\tilde{H}_{12} + 2/4\tilde{H}_1 + 3/4\tilde{H}_{11} + 2/4G_1 + 1/4G_2$. This leads to $0 < -\tilde{C}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{C}.(1/2\tilde{H}_1 + 1/2\tilde{H}_2) = 0$, a contradiction. So, $s = 1, \tilde{H}_{11}^2 = -3$.

Now $s = 1, t = 3, \tilde{H}_{11}^2 = -3$. To show that Case(4-2) of Theorem 1.1 occurs, it is sufficient to show that $\sigma = id$. Let $\sigma_2 : \tilde{S}_1 \rightarrow \tilde{T}$ be the blowing-up of the point $P_2 := \tilde{C} \cap \tilde{H}_2$ and let $L := \sigma_2^{-1}(P_2)$. Suppose to the contrary

that $\sigma \neq id$. Then by the hypothesis in Corollary 1.7, there is a smooth blowing-down $\sigma_1 : \tilde{S} \rightarrow \tilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we get $-L.(K_{\tilde{S}_1} + M^*) > 0$, where $M^* = 1/3\sigma_2'G_1 + 2/3\sigma_2'\tilde{H}_{11} + 2/3\sigma_2'\tilde{H}_1 + 1/3\sigma_2'\tilde{H}_{12} + 1/3\sigma_2'\tilde{C} + 1/3\sigma_2'\tilde{H}_{21} + 2/3\sigma_2'\tilde{H}_2 + 1/3\sigma_2'\tilde{H}_{22}$. Hence $0 < -L.(K_{\tilde{S}_1} + M^*) = 1 - L.(1/3\sigma_2'\tilde{C} + 2/3\sigma_2'\tilde{H}_2) = 0$. We reach a contradiction. Therefore, $\sigma = id$ and Case(4-2) of Theorem 1.1 occurs.

Case(5.3). \tilde{H}_{12} is not a tip of $\tilde{\Delta}_1$. Let H be the irreducible component of $\tilde{D} - \tilde{H}_1$ adjacent to \tilde{H}_{12} . Then $\tilde{D}^* \geq 1/7H + 2/7\tilde{H}_{12} + 3/7\tilde{H}_1 + 4/7\tilde{H}_{11} + 2/7G_1$. Note that H is a cross-section and $H.\tilde{\Sigma} = H.(K_{\tilde{T}} + \tilde{H}_{12}) = 1$ (cf. Lemma 1.6).

If \tilde{H}_{2j} is not a tip of $\tilde{\Delta}_2$ for $j = 1$ or 2 , say $j = 1$, then $\tilde{D}^* \geq 4/11\tilde{H}_{21} + 6/11\tilde{H}_2 + 3/11\tilde{H}_{22}$, and this leads to $0 < -\tilde{\Sigma}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{\Sigma}.(4/11\tilde{H}_{21} + 3/11\tilde{H}_{22} + 1/7H + 2/7G_1) = 1 - 4/11 - 3/11 - 1/7 - 2/7 < 0$, a contradiction. So, \tilde{H}_{2j} 's are tips of $\tilde{\Delta}_2$ and hence $\tilde{\Delta}_2 = \tilde{H}_{21} + \tilde{H}_2 + \tilde{H}_{22}$.

If G_1 or H is not a tip of $\tilde{\Delta}_1$ (resp. if $\tilde{H}_{11}^2 \leq -4$), then $\tilde{D}^* \geq 3/19H + 6/19\tilde{H}_{12} + 9/19\tilde{H}_1 + 12/19\tilde{H}_{11} + 8/19G_1$ or $\tilde{D}^* \geq 4/17H + 6/17\tilde{H}_{12} + 8/17\tilde{H}_1 + 10/17\tilde{H}_{11} + 5/17G_1$ (resp. $\tilde{D}^* \geq 2/11H + 4/11\tilde{H}_{12} + 6/11\tilde{H}_1 + 8/11\tilde{H}_{11} + 4/11G_1$) and hence $-\tilde{\Sigma}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{\Sigma}.(3/19H + 8/19G_1 + 1/4\tilde{H}_{21} + 1/4\tilde{H}_{22}) = 1 - 3/19 - 8/19 - 1/4 - 1/4 < 0$, or $\leq 1 - \tilde{\Sigma}.(4/17H + 5/17G_1 + 1/4\tilde{H}_{21} + 1/4\tilde{H}_{22}) = 1 - 4/17 - 5/17 - 1/4 - 1/4 < 0$ (resp. $\leq 1 - \tilde{\Sigma}.(2/11H + 4/11G_1 + 1/4\tilde{H}_{21} + 1/4\tilde{H}_{22}) = 1 - 2/11 - 4/11 - 1/4 - 1/4 < 0$). We reach a contradiction in any of the cases. So, $s = 1$, $\tilde{\Delta}_1 = H + \tilde{H}_{12} + \tilde{H}_1 + \tilde{H}_{11} + G_1$, $\tilde{H}_{11}^2 = -3$.

