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A general functional characterization of the microlocal singularities

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1. Introduction

As a generalization of a result of [7], we proved in [1] a functional characterization of the analytic wave front set of an hyperfunction. The main result of [1] is the following.

Denote by $\mathcal{A}(\mathbb{R}^n)$ the set of analytic functions on $\mathbb{R}^n$. Let $u \in \mathcal{A}'(\mathbb{R}^n \times \mathbb{R}^p)$ be an analytic functional and consider

$$Sg(x) = \int u(x,y)g(y)\,dy$$

for g analytic in a neighbourhood of the projection of $\mathrm{supp}(u)$ on $\mathbb{R}^p$. The result states that if $(x_0,\xi_0) \notin WF_a(Sg)$ for every $g \in \mathcal{A}(\mathbb{R}^p)$, then $((x_0,y),(\xi_0,0)) \notin WF_a(u)$ for every $y \in \mathbb{R}^p$. It characterizes special points of $WF_a(u)$ using the operator S. In this paper, we prove a similar result concerning any point of $WF_a(u)$.

Denote by $\mathcal{B}(\mathbb{R}^n)$ the set of all hyperfunctions on $\mathbb{R}^n$. It is known, [4], [9], that if $f, g \in \mathcal{B}(\mathbb{R}^n)$ and if $(x,\xi) \in WF_a(f)$ implies $(x,-\xi) \notin WF_a(g)$, then the product fg is well defined in $\mathcal{B}(\mathbb{R}^n)$. If in addition $\mathrm{supp}(f) \cap \mathrm{supp}(g)$ is compact, the product fg belongs to $\mathcal{A}'(\mathbb{R}^n)$, hence the pairing

$$\langle f, g\rangle = \int f(x)g(x)\,dx$$

is also well defined.

Let $u \in \mathcal{A}'(\mathbb{R}^n \times \mathbb{R}^p)$ and define as above

$$^t Sf(y) = \int u(x,y)f(x)\,dx \quad , \quad f \in \mathcal{A}(\mathbb{R}^n).$$

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Let $(y_0, \eta_0) \in \mathbb{R}^p \times \mathbb{R}^p \setminus \{0\}$ and assume that $(x, y_0, 0, -\eta_0) \notin WF_a(u)$ for every $x \in \mathbb{R}^n$. It follows from the composition properties of the analytic wave front set that $(y_0, -\eta_0) \notin WF_a({}^tSf)$ for any $f \in \mathcal{A}(\mathbb{R}^n)$, [4], [9]. Hence, using the formula

$$(Sg)(f) = \left\langle g, {}^tSf \right\rangle,$$

the operator S can be extended to any $g \in \mathcal{B}(\mathbb{R}^p)$ such that $WF_a(g) \subset \{(y_0, t\eta_0) : t > 0\}$.

In the same way, the operator S can also be extended to all $g \in \mathcal{B}(\mathbb{R}^p)$ such that $(x, y, 0, -\eta) \notin WF_a(u)$ for all $x \in \mathbb{R}^n$ and $(y, \eta) \in WF_a(g)$. Theorem 3 gives an explicit formula for this extension.

In this paper we prove the following result.

Theorem 1. *Let $u \in \mathcal{A}'(\mathbb{R}^n \times \mathbb{R}^p)$ and $y_0 \in \mathbb{R}^p$, $\eta_0 \in \mathbb{R}^p$ such that*

$$(x, y_0, 0, -\eta_0) \notin WF_a u$$

for every $x \in \mathbb{R}^n$. If $x_0 \in \mathbb{R}^n$, $\xi_0 \in \mathbb{R}^n$ and $(x_0, \xi_0) \notin WF_a(Sg)$ for every $g \in \mathcal{B}(\mathbb{R}^p)$ satisfying

$$WF_a g \subset \{(y_0, t\eta_0) : t > 0\}$$

then $(x_0, y_0, \xi_0, -\eta_0) \notin WF_a u$.

If $\eta_0 = 0$, this is the result of [1]. If the hypotheses are satisfied for ξ_0 and η_0, they are also satisfied for $t\xi_0$ and $s\eta_0$ for every $s, t \geq 0$. Hence, we also have $(x_0, y_0, t\xi_0, -s\eta_0) \notin WF_a u$ for every $s, t \geq 0$.

Conversely, if this last condition is satisfied, the composition properties of the analytic wave front set show that $(x_0, \xi_0) \notin WF_a(Sg)$ for every $g \in \mathcal{B}(\mathbb{R}^p)$ satisfying $WF_a g \subset \{(y_0, t\eta_0) : t > 0\}$.

In the proof of theorem 1, we use the closed graph theorem from a Fréchet space into a strictly webbed space as in [1].

2. Extension of the operator

Let $u \in \mathcal{A}'(\mathbb{R}^n \times \mathbb{R}^p)$. If g is analytic in a neighbourhood of the projection of $\operatorname{supp}(u)$ on $\mathbb{R}^p$ then

$$Sg(x) = \int u(x, y) g(y)\, dy$$

is the element of $\mathcal{A}'(\mathbb{R}^n)$ defined by $(Sg)(f) = u(f \otimes g)$ for every $f \in \mathcal{A}(\mathbb{R}^n)$.

Using the FBI transform, we give here an explicit formula for the extension of S to all $g \in \mathcal{B}(\mathbb{R}^p)$ such that $(x, y, 0, -\eta) \notin WF_a(u)$ for all $x \in \mathbb{R}^n$ and $(y, \eta) \in WF_a(g)$. It will be needed in the proof of theorem 1.

Let us first recall some definitions. If $u \in \mathcal{A}'(\mathbb{R}^n)$, the FBI *transform* of u is the family of holomorphic functions

$$Tu(z, \lambda) = u_{(x)}(e^{i\lambda\varphi(z,x)}), \quad z \in \mathbb{C}^n,\ \lambda > 0,$$

where φ is the quadratic polynomial $\varphi(z, x) = \frac{i}{2}(z - x)^2$. Using the continuity of u as a linear functional, it is easily seen that if $\mathrm{supp}(u) \subset \{x \in \mathbb{R}^n : |x| < a\}$, then for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|Tu(z, \lambda)| \le C_\varepsilon e^{\frac{\lambda}{2}\left((|\Im z|+\varepsilon)^2 - (|\Re z| - a)_+^2\right)}$$

if $z \in \mathbb{C}^n$ and $\lambda > 0$. See [5] or [10] for more details.

The *analytic wave front set* of an hyperfunction u in an open subset Ω of $\mathbb{R}^n$ is the subset $WF_a(u)$ of $T^*\Omega \setminus \{0\}$ defined by the condition $(x_0, \xi_0) \notin WF_a(u)$ if and only if there are $C, \varepsilon, r > 0$ and $v \in \mathcal{A}'(\mathbb{R}^n)$ that is equal to u near x_0 such that

$$|Tv(z, \lambda)| \le Ce^{\frac{\lambda}{2}(|\Im z|^2 - \varepsilon)}$$

if $|z - (x_0 - i\xi_0)| < r$ and $\lambda > 0$.

Lemma 2. *Let $u \in \mathcal{A}'(\mathbb{R}^n)$, $\rho > 0$ and $f \in \mathcal{O}(\{z \in \mathbb{C}^n : |\Im z| < \rho\})$. Assume that there are constants $C_k > 0$ such that $|f(z)| \le C_k(1 + |z|)^{-k}$. Then*

$$\begin{aligned} u(f) \;=\; & \frac{2^{-n-1}}{\pi^{3n/2}} \int_{\mathbb{R}^n} |\xi|^{n/2} e^{-|\xi|}\, d\xi \\ & \cdot \int_{\mathbb{R}^n} Tu(x - i\frac{\xi}{|\xi|}, |\xi|)(1 + i\frac{\xi.D_x}{|\xi|^2})\, Tf(x + i\frac{\xi}{|\xi|}, |\xi|)\, dx. \end{aligned}$$

The same formula holds with the differential operator $1 - i\xi.D_x/|\xi|^2$ acting on Tu and no derivatives on Tf.