Note that $r = 3$. Let E_1, E_2 (resp. E_3, E_4) be the (-1) -curves in S_2 (resp. S_3). Let $t_i + 2$ be the number of irreducible components of S_i . We may assume that $\tilde{H}_{21}.E_j = 1$ for $j = 1$ and 3 . Note that $8 = \overline{H}_{11}^2 = \tilde{H}_{11}^2 + 2 + (1+4) + (t_1+1) + (t_2+1)$. So, $t_1 + t_2 = 2$. Now $\overline{H}_{22}^2 = 2$ implies that $\tilde{H}_{22}.E_j = 1$ for $j = 2, 4$. But then it is impossible that $\overline{H}^2 = \overline{H}.\overline{H}_{22} = 2$. So, Case(5-3) is impossible.

This proves Claim(5).

In view of Claim(5), we may assume that

$$\tilde{H}_{11} \text{ is a tip of } \tilde{\Delta}_1.$$

Thus,

$$S_1 = \tilde{\Sigma} + E$$

where E is a (-1) -curve such that $E.\tilde{\Sigma} = 1$ and $E.\tilde{H}_{11} = S_1.\tilde{H}_{11} = 2$ (cf. Lemma 1.6 (5)). If $\tilde{H}_{11}^2 \leq -4$, then $\tilde{D}^* \geq 1/2\tilde{H}_{11}$ and $0 < -E.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - E.1/2\tilde{H}_{11} = 0$, a contradiction. So,

$$\tilde{H}_{11}^2 = -3.$$

CLAIM(6). Suppose that \tilde{H}_{12} is a tip. Then Case(4-3) of Theorem 1.1 occurs.

In this case, we have $r = 2$, i.e.,

$$S_i \quad (i = 0, 1, 2)$$

are all singular fibers of φ and

$$\tilde{\Delta}_1 = \tilde{H}_1 + \tilde{H}_{11} + \tilde{H}_{12}.$$

Hence $\tilde{\Sigma}$ meets only \tilde{H}_{2j} ($j = 1, 2$) in \tilde{D} (cf. Lemma 1.6 (5) and Claim(3)).

Write

$$S_2 = E_1 + \sum_{i=1}^t G_i + E_2$$

such that $E_1.G_1 = G_i.G_{i+1} = G_t.E_2 = 1$ ($i = 1, \dots, t-1$). We may assume that \tilde{H}_{2j} does not meet $\sum_i G_i$ for $j = 1$ or 2 , say $j = 1$. We may assume also that $\tilde{H}_{21}.E_1 = 1$. Then $\tilde{H}_{22}^2 = 2$ implies that either $t = 3$ and $\tilde{H}_{22}.E_2 = 1$, or $t = 4$ and $\tilde{H}_{22}.G_4 = 1$. Since $\tilde{H}_{11}^2 = 8$, we must have $t = 4$ and $\tilde{H}_{11}.E_j = 1$ for $j = 1$ and 2 . Now $\tilde{H}_{11}^2 = -3$,

$$\tilde{\Delta}_2 = \tilde{H}_{21} + \tilde{H}_2 + \tilde{H}_{22} + G_4 + G_3 + G_2 + G_1, \quad \text{and}$$

$$\tilde{\Delta}_1, \quad \tilde{\Delta}_2$$

are all connected components of \tilde{D} (cf. Lemma 1.1 (4) of Part I).

To prove that Case(4-3) of Theorem 1.1 occurs, it is sufficient to show that $\sigma = id$. Let $\sigma_2 : \tilde{S}_1 \rightarrow \tilde{T}$ be the blowing-up of the point $P_2 := \tilde{C} \cap \tilde{H}_2$

and set $L := \sigma_2^{-1}(P_2)$. Suppose to the contrary that $\sigma \neq id$. Then by the hypothesis in Corollary 1.7, there is a smooth blowing-down $\sigma_1 : \tilde{S} \rightarrow \tilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we get $-L.(K_{\tilde{S}_1} + M^*) > 0$, where $M^* = 1/2\sigma'_2\tilde{H}_{11} + 1/2\sigma'_2\tilde{H}_1 + 1/4\sigma'_2\tilde{H}_{12} + 1/4\sigma'_2\tilde{C} + 1/8\sigma'_2G_1 + 2/8\sigma'_2G_2 + 3/8\sigma'_2G_3 + 4/8\sigma'_2G_4 + 5/8\sigma'_2\tilde{H}_{22} + 6/8\sigma'_2\tilde{H}_2 + 3/8\sigma'_2\tilde{H}_{21}$. This leads to $-L.(K_{\tilde{S}_1} + M^*) = 1 - L.(1/4\sigma'_2\tilde{C} + 3/4\sigma'_2\tilde{H}_2) = 0$, a contradiction. Therefore $\sigma = id$ and Case(4-3) of Theorem 1.1 occurs.

This proves Claim(6).

CLAIM(7). Suppose that \tilde{H}_{12} is not a tip. Then either Theorem 1.1 (3) occurs or Case(4-4) of Theorem 1.1 occurs.

Then $r = 3$, i.e.,

$$S_i \ (i = 0, 1, 2, 3)$$

are all singular fibers of φ . Write

$$S_2 = E_1 + \sum_{i=1}^{t_1} G_i + E_2,$$

$$S_3 = E_3 + \sum_{i=t_1+1}^{t_1+t_2} G_i + E_4$$

such that $E_j^2 = -1$, $G_i^2 = -2$, $E_1.G_1 = G_{t_1}.E_2 = E_3.G_{t_1+1} = G_{t_1+t_2}.E_4 = G_i.G_{i+1} = 1$.

Let H be an irreducible component of $\tilde{\Delta}_1 - \tilde{H}_1$ adjacent to \tilde{H}_{12} . If H is not a tip of $\tilde{\Delta}_1$ then the intersection matrix of $\tilde{C} + \tilde{\Delta}_1$ and hence that of $C + T_2 + \Delta_1$ have a positive eigenvalue. So Theorem 1.1 (3) occurs. Thus we may assume that H is a tip of $\tilde{\Delta}_1$. Hence $\tilde{\Sigma}.H = 1$ and

$$\tilde{\Delta}_1 = \tilde{H}_1 + \tilde{H}_{11} + \tilde{H}_{12} + H.$$

Note that

$$\tilde{D}^* = 1/9H + 2/9\tilde{H}_{12} + 3/9\tilde{H}_1 + 4/9\tilde{H}_{11} + (\text{other terms}).$$

Now one may assume that $E_j.H = 1$ for $j = 2, 4$. Let $\varepsilon : \tilde{T} \rightarrow \Sigma_2$ be the smooth blowing-down of curves in the singular fibers of φ such that $\varepsilon(H)^2 = -2$. Then $\varepsilon(H_{2j})^2 = 2$ ($j = 1, 2$) and $\varepsilon(H_{11})^2 = 8$.

If $\tilde{H}_{11}.E_i = 2$ for $i = 1$ or 3 , say $i = 1$, then $S_2 = E_1 + E_2, S_3 = E_3 + E_4$ and $\tilde{H}_{11}.E_k = 1$ for $k = 3$ and 4 because $\varepsilon(H_{11})^2 = 8$. But then $\varepsilon(H_{2j})^2 \leq -2 + 3$ ($j = 1, 2$), a contradiction. If $\tilde{H}_{11}.E_i = 2$ for $i = 2$ or 4 , then $-E_i.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - E_i.(1/9H + 4/9\tilde{H}_{11}) = 1 - 1/9 - (4/9) \times 2 = 0$, a contradiction. So, $\tilde{H}_{11}.E_j = 1$ for $j = 1, 2, 3$ and 4 . Now $\varepsilon(H_{11})^2 = 8$ implies that $t_1 + t_2 = 3$.

If \tilde{H}_{2j} is not a tip of $\tilde{\Delta}_2$ for both $j = 1$ and 2 , then one may assume that $(t_1, t_2) = (1, 2)$ and $\tilde{H}_{21}.G_1 = 1$. Then it is impossible that $\varepsilon(H_{21})^2 = 2$. So, one may assume that \tilde{H}_{21} is a tip of $\tilde{\Delta}_2$.

Since $\varepsilon(H_{21})^2 = 2$, one may assume that $(t_1, t_2) = (1, 2)$ and $\tilde{H}_{21}.E_j = 1$ for $j = 2$ and 3 . Now $\varepsilon(H_{22}).\varepsilon(H_{21}) = 2$ implies that $\tilde{H}_{22}.E_1 = \tilde{H}_{22}.G_3 = 1$. So,

$$\tilde{\Delta}_2 = \tilde{H}_{21} + \tilde{H}_2 + \tilde{H}_{22} + G_3 + G_2,$$

and

$$\tilde{\Delta}_1, \tilde{\Delta}_2, G_1$$

are all connected components of \tilde{D} (cf. Lemma 1.1 (4), Part I).