Proof. Using the definition of the FBI transform, we get

$$\int_{\mathbb{R}^n} Tu(x - i\frac{\xi}{|\xi|}, |\xi|)(1 + i\frac{\xi.D_x}{|\xi|^2}) Tf(x + i\frac{\xi}{|\xi|}, |\xi|)\, dx$$

$$
\begin{aligned}
&= u_{(y)}(\int_{\mathbb{R}^n} f(w)\, dw \\
&\qquad \cdot \int_{\mathbb{R}^n} e^{-\frac{|\xi|}{2}(x-i\frac{\xi}{|\xi|}-y)^2-\frac{|\xi|}{2}(x+i\frac{\xi}{|\xi|}-w)^2}(2-\frac{i(x-w).\xi}{|\xi|})\, dx) \\
&= 2\pi^{n/2}|\xi|^{-n/2}e^{|\xi|}u_{(y)} \\
&\qquad \cdot(\int_{\mathbb{R}^n} f(w)e^{-i(y-w).\xi-\frac{|\xi|}{4}|y-w|^2}(1-\frac{i(y-w).\xi}{4|\xi|})\, dw).
\end{aligned}
$$

For $s > 0$ small, the integration path $w \mapsto w + is\xi/|\xi|$ shows that the inner integral is exponentially decreasing with respect to ξ. Hence, using the complex shift $\xi \to \xi - i|\xi|(y-w)/4$, we get

$$
\frac{2^{-n-1}}{\pi^{3n/2}}\int_{\mathbb{R}^n}|\xi|^{n/2}e^{-|\xi|}\, d\xi \int_{\mathbb{R}^n} Tu(x-i\frac{\xi}{|\xi|},|\xi|)(1+i\frac{\xi.D_x}{|\xi|^2})Tf(x+i\frac{\xi}{|\xi|},|\xi|)\, dx.
$$

$$
= (2\pi)^{-n}u_{(y)}(\int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} f(w)e^{-i(y-w).(\xi-\frac{i|\xi|}{4}(y-w))}(1-\frac{i(y-w).\xi}{4|\xi|})\, dw)
$$

$$
= (2\pi)^{-n}u_{(y)}((\mathcal{F}^-(\mathcal{F}^+f)(y)) = u(f)
$$

where $\mathcal{F}^{\pm}$ is the Fourier transform. □

We can now give the general definition of the operator S.

THEOREM 3. *Let $u \in \mathcal{A}'(\mathbb{R}^n \times \mathbb{R}^p)$, $g \in \mathcal{B}(\mathbb{R}^p)$ and let $K \times M$ be a compact set containing the support of u. Assume that $(x, y, 0, -\eta) \notin WF_a(u)$ for all $x \in \mathbb{R}^n$, $(y, \eta) \in WF_a(g)$ and consider $(Sg)(f)$ given by*

$$
\begin{aligned}
&\frac{2^{-p-1}}{\pi^{3p/2}}\int_{\mathbb{R}^p}|\eta|^{p/2}e^{-|\eta|}\, d\eta \\
&\qquad \cdot\int_{\mathbb{R}^p} Tg_0(y-i\frac{\eta}{|\eta|},|\eta|)\,(1+i\frac{\eta.D_y}{|\eta|^2})\,T({}^tSf)(y+i\frac{\eta}{|\eta|},|\eta|)\, dy
\end{aligned}
$$

for every $g_0 \in \mathcal{A}'(\mathbb{R}^p)$ that is equal to g in a neighbourhood of M and every f that is analytic near K.

Then the map $f \mapsto (Sg)(f)$ is independent of g_0 and is an analytic functional supported by K. Moreover, if g is analytic near M then $(Sg)(f) = u(f \otimes g)$.

PROOF. We first prove that the integral converges and defines an analytic functional supported by K. For every $\rho > 0$, let $K_\rho = \{z \in \mathbb{C}^n : \mathrm{d}(z, K) < \rho\}$.