Now $(t_1, t_2) = (1, 2)$ and $\tilde{H}_{11}^2 = -3$. To prove that Case(4-4) takes place, we have only to show that $\sigma = id$. Let $\sigma_2 : \tilde{S}_1 \rightarrow \tilde{T}$ be the blowing-up of the point $\tilde{C} \cap \tilde{H}_2$ and set $L := \sigma_2^{-1}(P_2)$. Suppose to the contrary that $\sigma \neq id$. Then by the hypothesis in Corollary 1.7, there is a smooth blowing-down $\sigma_1 : \tilde{S} \rightarrow \tilde{S}_1$ such that $\sigma = \sigma_2 \cdot \sigma_1$. Applying Lemma 1.4, we get $-L.(K_{\tilde{S}_1} + M^*) > 0$, where $M^* = 2/9\sigma'_2H + 4/9\sigma'_2\tilde{H}_{12} + 6/9\sigma'_2\tilde{H}_1 + 5/9\sigma'_2\tilde{H}_{11} + 3/9\sigma'_2\tilde{C} + 4/11\sigma'_2\tilde{H}_{21} + 8/11\sigma'_2\tilde{H}_2 + 6/11\sigma'_2\tilde{H}_{22} + 4/11\sigma'_2G_3 + 2/11\sigma'_2G_2$. This leads to $-L.(K_{\tilde{S}_1} + M^*) = 1 - L.(1/3\sigma'_2\tilde{C} + 8/11\sigma'_2\tilde{H}_2) = 1 - 1/3 - 8/11 < 0$, a contradiction. Therefore, $\sigma = id$ and Case(4-4) of Theorem 1.1 occurs.

This proves Claim(7) and also Lemma 1.10. \square

LEMMA 1.11. *Suppose that the case (2) in Lemma 1.8 occurs with $(\tilde{J}_a^2, \tilde{J}_b^2) = (-2, -3)$ or $(-2, -4)$ but the case in Corollary 1.7 does not occur. Then Theorem 1.1, (3) or (5) occurs.*

PROOF. By the hypothesis, $J_a = H_a$ and we may assume that $\tilde{H}_{a1}^2 \leq$

–3.

CLAIM(1). It is impossible that $\tilde{J}_b^2 = -4$.

We consider the case $\tilde{J}_b^2 = -4$. Since the case in Corollary 1.7 does not occur, we have $\sigma \neq id$. Let $\tau_i : \tilde{S}_i \rightarrow \tilde{T}$ be the blowing-up of the point $P_i := \tilde{C} \cap \tilde{J}_i$. Let $E_i := \tau_i^{-1}(P_i)$. Then for $t = a$ or b , there is a smooth blowing-down $\sigma_t : \tilde{S} \rightarrow \tilde{S}_t$ such that $\sigma = \tau_t \cdot \sigma_t$. Now we apply Lemma 1.4. In particular, we have $-E_t.(K_{\tilde{S}_t} + M^*) > 0$.

Case $t = a$. Then $M^* \geq 8/13\tau'_a\tilde{H}_a + 7/13\tau'_a\tilde{H}_{a1} + 4/13\tau'_a\tilde{H}_{a2} + 2/5\tau'_a\tilde{C} + 4/5\tau'_a\tilde{J}_b$. This leads to $0 < -E_a.(K_{\tilde{S}_t} + M^*) \leq 1 - E_a.(8/13\tau'_a\tilde{H}_a + 2/5\tau'_a\tilde{C}) = 1 - 8/13 - 2/5 < 0$, a contradiction. So this case is impossible.

Case $t = b$. Then $M^* \geq 1/4\tau'_b\tilde{C} + 1/2\tau'_b\tilde{H}_a + 1/2\tau'_b\tilde{H}_{a1} + 1/4\tau'_b\tilde{H}_{a2} + 3/4\tau'_b\tilde{J}_b$. This leads to $0 < -E_b.(K_{\tilde{S}_t} + M^*) \leq 1 - E_b.(1/4\tau'_b\tilde{C} + 3/4\tau'_b\tilde{J}_b) = 0$, a contradiction. So this case is also impossible.

This proves Claim(1).

Therefore, $\tilde{J}_b^2 = -3$.

CLAIM(2). $\tilde{\Delta}_a$ is a linear chain and the connected component of $\tilde{\Delta}_a - \tilde{H}_a$ containing \tilde{H}_{a2} is a (-2) -chain.

Since it is impossible that $\tilde{\Delta}_b$ is a linear chain with \tilde{J}_b as a tip (cf. the hypothesis $(*)$ after Lemma 1.3), we have $\tilde{D}^* \geq 1/2\tilde{J}_b$. We shall also show that if Claim(2) is false then $\tilde{D}^* \geq 1/2\tilde{H}_a$.

In fact, if $\tilde{\Delta}_a$ is a fork, such that either \tilde{H}_a is the central component or \tilde{H}_{a2} and the central component are contained in the same connected component of $\tilde{\Delta}_a - \tilde{H}_a$, then $\tilde{D}^* \geq 1/2\tilde{H}_{a1} + 1/2\tilde{H}_a + 1/4\tilde{H}_{a2}$. If $\tilde{\Delta}_a$ is a fork such that \tilde{H}_{a1} and the central component are contained in the same connected component of $\tilde{\Delta}_a - \tilde{H}_a$, then $\tilde{D}^* \geq 1/4\tilde{H}_{a2} + 2/4\tilde{H}_a + 3/4\tilde{H}_{a1}$. If $\tilde{\Delta}_a$ is a linear chain but the connected component of $\tilde{\Delta}_a - \tilde{H}_a$ containing \tilde{H}_{a2} is not a (-2) -chain, then $\tilde{D}^* \geq 1/2\tilde{H}_{a1} + 1/2\tilde{H}_a + 1/2\tilde{H}_{a2}$.

Now suppose Claim(2) is false. Then we have $\tilde{D}^* \geq 1/2\tilde{H}_a$ by the above arguments. This leads to $0 < -\tilde{C}.(K_{\tilde{T}} + \tilde{D}^*) \leq 1 - \tilde{C}.(1/2\tilde{H}_a + 1/2\tilde{J}_b) = 0$, a contradiction. So Claim(2) is true.

Thus, $\tilde{H}_{a2}^2 = -2$. If \tilde{J}_b is a tip of $\tilde{\Delta}_b$, i.e., if $J_b \neq H_b$, then Theorem 1.1 (3) occurs. Indeed, $\tilde{C} + \tilde{J}_b + \tilde{H}_a + \tilde{H}_{a2}$ is a support of a singular fiber of a \mathbf{P}^1 -fibration; hence the intersection matrices of $\tilde{C} + \tilde{J}_b + \tilde{\Delta}_a$ and $C + T_b + \Delta_a$ have a positive eigenvalue.

Therefore, we may assume that $J_b = H_b$. Since the case in Corollary 1.7 does not occur, there are two smooth blowing-downs $\sigma_1 : \tilde{S} \rightarrow \tilde{S}_1$, $\sigma_2 : \tilde{S}_1 \rightarrow \tilde{T}$ such that $\sigma = \sigma_2 \cdot \sigma_1$ and that :

- (1) $\sigma_1(T_a + C + T_b) = T'_a + E + T'_b$ where E is a (-1) -curve and $T'_i \leq \sigma_1(T_i)$,
- (2) $T'_a + \sigma_1(H_a) = \sum_{i=1}^s L_i$, $E.L_1 = L_i.L_{i+1} = 1$ ($i = 1, \dots, s-1$; $s \geq 1$), $L_s = \sigma_1(H_a)$, $L_1^2 = -t-1 \leq -3$, $L_j^2 = -2$ ($j > 1$),
- (3) $T'_b + \sigma_1(H_b) = \sum_{i=1}^t M_i$, $E.M_1 = M_i.M_{i+1} = 1$ ($i = 1, \dots, t-1$; $t \geq 2$), $M_t = \sigma_1(H_b)$, $M_j^2 = -2$ ($j < t$), $M_t^2 = -s-2 \leq -3$, and
- (4) σ_1 does not factorize through the blowing-up of the point $P_a := E \cap L_1$.

In particular, we see that $\sigma_1(\Delta_b)$ is a fork and hence $\tilde{\Delta}_b$ is a linear chain. Now we apply Lemma 1.4. In particular, we have $-E.(K_{\tilde{S}_1} + M^*) > 0$.

CLAIM(3). $\sigma_1 = id$. Hence $a = 2$, $b = 1$, $C = E$, $D_1 = M_1$, $D_2 = L_1$, $D_2^2 = -t-1 \leq -3$, $H_{21}^2 \leq -3$ and $T_1 = \sum_{i=1}^{t-1} M_i$ is a (-2) -twig.

Let $\tau_2 : \tilde{X} \rightarrow \tilde{S}_1$ be the blowing-up of the point $P_b := E \cap M_1$ and set $F := \tau_2^{-1}(P_b)$. Suppose that Claim(3) is false. Then by the definition of σ_1 (cf. the above condition(4)), there is a smooth blowing-down $\tau_1 : \tilde{S} \rightarrow \tilde{X}$ such that $\sigma_1 = \tau_2 \cdot \tau_1$. Now we apply Lemma 1.4. In particular, we have $-F.(K_{\tilde{X}} + N^*) > 0$, where $N = D$ if $\tau_1 = id$ and $N = \tau_1(D) - F$ otherwise.

Since $\tau_1(C + \Delta_1 + \Delta_2) - F$ can be contractible to quotient singularities (cf. Lemma 1.4), we have $s = 1$ or 2 , and if $s = 2$ then $t = 2$, $\tilde{H}_{a1}^2 = -3$ and $\tau_1(\Delta_a) = \tau_2'(E + \sum_i L_i) + \tau_1(H_{a1} + H_{a2})$.

Suppose $s = 1$. Then $N^* \geq (3t-2)/(6t-2)\tau_2'E + 2(3t-2)/(6t-2)\tau_1(H_a) + (4t-2)/(6t-2)\tau_1(H_{a1}) + (3t-2)/(6t-2)\tau_1(H_{a2}) + \sum_i(t+i)/(2t+1)\tau_2'(M_i) + t/(2t+1)\tau_1(H_{b1}) + t/(2t+1)\tau_1(H_{b2})$. This leads to

$$0 < -F.(K_{\tilde{X}} + N^*) \leq 1 - F.((3t-2)/(6t-2)\tau_2'E + (t+1)/(2t+1)\tau_2'M_1) = 1 - (3t-2)/(6t-2) - (t+1)/(2t+1) = (-t+2)/(6t-2)(2t+1) \leq 0,$$

because $t \geq 2$. We reach a contradiction.