Choose a compact set V such that $g_0 = g$ near V and $\mathrm{supp}(u)$ is in the interior of $\mathbb{R}^n \times V$. For every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|Tg_0(y - i\frac{\eta}{|\eta|}, |\eta|)| \leq C_\varepsilon e^{\frac{|\eta|}{2}(1+\varepsilon)}.$$

On the other hand, by the choice of V there are constants $\delta, C_\rho > 0$ such that

$$|(1 + i\frac{\eta.D_y}{|\eta|^2})T({}^tSf)(y + i\frac{\eta}{|\eta|}, |\eta|)| \leq C_\rho \|f\|_{K_\rho}(1 + |y|)e^{\frac{|\eta|}{2}(1-\delta(1+|y|)^2)}$$

for every $y \notin V$ and $\rho > 0$. Since

$$\int_{\mathbb{R}^p} |\eta|^{p/2}\, d\eta \int_{\mathbb{R}^p} e^{-\delta(1+|y|^2)|\eta|}(1 + |y|)\, dy < \infty,$$

for every $\delta > 0$, the integral extended to $\mathbb{R}^p \setminus V \times \mathbb{R}^p$ defines an analytic functional supported by K.

It remains to estimate the integral in a conic neighbourhood of every point $(y_0, \eta_0) \in V \times \mathbb{R}^p \setminus \{0\}$.

Assume that $(y_0, \eta_0) \in WF_a(g)$. Then $(x, y_0, 0, -\eta_0) \notin WF_a(u)$ for every x and from the composition properties of the analytic wave front set, we have $(y_0, -\eta_0) \notin WF_a({}^tSf)$. As in [1], we can strengthen this property. For every $m \in \mathbb{N}_0$, let

$$E_m = \{v \in \mathcal{A}'(M) : q_m(v) < \infty\}$$

with

$$q_m(v) = \sup_{|w-w_0| \leq 1/m, \lambda > 0} e^{-\frac{\lambda}{2}|\mathcal{I}w|^2 + \frac{\lambda}{m}}|Tv(w, \lambda)|$$

and $w_0 = y_0 + i\eta_0$. Endowed with the Fréchet semi-norms of $\mathcal{A}'(M)$ and with q_m, the linear space E_m is a Fréchet space. It follows that the subspace $E = \cup_m E_m$ of $\mathcal{A}'(M)$ has a strict web in which the sets with one index are the E_m. For every $\rho > 0$, ${}^tS$ maps $\mathcal{O}(K_\rho)$ in E and has a sequentially closed graph. Two applications of the localization theorem (see [2] or theorem 6 of [1]), show that there are constants $C, \delta > 0$ and $\rho' \in]0, \rho[$ such that

$$|T({}^tSf)(w, \lambda)| \leq C\|f\|_{K_{\rho'}} e^{\frac{\lambda}{2}(|\mathcal{I}w|^2 - \delta)}$$

for every $f \in \mathcal{O}(K_\rho)$, $\lambda > 0$ and $|w - w_0| < \delta$. Using the Cauchy's inequalities, we get

$$|(1+i\frac{\eta.D_y}{|\eta|^2})T(^tSf)(y+i\frac{\eta}{|\eta|},|\eta|)| \leq \frac{C'}{|\eta|}\|f\|_{K_{\rho'}}e^{\frac{|\eta|}{2}(1-\delta/2)}$$

in a conic neighbourhood of (y_0, η_0). Since

$$|Tg_0(y-i\frac{\eta}{|\eta|},|\eta|)| \leq C_\varepsilon e^{\frac{|\eta|}{2}(1+\varepsilon)}$$

we conclude in a neighbourhood of (y_0, η_0).

In the same way, if $(y_0, \eta_0) \notin WF_a(g)$ then Tg_0 is exponentially decreasing in a conic neighbourhood of this point and the other factor can be estimated by continuity.