Suppose that $s = 2$. Then $N^* \geq 9/23\tau'_2 E + 18/23\tau'_2(L_1) + 22/23\tau_1(H_a) + 15/23\tau_1(H_{a1}) + 11/23\tau_1(H_{a2}) + 10/16\tau'_2(M_1) + 14/16\tau_1(H_b) + 7/16\tau_1(H_{b1}) + 7/16\tau_1(H_{b2})$. This leads to

$$0 < -F.(K_{\tilde{X}} + N^*) \leq 1 - F.(9/23\tau'_2 E + 10/16\tau'_2 M_1) = 1 - 9/23 - 10/16 < 0.$$

We reach a contradiction.

So Claim(3) is true.

CLAIM(4). $s = 1$. Hence Δ_2 is a linear chain, $H_2 = D_2$ and $H_1^2 = -s - 2 = -3$.

Suppose $s \geq 3$. Then $s = 3, t = 2, H_{21}^2 = -3, D_2 = L_1, H_2 = L_3, D_1 = M_1, H_1 = M_2, H_2 = L_3, \Delta_2 = D_2 + L_2 + H_2 + H_{21} + H_{22}$ because Δ_2 is contractible to a quotient singularity. So, we have $D^* \geq 3/7D_1 + 6/7H_1 + 3/7H_{11} + 3/7H_{12} + 10/17D_2 + 13/17L_2 + 16/17H_2 + 11/17H_{21} + 8/17H_{22}$. This leads to $0 < -C.(K_{\tilde{S}} + D^*) \leq 1 - C.(3/7D_1 + 10/17D_2) = 1 - 3/7 - 10/17 < 0$, a contradiction.

Suppose $s = 2$. Then $D_2 = L_1, H_2 = L_2, D^* \geq \sum_i 2i/(2t+1)M_i + t/(2t+1)H_{11} + t/(2t+1)H_{12} + (7t-5)/(7t+1)D_2 + 4(2t-1)/(7t+1)H_2 + (5t-1)/(7t+1)H_{21} + 2(2t-1)/(7t+1)H_{22}$. This leads to $0 < -C.(K_{\tilde{S}} + D^*) \leq 1 - C.(2/(2t+1)D_1 + (7t-5)/(7t+1)D_2) = 1 - 2/(2t+1) - (7t-5)/(7t+1) = (4-2t)/(2t+1)(7t+1) \leq 0$, because $t \geq 2$. We reach a contradiction.

This proves Claim(4).

CLAIM(5). $t = 2, 3$. Hence $D_2^2 = -t - 1 = -3, -4$.

Note that $D^* \geq \sum_i i/(t+1)M_i + t/2(t+1)H_{11} + t/2(t+1)H_{12} + (6t-4)/(6t+1)D_2 + (4t-1)/(6t+1)H_{21} + (3t-2)/(6t+1)H_{22}$, where $D_2 = L_1$. So, $0 < -C.(K_{\tilde{S}} + D^*) \leq 1 - C.(1/(t+1)D_1 + (6t-4)/(6t+1)D_2) = 1 - 1/(t+1) - (6t-4)/(6t+1) = (4-t)/(t+1)(6t+1)$. Hence $t \leq 3$. This proves Claim(5).

CLAIM(6). Theorem 1.1 (5) occurs.

Consider first the case $D_2^2 = -t - 1 = -3$. Then $D_1 = M_1, H_1 = M_2, D_2 = H_2, D^* \geq 1/3D_1 + 2/3H_1 + 1/3H_{11} + 1/3H_{12} + 7/13H_{21} + 8/13D_2 +$

$4/13H_{22}$. If H_{21} is not a tip (resp. H_{22} is not a tip, or $H_{21}^2 \leq -4$), then $D^* \geq 2/3H_{21} + 2/3D_2 + 1/3H_{22}$ (resp. $D^* \geq 5/9H_{21} + 6/9D_2 + 4/9H_{22}$, or $D^* \geq 2/3H_{21} + 2/3D_2 + 1/3H_{22}$). Any of the three cases leads to $0 < -C.(K_{\tilde{S}} + D^*) \leq 1 - C.(1/3D_1 + 2/3D_2) = 0$, a contradiction. Thus, $\Delta_2 = D_2 + H_{21} + H_{22}$ and $H_{21}^2 = -3$. So, Δ_2 is as described in Figure 5 or 6.

Let T'_1, T''_1 be twigs of Δ_1 containing H_{11}, H_{12} , respectively. If both T'_1 and T''_1 have more than one irreducible components (resp. T'_1 or T''_1 , say T'_1 has more than two irreducible components), then $D^* \geq 3/7D_1 + 6/7H_1 + 4/7H_{11} + 4/7H_{12}$ (resp. $D^* \geq 2/5D_1 + 4/5H_1 + 3/5H_{11} + 2/5H_{12}$). Any of the two cases leads to $0 < -C.(K_{\tilde{S}} + D^*) \leq 1 - C.(2/5D_1 + 8/13D_2) = 1 - 2/5 - 8/13 < 0$, a contradiction.

To show that $C + \Delta_1 + \Delta_2$ is as described in Figure 5 or 6, it remains to show that $\Delta_1 - H_1$ consists of only (-2) -curves. Indeed, if $H_{1j}^2 \leq -3$ for $j = 1$ or 2 , say $j = 1$, then $D^* \geq 2/5D_1 + 4/5H_1 + 3/5H_{11} + 2/5H_{12}$ and a contradiction is derived as in the above paragraph. Note that $H := \Delta_1 - (D_1 + H_1 + H_{11} + H_{12})$ is zero or a single curve. It remains to show that $H^2 = -2$ if $H \neq 0$. Indeed, suppose $H^2 \leq -3$ and suppose, without loss of generality, $H \leq T'_1$. Then $D^* \geq 3/7D_1 + 6/7H_1 + 4/7H + 5/7H_{11} + 3/7H_{12}$, and we reach again a contradiction as in the above paragraph.

We have proved that $C + \Delta_1 + \Delta_2$ is as described in Figure 5 if $D_2^2 = -3$.

Now we consider the case $D_2^2 = -4$. Let $\gamma_1 : \tilde{S} \rightarrow \tilde{X}$ be the blowing-down of C . Let $\gamma_2 : \tilde{X} \rightarrow \tilde{T}$ be the smooth blowing-down such that $\sigma = \gamma_2 \cdot \gamma_1$. Now we apply Lemma 1.4. In particular, we have $-F.(K_{\tilde{X}} + N^*) > 0$ where $F = \gamma_1(D_1)$ is a (-1) -curve and $N = \gamma_1(D) - F$.

Now F meets a (-2) -curve $\gamma(M_2)$ and a (-3) -curve $\gamma(D_2)$. By making use of the latter inequality for F and by the arguments for the case $D_2^2 = -3$, we can also prove that $\gamma(\Delta_1 - D_1), \gamma(\Delta_2)$ have the same weighted dual graphs as Δ_1, Δ_2 , respectively in Figure 5. To verify that $C + \Delta_1 + \Delta_2$ is as described in Figure 6, it remains to show that $H := \Delta_1 - (D_1 + M_2 + H_1 + H_{11} + H_{12}) = 0$. Suppose $H \neq 0$, say H is adjacent to H_{11} . Then $D^* \geq 2/7D_1 + 4/7M_2 + 6/7H_1 + 2/7H + 4/7H_{11} + 3/7H_{12} + 11/19H_{21} + 14/19D_2 + 7/19H_{21}$. This leads to $0 < -C.(K_{\tilde{S}} + D^*) \leq 1 - C.(2/7D_1 + 14/19D_2) = 1 - 2/7 - 14/19 < 0$, a contradiction.

This proves Claim(6) and hence Lemma 1.11. \square

LEMMA 1.12. *In the Case (3) of Theorem 1.1, $\pi_1(S^o)$ is finite.*

PROOF. The argument in this case is similar to the proof of Lemma 6.24 at the end of Part I. We can assume that the intersection matrix of $C + T_1 + \Delta_2$ has a positive eigenvalue. Let $T_1 = B_1 + L_2 + \cdots + L_r$ be the twig. If U is a nice tubular neighborhood of $C + T_1 + \Delta_2$, then it is easy to see that $U - D$ has $N - D$ as a strong deformation retract, where N is a tubular neighborhood of $C + \Delta_2$. Now the rest of the argument is exactly as in the proof of Lemma 6.24 in Part I. \square

LEMMA 1.13. *Suppose that Theorem 1.1 (4) occurs. Then $\pi_1(S^o)$ is finite.*

PROOF. We will use the description of $C + \Delta_1 + \Delta_2$ in Figure 1, 2, 3 or 4. As before, the intersection matrix of $C + \Delta_1 + \Delta_2$ has a positive eigenvalue and by Lemma 1.10 of Part I we have a surjection $\pi_1(U - \Delta_1 - \Delta_2) \rightarrow \pi_1(S^o)$, where U is a small neighborhood of $C \cup \Delta_1 \cup \Delta_2$. We will use the presentation of $\pi_1(U - \Delta_1 - \Delta_2)$ given by Mumford in [3].