Let us show that the definition of Sg is independent of g_0. We have to show that the right hand side is equal to 0 if $\mathrm{supp}(g_0) \cap M = \emptyset$. If $\varepsilon > 0$, consider

$$g_\varepsilon(x) = (2\pi\varepsilon)^{-p/2} g_{0(y)}(e^{-\frac{1}{2\varepsilon}|x-y|^2}).$$

By lemma 2, we get

$$\begin{aligned} u(f \otimes g_\varepsilon) &= (^tSf)(g_\varepsilon) \\ &= \frac{2^{-p-1}}{\pi^{3p/2}} \int_{\mathbb{R}^p} |\eta|^{p/2} e^{-|\eta|}\, d\eta \\ &\quad \cdot \int_{\mathbb{R}^p} Tg_\varepsilon(y-i\frac{\eta}{|\eta|},|\eta|)\,(1+i\frac{\eta.D_y}{|\eta|^2})\,T(^tSf)(y+i\frac{\eta}{|\eta|},|\eta|)\,dx. \end{aligned}$$

This expression converges to 0 with ε since $f \otimes g_\varepsilon$ uniformly converges to 0 in a complex neighbourhood of $\mathrm{supp}(u)$. Let us show that the integral converges to the right hand side in the definition. Choose a compact neighbourhood W of M such that $\mathrm{supp}(g_0) \cap W = \emptyset$. Since

$$Tg_\varepsilon(z,\lambda) = (1+\varepsilon\lambda)^{-p/2} Tg_0(z, \frac{\lambda}{1+\varepsilon\lambda}),$$

we have the estimation

$$|Tg_\varepsilon(y-i\frac{\eta}{|\eta|},|\eta|)| \leq C_\delta e^{\frac{|\eta|}{2}(1+\delta)}.$$

There are constants $C, \rho > 0$ such that

$$|(1 + i\frac{\eta.D_y}{|\eta|^2})\, T({}^tSf)(y + i\frac{\eta}{|\eta|}, |\eta|)| \leq C(1+|y|)e^{\frac{|\eta|}{2}(1-\rho(1+|y|^2))}.$$

if $y \notin W$. So we can conclude in this case. In the same way, for the integral on W, we use the estimates

$$|Tg_\varepsilon(y - i\frac{\eta}{|\eta|}, |\eta|)| \leq Ce^{\frac{|\eta|}{2(1+\varepsilon|\eta|)}(1-\rho)} \leq Ce^{\frac{|\eta|}{2}(1-\rho)}$$

and

$$|(1 + i\frac{\eta.D_y}{|\eta|^2})\, T({}^tSf)(y + i\frac{\eta}{|\eta|}, |\eta|)| \leq C_\delta(1+|y|)e^{\frac{|\eta|}{2}(1+\delta)}.$$

Finally, assume that g is analytic in a neighbourhood of M and apply lemma 2 to $({}^tSf)(g_\varepsilon)$ with

$$g_\varepsilon(x) = (2\pi\varepsilon)^{-p/2}\int_\omega g(y)e^{-\frac{1}{2\varepsilon}|x-y|^2}\,dy$$

where ω is a relatively compact open set such that $M \subset \omega$ and g is analytic near $\overline{\omega}$. The function that is equal to 1 in ω and 0 outside is denoted by χ_ω. Using a complex deformation to evaluate the integral over ω, we obtain the estimate

$$\begin{aligned} |Tg_\varepsilon(y - i\frac{\eta}{|\eta|}, |\eta|)| &= (1+\epsilon|\eta|)^{-p/2}|\int_\omega g(x)e^{-\frac{|\eta|}{2(1+\varepsilon|\eta|)}(y-i\frac{\eta}{|\eta|}-x)^2}\,dx| \\ &\leq Ce^{\frac{|\eta|}{2(1+\varepsilon|\eta|)}(1-\delta)} \leq Ce^{\frac{|\eta|}{2}(1-\delta)} \end{aligned}$$

for some $\delta > 0$ and y near M. Hence, as before we can say that $u(f \otimes g_\varepsilon)$ converges to $S(g\chi_\omega)(f) = (Sg)(f)$ when ε goes to 0. Moreover from lemma 9.1.2 of [4], it follows that g_ε uniformly converges to g in a complex neighbourhood of M. This proves that theorem 3 extends the previous definition of S. $\square$

3. Proof of theorem 1

We may assume that $\xi_0 \neq 0$, $\eta_0 \neq 0$ and $(x_0, y_0) \in \mathrm{supp}(u)$. Let $B(r)$ be an open ball such that $\mathrm{supp}(u) \subset \mathbb{R}^n \times B(r)$ and $V = B(R)$ with $R > r$. Let C_m, $m \in \mathbb{N}_0$, be a fundamental sequence of compact sets of

$$C = \{w \in \mathbb{C}^p : |\Re w| \leq r,\ |\Im w| \geq 1\} \setminus \{y_0 - it\eta_0 : t > 0\}$$

and $w_0 = y_0 + i\eta_0$. There is a sequence $\delta_m > 0$ such that

$$\lambda\mu|\Re w - \Re w'|^2 + |\lambda\Im w + \mu\Im w'|^2 \geq \delta_m(\lambda+\mu)^2$$

if $w' \in C_m$, $\lambda, \mu > 0$ and $|w - w_0| \leq \delta_m$. Choose a decreasing sequence r_m such that $0 < r_m < \delta_m$ and consider

$$F = \{g \in \mathcal{A}'(\overline{V}) : q_m(g) < \infty \text{ for every } m \geq 1\}$$

with

$$q_m(g) = \sup_{w \in C_m, \mu > 0} |Tg(w,\mu)| e^{-\frac{\mu}{2}(|\Im w|^2 - r_m)}.$$

As the balls $\{g \in \mathcal{A}'(\overline{V}) : q_m(g) \leq r\}$ are closed in $\mathcal{A}'(\overline{V})$ for every r and m, the space F endowed with the semi-norms induced by $\mathcal{A}'(\overline{V})$ and the semi-norms q_m is a Fréchet space. Moreover, if $g \in F$ then $WF_a(g) \cap (B(r) \times \mathbb{R}^p) \subset \{(y_0, t\eta_0) : t > 0\}$.

Let K be the projection of $\mathrm{supp}(u)$ on $\mathbb{R}^n$ and $E_m = \{f \in \mathcal{A}'(K) : p_m(f) < +\infty\}$ with

$$p_m(f) = \sup_{|z - z_0| < 1/m, \lambda > 0} e^{-\frac{\lambda}{2}|\Im z|^2} |Tf(z,\lambda)|$$

and $z_0 = x_0 - i\xi_0$. For every $m \in \mathbb{N}_0$, E_m is a Fréchet space for the semi-norms induced by $\mathcal{A}'(K)$ and p_m. Hence, it has a strict web for which the sets with one index are closed balls of p_m. Let $E = \cup_{m=1}^{\infty} E_m$ with the topology induced by $\mathcal{A}'(K)$. This space has a strict web for which the sets with one index are the spaces E_m.

The operator S maps F into E. Indeed, since the sheaf of microfunctions is flabby hence supple, any $g \in F$ can be written $g = g_0 + g_1$ with $g_0, g_1 \in \mathcal{B}(\mathbb{R}^p)$ and

$$WF_a(g_0) \subset \{(y_0, t\eta_0) : t > 0\} \ , \ \ WF_a g_1 \subset \overline{V} \setminus B(r) \times \mathbb{R}^p.$$

By hypothesis, we have $Sg_0 \in E$. On the other hand, by the main result of [1], the point $((x_0, y), (\xi_0, 0))$ does not belong to $WF_a(u)$ for every $y \in \mathbb{R}^p$. Since g_1 is analytic near the projection of $\mathrm{supp}(u)$, it follows that $(x_0, \xi_0) \notin WF_a(Sg_1)$.

Using the explicit expression of Sg given in the previous paragraph, it is easily seen that $S : F \to E$ is linear and has a sequentially closed graph.