Case (4-1). Then $\pi_1(\partial U)$ is given by generators $e_0, e_1, e_{11}, e_{12}, e_2, e_{21}, e_{22}, g_1, g_2$ corresponding to $C, H_1, H_{11}, H_{12}, H_2, H_{21}, H_{22}, G_1, G_2$ respectively and the following relations (cf. Figure 1) :

$$\begin{aligned} 1 &= e_{11}^{-3} e_1 = e_{12}^{-2} e_1 = e_{11} e_{12} e_1^{-2} e_0 = e_1 e_0^{-1} e_2 \\ &= e_0 e_2^{-3} e_{21} e_{22} = g_1 e_{21}^{-2} e_2 = g_1^{-2} e_{21} = g_2 e_{22}^{-2} e_2 = g_2^{-2} e_{22}. \end{aligned}$$

Now $\pi_1(U - D)$ is obtained by putting $e_0 = 1$ in the relations above. Hence in $\pi_1(U - D)$, we have

$$\begin{aligned} e_1 &= e_{12}^2, \quad e_2 = e_1^{-1} = e_{12}^{-2}, \quad e_{11} = e_1^2 e_{12}^{-1} = e_{12}^3, \quad e_{22} = g_2^2, \\ e_{21} &= e_2^3 e_{22}^{-1} = e_{12}^{-6} g_2^{-2}, \quad g_1 = e_2^{-1} e_{21}^2 = e_{12}^2 (e_{12}^{-6} g_2^{-2})^2, \\ e_{12}^2 &= e_1 = e_{11}^3 = e_{12}^9, \quad e_{12}^7 = 1, \\ e_{12}^2 &= e_2^{-1} = g_2 e_{22}^{-2} = g_2^{-3}, \quad e_{12} = e_{12}^{-6} = g_2^9. \end{aligned}$$

Here 7 is the absolute value of the determinant of the intersection matrix of Δ_1 . The above relation shows that all the generators of $\pi_1(U - D)$ can

be expressed in terms of g_2 and $g_2^{63} = e_{12}^7 = 1$. Hence $\pi_1(U - D)$ is a finite cyclic group generated by g_2 . Thus $\pi_1(S^o)$ is a finite cyclic group in this case.

Case (4-2). We argue exactly as above. The determinant of $\Delta_1 = \pm 11$ and $\pi_1(U - D)$ is generated by e_{21} (corresponding to H_{21}) (cf. Figure 2). Again $\pi_1(U - D)$ is a finite cyclic group.

Case (4-3). Then the determinant of $\Delta_1 = \pm 7$ (cf. Figure 3). In this case $\pi_1(U - D)$ is a finite group generated by g_1 (corresponding to G_1).

In the above cases, the crucial fact used was the linearity of Δ_1, Δ_2 .

Case (4-4). Then the determinants of Δ_1, Δ_2 are $\pm 9, \pm 14$ respectively (both non-primes) (cf. Figure 4). In this case we use the (-1) -curve E in the singular fiber S_1 . Now $E + \Delta_1$ supports a divisor with a positive self-intersection. E intersects only the curve H_{11} from Δ_1 ($E.H_{11} = 2$) which is a tip of the linear chain Δ_1 . Now the argument used for the case $|K + C + D| \neq \phi$ in Part **I**, using Lemma 1.14 in Part **I**, proves that $\pi_1(S^o)$ is a finite group.

This proves Lemma 1.13. \square

LEMMA 1.14. *Suppose that Theorem 1.1 (5) occurs. Then $\pi_1(S^o)$ is finite.*

PROOF. In Case (5-5) of Theorem 1.1, the determinant of $\Delta_2 = \pm 13$ and Δ_2 is linear (whether $H = \phi$ or $\neq \phi$). In Case (5-6) of Theorem 1.1, the determinant of $\Delta_2 = \pm 19$ and Δ_2 is linear (cf. Figures 5, 6).

If U is a tubular neighborhood of $C \cup \Delta_1 \cup \Delta_2$, then using Mumford's presentation we see that $\pi_1(U - D)$ is a homomorphic image of $\pi_1(U_1 - \Delta_1)$, where U_1 is a small tubular neighborhood of Δ_1 . Since Δ_1 defines a quotient singular point, we deduce the finiteness of $\pi_1(S^o)$. Lemma 1.14 is proved. \square

Thus we have proved Theorem 1.1 and also the Main Theorem.

REMARK. By [6, 7, 8], we see that our main theorem is still true with the ampleness of the anti-canonical divisor $-K_S$ replaced by the weaker nef and bigness, but it is not true any more if either one replaces the ampleness

of $-K_S$ by that the anti-Kodaira dimension equals two, or one lets S have worse log canonical singularities.

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(Received May 13, 1994)

(Revised July 29, 1994)

R. V. Gurjar
 School of Mathematics
 Tata Institute of Fundamental Research
 Bombay 400 005
 INDIA
 e-mail: gurjar@tifrvax.tifr.res.in

D.-Q. Zhang
 Department of Mathematics
 National University of Singapore
 SINGAPORE
 e-mail: matzdq@math.nus.sg