With two applications of the localization theorem of [2], we get $C, \varepsilon > 0$, $m \in \mathbb{N}_0$ and a neighbourhood W of $\overline{V}$ in $\mathbb{C}^p$ such that

$$|T(Sg)(z,\lambda)| \leq Ce^{\frac{\lambda}{2}|\Im z|^2}(q_m(g) + \sup_{h\in\mathcal{O}(\mathbb{C}^p),\|h\|_W\leq 1} |g(h)|)$$

if $|z - z_0| < \varepsilon$, $\lambda > 0$ and $g \in F$.

With

$$f_{\lambda,z}(x) = e^{-\lambda(z-x)^2/2} \quad , \quad g_{\lambda,w}(y) = e^{-\lambda(w-y)^2/2}$$

we have

$$Tu(z,w,\lambda) = T(Sg_{\lambda,w})(z,\lambda).$$

It follows that

$$|Tu(z,w,\lambda)| \leq Ce^{\frac{\lambda}{2}|\Im z|^2}(q_m(g_{\lambda,w}\chi_V) + \sup_{\|h\|_W\leq 1} |\int_V g_{\lambda,w}(y)h(y)\,dy|)$$

if $|z - z_0| < \varepsilon$.

We have to show that the right hand side is exponentially decreasing with respect to the weight $\frac{\lambda}{2}(|\Im z|^2 + |\Im w|^2)$ near (z_0, w_0).

We first evaluate $q_m(g_{\lambda,w}\chi_V)$. We have

$$\begin{aligned} T(g_{\lambda,w}\chi_V)(w',\mu) &= (\frac{2\pi}{\lambda+\mu})^{n/2}e^{-\frac{\lambda\mu}{2(\lambda+\mu)}(w-w')^2} \\ &\quad -\int_{\mathbb{R}^p\setminus V} e^{-\frac{\mu}{2}(w'-y)^2-\frac{\lambda}{2}(w-y)^2}\,dy. \end{aligned}$$

In the estimation of $Tu(z,w,\lambda)$, the first term of the previous equality gives the exponential of

$$\frac{\lambda}{2}|\Im z|^2 - \frac{\mu}{2}(|\Im w'|^2 - r_m) - \frac{\lambda\mu}{2(\lambda+\mu)}|\Re w - \Re w'|^2 + \frac{\lambda\mu}{2(\lambda+\mu)}|\Im w - \Im w'|^2$$

$$= \frac{\lambda}{2}(|\Im z|^2 + |\Im w|^2) - \frac{\lambda\mu}{2(\lambda+\mu)}|\Re w - \Re w'|^2 - \frac{|\lambda\Im w + \mu\Im w'|^2}{2(\lambda+\mu)} + \frac{r_m\mu}{2}$$

$$\leq \frac{\lambda}{2}(|\Im z|^2 + |\Im w|^2) - \frac{\delta_m}{2}(\lambda+\mu) + \frac{r_m\mu}{2}$$

if $|w - w_0| < \delta_m$, $w' \in C_m$ and $\mu > 1$. This term is exponentially decreasing with respect to λ. Consider now the integral over $\mathbb{R}^p\setminus V$. Since $|\Re w|, |\Re w'| \leq r$ and $|y| > R$, the exponent can be estimated by

$$\frac{\lambda}{2}|\Im w|^2 + \frac{\mu}{2}|\Im w'|^2 - \frac{\lambda+\mu}{2}(R-r)^2.$$

If $r_m < (R-r)^2$, this shows that this term is also exponentially decreasing.

Finally, using a complex shift $y \to y + it\Im w\chi(y)$ where $\chi \in C_0^\infty(V)$ is equal to 1 in a neighbourhood of $\Re w_0$ and $t > 0$ is small, we get the estimate

$$\sup_{\|h\|_W \leq 1} \left| \int_V g_{\lambda,w}(y)h(y)\,dy \right| \leq C' e^{\frac{\lambda}{2}(|\Im w|^2 - \varepsilon')}.$$

This proves the theorem. □

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